

Categoricity transfer in Simple Finitary Abstract Elementary Classes

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August 15, 2008

Abstract

We continue to study finitary abstract elementary classes, defined in [7]. We introduce a concept of weak κ -categoricity and an f-primary model in a \aleph_0 -stable simple finitary AEC with the extension property, and gain the following theorem: Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be a simple finitary AEC, weakly categorical in some uncountable κ . Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ is weakly categorical in each $\lambda \geq \min\{\kappa, \beth_{(2^{\aleph_0})^+}\}$. We have that if the class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is also LS(\mathbb{K})-tame, weak κ -categoricity is equivalent with κ -categoricity in the usual sense.

We also discuss the relation between finitary AECs and some other non-elementary frameworks and give several examples.

1 Introduction

This paper continues to study finitary abstract elementary classes, which were studied before in the papers Hyttinen, Kesälä [7], [9]. Especially, we give a categoricity transfer theorem for these classes with the additional assumption of simplicity. A previous version of this paper is published in the Ph.D. thesis by Kesälä [8]. However, after we learned the results in Kueker [14], we added to this paper a thorough account on the connections between our framework and *atomic AECs*, see section 2.3. We also added a section with examples of different classes definable in $L_{\omega_1\omega}$, clarifying the relations between properties such as excellence, homogeneity, tameness and simplicity.

The framework of *finitary abstract elementary classes* refines the framework of AECs defined by Shelah in [22]. A finitary class is an abstract elementary class with joint embedding, amalgamation over models, arbitrarily large models, countable downward Löwenheim-Skolem number and *finite character*. The novel property is finite character, which gives a finitary condition for the \mathbb{K} -elementary submodel relation $\preceq_{\mathbb{K}}$ for two models $\mathcal{A} \subseteq \mathcal{B}$ in an abstract elementary class $(\mathbb{K}, \preceq_{\mathbb{K}})$: $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$ if and only if for each finite tuple $\bar{a} \in \mathcal{A}$ the Galois type of \bar{a} calculated in \mathcal{A} is the same as the Galois type calculated in \mathcal{B} . Finite character holds if $\preceq_{\mathbb{K}}$ is given by any fragment of the formal language $L_{\infty\omega}$. Furthermore,

*Partially supported by the Academy of Finland, grant 1106753.

†Supported by the graduate school MALJA, Academy of Finland, grant 119813 and the Austrian Science Fund FWF

2000 Mathematics subject classification: Primary 03C45; Secondary 03C52.

our framework generalizes both homogeneous and excellent classes. Due to finite character, many of the tools present in these classes generalize to our framework. Our original motivation was to concentrate on *finite dependencies* for abstract elementary classes to understand the *geometric* properties of non-elementary classes of structures. That is why we decided to focus on *weak types* and *simplicity* implying an independence calculus over arbitrary sets and not only models. One main motivation were the geometric studies for homogeneous and excellent classes in Hyttinen, Lessmann and Shelah [12].

In addition of studying a finitary class $(\mathbb{K}, \preceq_{\mathbb{K}})$, we study the class $(\mathbb{K}^{\omega}, \preceq_{\mathbb{K}})$, which is the class of \aleph_0 -saturated models of \mathbb{K} . Under \aleph_0 -stability, these classes are equal if and only if the class \mathbb{K} is \aleph_0 -categorical. The class \mathbb{K}^{ω} is usually even better behaved: splitting works in \mathbb{K}^{ω} and we get the full Morley theorem for a simple, tame \mathbb{K}^{ω} . Furthermore, Kueker [14] shows that \mathbb{K}^{ω} is definable with a sentence in $L_{\omega_1\omega}$, which makes this class very close to the framework of \aleph_0 -stable atomic AECs, previously studied by Shelah and Baldwin, see the book [1]. What exactly is the relation, is clarified in section 2.3. For example, we can see that splitting is the same notion in both frameworks, but some properties of splitting, mainly the extension property, are formulated differently, which turns out to be crucial. We also study the relation between \mathbb{K} and \mathbb{K}^{ω} with concrete examples in section 6.

We work with two notions of independence: the notion \downarrow^s is based on splitting and the notion \downarrow is based on Lascar-splitting and has a built-in extension property. Good behaviour for \downarrow^s follows from \aleph_0 -stability but only over \aleph_0 -saturated models. The additional assumption of simplicity gives good behaviour for the notion \downarrow over *arbitrary sets*, and we show in Theorem 4.9 that this is the only possible notion satisfying certain properties over sets. It is important to keep in mind which concepts in this paper refer to \downarrow^s and which refer to \downarrow : U -rank, Morley-sequence and the extension property refer to \downarrow^s and simplicity and f -isolation refer to \downarrow . We also study the relation between these notions: already in [7] we showed that \aleph_0 -stability, extension property and finite U -rank imply simplicity. Now we give some more thought to the extension property for non-splitting in section 4.2 and show that it is implied by categoricity and simplicity (Proposition 4.26). We also know it follows from tameness, but we would be interested if it would follow already from \aleph_0 -stability or simplicity. A weaker version is used for atomic AECs and this version follows already from \aleph_0 -stability, but this version is not enough for our theory. The extension property for non-splitting implies that \downarrow^s and \downarrow agree over \aleph_0 -saturated models. In the superstable case [9] we cannot use splitting and hence only use the notion \downarrow .

We introduce two notions of isolation in \aleph_0 -stable finitary classes with the extension property. The notion of *weak isolation* gives a notion of a primary model of the form $\mathcal{A}[\bar{a}]$, which is a constructible model over a tuple \bar{a} and a countable \aleph_0 -saturated model \mathcal{A} . These models are used to study properties of the U -rank. In Proposition 3.17 we show that if both $U(\bar{a}/\mathcal{A})$ and $U(\bar{b}/\mathcal{A})$ are finite for a countable \aleph_0 -saturated model \mathcal{A} , then

$$U(\bar{a}\bar{b}/\mathcal{A}) \leq U(\bar{a}/\mathcal{A}) + U(\bar{b}/\mathcal{A}).$$

The notion of *f -isolation* gives f -primary models. Simplicity implies that there are f -primary models over sets of the form $\mathcal{A} \cup B$, where \mathcal{A} is an \aleph_0 -saturated model and B is an arbitrary set. These models are used in the proof of the categoricity transfer theorem.

We introduce also a concept of weak κ -categoricity, which means that each model of size κ is weakly saturated, and show the following theorem (Theorem 5.11):

Theorem 1.1 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple finitary AEC and weakly categorical in some uncountable cardinal κ . Then*

1. $((\mathbb{K})^\omega, \preceq_{\mathbb{K}})$ is weakly categorical in each uncountable κ and
2. $(\mathbb{K}, \preceq_{\mathbb{K}})$ is weakly categorical in each λ such that $\lambda \geq \min\{\kappa, \beth_{(2^{\aleph_0})^+}\}$.

Weak categoricity is always implied by categoricity in finitary AECs, but we need *tameness* to show that the two notions are equal. As a corollary we get the following (Corollary 5.14):

Corollary 1.2 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple LS(\mathbb{K})-tame finitary AEC categorical in some uncountable κ . Then*

1. $((\mathbb{K})^\omega, \preceq_{\mathbb{K}})$ is categorical in each uncountable κ and
2. $(\mathbb{K}, \preceq_{\mathbb{K}})$ is categorical in each λ such that $\lambda \geq \min\{\kappa, \beth_{(2^{\aleph_0})^+}\}$.

The last corollary is another partial result towards a categoricity transfer for abstract elementary classes. Previous results for AECs is the very general theorem by Shelah [23] for AECs with amalgamation and joint embedding and the transfer theorems from lower cardinals with an additional assumption of tameness by Grossberg and VanDieren [4], [5], [3] and Lessmann [18]. All of these assume the categoricity cardinal to be a successor. We do not need such an assumption.

The theorem can also be seen as a categoricity transfer theorem for $L_{\omega_1\omega}$. The theorem for an arbitrary sentence in $L_{\omega_1\omega}$ by Shelah in [21] assumes categoricity in infinitely many cardinals \aleph_n for $n < \omega$ and also needs some set-theoretical assumptions. The full Morley theorem holds for sentences in $L_{\omega_1\omega}$ which are either *homogeneous* (see Keisler [13] or Shelah [19]) or *excellent* (see Shelah [21], Lessmann [16] or Baldwin [1]). The only found counterexamples for categoricity transfer for sentences $L_{\omega_1\omega}$ with amalgamation, joint embedding and arbitrary large models are given by the construction found by Hart and Shelah [6] and further studied by Baldwin and Kolesnikov [2]: the examples are categorical in $\aleph_0, \dots, \aleph_k$ for a given $k < \omega$, but not in \aleph_{k+1} . These examples are not tame. In section 6 we show that they are neither simple, but have the extension property for non-splitting.

Since the full Morley theorem holds also in the homogeneous framework and for excellent atomic AECs, it is reasonable to ask what is the relation between these theorems. Although tame, finitary AECs include both the homogeneous and excellent framework, our theorem assumes also *simplicity*, which makes our theorem incomparable. We discuss the relations between different frameworks with some examples in section 6. For an atomic AEC, our theorem gives one method to determine it's excellence: categoricity in a single uncountable cardinal transfers to all uncountable cardinals and hence under certain set-theoretic assumptions the class will be excellent, see Corollary 5.15.

At the moment there do not exist a resourced library of examples of abstract elementary classes, which makes the study on the relations between different properties very difficult. We do not have an example of a simple, \aleph_0 -stable and \aleph_0 -categorical finitary class which would not be excellent, or even one that would not be tame. We do have examples of non-excellence when we drop either the \aleph_0 -stability or the \aleph_0 -categoricity assumption, see examples 6.21 and 6.4. However, showing for example that every \aleph_0 -stable, \aleph_0 -categorical simple finitary class would be excellent would involve much better understanding of the model theory of finitary AECs. Furthermore, there are different possible notions for 'excellence', see the discussion after Question 6.28.

All the examples in section 6 are definable with a single sentence in $L_{\infty\omega}$, usually even a sentence in $L_{\omega_1\omega}$, but it is an open question whether a finitary class always is. Kueker

[14] shows that a finitary class is always closed under $L_{\infty\omega}$ -equivalence. For more discussion and open questions on definability issues, see Kueker [14].

2 Finitary abstract elementary classes

Let τ be a countable vocabulary. We recall the definitions of an abstract elementary class, amalgamation, joint embedding and finite character.

Definition 2.1 A class of τ -structures $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an abstract elementary class if

1. Both \mathbb{K} and the binary relation $\preceq_{\mathbb{K}}$ are closed under isomorphism.
2. If $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$, then \mathcal{A} is a substructure of \mathcal{B} .
3. $\preceq_{\mathbb{K}}$ is a partial order on \mathbb{K} .
4. If $\langle \mathcal{A}_i : i < \delta \rangle$ is an $\preceq_{\mathbb{K}}$ -increasing chain:
 - (a) $\bigcup_{i < \delta} \mathcal{A}_i \in \mathbb{K}$;
 - (b) for each $j < \delta$, $\mathcal{A}_j \preceq_{\mathbb{K}} \bigcup_{i < \delta} \mathcal{A}_i$
 - (c) if each $\mathcal{A}_i \preceq_{\mathbb{K}} \mathcal{M} \in \mathbb{K}$, then $\bigcup_{i < \delta} \mathcal{A}_i \preceq_{\mathbb{K}} \mathcal{M}$.
5. If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$, $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{C}$, $\mathcal{B} \preceq_{\mathbb{K}} \mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$.
6. There is a Löwenheim-Skolem number $\text{LS}(\mathbb{K})$ such that if $\mathcal{A} \in \mathbb{K}$ and $B \subset \mathcal{A}$ a subset, there is $\mathcal{A}' \in \mathbb{K}$ such that $B \subset \mathcal{A}' \preceq_{\mathbb{K}} \mathcal{A}$ and $|\mathcal{A}'| = |B| + \text{LS}(\mathbb{K})$.

When $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$, we say that \mathcal{B} is an \mathbb{K} -extension of \mathcal{A} and \mathcal{A} is an \mathbb{K} -submodel of \mathcal{B} . If $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ and $f : \mathcal{A} \rightarrow \mathcal{B}$ an embedding such that $f(\mathcal{A}) \preceq_{\mathbb{K}} \mathcal{B}$, we say that f is a \mathbb{K} -embedding.

Definition 2.2 (Amalgamation) We say that $(\mathbb{K}, \preceq_{\mathbb{K}})$ has the amalgamation property, if it satisfies the following:

If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$, $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$, $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{C}$ and $\mathcal{B} \cap \mathcal{C} = \mathcal{A}$, there is $\mathcal{D} \in \mathbb{K}$ and a map $f : \mathcal{B} \cup \mathcal{C} \rightarrow \mathcal{D}$ such that $f \upharpoonright \mathcal{B}$ and $f \upharpoonright \mathcal{C}$ are \mathbb{K} -embeddings.

Definition 2.3 (Joint embedding) We say that $(\mathbb{K}, \preceq_{\mathbb{K}})$ has the joint embedding property if for every $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ there is $\mathcal{C} \in \mathbb{K}$ and \mathbb{K} -embeddings $f : \mathcal{A} \rightarrow \mathcal{C}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$.

To define finite character we use the following concept of \mathcal{A} -Galois type.

Definition 2.4 (\mathcal{A} -Galois type) For $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ and $\bar{a} \in \mathcal{A}, \bar{b} \in \mathcal{B}$ we say

$$\text{tp}^{\mathfrak{g}}(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^{\mathfrak{g}}(\bar{b}/\emptyset, \mathcal{B})$$

if there is $\mathcal{C} \in \mathbb{K}$ and \mathbb{K} -embeddings $f : \mathcal{A} \rightarrow \mathcal{C}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$ such that $f(\bar{a}) = g(\bar{b})$.

With finite character, we can decide whether a model is a \mathbb{K} -submodel of another model by only looking at all finite parts of it.

Definition 2.5 (Finite character) We say that an AEC $(\mathbb{K}, \preceq_{\mathbb{K}})$ has finite character, if it satisfies the following: If, $\mathcal{A}, \mathcal{B} \in \mathbb{K}$, $\mathcal{A} \subset \mathcal{B}$, and for each finite $\bar{a} \in \mathcal{A}$ we have that $\text{tp}^{\mathfrak{g}}(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^{\mathfrak{g}}(\bar{a}/\emptyset, \mathcal{B})$, then $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$.

Finite character implies that if $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$ and $f : \mathcal{A} \rightarrow \mathcal{B}$ is a mapping, then f is a \mathbb{K} -embedding if and only if

$$\text{tp}^{\mathfrak{g}}(\bar{a}/\emptyset, \mathcal{B}) = \text{tp}^{\mathfrak{g}}(f(\bar{a})/\emptyset, \mathcal{B})$$

for each $\bar{a} \in \mathcal{A}$.

In [7], the authors defined a finitary abstract elementary class to be an abstract elementary class with arbitrary large models, $\text{LS}(\mathbb{K})$ being \aleph_0 , disjoint amalgamation, a prime model over \emptyset and finite character. In this paper we study a class with weaker assumptions, but still call it a finitary abstract elementary class, since all the main theorems of [7] hold for this class also. For completeness, we will reprove some of the results with the weaker assumptions, but skip those proofs which will be exactly the same. The main difference is that we are not able to construct a monster model with as good homogeneity properties as in [7]. For more details, see the thesis [8].

Definition 2.6 (Finitary abstract elementary class) We say that an abstract elementary class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is finitary, if it satisfies the following:

1. $\text{LS}(\mathbb{K}) = \aleph_0$.
2. $(\mathbb{K}, \preceq_{\mathbb{K}})$ has arbitrarily large models,
3. $(\mathbb{K}, \preceq_{\mathbb{K}})$ has the amalgamation property,
4. $(\mathbb{K}, \preceq_{\mathbb{K}})$ has the joint embedding property and
5. $(\mathbb{K}, \preceq_{\mathbb{K}})$ has finite character.

From now on we will assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a finitary abstract elementary class. We will also consider \aleph_0 -stability and the extension property. Both of these will be implied by κ -categoricity above the Hanf number H of abstract elementary classes with $\text{LS}(\mathbb{K}) = \aleph_0$. We have that $\text{H} = \beth_{(2^{\aleph_0})^+}$.

In [7], we proved a stronger version of the representation theorem of Shelah's using disjoint amalgamation and prime model. Here we use a version which is a special case of the original one in [22]. The reader should look for details in [7] or [24].

Definition 2.7 For $n, k < \omega$, let F_n^k be a k -ary function symbol, $\tau^* = \tau \cup \{F_n^k : n, k < \omega\}$ and \mathbb{K}^* be the class of all τ^* -structures such that for $\mathcal{A} \in \mathbb{K}^*$:

1. $\mathcal{A} \upharpoonright_{\tau} \in \mathbb{K}$,
2. For all $\bar{a} \in \mathcal{A}$, $\mathcal{A}_{\bar{a}} = \{(F_n^{lg(\bar{a})})^{\mathcal{A}}(\bar{a}) : 0 < n < \omega\}$, is such that
 - (a) $\mathcal{A}_{\bar{a}} \in \mathbb{K}$ and $\mathcal{A}_{\bar{a}} \preceq_{\mathbb{K}} \mathcal{A} \upharpoonright_{\tau}$,
 - (b) if $\bar{b} \subset \bar{a}$ then $\bar{b} \in \mathcal{A}_{\bar{a}} \subset \mathcal{A}_{\bar{a}}^1$.

¹Here $\bar{b} \subset \bar{a}$ means that $lg(\bar{b}) \leq lg(\bar{a})$ and the members of the tuple \bar{b} are contained in the set of members of \bar{a} , i.e. when $\bar{b} = (b_0, \dots, b_k)$ and $\bar{a} = (a_0, \dots, a_n)$, $\{b_0, \dots, b_k\} \subset \{a_0, \dots, a_n\}$.

Lemma 2.8 (Shelah) *If $\mathcal{A} \in \mathbb{K}^*$ and $B \subset \mathcal{A}$ a subset such that B is closed under functions F_n^k , then $B \upharpoonright \tau \in \mathbb{K}$ and $B \upharpoonright \tau \preceq_{\mathbb{K}} \mathcal{A} \upharpoonright \tau$.*

Lemma 2.9 (Shelah) *For every $\mathcal{A} \in \mathbb{K}$ there is $\mathcal{A}^* \in \mathbb{K}^*$ such that $\mathcal{A}^* \upharpoonright \tau = \mathcal{A}$. Furthermore, if $\mathcal{A}_0 \preceq_{\mathbb{K}} \mathcal{A}$ we can choose \mathcal{A}^* and \mathcal{A}_0^* such that \mathcal{A}_0^* is a τ^* -submodel of \mathcal{A}^* .*

Theorem 2.10 (Monster model) *Let μ be a cardinal. There is $\mathfrak{M} \in \mathbb{K}$ such that:*

1. **Universality:** \mathfrak{M} is μ -universal, that is for all $\mathcal{A} \in \mathbb{K}$, $|\mathcal{A}| < \mu$, there is a \mathbb{K} -embedding $f : \mathcal{A} \rightarrow \mathfrak{M}$.
2. **\mathbb{K} -homogeneity:** For all $\mathcal{A} \preceq_{\mathbb{K}} \mathfrak{M}$ such that $|\mathcal{A}| < \mu$ and mappings $f : \mathcal{A} \rightarrow \mathfrak{M}$ such that for all finite tuples $\bar{a} \in \mathcal{A}$

$$\text{tp}_{\mathfrak{M}}^{\mathfrak{g}}(\bar{a}/\emptyset) = \text{tp}_{\mathfrak{M}}^{\mathfrak{g}}(f(\bar{a})/\emptyset),$$

there is $g \in \text{Aut}(\mathfrak{M})$ extending f .

From the property 2 it follows that for any $\mathcal{A} \preceq_{\mathbb{K}} \mathfrak{M}$ an \mathbb{K} -embedding $f : \mathcal{A} \rightarrow \mathfrak{M}$ extends to an automorphism of \mathfrak{M} . We assume that we are in the monster model.

We know by Lemma 2.9 that $\mathfrak{M} = \mathfrak{M}^* \upharpoonright \tau$ for some \mathfrak{M}^* in the extended language τ^* , but unlike in [7], we are not able to prove that \mathfrak{M}^* would be a homogeneous model. For any model N of τ^* and a set $A \subset N$ denote by $SH^N(A)$ the closure of A under the functions of τ^* . For the monster model we abbreviate that $SH^{\mathfrak{M}^*}(A) = SH(A)$. By Lemma 2.8, for each subset $A \subset \mathfrak{M}$, always $SH(A) \upharpoonright \tau \preceq_{\mathbb{K}} \mathfrak{M}$. Also Lemma 2.9 gives that for any particular model $\mathcal{A} \preceq_{\mathbb{K}} \mathfrak{M}$ we may define \mathfrak{M}^* so that $SH(\mathcal{A}) = \mathcal{A}$. We will take advantage of this in Lemma 2.26. Thus we remark that although we fix a monster model \mathfrak{M} , there is no reason to fix any particular extension $\mathfrak{M}^* \in \mathbb{K}^*$.

We recall the definitions of Galois type and weak type from [7]. Galois type is the usual notion, except that we define it over arbitrary sets, not only models. Weak type has a built-in finite character.

Definition 2.11 (Galois type) *We write $\text{tp}^{\mathfrak{g}}(\bar{a}/A) = \text{tp}^{\mathfrak{g}}(\bar{b}/A)$ if there is $f \in \text{Aut}(\mathfrak{M}/A)$ such that $f(\bar{a}) = \bar{b}$.*

We have that the Galois type over \emptyset agrees with the type $\text{tp}^{\mathfrak{g}}(\bar{a}/\emptyset, \mathfrak{M})$ of definition 2.4.

Definition 2.12 (Weak type) *We write $\text{tp}^{\mathfrak{w}}(\bar{a}/A) = \text{tp}^{\mathfrak{w}}(\bar{b}/A)$ if $\text{tp}^{\mathfrak{g}}(\bar{a}/B) = \text{tp}^{\mathfrak{g}}(\bar{b}/B)$ for each finite $B \subset A$.*

Clearly weak type and Galois type agree over finite sets.

2.1 Indiscernible sequences and Ehrenfeucht-Mostowski models

In [7] we used the homogeneity of \mathfrak{M}^* to find suitable indiscernible sequences. The following Proposition does the same in this context. The construction is due to Shelah and a detailed proof of this formulation can be found in [8].

Proposition 2.13 (Shelah) *Let $(\bar{b}_i)_{i < \mathbb{H}}$ be a sequence of distinct tuples, let A be a countable set and let $(I, <)$ be a linear ordering. Then there is a sequence $(\bar{a}_i)_{i \in I}$ and for each suborder $J \subset I$ a model $EM(J \cup A) \in \mathbb{K}^*$, $EM(J \cup A) \upharpoonright \tau \preceq_{\mathbb{K}} \mathfrak{M}$, with the following properties*

1. *When $J \subset J' \subset I$,*

- (a) $A \cup (\bar{a}_i)_{i \in J} \subset EM(J \cup A)$,
- (b) *each element of $EM(J \cup A)$ is of the form $t(\bar{d})$, where t is a term of τ^* and $\bar{d} \in J \cup A$,*
- (c) $EM(J \cup A) \subset EM(J' \cup A)$ *a τ^* -submodel and*
- (d) $|EM(J \cup A)| = |J| + \aleph_0$.

2. *For every finite $i_0 < \dots < i_n$ there are $j_0 < \dots < j_n < \mathbb{H}$ such that*

$$\text{tp}^g(\bar{a}_{i_0}, \dots, \bar{a}_{i_n}/A) = \text{tp}^g(\bar{b}_{j_0}, \dots, \bar{b}_{j_n}/A).$$

Furthermore, there is a τ^ -isomorphism $f : EM(\bar{a}_{i_0}, \dots, \bar{a}_{i_n} \cup A) \rightarrow SH(\bar{b}_{j_0}, \dots, \bar{b}_{j_n} \cup A)$ over A mapping \bar{a}_{i_k} to \bar{b}_{j_k} for each $0 \leq k \leq n$.*

3. *If $J \subset I$ and $f : J \rightarrow I$ is order-preserving,*

$$\text{tp}^g((\bar{a}_i)_{i \in J}/A) = \text{tp}^g((\bar{a}_{f(i)})_{i \in J}/A).$$

Furthermore, there is a τ^ -isomorphism*

$$F : EM((\bar{a}_i)_{i \in J} \cup A) \rightarrow EM((\bar{a}_j)_{j \in f(J)} \cup A)$$

over A mapping \bar{a}_i to $\bar{a}_{f(i)}$ for each $i \in J$.

4. *For any linear order I' extending I we can extend the sequence to $(\bar{a}_i)_{i \in I'}$ with coherent models $EM(J \cup A)$, for suborders $J \subset I'$, with property 3.*

When \bar{a} is an n -tuple in $EM(I \cup A)$ as in the previous theorem, and $I = (i)_{i \in I} \subset \mathfrak{M}$ a linear order, we have that

$$\bar{a} = (t_1(i_0^1, \dots, i_{k_0}^1, \bar{a}_0), \dots, t_{n-1}(i_0^{n-1}, \dots, i_{k_{n-1}}^{n-1}, \bar{a}_{n-1})),$$

for some $\{i_0^1, \dots, i_{k_0}^1, \dots, i_0^{n-1}, \dots, i_{k_{n-1}}^{n-1}\} \subset I$ and $\{\bar{a}_0, \dots, \bar{a}_{n-1}\} \subset A$. We want to use a shorter notation and write

$$\bar{a} = \bar{t}(i_0, \dots, i_k, \bar{a}),$$

where $(i_0, \dots, i_k) = (i_0^1, \dots, i_{k_0}^1, \dots, i_0^{n-1}, \dots, i_{k_{n-1}}^{n-1})$ and $\bar{a} = (\bar{a}_0, \dots, \bar{a}_n)$. We say that \bar{a} is generated by the sequence \bar{t} of terms from (i_0, \dots, i_k) and \bar{a} .

We recall the definition of a strongly A -indiscernible sequence from [7].

Definition 2.14 (Strong indiscernibility) *We say that a sequence $(\bar{a}_i)_{i < \alpha}$ of tuples is strongly indiscernible over A , or strongly A -indiscernible, if for every ordinal $\lambda \geq \alpha$ there is a sequence $(\bar{a}_i)_{i < \lambda}$ extending $(\bar{a}_i)_{i < \alpha}$ such that for any order-preserving partial $f : \lambda \rightarrow \lambda$, there is $F \in \text{Aut}(\mathfrak{M}/A)$ such that $F(\bar{a}_i) = \bar{a}_{f(i)}$ for all $i \in \text{dom}(f)$.*

By Proposition 2.13 we easily see the following.

Proposition 2.15 Let $(\bar{b}_i)_{i < H}$ be a sequence of distinct tuples and let A be a countable set. There is a strongly A -indiscernible sequence $(\bar{a})_{i < \omega}$ such that for each $n < \omega$ there are $i_0 < \dots < i_n < H$ such that

$$\text{tp}^q(\bar{a}_0, \dots, \bar{a}_n/A) = \text{tp}^g(\bar{b}_{i_0}, \dots, \bar{b}_{i_n}/A).$$

Corollary 2.16 Assume that X is a set of k -tuples from \mathfrak{M} such that $|X| > H$, A a countable set and n is a natural number. There exists a strongly A -indiscernible sequence $(\bar{a}_i)_{i < \omega}$ such that $\{\bar{a}_0, \dots, \bar{a}_n\} \subset X$.

In the following we define a technical concept called a tidy sequence. The concept will help us to form indiscernible sequences of tuples, where each element of a tuple is contained in another indiscernible sequence.

Definition 2.17 (Tidy sequence) Let $i_0^\alpha < \dots < i_n^\alpha < \kappa$ for each $\alpha < \omega_1$, where κ is a cardinal. We say that the sequence $(i_0^\alpha, \dots, i_n^\alpha)_{\alpha < \omega_1}$ is tidy, if for each $0 \leq k \leq n$ one of the following holds.

1. The index at k is constant, that is, $i_k^\alpha = \beta < \kappa$ is fixed for each $\alpha < \omega_1$.
2. The index at k is included in some m -block, that is, $k \in \{p, p+1, \dots, p+m\}$ such that
 - (a) $p+m > n$ or for each $\beta < \omega_1$, we have $i_{p+m+1}^\beta \geq \sup\{i_{p+m}^\alpha : \alpha < \omega_1\}$,
 - (b) $p-i < 0$ or for each $\beta < \omega_1$, we have $i_{p-1}^\beta < \min\{i_p^\alpha : \alpha < \omega_1\}$ and
 - (c) for each $\alpha < \beta < \omega_1$, we have $i_p^\alpha < \dots < i_{p+m}^\alpha < i_p^\beta < \dots < i_{p+m}^\beta$.

We note that, in the previous definition, the index k is said to be in a 0-block, if whenever $\alpha < \beta$ we have $i_k^\alpha < i_k^\beta$, all indexes i_{k-1}^β are smaller than i_k^0 and all indexes i_{k+1}^β are greater or equal to $\sup\{i_k^\alpha : \alpha < \omega_1\}$.

Lemma 2.18 Assume that $i_0^\alpha < \dots < i_n^\alpha < \kappa$ for all $\alpha < \omega_1$, where κ is a cardinal. There is an uncountable subsequence $(i_0^{\alpha_j}, \dots, i_n^{\alpha_j})_{j < \omega_1}$ such that it is tidy.

Proof: We first claim that whenever $(i_\alpha)_{\alpha < \omega_1}$ are different indexes such that $i_\alpha < \kappa$, there is a subsequence $(i_{\alpha_j})_{j < \omega_1}$ such that $j < j' < \omega_1$ implies $i_{\alpha_j} < i_{\alpha_{j'}}$. We prove the claim by choosing for $j < \omega_1$ such α_j that

$$i_{\alpha_j} = \min\{i_\alpha < \kappa : \alpha > \sup\{\alpha_{j'} : j' < j\} \text{ and } i_\alpha > \sup\{i_{\alpha_{j'}} : j' < j\}\}.$$

Now since for any countable j there are still uncountably many i_α to choose from, the above set is nonempty.

We prove the lemma by induction on n . First the case when $n = 0$. Either there are uncountably many same indexes i_0^α or if not, there are uncountably many different. In the first case we find a constant subsequence, and in the second case we find a 0-block by the previous claim.

Then assume that we have shown the lemma for n , and need to find a tidy subsequence of $(i_0^\alpha, \dots, i_{n+1}^\alpha)_{\alpha < \omega_1}$. By induction, we may assume that $(i_0^\alpha, \dots, i_n^\alpha)_{\alpha < \omega_1}$ is tidy. First we look at the case where there are uncountably many $\alpha < \omega_1$ such that $i_{n+1}^\alpha \geq \sup\{i_n^\alpha : \alpha < \omega_1\}$.

If so, we do as in the case $n = 0$. Otherwise there must be a subsequence $(\alpha_k)_{k < \omega_1}$ of different indexes such that

$$\sup\{i_{n+1}^{\alpha_k} : k < \omega_1\} = \sup\{i_n^\alpha : \alpha < \omega_1\}.$$

By reenumerating the sequence we write $\alpha_k = \gamma < \omega_1$. By the previous claim we may assume that $\gamma' < \gamma < \omega_1$ implies $i_{n+1}^{\gamma'} < i_{n+1}^\gamma$. Since the index at n can't be a constant, there is $m < \omega$ and an m -block for indexes $p, \dots, p+m$ such that $p+m = n$. But now we define the subsequence $(i_0^{\alpha_j}, \dots, i_{n+1}^{\alpha_j})_{\beta < \omega_1}$ of $(i_0^\gamma, \dots, i_{n+1}^\gamma)_{\gamma < \omega_1}$ as follows. For each countable j , let α_j be such that

$$i_p^{\alpha_j} = \min\{i_p^\gamma < \kappa : i_p^\gamma > i_{n+1}^{\alpha_{j'}} \text{ for each } j' < j\}.$$

Again the minimum exists as a minimum of a non-empty subset of a well-order. Now $(i_p^{\alpha_j}, \dots, i_{p+m}^{\alpha_j}, i_{n+1}^{\alpha_j})_{j < \omega_1}$ forms an $m+1$ -block and thus the sequence $(i_0^{\alpha_j}, \dots, i_{n+1}^{\alpha_j})_{j < \omega_1}$ is tidy. \square

Lemma 2.19 *Let $EM((\bar{a}_i)_{i < \kappa} \cup A)$ be as in Proposition 2.13, for a sequence $(\bar{a}_i)_{i < \kappa}$ and a countable set A . Let*

$$\bar{b}_\alpha = \bar{t}(\bar{a}_{i_0^\alpha}, \dots, \bar{a}_{i_n^\alpha}, \bar{a})_{\alpha < \omega_1}$$

be a sequence of tuples such that \bar{t} is a sequence of terms of τ^ , $\bar{a} \in A$ fixed and $i_0^\alpha < \dots < i_n^\alpha < \kappa$ for each $\alpha < \omega_1$. Then there exists an uncountable subsequence $(\bar{b}_\beta)_{\beta < \omega_1}$ such that it is a strongly A -indiscernible sequence.*

Proof: By Lemma 2.18 we find a subsequence $(i_0^\beta, \dots, i_n^\beta)_{\beta < \omega_1}$ which is tidy. We want to show that the sequence $(\bar{b}_\beta)_{\beta < \omega_1}$ is strongly A -indiscernible.

Let $\lambda \geq \omega_1$ be an ordinal. We want to extend the sequence to a suitable sequence $(i_0^\beta, \dots, i_n^\beta)_{\beta < \lambda}$. First by 4. of Proposition 2.13, we can extend $(\bar{a}_i)_{i < \kappa}$ to $(\bar{a}_i)_{i \in I}$, where I is a $|\lambda|^+$ -dense linear order extending κ .

For an index k there are two different cases.

1. Suppose $i_k^\gamma = \beta$ is constant for all $\gamma < \omega_1$.
2. Suppose that $(i_k^\gamma, \dots, i_{k+m}^\gamma)_{\gamma < \omega_1}$ forms an m -block.

For the case in 1., take $i_k^\alpha = \beta$ for all $\alpha < \lambda$. In the case of 2., we use $|\lambda|^+$ -density to extend the m -block to $(i_k^\alpha, \dots, i_{k+m}^\alpha)_{\alpha < \lambda}$ such that for $\gamma' < \gamma < \lambda$ we have

$$i_k^{\gamma'} < \dots < i_{k+m}^{\gamma'} < i_k^\gamma < \dots < i_{k+m}^\gamma < \min\{i_{k+m+1}^\alpha : \alpha < \omega_1\}$$

Finally, let $f : \lambda \rightarrow \lambda$ be partial and order-preserving. We have that the mapping $i_k^\beta \mapsto i_k^{f(\beta)}$, for all $0 \leq k \leq n$ and $\beta \in \text{dom}(f)$, preserves the ordering of I . Thus by 3. of Proposition 2.13, there is a τ^* -isomorphism

$$F : EM((\bar{a}_0^\beta, \dots, \bar{a}_n^\beta)_{\beta \in \text{dom}(f)} \cup A) \rightarrow EM((\bar{a}_0^{f(\beta)}, \dots, \bar{a}_n^{f(\beta)})_{\beta \in \text{rng}(f)} \cup A)$$

fixing A pointwise and mapping $(\bar{a}_{i_0^\beta}, \dots, \bar{a}_{i_n^\beta})$ to $(\bar{a}_{i_0^{f(\beta)}}, \dots, \bar{a}_{i_n^{f(\beta)}})$ for all $\beta \in \text{dom}(f)$. This mapping F extends to an automorphism of \mathfrak{M} mapping \bar{b}_β to $\bar{b}_{f(\beta)}$ for all $\beta \in \text{dom}(f)$. \square

2.2 Categoricity and stability

In [7], we had two different notions of saturation. A model \mathcal{A} is weakly κ -saturated, if the following holds: Let $B \subset \mathcal{A}$, $|B| < \kappa$ and $\bar{b} \in \mathfrak{M}$. Then $\text{tp}^w(\bar{b}/B)$ is realized in \mathcal{A} . A model \mathcal{A} is weakly saturated, if it is weakly $|\mathcal{A}|$ -saturated. Similarly a model is κ -saturated or saturated, if the same holds for Galois types. Clearly weak \aleph_0 -saturation is the same notion as \aleph_0 -saturation. We discussed also about κ -categoricity for a cardinal κ , that is, each model of size κ being isomorphic. First we recall the notion of \aleph_0 -stability.

Definition 2.20 (\aleph_0 -stability) *We say that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is \aleph_0 -stable if for all countable A and \bar{a}_i for $i < \omega_1$, there are $i < j < \omega_1$ such that $\text{tp}^w(\bar{a}_i/A) = \text{tp}^w(\bar{a}_j/A)$.*

Although our notion of \aleph_0 -stability refers to weak types, we will not call it weak \aleph_0 -stability. We recall an important Theorem from [7], there called Theorem 3.12. This implies that our notion of \aleph_0 -stability implies also \aleph_0 -stability respect to Galois types.

Theorem 2.21 *Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an \aleph_0 -stable finitary AEC. Assume that \mathcal{A} is a countable model and $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$. Then there is $f \in \text{Aut}(\mathfrak{M}/\mathcal{A})$ such that $f(\bar{a}) = \bar{b}$.*

We recall a well-known fact about Galois types. The result can be found in [23], see [17] for an easy proof.

Lemma 2.22

1. *Let $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$, $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}'$, $|\mathcal{A}| < |\mathcal{B}'| \leq \kappa$ and \mathcal{B} be κ -saturated. Then there is an automorphism $f \in \text{Aut}(\mathfrak{M}/\mathcal{A})$ such that $f(\mathcal{B}') \preceq_{\mathbb{K}} \mathcal{B}$.*
2. *Two saturated models $\mathcal{B}_1, \mathcal{B}_2$ containing \mathcal{A} , such that $|\mathcal{A}| < |\mathcal{B}_1| = |\mathcal{B}_2|$, are isomorphic over \mathcal{A} .*

We have that \aleph_0 -stability of the class $(\mathbb{K}, \preceq_{\mathbb{K}})$ implies that there are countable \aleph_0 -saturated models. Since any two countable \aleph_0 -saturated models are isomorphic, we gain from the previous theorem that any two κ -saturated models of size κ are isomorphic. Then, under \aleph_0 -stability, if all models of size κ are κ -saturated, the class is categorical in κ . This gives raise to the following concept of weak categoricity.

Definition 2.23 (Weak categoricity) *We say that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is weakly κ -categorical if each model of size κ is weakly saturated.*

Theorem 2.24 *Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be a finitary abstract elementary class, which is weakly κ -categorical for some uncountable κ . Then it is \aleph_0 -stable.*

Proof: Let A be a countable set. By proposition 2.13 there is a model $EM((a_i)_{i < \kappa} \cup A)$ of size κ such that each element is of the form $t(\bar{a}_{i_0}, \dots, \bar{a}_{i_n}, \bar{a})$ for a term t of τ^* , $\{i_0, \dots, i_n\} \subset \kappa$ and $\bar{a} \in A$, and each partial order-preserving $f : \kappa \rightarrow \kappa$ induces $F \in \text{Aut}(\mathfrak{M}/A)$ mapping \bar{a}_i to $\bar{a}_{f(i)}$ for each $i \in \text{dom}(f)$.

We denote $\kappa = (\bar{a}_i)_{i < \kappa}$. Since $|EM(\kappa \cup A)| = \kappa$, it is saturated. Each weak type over A is realized in $SH(\kappa \cup A)$. We assume that $(\bar{b}_i)_{i < \omega_1}$ are elements of $EM(\kappa \cup A)$. To prove the theorem, we should find $i < j < \omega_1$ such that $\text{tp}^w(\bar{b}_i/A) = \text{tp}^w(\bar{b}_j/A)$.

Since τ^* and \mathcal{A} are countable, we may assume that each \bar{b}_i is of the form $\bar{t}(\bar{a}_{i_0}, \dots, \bar{a}_{i_n}, \bar{a})$ for a fixed sequence t of terms of τ^* and fixed $\bar{a} \in \mathcal{A}$. Furthermore, we may assume that each $\{j_0, \dots, j_n\}$ have the same order in κ . But then $\text{tp}^w(\bar{b}_i/A) = \text{tp}^w(\bar{b}_j/A)$ for all $i < j < \omega_1$. \square

It is a result of Shelah's, that κ -categoricity also implies \aleph_0 -stability. The proof is similar to the previous proof. We assume that there would be $(\bar{a}_i)_{i < \omega_1}$ with different type over A . Both A and $(\bar{a}_i)_{i < \omega_1}$ are included in some model \mathcal{B} of size κ . But κ -categoricity implies that \mathcal{B} is isomorphic to a model of type $EM(\kappa) = EM(\kappa \cup \emptyset)$ as above. But any countable subset A' of $EM(\kappa)$ is generated by a countable well-order $I \subset \kappa$, and terms $t(\bar{a})$ and $t(\bar{b})$ have the same type over A' if and only if there is a partial order-preserving $f : \kappa \rightarrow \kappa$ fixing I and mapping \bar{a} to \bar{b} . Thus we can see that there can be only countable many different Galois types over any countable subset of $EM(\kappa)$, a contradiction. By combining these two results we get the following.

Lemma 2.25 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a finitary AEC. Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ is weakly \aleph_1 -categorical if and only if it is \aleph_1 -categorical.*

Proof: Both \aleph_1 -categoricity and weak \aleph_1 -categoricity imply \aleph_0 -stability. By Theorem 2.21, a weakly \aleph_1 -saturated model is also \aleph_1 -saturated respect to Galois types. Let \mathcal{A}_1 and \mathcal{B}_2 be two saturated models of size \aleph_1 . By \aleph_0 -stability we may assume that they have a common \aleph_0 -saturated countable submodel, and then by Lemma 2.22 are isomorphic.

We prove the other direction. By \aleph_0 -stability, we can build a (weakly) \aleph_1 -saturated model as an increasing chain of countable models \mathcal{A}_i , $i < \omega_1$, such that each (weak) type over \mathcal{A}_i is realized in \mathcal{A}_{i+1} . Then by \aleph_1 -categoricity, each model of size \aleph_1 is isomorphic to this model. \square

We state another corollary of Proposition 2.13.

Lemma 2.26 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is weakly categorical in an uncountable cardinal κ . Then each model of size \aleph_1 is \aleph_1 -saturated.*

Proof: Let $\mathcal{A} = (b_i)_{i < \aleph_1}$ be a model of size \aleph_1 and $A \subset \mathcal{A}$ a countable set. By Lemma 2.9 we may assume that $SH(\mathcal{A}) = \mathcal{A}$. Let $\text{tp}^w(\bar{a}/A)$ be a weak type over A . Then let $EM(I \cup A)$, for each $I \subset \kappa$, be as in Proposition 2.13. Now $A \subset EM(\kappa \cup A)$ and $EM(\kappa \cup A)$ is \aleph_1 -saturated by weak κ -categoricity.

The type $\text{tp}^w(\bar{a}/A)$ is realized in $EM(i_0, \dots, i_n \cup A)$ for some finite $i_0 < \dots < i_n < \kappa$. By condition 2 of Proposition 2.13 there are $j_0 < \dots < j_n < \aleph_1$ and a τ^* -isomorphism

$$f : EM(i_0, \dots, i_n \cup A) \rightarrow SH(b_{j_0}, \dots, b_{j_n} \cup A),$$

fixing A pointwise. Now $\text{tp}^w(\bar{a}/A)$ is realized in $SH(b_{j_0}, \dots, b_{j_n} \cup A) \upharpoonright \tau \preceq_{\mathbb{K}} \mathcal{A}$. \square

2.3 A presentation theorem for small Finitary Abstract Elementary Classes

The result by David Kueker [14] states that any finitary abstract elementary class is closed under $L_{\infty\omega}$ -equivalence. If the class is in addition \aleph_0 -categorical, we can use the Scott

sentence of the one countable model to define the class in $L_{\omega_1\omega}$. It is known (see for example the book [13]) that the class of models of a complete sentence in $L_{\omega_1\omega}$ can be written as a class of all atomic models of a countable first-order theory in an extended vocabulary. Hence an \aleph_0 -categorical finitary AEC can be presented as a class of atomic models of a countable first-order theory. This is relevant especially if we study the class of \aleph_0 -saturated models of an \aleph_0 -stable finitary AEC, since this class is \aleph_0 -categorical. We give an alternative proof of the result in this context emphasizing the relation between the two countable vocabularies. Hence we are able to see how the behaviour of types, especially stability and splitting, is preserved in the presentation.

We define that a finitary class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is *small* if only countable many Galois types of finite tuples over the empty set realized in model of \mathbb{K} , or equivalently, realized in the monster model \mathfrak{M} of $(\mathbb{K}, \preceq_{\mathbb{K}})$. We denote

$$\text{ga-S}_n(\emptyset) = \{p := \text{tp}^{\mathfrak{g}}(\bar{a}/\emptyset) : |\bar{a}| = n, \bar{a} \in \mathfrak{M}\}.$$

If \mathbb{K} is small, the set $\text{ga-S}_n(\emptyset)$ is countable for each $n < \omega$. We also denote that

$$(\mathbb{K})^\omega = \{\mathcal{A} \in \mathbb{K} : \mathcal{A} \text{ is } \aleph_0\text{-saturated}\}.$$

In $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a small finitary class, also $((\mathbb{K})^\omega, \preceq_{\mathbb{K}})$ is a finitary class. Especially, it has a countable Löwenheim-Skolem number. Furthermore, it is \aleph_0 -categorical.

Theorem 2.27 *Assume that \mathbb{K} is a small finitary class of τ -structures, where τ is a countable vocabulary. There is a countable $\tau^* \supseteq \tau$, a first-order τ^* -theory T^* and a countable set Γ of T^* -types such that*

1. every $\mathcal{A} \in (\mathbb{K})^\omega$ can be expanded to a τ^* -structure $\mathcal{A}^* \models T^*$ omitting Γ and
2. for every $\mathcal{A}^* \models T^*$ omitting Γ , $\mathcal{A}^* \upharpoonright \tau \in (\mathbb{K})^\omega$.
3. Furthermore, if $\mathcal{B}^*, \mathcal{A}^* \models T^*$ and \mathcal{A}^* omits Γ , we have that

$$\mathcal{B}^* \text{ is an } \tau^*\text{-substructure of } \mathcal{A}^*$$

if and only if

$$\mathcal{B}^* \upharpoonright \tau \preceq_{\mathbb{K}} \mathcal{A}^* \upharpoonright \tau.$$

Proof: We let $\tau^* = \tau \cup \{S_p : p \in \text{ga-S}_n(\emptyset), 0 < n < \omega\}$, where each S_p , $p \in \text{ga-S}_n(\emptyset)$ is an n -ary predicate. Since \mathbb{K} is small, the language τ^* is countable.

First we show how to expand a structure $\mathcal{A} \in \mathbb{K}$ to an τ^* -structure \mathcal{A}^* . We may assume that $\mathcal{A} \preceq_{\mathbb{K}} \mathfrak{M}$ (Otherwise induce the structure from a \mathbb{K} -embedded structure in \mathfrak{M}). Let $n < \omega$ and $p \in \text{ga-S}_n(\emptyset)$. We define for each n -tuple $\bar{a} \in \mathcal{A}$ that

$$\mathcal{A}^* \models S_p(\bar{a}) \text{ if and only if } \bar{a} \text{ realizes } p \text{ in } \mathfrak{M}.$$

For two types $p \in \text{ga-S}_n(\emptyset)$, $q \in \text{ga-S}_{n+k}(\emptyset)$ we define that

$$q \upharpoonright n = p$$

if and only if for each tuple $\bar{a} = (a_1, \dots, a_n, \dots, a_k) \in \mathfrak{M}$, if \bar{a} realizes q , then (a_1, \dots, a_n) realizes p . Now we also know how to expand the monster model \mathfrak{M} to an τ^* -structure \mathfrak{M}^* , and we can define the theory T^* .

$$T^* =$$

$$\{\forall \bar{x} \exists \bar{y} (S_p(\bar{x}) \rightarrow S_q(\bar{x}, \bar{y})) : p \in \text{ga-S}_n(\emptyset), q \in \text{ga-S}_{n+k}(\emptyset), q \upharpoonright n = p, 0 < n, k < \omega\}$$

\(\cup\)

$$\{\forall \bar{x} (S_p(\bar{x}) \rightarrow \phi(\bar{x})) : \phi(\bar{x}) \in \tau^* \text{ atomic or negated atomic s.t. } \mathfrak{M}^* \models S_p(\bar{x}) \rightarrow \phi(\bar{x})\}$$

Finally we define the set of types $\Gamma = \bigcup_{n < \omega} \Gamma_n$ such that

$$\Gamma_n = \{\neg S_p(\bar{x}) : p \in \text{ga-S}_n(\emptyset)\}.$$

We clearly see that if a model $\mathcal{A} \in \mathbb{K}$ is \aleph_0 -saturated, then $\mathcal{A}^* \models T^*$ and \mathcal{A}^* omits each Γ_n . This holds also for the monster model \mathfrak{M} , since it is \aleph_0 -saturated.

We introduce the following notation: Let

$$\mathbb{K}^* = \{\mathcal{A}^* \text{ an } \tau^* \text{-structure} : \mathcal{A}^* \models T^* \text{ and } \mathcal{A}^* \text{ omits } \Gamma\},$$

where τ^*, T^* and Γ are as above. We also write that

$$\mathfrak{M} \models p(\bar{a})$$

if the tuple $\bar{a} \in \mathfrak{M}$ realizes $p \in \text{ga-S}_n(\emptyset)$ in \mathfrak{M} for $n = |\bar{a}|$. Hence

$$\mathfrak{M} \models p(\bar{a}) \text{ if and only if } \mathfrak{M}^* \models S_p(\bar{a}).$$

Instead of 2., we prove a stronger claim:

- 2'. for every $\mathcal{A}^* \in \mathbb{K}^*$ there is a \mathbb{K} -embedding $f : \mathcal{A}^* \upharpoonright \tau \rightarrow \mathfrak{M}$ such that for every finite tuple $\bar{a} \in \mathcal{A}^*$ and $p \in \text{ga-S}_n(\emptyset)$,

$$\mathcal{A}^* \models S_p(\bar{a}) \text{ if and only if } \mathfrak{M} \models p(f(\bar{a})).$$

Now we prove 2' by induction of the size of the models $\mathcal{A} \in \mathbb{K}$, $\mathcal{A}^* \in \mathbb{K}^*$. Since $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a small finitary class, there exists a countable \aleph_0 -saturated model.

To prove 2'. for countable models, Let $\mathcal{A}^* \in \mathbb{K}^*$ be countable. Let $\mathcal{A}' \preceq_{\mathbb{K}} \mathfrak{M}$ be countable and ω saturated. We construct increasing partial finite maps $f_k : \mathcal{A}^* \rightarrow \mathcal{A}'$ such that for each $\bar{a} \in \text{dom}(f_k)$ and each $p \in \text{ga-S}_n(\emptyset)$,

$$\mathcal{A}^* \models S_p(\bar{a}) \text{ if and only if } \mathfrak{M} \models p(f_k(\bar{a})),$$

and $\bigcup_{k < \omega} \text{dom}(f_k) = \mathcal{A}^*$ and $\bigcup_{k < \omega} \text{rng}(f_k) = \mathcal{A}'$.

Let f_0 be the empty function. Assume that we have defined f_k and define f_{k+1} adding $a \in \mathcal{A}^*$ to the domain and $b \in \mathcal{A}'$ to the range as follows. Let $\bar{c} = \text{dom}(f_k) \in \mathcal{A}^*$. Since \mathcal{A}^* omits Γ , there is $S_p \in \tau^*$ such that $\mathcal{A}^* \models S_p(\bar{c}, a)$. Denote $p' = p \upharpoonright |\bar{c}|$. Since

$$\mathcal{A}^* \models \forall \bar{x}, y (S_p(\bar{x}, y) \rightarrow S_{p'}(\bar{x})),$$

we have that $\mathcal{A}^* \models S_{p'}(\bar{c})$. By induction, $f_k(\bar{c})$ realizes p' in \mathfrak{M} . Since \mathcal{A}' is \aleph_0 -saturated, there is $a' \in \mathcal{A}'$ such that $(f_k(\bar{c}), a')$ realizes p . Then let $q = \text{tp}^{\mathfrak{M}}(f_k(\bar{c}), a', b)$. Since

$$\mathcal{A}^* \models \forall \bar{x}, y \exists z (S_p(\bar{x}, y) \rightarrow S_q(\bar{x}, y, z)),$$

there is $b' \in \mathcal{A}^*$ such that $\mathcal{A}^* \models S_q(\bar{c}, a, b')$. We define $f_{k+1}(\bar{c}, a, b') = (f_k(\bar{c}), a', b)$. Now for any $p'' \in \text{ga-S}_n(\emptyset)$ and $\bar{d} \in \text{dom}(f_{k+1})$, since $\mathcal{A}^* \models T^*$, $\mathcal{A}^* \models S_{p''}(\bar{d})$ if and only if $\forall \bar{z}(S_q(\bar{z}) \rightarrow S_{p''}(\bar{z}')) \in T^*$ if and only if $q \upharpoonright \bar{z}' = p''$ if and only if $\mathfrak{M} \models p''(f_{k+1}(\bar{d}))$.

The union

$$f = \bigcup_{k < \omega} f_k$$

is a τ -isomorphism between \mathcal{A}^* and \mathcal{A}' and hence $\mathcal{A} = \mathcal{A}^* \upharpoonright \tau \in \mathbb{K}$. Also \mathcal{A} is \aleph_0 -saturated.

Now we assume that 2'. hold for all models of size λ' for $\aleph_0 \leq \lambda' < \lambda$ and prove 2'. for models of size λ . Assume that $\mathcal{A}^* \in \mathbb{K}^*$ and $|\mathcal{A}^*| = \lambda$. There are models $\mathcal{A}_i \in \mathbb{K}^*$ such that

$$\mathcal{A}^* = \bigcup_{i < \omega} \mathcal{A}_i^*,$$

$|\mathcal{A}_i| < \lambda$ and \mathcal{A}_i^* is τ^* -substructure of \mathcal{A}_j^* for each $i < j < \lambda$. We may also take the chain to be continuous. Denote $\mathcal{A}^* \upharpoonright \tau = \mathcal{A}$ and $\mathcal{A}_i^* \upharpoonright \tau = \mathcal{A}_i$ for each $i < \lambda$. By induction, for each $i < \lambda$ there is $\mathcal{B}_i \preceq_{\mathbb{K}} \mathfrak{M}$ and a \mathbb{K} -embedding $f_i : \mathcal{A}_i \rightarrow \mathcal{B}_i$ such that for all $\bar{a} \in \mathcal{A}_i$ and $p \in \text{ga-S}_n(\emptyset)$,

$$\mathcal{A}_i^* \models S_p(\bar{a}) \text{ if and only if } \mathfrak{M} \models p(f_i(\bar{a})).$$

We claim that we can choose \mathcal{B}'_i as above such that $\mathcal{B}'_i \preceq_{\mathbb{K}} \mathcal{B}'_j$ for each $i < j < \lambda$. Then we are done with the proof of 2'.

We prove the claim by induction on $i < \lambda$. On limit ordinals we can take unions. Assume that we have chosen \mathcal{B}'_i for $i \leq \alpha$ and Let $\mathcal{B}'_{\alpha+1}$ be as above. Now the mapping $f_{\alpha+1} \circ f_{\alpha}^{-1} : \mathcal{B}'_{\alpha} \rightarrow \mathcal{B}'_{\alpha+1}$ is an embedding. Furthermore, it is a \mathbb{K} -embedding by finite character, since $\mathcal{B}'_{\alpha} \preceq_{\mathbb{K}} \mathfrak{M}$ and for each $\bar{a} \in \mathcal{B}'_{\alpha}$ $p \in \text{ga-S}_n(\emptyset)$,

$$\mathfrak{M} \models (p(\bar{a}))$$

$$\text{if and only if } \mathcal{A}_{\alpha}^* \models S_p(f_{\alpha}^{-1}(\bar{a}))$$

$$\text{if and only if } \mathcal{A}_{\alpha+1}^* \models S_p(f_{\alpha}^{-1}(\bar{a}))$$

$$\text{if and only if } \mathfrak{M} \models (p((f_{\alpha+1} \circ f_{\alpha}^{-1})(\bar{a}))).$$

Then $f_{\alpha+1} \circ f_{\alpha}^{-1}$ extends to an automorphism f of \mathfrak{M} . We can take $\mathcal{B}'_{\alpha+1} = f^{-1}(\mathcal{B}'_{\alpha+1})$. Then since $(f_{\alpha+1} \circ f_{\alpha}^{-1})(\mathcal{B}'_{\alpha}) \preceq_{\mathbb{K}} \mathcal{B}'_{\alpha+1}$, we have that

$$\mathcal{B}'_{\alpha} = (f^{-1} \circ f_{\alpha+1} \circ f_{\alpha}^{-1})(\mathcal{B}'_{\alpha}) \preceq_{\mathbb{K}} \mathcal{B}'_{\alpha+1}.$$

To prove 3., let $\mathcal{B}^* \models T^*$ be a substructure of \mathcal{A}^* . Then also \mathcal{B}^* omits Γ . Hence $\mathcal{B} = \mathcal{B}^* \upharpoonright \tau \in (\mathbb{K})^{\omega}$ and as in the previous claim, there is a \mathbb{K} -embedding $g : \mathcal{B} \rightarrow \mathfrak{M}$ such that for each $\bar{a} \in \mathcal{B}$ and each $p \in \text{ga-S}_n(\emptyset)$,

$$\mathcal{B}^* \models S_p(\bar{a}) \text{ if and only if } \mathfrak{M} \models p(g(\bar{a})).$$

Also, there is a \mathbb{K} -embedding $f : \mathcal{A} \rightarrow \mathfrak{M}$ such that

$$\mathcal{A}^* \models S_p(\bar{a}) \text{ if and only if } \mathfrak{M} \models p(f(\bar{a})).$$

To show that $\mathcal{B} \preceq_{\mathbb{K}} \mathcal{A}$ it is enough to show that $f(\mathcal{B}) \preceq_{\mathbb{K}} \mathfrak{M}$. Hence then also $f(\mathcal{B}) \preceq_{\mathbb{K}} f(\mathcal{A})$, the claim follows, since $\preceq_{\mathbb{K}}$ is closed under isomorphism. We show this by showing that the embedding $f \circ g^{-1} : g(\mathcal{B}) \rightarrow \mathfrak{M}$ is a \mathbb{K} -embedding.

For all finite tuples $g(\bar{a}) \in g(\mathcal{B})$, and types $p \in \text{ga-S}_n(\emptyset)$, $\mathfrak{M} \models p(g(\bar{a}))$ if and only if $\mathcal{B}^* \models S_p(\bar{a})$ if and only if $\mathcal{A}^* \models S_p(\bar{a})$ if and only if $\mathfrak{M} \models p(f(\bar{a}))$ if and only if $\mathfrak{M} \models p((f \circ g^{-1})(g(\bar{a})))$. Hence this holds for all $g(\bar{a}) \in g(\mathcal{B})$ and $g(\mathcal{B}) \preceq_{\mathbb{K}} \mathfrak{M}$, we get that $g^{-1} : g(\mathcal{B}) \rightarrow \mathfrak{M}$ is a \mathbb{K} -embedding. This proves 3. \square

By looking at the previous proof, we see that the two classes \mathbb{K}^* and $(\mathbb{K})^\omega$ will have exactly the same number of models in each cardinality κ . For a class of τ -structures \mathbb{K} , we denote $|Mod(\mathbb{K}, \kappa)|$ to be the number of τ -structures in \mathbb{K} of size κ , up to τ -isomorphism.

Remark 2.28 *Let \mathbb{K} and \mathbb{K}^* be as in the previous theorem. Then for each cardinal κ ,*

$$|Mod((\mathbb{K})^\omega, \kappa)| = |Mod(\mathbb{K}^*, \kappa)|.$$

Proof: Clearly if $\mathcal{A}_1, \mathcal{A}_2 \in (\mathbb{K})^\omega$ are not τ -isomorphic, then the expansions $\mathcal{A}_1^*, \mathcal{A}_2^* \in \mathbb{K}^*$ are not τ^* -isomorphic. Hence $|Mod((\mathbb{K})^\omega, \kappa)| \leq |Mod(\mathbb{K}^*, \kappa)|$. To prove that $|Mod((\mathbb{K})^\omega, \kappa)| \geq |Mod(\mathbb{K}^*, \kappa)|$ we claim that the reducts of two non-isomorphic τ^* -structures in \mathbb{K}^* cannot be τ -isomorphic.

Assume that τ -structures $\mathcal{A}_1, \mathcal{A}_2 \in (\mathbb{K})^\omega$ have τ^* -expansions $\mathcal{A}_1^*, \mathcal{A}_2^* \in \mathbb{K}^*$. To prove the claim, we show that any τ -isomorphism

$$f : \mathcal{A}_1 \rightarrow \mathcal{A}_2$$

is also a τ^* -isomorphism between \mathcal{A}_1^* and \mathcal{A}_2^* . Item 2' from the proof of Theorem 2.27 gives \mathbb{K} -embeddings $f_i : \mathcal{A}_i \rightarrow \mathfrak{M}$ such that for every $\bar{a} \in \mathcal{A}_i$ and $p \in \text{ga-S}_n(\emptyset)$,

$$\mathcal{A}_i \models S_p(\bar{a}) \text{ if and only if } \mathfrak{M} \models p(f_i(\bar{a})).$$

Also $f_2 \circ f \circ f_1^{-1}$ gives a τ -isomorphism between $f_1(\mathcal{A}_1), f_2(\mathcal{A}_2) \preceq_{\mathbb{K}} \mathfrak{M}$. Then for any tuple $\bar{a} \in \mathcal{A}_1$ and any $S_p \in \tau^* \setminus \tau$,

$$\mathcal{A}_1 \models S_p(\bar{a}) \text{ if and only if}$$

$$\mathfrak{M} \models p(f_1(\bar{a})) \text{ if and only if}$$

$$\mathfrak{M} \models p(f_2 \circ f \circ f_1^{-1}(f_1(\bar{a}))) \text{ if and only if}$$

$$\mathfrak{M} \models p(f_2(f(\bar{a}))) \text{ if and only if}$$

$$\mathcal{A}_2 \models S_p(f(\bar{a})).$$

We conclude that f is a τ^* -isomorphism between \mathcal{A}_1^* and \mathcal{A}_2^* . \square

Similarly, there is a close connection between the different notions of type.

Theorem 2.29 *Let $\mathcal{A}^* \subseteq \mathfrak{M}^*$, $\mathcal{A}^* \upharpoonright \tau = \mathcal{A} \preceq_{\mathbb{K}} \mathfrak{M}$ and $\bar{a}, \bar{b} \in \mathfrak{M}$. Then the following are equivalent:*

1. $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$
2. for all atomic τ^* -formulas $\phi(\bar{x}, \bar{y})$ and $\bar{c} \in \mathcal{A}$,

$$\mathfrak{M}^* \models \phi(\bar{a}, \bar{c}) \text{ if and only if } \mathfrak{M}^* \models \phi(\bar{b}, \bar{c})$$

3. for all τ^* -formulas $\phi(\bar{x}, \bar{y})$ and $\bar{c} \in \mathcal{A}$,

$$\mathfrak{M}^* \models \phi(\bar{a}, \bar{c}) \text{ if and only if } \mathfrak{M}^* \models \phi(\bar{b}, \bar{c}).$$

Proof: We have that $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$ if and only if for all finite $\bar{c} \in \mathcal{A}$, $\text{tp}^g(\bar{a}, \bar{c}/\emptyset) = \text{tp}^g(\bar{b}, \bar{c}/\emptyset)$. This holds if and only if $\mathfrak{M}^* \models S_p(\bar{a}, \bar{c}) \leftrightarrow S_p(\bar{b}, \bar{c})$ for all $\bar{c} \in \mathcal{A}$, $p \in \text{ga-S}_n(\emptyset)$. Hence 1 follows from 2. Clearly also 2 follows from 3. We prove that 3. follows from 1.

Assume that 1. holds. Let $\bar{c} \in \mathcal{A}$ be a finite tuple. There is an τ -automorphism $f \in \text{Aut}(\mathfrak{M})$ such that $f(\bar{a}, \bar{c}) = (\bar{b}, \bar{c})$. Also for all $S_p \in \tau^*$, $\bar{d} \in \mathfrak{M}$ we have that $\mathfrak{M}^* \models S_p(\bar{d})$ if and only if $\mathfrak{M} \models p(\bar{d})$ if and only if $\mathfrak{M} \models p(f(\bar{d}))$ if and only if $\mathfrak{M}^* \models S_p(f(\bar{d}))$. We get that f is also an τ^* -automorphism. Hence for all formulas $\phi(\bar{x}, \bar{y}) \in \tau^*$ and $\bar{c} \in \mathcal{A}$,

$$\mathfrak{M}^* \models \phi(\bar{d}) \text{ if and only if } \mathfrak{M}^* \models \phi(f(\bar{d})).$$

We have shown 3. □

We recall from [7] that a finitary AEC is \aleph_0 -stable with respect to weak types if and only if it is \aleph_0 -stable with respect to Galois types. We get the following corollary.

Corollary 2.30 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a finitary abstract elementary class. The following are equivalent.*

1. $(\mathbb{K}, \preceq_{\mathbb{K}})$ is \aleph_0 -stable with respect to weak types.
2. $(\mathbb{K}, \preceq_{\mathbb{K}})$ is \aleph_0 -stable with respect to Galois types.
3. $(\mathbb{K}, \preceq_{\mathbb{K}})$ is small and the class $(\mathbb{K}^*, \preceq_{\tau^*})$ is \aleph_0 -stable with respect to types in the language τ^* .

Finally we see that the class \mathbb{K}^* is actually atomic. That is, omitting the types Γ we omit all non-isolated types.

Theorem 2.31 $\mathbb{K}^* = \{\mathcal{A}^* \text{ an } \tau^*\text{-structure} : \mathcal{A}^* \models T^* \text{ and } \mathcal{A}^* \text{ atomic.}\}$.

Proof: Let $\mathcal{A}^* \in \mathbb{K}^*$. By 2'. of Theorem 2.27, we may assume that $\mathcal{A}^* \preceq_{\mathbb{K}} \mathfrak{M}$. Let $\bar{a} \in \mathcal{A}^*$. There is $S_p \in \tau^*$ such that $\mathcal{A}^* \models S_p(\bar{a})$. We claim that the formula $S_p(\bar{x})$ isolates the τ^* -type of \bar{a} .

Assume that \bar{b} is another tuple in \mathfrak{M}^* . Then $\mathfrak{M}^* \models S_p(\bar{b})$ if and only if there is an τ^* -automorphism of \mathfrak{M}^* mapping \bar{a} to \bar{b} . (See the proof of Theorem 2.29.) Hence \bar{b} realizes the τ^* -type of \bar{a} if and only if it realizes $S_p(\bar{x})$. □

3 Splitting

We recall the definitions of splitting and the notion of independence based on splitting from [7].

Definition 3.1 (Splitting) We say that the weak type $\text{tp}^w(\bar{a}/A)$ splits over finite $B \subset A$ if there are $\bar{c}, \bar{d} \in A$ such that

$$\begin{aligned} \text{tp}^w(\bar{c}/B) &= \text{tp}^w(\bar{d}/B) \text{ but} \\ \text{tp}^w(\bar{c}/B \cup \{\bar{a}\}) &\neq \text{tp}^w(\bar{d}/B \cup \{\bar{a}\}). \end{aligned}$$

We say that such \bar{c}, \bar{d} witness the fact.

Definition 3.2 (Independence) We write that

$$\bar{a} \downarrow_A^s B$$

if $\text{tp}^w(\bar{a}/A \cup B)$ does not split over a finite subset of A .

As in Theorem 3.17 of [7], we can now prove the following properties of \downarrow^s over \aleph_0 -saturated models.

Theorem 3.3 Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an \aleph_0 -stable finitary AEC.

1. **Monotonicity** If $A \subset B \subset C \subset D$ and $\bar{a} \downarrow_A^s D$, then $\bar{a} \downarrow_B^s C$.
2. **Invariance** If f is an automorphism of \mathfrak{M} , $\bar{a} \downarrow_A^s B$ if and only if $f(\bar{a}) \downarrow_{f(A)}^s f(B)$.
3. **Local character** For each model \mathcal{A} and a finite sequence \bar{a} there is finite $E \subset \mathcal{A}$ such that $\bar{a} \downarrow_E^s \mathcal{A}$.
4. **Countable extension** Let \mathcal{A} be a countable \aleph_0 -saturated model. Let B be countable containing \mathcal{A} . For each \bar{a} there is \bar{b} realizing $\text{tp}^w(\bar{a}/\mathcal{A})$ such that $\bar{b} \downarrow_{\mathcal{A}}^s B$. Moreover, if $\text{tp}^w(\bar{a}/\mathcal{A})$ does not split over the finite subset E , then $\text{tp}^w(\bar{b}/B)$ does not split over the finite set E .
5. **Stationarity** Assume \mathcal{A} is an \aleph_0 -saturated model and $A \subset B$. If $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$, $\bar{a} \downarrow_{\mathcal{A}}^s B$ and $\bar{b} \downarrow_{\mathcal{A}}^s B$, then $\text{tp}^w(\bar{a}/B) = \text{tp}^w(\bar{b}/B)$.
6. **Transitivity** Let $A \subset \mathcal{B} \subset C$ and \mathcal{B} be an \aleph_0 -saturated model. Then $\bar{a} \downarrow_A^s C$ if and only if $\bar{a} \downarrow_A^s \mathcal{B}$ and $\bar{a} \downarrow_{\mathcal{B}}^s C$.
7. **Finite character** Let E be finite and $E \subset B$. Then $\bar{a} \downarrow_E^s B$ if and only if $\bar{a} \downarrow_{E'}^s B_0$ for every finite $B_0 \subset B$.

The same holds if instead of E we have an \aleph_0 -saturated model \mathcal{A} .

This list does not include the extension property for types over models of arbitrary size nor symmetry. In [7] we prove symmetry over \aleph_0 -saturated models from this extension property using the assumption of disjoint amalgamation. Furthermore, in the thesis [8], we do the same without assuming disjoint amalgamation.

However, splitting has also been studied in the context of *atomic AECs*, that is, atomic models of a countable first-order theory. Most of this work due to Shelah and is collected in Baldwin's book [1], see chapters 19 and 20. Since the \aleph_0 -saturated models of an \aleph_0 -stable finitary AEC can be presented as an atomic AEC (see section 2.3), we can use the results for atomic AECs also in this context. Especially, symmetry follows but the extension property for non-splitting does not.

Remark 3.4 Assume that \mathcal{A} is an \aleph_0 -saturated model of an \aleph_0 -stable finitary AEC in the vocabulary τ and \mathcal{A}^* is the τ^* -structure in the corresponding atomic AEC as in section 2.3. Let \bar{a} be a tuple in the monster model. The following are equivalent.

1. The weak type $\text{tp}^w(\bar{a}/\mathcal{A})$ splits over the finite set $E \subset \mathcal{A}$.
2. The atomic τ^* -type $\text{tp}^*(\bar{a}/\mathcal{A}^*)$ $*$ -splits over the finite set $E \subset \mathcal{A}^*$ as in the definition 20.4 of [1].

In Theorem 20.9 of [1] there is a version of the extension property for non-splitting for atomic AECs. Unfortunately, this theorem does not imply the extension property for our context. The theorem states that there is a non-splitting extension which is an *atomic type*. However, this type is not necessarily realized in the monster model. Also for finitary AECs there exists a non-splitting extension which is an *abstract type* (see section 4.2) but we do not know whether the type is realized in the monster. For our purposes we need the non-splitting extension to be found in the monster and that is why we often state the extension property for non-splitting as a separate assumption. However, symmetry over \aleph_0 -saturated models follows also without the extension property for non-splitting, just using the symmetry for atomic AECs.

Lemma 3.5 (Symmetry) Assume that \mathcal{A} is an \aleph_0 -saturated model and \bar{a}, \bar{b} are finite tuples in the monster model. Then

$$\bar{a} \downarrow_{\mathcal{A}}^s \bar{b} \text{ if and only if } \bar{b} \downarrow_{\mathcal{A}}^s \bar{a}.$$

Proof: By assumption, $\text{tp}^w(\bar{a}/\mathcal{A})$ does not split over some finite $E \subset \mathcal{A}$.

By the results in section 2.3, the atomic monster model \mathfrak{M}^* in the extended vocabulary τ^* is \aleph_0 -stable in the atomic AEC \mathbb{K}^* and the expansion \mathcal{A}^* to τ^* is an atomic τ^* -elementary substructure of \mathfrak{M}^* . Hence the atomic τ^* -type $\text{tp}^*(\bar{a}/\mathcal{A}^*)$ does not $*$ -split over E by remark 3.4.

We can use Theorem 20.21 (Symmetry) of [1] to conclude that $\text{tp}^*(\bar{b}/\mathcal{A}^*)$ does not $*$ -split over some finite $E' \subset \mathcal{A}^*$. Then by Remark 3.4, $\text{tp}^w(\bar{b}/\mathcal{A})$ does not split over the same $E' \subset \mathcal{A}$. This proves the lemma. \square

We formulate our assumption of the extension property for non-splitting. The phrase ‘extension property’ will from now on refer to this definition. We note that this property is about *non-splitting* extensions for types over *models*. When studying simplicity, we will also have a stronger notion of independence and in that context we also talk about free extensions of types over arbitrary sets.

Definition 3.6 (Extension property) We say that $(\mathbb{K}, \preceq_{\mathbb{K}})$ has the λ -extension property if the following holds:

Let \mathcal{A} be an \aleph_0 -saturated model and let B contain \mathcal{A} , $|B| < \lambda$. Assume that $\text{tp}^w(\bar{a}/\mathcal{A})$ does not split over finite subset E . Then there exists \bar{b} realizing $\text{tp}^w(\bar{a}/\mathcal{A})$ such that $\text{tp}^w(\bar{b}/B)$ does not split over E .

We say that $(\mathbb{K}, \preceq_{\mathbb{K}})$ has the extension property if it has the λ -extension property for all cardinals λ .

See the thesis [8] for the proof of the following theorem, which uses an idea by Shelah. We remark that weak categoricity is not enough for this proof.

Theorem 3.7 (Extension property) *Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be a finitary AEC, categorical in some $\kappa \geq \mathfrak{H}$. Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ has the extension property.*

The following can be shown as Theorem 4.9 in [7]. Since the proof only uses symmetry, we do not need to assume the extension property for non-splitting.

Theorem 3.8 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable finitary AEC. Then there is a weakly saturated model in every infinite cardinal λ .*

The previous theorem shows, that for any infinite κ and $(\mathbb{K}, \preceq_{\mathbb{K}})$ a finitary AEC, κ -categoricity implies weak κ -categoricity.

3.1 Primary models and the U -rank

We will define a concept of a primary model $\mathcal{A}[\bar{a}]$, which is a constructible model over a countable model \mathcal{A} and a finite tuple \bar{a} . A similar construction was used in [7] in the proof of Theorem 3.12, to show that weak types and Galois types agree over countable models. Under simplicity, we will define a concept of an f-primary model in section 5, which is constructible over $\mathcal{A} \cup B$, where B is an arbitrary set. Throughout this section we will assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable finitary AEC.

We recall from [7] the definition of a type being weakly isolated.

Definition 3.9 (Weakly isolated type) *We say that a type $\text{tp}^w(\bar{b}/\mathcal{A} \cup \bar{a})$ is weakly isolated over finite $A' \cup \bar{a}$, if whenever \bar{d} realizes $\text{tp}^w(\bar{b}/A' \cup \bar{a})$, then \bar{d} realizes $\text{tp}^w(\bar{b}/\mathcal{A} \cup \bar{a})$.*

As in [7], we can use weak \aleph_0 -stability to show the following.

Lemma 3.10 *Assume \mathcal{A} is a countable model, A a finite subset of \mathcal{A} and \bar{a} be given. Then for each \bar{b} there are \bar{c} and a finite $A' \subset \mathcal{A}$ such that*

- i) \bar{c} realizes $\text{tp}^w(\bar{b}/A \cup \bar{a})$ and*
- ii) $\text{tp}^w(\bar{c}/\mathcal{A} \cup \bar{a})$ is weakly isolated over $A' \cup \bar{a}$.*

In Lemma 3.8 of [7] we studied a countable set A having a so called \aleph_0 -saturation property, i.e. for every finite $A' \subset A$ and \bar{a} , the type $\text{tp}^w(\bar{a}/A')$ is realized in A . One consequence of finite character was that a countable set A having this property is a $\preceq_{\mathbb{K}}$ -substructure of the monster model i.e. a model. We can easily see that this generalizes to larger sets also, since an uncountable set with \aleph_0 -saturation property can always be written as an increasing union of smaller sets with this property. Thus we write the fact as follows.

Proposition 3.11 *Assume that A is a set with the following property: For any finite $A' \subset A$ and $\bar{a} \in \mathfrak{M}$ the type $\text{tp}^w(\bar{a}/A')$ is realized in A . Then $A \preceq_{\mathbb{K}} \mathfrak{M}$.*

Definition 3.12 (Primary model) *Assume that $\mathcal{A} = \{\bar{a}_i : i < \omega\}$ is a countable model and \bar{a} a finite tuple. Let $\bar{b}_i, i < \omega$, be increasing and $A_i, i < \omega$, finite such that:*

1. $\bar{b}_0 = \bar{a}$ and $\text{tp}^w(\bar{a}/\mathcal{A})$ does not split over finite subset A_0 .
2. $\bar{a}_n \cup A_n \subset A_{n+1} \subset \mathcal{A}$.
3. $\text{tp}^w(\bar{b}_{n+1}/\mathcal{A} \cup \bar{a})$ is weakly isolated over $A_n \cup \bar{a}$.
4. $\text{tp}^w(\bar{b}_{n+1}/\mathcal{A})$ does not split over A_n .
5. $\mathcal{A} \cup \bigcup_{i < \omega} \bar{b}_n$ is an \aleph_0 -saturated model.

We call this model $\mathcal{A} \cup \bigcup_{i < \omega} \bar{b}_n$ the primary model $\mathcal{A}[\bar{a}]$.

We remark that since both \mathcal{A} and $\mathcal{A}[\bar{a}]$ are \mathbb{K} -submodels of \mathfrak{M} and $\mathcal{A} \subset \mathcal{A}[\bar{a}]$, also $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{A}[\bar{a}]$.

Lemma 3.13 *Assume that \mathcal{A} is countable. Then there exists a model $\mathcal{A}[\bar{a}]$ with properties (1) - (5) of Definition 3.12. If also $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$, $\bar{a} \in \mathcal{B}$ and \mathcal{B} is \aleph_0 -saturated, we can take $\mathcal{A}[\bar{a}] \preceq_{\mathbb{K}} \mathcal{B}$.*

Proof: Let $\mathcal{A} = \{\bar{a}_n : n < \omega\}$. By induction, we construct an increasing sequence of tuples $\bar{b}_n \in \mathcal{B}$, and finite sets $A_n \subset A$ $n < \omega$, such that they satisfy properties (1)- (5) of Definition 3.12.

For $n = 0$, do as (1), using local character of splitting. Assume that $\bar{b}_j \in \mathcal{B}$ and $A_j \subset A$ have been constructed for $j \leq n$. By \aleph_0 -stability, we can find $\{c_i^j : i < \omega, j \leq n\}$ realizing all the Galois types over $A_j \cup \bar{b}_j$, for $j \leq n$. Let $\bar{d}_n = (c_i^j)_{i,j \leq n}$. By Lemma 3.10 there exists A'_{n+1} finite with $A_n \subseteq A_{n+1} \subseteq \mathcal{A}$ and there exists $\bar{b}' \hat{\sim} \bar{d}'$ realizing $\text{tp}^w(\bar{b}_n \hat{\sim} \bar{d}_n / A_n)$ such that $\text{tp}^w(\bar{b}' \hat{\sim} \bar{d}' / \mathcal{A} \cup \bar{a})$ is weakly isolated over $A'_{n+1} \cup \bar{a}$. Since $\text{tp}^w(\bar{b}' / A'_{n+1} \cup \bar{a}) = \text{tp}^w(\bar{b}_n / A'_{n+1} \cup \bar{a})$ by induction hypothesis, we may assume that $\bar{b}' = \bar{b}_n$. Then since \mathcal{B} is \aleph_0 -saturated and $A'_{n+1} \cup \bar{a} \cup \bar{b}_n$ is in \mathcal{B} , we may also assume that \bar{d}' is in \mathcal{B} . Let $\bar{b}_{n+1} = \bar{b}_n \hat{\sim} \bar{d}' \in \mathcal{B}$.

By local character there is finite $E \subset \mathcal{A}$ such that $\text{tp}^w(\bar{b}_{n+1}/\mathcal{A})$ does not split over E . We take $A_{n+1} = A'_{n+1} \cup E \cup \bar{a}_n$. Then (1), (2), (3) are satisfied. Finite character is used to show (4) in the form of Lemma 3.11. Let $c \in \mathfrak{M}$ and $B \subseteq \mathcal{A} \cup \bigcup_{n < \omega} \bar{b}_n$ finite. By (2), $\mathcal{A} = \bigcup_{n < \omega} A_n$, so there exists $n < \omega$ such that $B \subseteq A_n \cup \bar{b}_n$. Then $\text{tp}^w(c/A_n \cup \bar{b}_n)$ is realized by some c_n^j , and hence belongs to \bar{b}_{j+1} for some j . We are done with the construction. \square

Definition 3.14 (Domination) *We say that a set A dominates a model \mathcal{B} over a model \mathcal{A} , if for every tuple \bar{c} ,*

$$\bar{c} \downarrow_{\mathcal{A}} A \Leftrightarrow \bar{c} \downarrow_{\mathcal{A}} \mathcal{B}.$$

In the following lemma we see that \bar{a} dominates the primary model $\mathcal{A}[\bar{a}]$ over \mathcal{A} , when \mathcal{A} is \aleph_0 -saturated.

Lemma 3.15 *Let \mathcal{A} be a countable \aleph_0 -saturated model and \bar{c}, \bar{a} finite. Then*

$$\bar{c} \downarrow_{\mathcal{A}}^s \bar{a} \text{ if and only if } \bar{c} \downarrow_{\mathcal{A}}^s \mathcal{A}[\bar{a}].$$

Proof: By monotonicity, the other direction is clear. We assume to the contrary, that $\bar{c} \downarrow_{\mathcal{A}}^s \bar{a}$ and $\bar{c} \not\downarrow_{\mathcal{A}}^s \mathcal{A}[\bar{a}]$. Let $\mathcal{A}[\bar{a}] = \mathcal{A} \cup \bigcup_{i < \omega} \bar{b}_i$ be as in the definition of a primary model, where \bar{b}_n are increasing and $\bar{b}_0 = \bar{a}$. Thus, by finite character of splitting, there is n such that $\bar{c} \not\downarrow_{\mathcal{A}}^s \bar{b}_{n+1}$. Then, by symmetry, $\bar{b}_{n+1} \not\downarrow_{\mathcal{A}}^s \bar{c}$ and thus $\text{tp}^w(\bar{b}_{n+1}/\mathcal{A} \cup \bar{c})$ splits over any finite $E \subset \mathcal{A}$.

Let $A_n \subset \mathcal{A}$ be the finite set such that $\bar{b}_n \downarrow_{A_n}^s \mathcal{A}$ and $\text{tp}^w(\bar{b}_{n+1}/\mathcal{A})$ is weakly isolated over A_n . We may find a finite B such that $A_n \subset B \subset \mathcal{A}$ and $\text{tp}^w(\bar{b}_{n+1}/B \cup \bar{c})$ splits over A_n . By assumption and symmetry, $\bar{a} \downarrow_{\mathcal{A}}^s \bar{c}$, and since $\bar{a} \downarrow_{A_n}^s \mathcal{A}$, we get by transitivity and monotonicity that $\bar{a} \downarrow_B^s \mathcal{A} \cup \bar{c}$.

Since \mathcal{A} is \aleph_0 -saturated, there is $\bar{c}' \in \mathcal{A}$ such that $\text{tp}^w(\bar{c}'/B) = \text{tp}^w(\bar{c}/B)$. But $\text{tp}^w(\bar{a}/\mathcal{A} \cup \bar{c})$ does not split over B , and thus we get an automorphism $f \in \text{Aut}(\mathfrak{M}/B \cup \bar{a})$ such that $f(\bar{c}) = \bar{c}'$. By invariance, $\text{tp}^w(f(\bar{b}_{n+1})/B \cup \bar{c}')$ splits over A_n . On the other hand $\text{tp}^w(\bar{b}_{n+1}/B \cup \bar{c}')$ does not split over A_n , and thus

$$\text{tp}^w(\bar{b}_{n+1}/B \cup \bar{c}') \neq \text{tp}^w(f(\bar{b}_{n+1})/B \cup \bar{c}').$$

But $\text{tp}^w(\bar{b}_{n+1}/A_n \cup \bar{a}) = \text{tp}^w(f(\bar{b}_{n+1})/A_n \cup \bar{a})$, and this is a contradiction, since $\text{tp}^w(\bar{b}_{n+1}/\mathcal{A} \cup \bar{a})$ was weakly isolated over $A_n \cup \bar{a}$. \square

In [7], we defined a concept of U -rank. With primary models we can show that if both $U(\bar{a}/\mathcal{A})$ and $U(\bar{b}/\mathcal{A})$ are finite for a countable \aleph_0 -saturated model \mathcal{A} , then

$$U(\bar{a}\bar{b}/\mathcal{A}) \leq U(\bar{a}/\mathcal{A}) + U(\bar{b}/\mathcal{A}).$$

In the case of [7] we proved that one sufficient condition for simplicity for an \aleph_0 -stable finitary AEC with extension property was the class having a finite U -rank. We defined that $(\mathbb{K}, \preceq_{\mathbb{K}})$ has finite U -rank, if for each finite sequence \bar{a} ,

$$\sup\{U(\bar{a}/\mathcal{A}) : \mathcal{A} \in \mathbb{K} \text{ countable and } \aleph_0\text{-saturated}\} < \aleph_0.$$

We also mentioned that it is enough to study, instead of arbitrary finite tuples \bar{a} , only singletons a . The previous equation shows this claim. As we will see in the following sections, we can develop the same theory of simplicity as in [7] for the class with assumptions of Definition 2.6.

We recall the definition of U -rank.

Definition 3.16 *Let \mathcal{A} be countable and \aleph_0 -saturated model. Define U -rank of \bar{a} over \mathcal{A} , $U(\bar{a}/\mathcal{A})$, by induction:*

1. Always $U(\bar{a}/\mathcal{A}) \geq 0$.
2. $U(\bar{a}/\mathcal{A}) \geq \beta + 1$ iff there is countable \aleph_0 -saturated model \mathcal{B} such that $\mathcal{A} \subset \mathcal{B}$, $U(\bar{a}/\mathcal{B}) \geq \beta$ and $\bar{a} \not\downarrow_{\mathcal{A}}^s \mathcal{B}$

For a countable \aleph_0 -saturated model \mathcal{A} , define

$$U(\bar{a}/\mathcal{A}) = \min\{\alpha : U(\bar{a}/\mathcal{A}) \not\geq \alpha + 1\}$$

if such an ordinal exists. Then define U -rank for arbitrary \aleph_0 -saturated model \mathcal{A} as

$$U(\bar{a}/\mathcal{A}) = \min\{U(\bar{a}/\mathcal{A}') : \mathcal{A}' \subset \mathcal{A} \text{ countable } \aleph_0\text{-saturated model.}\}$$

For finite \bar{a} and a set A , define

$$U(\bar{a}/A) = \sup\{U(\bar{b}/\mathcal{A}) : \text{tp}^w(\bar{b}/A) = \text{tp}^w(\bar{a}/A), A \subset \mathcal{A} \text{ and } \mathcal{A} \text{ } \aleph_0\text{-saturated}\}.$$

Finally we prove the equation.

Proposition 3.17 *Assume that \mathcal{A} is a countable \aleph_0 -saturated model and both $U(\bar{a}/\mathcal{A})$ and $U(\bar{b}/\mathcal{A})$ are finite. Then we have that*

$$U(\bar{a}\bar{b}/\mathcal{A}) \leq U(\bar{a}/\mathcal{A}) + U(\bar{b}/\mathcal{A}).$$

Proof: Let $U(\bar{a}/\mathcal{A}) = n$ and $U(\bar{b}/\mathcal{A}) = m$ for finite n and m . We assume the contrary, that $U(\bar{a}\bar{b}/\mathcal{A}) > m + n$. Then by the definition of U -rank there are countable and \aleph_0 -saturated \mathcal{A}_i for $0 \leq i \leq m + n + 1$ such that $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ and $\bar{a}\bar{b} \not\downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$ for each i . Using Lemma 3.13 we define inductively, starting from $i = m + n + 1$, models \mathcal{B}_i for $m + n + 1 \geq i \geq 0$ as follows:

1. $\mathcal{B}_{m+n+1} = \mathcal{A}_{m+n+1}[\bar{b}]$,
2. $\mathcal{B}_i \subset \mathcal{B}_{i+1}$ and $\mathcal{B}_i = \mathcal{A}_i[\bar{b}]$.

Let $0 \leq i \leq m + n + 1$. We claim that either $\bar{a} \not\downarrow_{\mathcal{B}_i} \mathcal{B}_{i+1}$ or $\bar{b} \not\downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$. To prove the claim, assume that neither holds. By symmetry and monotonicity, $\mathcal{A}_{i+1} \downarrow_{\mathcal{B}_i} \bar{a}$. On the other hand by symmetry $\mathcal{A}_{i+1} \downarrow_{\mathcal{A}_i} \bar{b}$, and by dominance, $\mathcal{A}_{i+1} \downarrow_{\mathcal{A}_i} \mathcal{B}_i$. We get from transitivity that $\mathcal{A}_{i+1} \downarrow_{\mathcal{A}_i} \bar{a} \cup \mathcal{B}_i$, but then $\bar{a} \cup \bar{b} \downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$ by monotonicity and symmetry, a contradiction. This proves the claim.

Now there must be either $n+1$ indexes i such that $\bar{a} \not\downarrow_{\mathcal{B}_i} \mathcal{B}_{i+1}$ or $m+1$ indexes i such that $\bar{b} \not\downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$. Since $\mathcal{A} \subset \mathcal{A}_i$ and $\mathcal{A} \subset \mathcal{B}_i$ for each i , both cases give a contradiction with the definition of U -rank. \square

3.2 Morley sequences

We also recall the definition of a Morley-sequence and the proof of the following Lemma, which is a small strengthening to Lemma 5.3 in [7], proved using Fodor's Lemma.

Definition 3.18 (Morley sequence) *Suppose \mathcal{A} is an \aleph_0 -saturated model. We say that $(\bar{a}_i)_{i < \alpha}$ is a Morley sequence over \mathcal{A} if for each $i < j < \alpha$, $\text{tp}^w(\bar{a}_i/\mathcal{A}) = \text{tp}^w(\bar{a}_j/\mathcal{A})$ and for each $i < \alpha$, $\bar{a}_i \downarrow_{\mathcal{A}}^s \bigcup_{j < i} \bar{a}_j$.*

Lemma 3.19 *Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an \aleph_0 -stable finitary AEC with the extension property. Let E be a set, I a sequence of tuples such that $i \neq j$ implies $\bar{a}_i \neq \bar{a}_j$, and $\aleph_0 + |E| \leq \lambda < |I|$. Then there is an \aleph_0 -saturated model \mathcal{A} of size λ containing E and subsequence $(\bar{a}_i)_{i < \lambda^+} \subset I$ such that it is a Morley sequence over \mathcal{A} .*

Furthermore, there is finite $E' \subset \mathcal{A}$ such that $\text{tp}^w(\bar{a}_i/\mathcal{A} \cup \bigcup_{j < i} \bar{a}_j)$ does not split over E' for all $i < \lambda^+$. In addition, if $I \cup E \subset \mathcal{B}$ for some \aleph_0 -saturated model \mathcal{B} , we may choose $\mathcal{A} \subset \mathcal{B}$.

Also the following is proved in Lemma 5.2 of [7].

Lemma 3.20 *Let $(\bar{a}_i)_{i < \alpha}$ be a Morley sequence over an \aleph_0 -saturated model \mathcal{A} . Then for every n and $i_0 < \dots < i_n < \alpha$ we have that $\text{tp}^w(\bar{a}_0, \dots, \bar{a}_n/\mathcal{A}) = \text{tp}^w(\bar{a}_{i_0}, \dots, \bar{a}_{i_n}/\mathcal{A})$.*

Using Corollary 2.15 instead of the homogeneous model \mathfrak{M}^* we can prove the following two results as Lemma 5.8 and Corollary 5.9 of [7].

Definition 3.21 *We say that two sequences $(\bar{a}_i)_{i < \alpha}$ and $(\bar{b}_j)_{j < \beta}$ are equivalent over E , if for every finite n we have that $\text{tp}^g(\bar{a}_0, \dots, \bar{a}_n/E) = \text{tp}^g(\bar{b}_0, \dots, \bar{b}_n/E)$.*

Lemma 3.22 *Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an \aleph_0 -stable finitary AEC with the extension property. Let E be countable. The following are equivalent:*

1. *A sequence $(\bar{a}_i)_{i < \omega}$ is E -equivalent to a sequence $(\bar{b}_i)_{i < \omega}$, which is a Morley sequence over some countable \aleph_0 -saturated model \mathcal{A} containing E .*
2. *A sequence $(\bar{a}_i)_{i < \omega}$ is E -equivalent to a strongly E -indiscernible sequence $(\bar{b}_i)_{i < \omega}$.*

Corollary 3.23 *Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an \aleph_0 -stable finitary AEC with the extension property. Let E be countable and I an uncountable sequence of distinct tuples. Then for any $n < \omega$ there is a subsequence $(\bar{a}_0, \dots, \bar{a}_{n-1}) \subset I$, which is a beginning of a strongly E -indiscernible sequence.*

4 Simplicity

In this section we recall the definitions of strong splitting, independence, simplicity and U -rank from [7]. We also recall some results about these and prove some more properties, which were not needed there but will be of use in this paper. For example, we will show that simplicity implies extensible U -rank. Throughout this section, if not mentioned otherwise, we will assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable finitary AEC with the extension property. We will restate these assumptions in each theorem, but not necessarily in each lemma.

In the last subsection 4.2 we will give two equivalent conditions for an \aleph_0 -stable finitary AEC to have the extension property, and show that the extension property is implied by simplicity and weak categoricity in any uncountable cardinal.

Definition 4.1 (Lascar strong type) *We say that \bar{a} and \bar{b} have the same Lascar strong type over E , written*

$$\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E),$$

if $\ell(\bar{a}) = \ell(\bar{b})$ and $\mathbf{E}(\bar{a}, \bar{b})$ holds for any E -invariant equivalence relation \mathbf{E} of $\ell(\bar{a})$ -tuples with a bounded number of classes.

Definition 4.2 *We say that $\text{tp}^w(\bar{a}/A)$ Lascar-splits over finite E if there is a strongly E -indiscernible sequence $(\bar{a}_i)_{i < \omega}$ such that $\bar{a}_0, \bar{a}_1 \in A$ and $\text{tp}^g(\bar{a}_0/E \cup \{\bar{a}\}) \neq \text{tp}^g(\bar{a}_1/E \cup \{\bar{a}\})$.*

Definition 4.3 (Independence) *We say that \bar{a} is independent of B over C , write*

$$\bar{a} \downarrow_C B,$$

if there is finite $E \subset C$ such that for all D containing $C \cup B$ there is \bar{b} such that $\text{tp}^w(\bar{b}/B \cup C) = \text{tp}^w(\bar{a}/B \cup C)$ and $\text{tp}^w(\bar{b}/D)$ does not Lascar-split over E . We then write

$$A \downarrow_C B,$$

if $\bar{a} \downarrow_C B$ for every finite tuple $\bar{a} \in A$.

Definition 4.4 (Simplicity) We say that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple if for each \bar{a} and B there is finite $E \subset B$ such that $\bar{a} \downarrow_E B$.

The main theorem about simplicity, namely Theorem 7.11 of [7], holds also in this context. In [7] it is also marked out which partial properties hold without simplicity. In this section we assume simplicity and do not distinguish which results hold also without simplicity.

We should also discuss the role of the extension property for non-splitting in this section. In [9] we get a similar list of properties using simplicity and superstability, without referring to splitting at all. The only difference is that we get stationarity of weak types over *a-saturated* models instead of \aleph_0 -saturated as in 9. \aleph_0 -stability and the extension property of splitting imply that a model is *a-saturated* if and only if it is \aleph_0 -saturated. We also need the extension property of splitting to prove Lemma 4.6, that the notions \downarrow and \downarrow^s agree over \aleph_0 -saturated models. Since the U -rank is defined in terms of \downarrow^s and simplicity in terms of \downarrow , we really need to assume the extension property for non-splitting for the results in this section.

Theorem 4.5 Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an \aleph_0 -stable finitary AEC with the extension property. Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Then, \downarrow satisfies the following properties:

1. **Invariance:** If $A \downarrow_C B$, then $f(A) \downarrow_{f(C)} f(B)$ for an $f \in \text{Aut}(\mathfrak{M})$.
2. **Finite character:** $A \downarrow_C B$ if and only if $\bar{a} \downarrow_C \bar{b}$ for every finite $\bar{a} \in A$ and $\bar{b} \in B$.
3. **Monotonicity:** If $A \downarrow_C B$ and $C \subset D \subset C \cup B$ then $A \downarrow_C D$ and $A \downarrow_D B$.
4. **Local character:** For any finite \bar{a} and any B there exists a finite $E \subset B$ such that $\bar{a} \downarrow_E B$.
5. **Extension:** For any \bar{a} , C and B containing C there is \bar{b} such that $\text{tp}^w(\bar{b}/C) = \text{tp}^w(\bar{a}/C)$ and $\bar{b} \downarrow_C B$.
6. For any finite C , \bar{a} and any B containing C , there is \bar{b} such that $\text{Lstp}(\bar{b}/C) = \text{Lstp}(\bar{a}/C)$ and $\bar{b} \downarrow_C B$.
7. **Symmetry:** $A \downarrow_C B$ if and only if $B \downarrow_C A$.
8. **Transitivity:** Let $B \subset C \subset D$. If $A \downarrow_B C$ and $A \downarrow_C D$, then $A \downarrow_B D$.
9. **Stationarity over \aleph_0 -saturated models:** Let \mathcal{A} be an \aleph_0 -saturated model. If $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$, $\bar{a} \downarrow_{\mathcal{A}} B$ and $\bar{b} \downarrow_{\mathcal{A}} B$, then $\text{tp}^w(\bar{a}/B \cup \mathcal{A}) = \text{tp}^w(\bar{b}/B \cup \mathcal{A})$.
10. **Stationarity of Lascar strong types:** If $\text{Lstp}(\bar{a}/C) = \text{Lstp}(\bar{b}/C)$, $\bar{a} \downarrow_C B$ and $\bar{b} \downarrow_C B$, then $\text{tp}^w(\bar{a}/B \cup C) = \text{tp}^w(\bar{b}/B \cup C)$.
11. **Pairs lemma:** Let $A \subset B$, $\bar{a} \downarrow_A B$ and $\bar{b} \downarrow_{A \cup \bar{a}} B$. Then $\bar{a} \cup \bar{b} \downarrow_A B$.

Also as in in [7] we gain the following lemma for an \aleph_0 -stable finitary AEC with the extension property.

Lemma 4.6 Let \mathcal{A} be an \aleph_0 -saturated model. Then $\bar{a} \downarrow_{\mathcal{A}} B$ if and only if $\bar{a} \downarrow_{\mathcal{A}}^s B$.

If $\bar{a} \notin \mathcal{A}$ and \mathcal{A} is an \aleph_0 -saturated model, we have that $\bar{a} \not\downarrow_{\mathcal{A}}^s \bar{a}$. From the previous lemma it follows that also $\bar{a} \not\downarrow_{\mathcal{A}} \bar{a}$.

We can prove also another property called Left Transitivity, which shows that the result of Pairs Lemma is actually an equivalence.

Proposition 4.7 (Left transitivity) *Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be simple. If $\bar{a} \cup \bar{b} \downarrow_A B$, then $\bar{b} \downarrow_{A \cup \bar{a}} B$.*

Proof: By symmetry, $B \downarrow_A \bar{a} \cup \bar{b}$ and then by monotonicity, $B \downarrow_{A \cup \bar{a}} \bar{b}$. By symmetry again, $\bar{b} \downarrow_{A \cup \bar{a}} B$. \square

We recall also the proof of the following from Proposition 5.29 of [7].

Proposition 4.8 *Let \mathcal{A} be an \aleph_0 -saturated model. Then \mathcal{A} is also a-saturated, i.e. for every finite subset $A \subset \mathcal{A}$ and \bar{a} the Lascar strong type $\text{Lstp}(\bar{a}/A)$ is realized in \mathcal{A} .*

We prove one more result about our notion of simplicity, namely if a finitary AEC admits any notion of independence which satisfies a list of properties over sets, then it has to be simple with respect to our notion. This proof is analogous to the proof of Theorem 3.9 in [11].

Theorem 4.9 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a finitary AEC and let \downarrow^{abs} be a notion satisfying*

1. **Invariance:** *If $A \downarrow_C^{abs} B$, then $f(A) \downarrow_{f(C)}^{abs} f(B)$ for an $f \in \text{Aut}(\mathfrak{M})$.*
2. **Finite character:** *$A \downarrow_C^{abs} B$ if and only if $\bar{a} \downarrow_C^{abs} \bar{b}$ for every finite $\bar{a} \in A$ and $\bar{b} \in B$.*
3. **Monotonicity:** *If $A \downarrow_C^{abs} B$ and $C \subset D \subset C \cup B$ then $A \downarrow_C^{abs} D$ and $A \downarrow_D^{abs} B$.*
4. **Local character:** *For any finite \bar{a} and any B there exists a finite $E \subset B$ such that $\bar{a} \downarrow_E^{abs} B$.*
5. **Extension:** *For any \bar{a} , C and B containing C there is \bar{b} such that $\text{tp}^w(\bar{b}/C) = \text{tp}^w(\bar{a}/C)$ and $\bar{b} \downarrow_C^{abs} B$.*
8. **Transitivity:** *Let $B \subset C \subset D$. If $A \downarrow_B^{abs} C$ and $A \downarrow_C^{abs} D$, then $A \downarrow_B^{abs} D$.*
12. **Bounded number of free extensions:** *There is a cardinal λ such that there are at most λ many free extensions of a type over a finite set, that is, if C is finite and \bar{a}_i , $i < \lambda^+$ realize $\text{tp}^w(\bar{a}/C)$ such that $\bar{a}_i \downarrow_C^{abs} B$ for each $i < \lambda^+$, then there are $i, j < \lambda^+$ such that $\text{tp}^w(\bar{a}_i/B) = \text{tp}^w(\bar{a}_j/B)$.*

Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple on the sense of Definition 4.4. Furthermore, for all sets A, B, C ,

$$A \downarrow_C^{abs} B \text{ if and only if } A \downarrow_C B,$$

where \downarrow is the notion from Definition 4.4.

Proof: Let \downarrow^{abs} be the notion assumed to satisfy the properties listed in the theorem. We prove that for all sets sets A, C, B

$$A \downarrow_C^{abs} B \text{ if and only if } A \downarrow_C B.$$

Hence by local character of \downarrow^{abs} , the class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple in the sense of Definition 4.4. First we prove a claim towards the result.

Claim 4.10 *Let C be finite. If $\bar{a} \downarrow_C^{abs} B$, then $\text{tp}^w(\bar{a}/C \cup B)$ does not Lascar-split over C .*

Proof: Assume to the contrary, that $\text{tp}^w(\bar{a}/C \cup B)$ does Lascar-split over C and let $(\bar{b}_i)_{i < \omega}$ be strongly C -indiscernible with $\bar{b}_0, \bar{b}_1 \in C \cup B$ and

$$\text{tp}^g(\bar{b}_0/C \cup \bar{a}) \neq \text{tp}^g(\bar{b}_1/C \cup \bar{a}).$$

Let $\kappa = \max\{\mathbb{H}, \lambda^+\}$, where λ^+ is as in item 12. We may extend the strongly C -indiscernible sequence to $(\bar{b}_i)_{i < \kappa}$. By extension, there is \bar{a}' realizing $\text{tp}^w(\bar{a}/C \cup B)$ such that $\bar{a}' \downarrow_C^{abs} B \cup (\bar{b}_i)_{i < \kappa}$. Since now also $\text{tp}^g(\bar{b}_0/C \cup \bar{a}') \neq \text{tp}^g(\bar{b}_1/C \cup \bar{a}')$, we may assume that

$$\bar{a} \downarrow_C^{abs} B \cup (\bar{b}_i)_{i < \kappa}.$$

In addition, we may assume that

$$\text{tp}^g(\bar{b}_i/C \cup \bar{a}) = \text{tp}^g(\bar{b}_1/C \cup \bar{a})$$

for each $0 < i < \kappa$. This can be done, since there are κ many i with \bar{b}_i satisfying the same Galois type over $C \cup \bar{a}$ and we must have that this type differs from either $\text{tp}^g(\bar{b}_0/C \cup \bar{a})$ or $\text{tp}^g(\bar{b}_1/C \cup \bar{a})$. We may move to a subsequence, where two first elements have different, and the rest the same, Galois type over $C \cup \bar{a}$.

Let $f_m \in \text{Aut}(\mathfrak{M}/C)$ be the automorphism given by strong indiscernibility mapping \bar{b}_i to \bar{b}_{m+i} for each $i < \kappa$. Denote $\bar{a}'_m = f_m(\bar{a})$. By Invariance we get that

$$\bar{a}'_m \downarrow_C^{abs} B \cup (\bar{b}_i)_{m \leq i < \kappa}.$$

Furthermore by Extension there is \bar{a}_m realizing $\text{tp}^w(\bar{a}'_m/C \cup B \cup (\bar{b}_i)_{m \leq i < \kappa})$ such that

$$\bar{a}_m \downarrow_C^{abs} B \cup (\bar{b}_i)_{0 \leq i < \kappa}.$$

Now we show that when $i < j < \kappa$,

$$\text{tp}^g(\bar{a}_i/C \cup \bar{b}_i) \neq \text{tp}^g(\bar{a}_j/C \cup \bar{b}_i),$$

and hence

$$\text{tp}^w(\bar{a}_i/C \cup B \cup (\bar{b}_i)_{0 \leq i}) \neq \text{tp}^w(\bar{a}_j/C \cup B \cup (\bar{b}_i)_{0 \leq i}).$$

This will prove the claim by giving more than λ many free extensions of $\text{tp}^w(\bar{a}/C)$ and hence contradicting 12. To show this, assume the contrary that $g \in \text{Aut}(\mathfrak{M}/C \cup \bar{b}_i)$ maps \bar{a}_i to \bar{a}_j . Let $h_m \in \text{Aut}(\mathfrak{M}/C \cup \bar{b}_i)$ map \bar{a}'_m to \bar{a}_m for $m \in \{i, j\}$. The automorphism

$$f_j^{-1} \circ h_j^{-1} \circ g \circ h_i \circ f_i$$

fixes $C \cup \bar{a}$ pointwise and maps \bar{b}_0 to some \bar{b}_m , where $m > 0$. This is a contradiction. \square

To prove that the two notions \downarrow^{abs} and \downarrow are equivalent, it is enough to show that for all finite tuples \bar{a} and all sets C, B ,

$$\bar{a} \downarrow_C^{abs} B \text{ if and only if } \bar{a} \downarrow_C B.$$

This is due to the definition of \downarrow and the finite character of \downarrow^{abs} . First we assume that $\bar{a} \downarrow_C^{abs} B$ and show that then also $\bar{a} \downarrow_C B$. Using local character and transitivity, we choose

finite $E \subset C$ such that $\bar{a} \downarrow_E^{abs} C \cup B$. Let D contain $C \cup B$. By extension there is \bar{a}' realizing $\text{tp}^w(\bar{a}/C \cup B)$ such that $\bar{a}' \downarrow_E^{abs} D$. By Claim 4.10, $\text{tp}^w(\bar{a}'/D)$ does not Lascar-split over E . Hence $\bar{a} \downarrow_C B$ by definition.

Then we assume that $\bar{a} \downarrow_C B$ and show that $\bar{a} \downarrow_C^{abs} B$. We assume to the contrary, that $\bar{a} \not\downarrow_C^{abs} B$. By finite character of \downarrow^{abs} we have that $\bar{a} \not\downarrow_C^{abs} \bar{b}$ for some finite $\bar{b} \in B$. By the definition of \downarrow , there is finite $E \subset C$ such that $\bar{a} \not\downarrow_E \bar{b}$. By monotonicity, $\bar{a} \not\downarrow_E^{abs} \bar{b}$.

Then we claim that the type $\text{tp}^w(\bar{b}/E)$ must be unbounded. If not, the set

$$B^* = \{\bar{b}' \in \mathfrak{M} : \text{tp}^w(\bar{b}'/E) = \text{tp}^w(\bar{b}/E)\}$$

would be bounded. Then by extension there would be \bar{a}' realizing $\text{tp}^w(\bar{a}/E)$ such that $\bar{a}' \downarrow_E^{abs} B^*$, and since E is finite, there would be $f \in \text{Aut}(\mathfrak{M}/E)$ mapping \bar{a} to \bar{a}' . But since f fixes B^* setwise, we would get $\bar{a} \downarrow_E^{abs} B^*$ by invariance and furthermore $\bar{a} \downarrow_E^{abs} \bar{b}$ by monotonicity. Hence the type $\text{tp}^w(\bar{b}/E)$ is unbounded.

Now, using Corollary 2.16, extension and invariance, we can choose a non-trivial strongly E -indiscernible sequence $(b_i)_{i < \omega}$ such that $\bar{b}_0 = \bar{b}$ and

$$\bar{b}_i \downarrow_E^{abs} \bigcup_{j < i} \bar{b}_j$$

for each $i < \omega$. Since $\bar{a} \downarrow_E \bar{b}_0$, there is \bar{a}' realizing $\text{tp}^w(\bar{a}/E \cup \bar{b}_0)$ such that the type $\text{tp}^w(\bar{a}'/E \cup \{\bar{b}_j : i < \omega\})$ does not Lascar-split over E . That implies that

$$\text{tp}^w(\bar{a}'\bar{b}_0/E) = \text{tp}^w(\bar{a}'\bar{b}_i/E)$$

for each $i < \omega$. By invariance,

$$\bar{a}' \not\downarrow_E^{abs} \bar{b}_i$$

for each $i < \omega$ and furthermore by symmetry and transitivity,

$$\bar{a}' \not\downarrow_{E \cup \{\bar{b}_j : j < i\}}^{abs} \bar{b}_i$$

for each $i < \omega$. This contradicts the local character of \downarrow^{abs} . \square

We remark that the notion \downarrow of Definition 4.4 satisfies also the property 12 (Bounded number of free extensions) when $(\mathbb{K}, \preceq_{\mathbb{K}})$ is \aleph_0 -stable and satisfies the extension property for non-splitting. This follows from the Stationarity of Lascar strong types and that the number of Lascar strong types over a finite set is at most \aleph_0 . Also in the simple, superstable case in [9] we have the property 12. There the number of Lascar strong types is strictly less than the number H and we also have stationarity for Lascar strong types. There we do not assume the extension property for non-splitting.

4.1 Extensible U -rank

In this section we assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable finitary AEC with the extension property for non-splitting. It is necessary to assume the extension property, since only under that assumption we can prove that the two notions \downarrow^s and \downarrow^s agree over \aleph_0 -saturated models. We recall the definition of extensible U -rank.

Definition 4.11 (Extensible U -rank) We say that $(\mathbb{K}, \preceq_{\mathbb{K}})$ has extensible U -rank if for each finite E , each \bar{a} and each \aleph_0 -saturated model \mathcal{A} containing E there is \bar{b} such that $\text{tp}^w(\bar{a}/E) = \text{tp}^w(\bar{b}/E)$ and $U(\bar{b}/\mathcal{A}) = U(\bar{a}/E)$.

In Theorem 6.17 of [7] we gained the the following result.

Theorem 4.12 Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a finitary AEC, stable in \aleph_0 , with extension property and extensible U -rank. Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple.

Here we add that simplicity also implies extensible U -rank. The proof is again similar to the one in [11]. First we need a technical lemma for the proof.

Lemma 4.13 Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Let \mathcal{A} be countable \aleph_0 -saturated model and $E \subset \mathcal{A}$ finite such that $\mathcal{A} \downarrow_E \bar{a}$ and B a set. Then there is $f \in \text{Aut}(\mathfrak{M}/E \cup \bar{a})$ such that $f(\mathcal{A}) \downarrow_E \bar{a} \cup B$.

Proof: Write $\mathcal{A} = \bigcup_{n < \omega} A_n$ as an increasing union of finite sets such that $E \subset A_0$. Then $A_n \downarrow_E \bar{a}$ for each $n < \omega$. We define by induction on n mappings $f_n \in \text{Aut}(\mathfrak{M}/E \cup \bar{a})$ such that $m < n$ implies $f_n \upharpoonright A_m = f_m \upharpoonright A_m$ and $f_n(A_n) \downarrow_E \bar{a} \cup \mathcal{B}$ for each $n < \omega$.

First we get f_0 from extension. Assume we have defined f_n . By invariance $f_n(A_{n+1}) \downarrow_E \bar{a}$ and by Left Transitivity we get that $f_n(A_{n+1}) \downarrow_{E \cup f_n(A_n)} \bar{a}$. Again by extension there is $g \in \text{Aut}(\mathfrak{M}/E \cup f_n(A_n) \cup \bar{a})$ such that $g(f_n(A_{n+1})) \downarrow_{E \cup f_n(A_n)} B \cup \bar{a}$. By induction and Pairs Lemma also $g(f_n(A_{n+1})) \downarrow_E B \cup \bar{a}$. We can take $f_{n+1} = g \circ f_n$.

Finally $\bigcup_{n < \omega} f_n \upharpoonright A_n : \mathcal{A} \rightarrow \mathfrak{M}$ extends to an automorphism $f' \in \text{Aut}(\mathfrak{M}/E)$. By the construction $f'(\mathcal{A}) \downarrow_E B \cup \bar{a}$. But now $f' \circ f_n^{-1} \in \text{Aut}(\mathfrak{M}/f'(A_n))$ maps \bar{a} to $f'(\bar{a})$ for each $n < \omega$, and we get that $\text{tp}^w(\bar{a}/f'(\mathcal{A})) = \text{tp}^w(f'(\bar{a})/f'(\mathcal{A}))$. By Theorem 2.21 there is $f'' \in \text{Aut}(\mathfrak{M}/f'(\mathcal{A}))$ such that $f''(f'(\bar{a})) = \bar{a}$. Since $f''(f'(\mathcal{A})) = f'(\mathcal{A})$, we can take $f = f'' \circ f' \in \text{Aut}(\mathfrak{M}/E \cup \bar{a})$. \square

We also recall the proofs of the following facts from [7].

Theorem 4.14 Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an \aleph_0 -stable finitary AEC with the extension property. Assume that $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$ are \aleph_0 -saturated. Then

1. $U(\bar{a}/\mathcal{B}) \leq U(\bar{a}/\mathcal{A})$ and
2. $\bar{a} \downarrow_{\mathcal{A}}^s \mathcal{B}$ if and only if $U(\bar{a}/\mathcal{A}) = U(\bar{a}/\mathcal{B})$.

Lemma 4.15 Assume that E is finite. Then

$$U(\bar{a}/E) = \sup\{U(\bar{a}/\mathcal{A}) : \mathcal{A} \text{ } \aleph_0\text{-saturated and countable, } E \subset \mathcal{A}\}.$$

The following proposition is similar to the one in [11].

Proposition 4.16 Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Let C be finite, \mathcal{A} an \aleph_0 -saturated model containing E and \bar{b} a tuple. Then $\bar{b} \downarrow_E \mathcal{A}$ implies that $U(\bar{b}/E) = U(\bar{b}/\mathcal{A})$.

Proof: By definition we have that

$$U(\bar{b}/\mathcal{A}) = \min\{U(\bar{b}/\mathcal{A}') : \mathcal{A}' \subset \mathcal{A}, \mathcal{A}' \aleph_0\text{-saturated and countable}\}.$$

Let \mathcal{A}' be countable and \aleph_0 -saturated such that $U(\bar{b}/\mathcal{A}') = U(\bar{b}/\mathcal{A})$. By Theorem 4.14(1) we may assume that $E \subset \mathcal{A}'$. We need to show that $U(\bar{b}/E) = U(\bar{b}/\mathcal{A}')$.

We prove that for every countable \aleph_0 -saturated model \mathcal{A} containing E such that $\bar{b} \downarrow_E \mathcal{A}$, $U(\bar{b}/E) \geq \alpha$ implies that $U(\bar{b}/\mathcal{A}) \geq \alpha$. It is enough to prove this for successor cardinals. Assume that $U(\bar{b}/E) \geq \alpha + 1$. By Lemma 4.15 there is some countable \aleph_0 -saturated \mathcal{B} such that $E \subset \mathcal{B}$ and $U(\bar{b}/E) = U(\bar{b}/\mathcal{B})$. By symmetry, $\mathcal{A} \downarrow_E \bar{b}$ and we get from Lemma 4.13 an automorphism $f \in \text{Aut}(\mathfrak{M}/E \cup \bar{b})$ such that $f(\mathcal{A}) \downarrow_E \mathcal{B} \cup \bar{b}$. Since $U(\bar{b}/\mathcal{A}) = U(\bar{b}/f(\mathcal{A}))$, we may assume that $\mathcal{A} \downarrow_E \mathcal{B} \cup \bar{b}$. From monotonicity and symmetry we get that $\bar{b} \downarrow_{\mathcal{B}} \mathcal{A} \cup \mathcal{B}$.

Now let \mathcal{B}' be countable and \aleph_0 -saturated containing both \mathcal{A} and \mathcal{B} . By extension there is \bar{b}' realizing $\text{tp}^w(\bar{b}/\mathcal{A} \cup \mathcal{B})$ such that $\bar{b}' \downarrow_{\mathcal{B}} \mathcal{B}'$. By Theorem 4.14 and since weak type preserves U -rank we get that $U(\bar{b}/\mathcal{A}) = U(\bar{b}'/\mathcal{A}) \geq U(\bar{b}'/\mathcal{B}') = U(\bar{b}'/\mathcal{B}) = U(\bar{b}/\mathcal{B}) \geq \alpha + 1$. \square

Finally we get the result.

Corollary 4.17 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ has extensible U -rank.*

Proof: Let E be finite, \bar{a} a tuple and \mathcal{A} an \aleph_0 -saturated model containing E . By extension, there is \bar{b} such that $\text{tp}^w(\bar{b}/E) = \text{tp}^w(\bar{a}/E)$ and $\bar{b} \downarrow_E \mathcal{A}$. The previous proposition implies that this \bar{b} is the one needed for the definition of extensible U -rank, since now $U(\bar{a}/E) = U(\bar{b}/E) = U(\bar{b}/\mathcal{A})$. \square

Simplicity implies the result of Theorem 4.14 holds also for finite sets $A \subset B$. For this we need another technical lemma.

Lemma 4.18 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Assume that \bar{a} is a tuple, A a finite set and \mathcal{B} an \aleph_0 -saturated model containing A . Then there is a countable \aleph_0 -saturated model $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$ such that $A \subset \mathcal{A}$ and $\mathcal{A} \downarrow_A \bar{a}$.*

Proof: We define finite sets $B_i \subset \mathcal{B}$ such that $B_0 = A$ and $B_i \downarrow_A \bar{a}$ for each $i < \omega$ and $\bigcup_{i < \omega} B_i$ has the \aleph_0 -saturation property of Proposition 3.11. Then we have that $\mathcal{A} = \bigcup_{i < \omega} B_i \preceq_{\mathbb{K}} \mathcal{B}$ and $\mathcal{A} \downarrow_A \bar{a}$. Simultaneously we define weak types p_i with domain B_i for each $i < \omega$.

First we let $B_0 = A \subset \mathcal{B}$. Then $B_0 \downarrow_A \bar{a}$ by simplicity and symmetry. Assume we have defined B_n . By, \aleph_0 -stability, there are countably many different types over B_n . Let $(\bar{b}_i^n)_{i < \omega}$ be representatives of all types over B_n . Let p_n be the type of the conjunction of tuples \bar{b}_i^n for $i, j \leq n$ over B_n .

Let $B \subset \mathcal{B}$ be finite such that $\bar{a} \cup B_n \downarrow_B \mathcal{B}$. By simplicity and extension there is B_{n+1} satisfying the type p_n such that $B_{n+1} \downarrow_{B_n} B_n \cup B$. Since \mathcal{B} is \aleph_0 -saturated, we may assume that $B_{n+1} \subset \mathcal{B}$. Now, by symmetry, $B_{n+1} \downarrow_B \bar{a} \cup B_n$. By monotonicity, $B_{n+1} \downarrow_{B_n \cup B} \bar{a}$ and by transitivity, $B_{n+1} \downarrow_{B_n} B \cup \bar{a}$. We have that $B_{n+1} \downarrow_{B_n \cup A} \bar{a}$ and $B_n \downarrow_A \bar{a}$, and since by Pairs Lemma $B_{n+1} \downarrow_A \bar{a}$.

We can easily see that $\bigcup_{i < \omega} B_i$ has the \aleph_0 -saturation property, and thus is a model by Proposition 3.11. \square

Proposition 4.19 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Let $A \subset B$ be finite sets. Then*

1. *always $U(\bar{a}/B) \leq U(\bar{a}/A)$ and*
2. *$U(\bar{a}/A) = U(\bar{a}/B)$ if and only if $\bar{a} \downarrow_A B$.*

Proof: Item 1 is clear from the definition. We prove item 2. Let \mathcal{B}' be countable and \aleph_0 -saturated containing B . By extension there is an automorphism $f \in \text{Aut}(\mathfrak{M}/B)$ such that $f(\bar{a}) \downarrow_B \mathcal{B}'$. Denote $\mathcal{B} = f^{-1}(\mathcal{B}')$. Now \mathcal{B} is a countable \aleph_0 -saturated model such that $A \subset B \subset \mathcal{B}$ and by invariance $\bar{a} \downarrow_B \mathcal{B}$. Proposition 4.16 says that $U(\bar{a}/B) = U(\bar{a}/\mathcal{B})$. Furthermore, Lemma 4.18 gives a countable \aleph_0 -saturated $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$ such that $A \subset \mathcal{A}$ and $\mathcal{A} \downarrow_A \bar{a}$. By symmetry, $\bar{a} \downarrow_A \mathcal{A}$ and again by Proposition 4.16, $U(\bar{a}/A) = U(\bar{a}/\mathcal{A})$.

Assume that $U(\bar{a}/A) = U(\bar{a}/B)$ and thus $U(\bar{a}/\mathcal{A}) = U(\bar{a}/\mathcal{B})$. Theorem 4.14 gives that $\bar{a} \downarrow_{\mathcal{A}} \mathcal{B}$, and by transitivity $\bar{a} \downarrow_A \mathcal{B}$ and by monotonicity $\bar{a} \downarrow_A B$.

Then assume that $\bar{a} \not\downarrow_A B$. By the previous reasoning we must have that $\bar{a} \not\downarrow_{\mathcal{A}} \mathcal{B}$. Then similarly by Theorem 4.14, $U(\bar{a}/A) > U(\bar{a}/B)$. \square

The following corollary will be used to prove the existence of f-primary models in section 5.

Corollary 4.20 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Let \bar{a}_i be tuples and A_i finite sets such that $A_i \subset A_{i+1}$ and $\text{tp}^w(\bar{a}_{i+1}/A_i) = \text{tp}^w(\bar{a}_i/A_i)$ for each $i < \omega$. Then there is $i < \omega$ such that $\bar{a}_{i+1} \downarrow_{A_i} A_{i+1}$.*

Proof: Assume to the contrary that $\bar{a}_{i+1} \not\downarrow_{A_i} A_{i+1}$ for each $i < \omega$. Then by the previous Proposition $U(\bar{a}_i/A_i)_{i < \omega}$ is a strictly decreasing sequence of ordinals, a contradiction. \square

4.2 Further comments on the extension property for non-splitting

In this section we study some equivalent and some sufficient conditions for the extension property for non-splitting.

We will show that simplicity together with weak κ -categoricity imply the extension property. First we write two equivalent conditions for an \aleph_0 -stable finitary AEC to satisfy the extension property. We define a concept of an abstract type p . We say that p is a *finitely realized weak type* over B , if p is a collection

$$p = \{\bar{a}_A : A \subset B \text{ finite}\},$$

such that for any finite $A \subset A' \subset B$, $\text{tp}^w(\bar{a}_{A'}/A) = \text{tp}^w(\bar{a}_A/A)$. Two types p and p' are equal if when $\bar{a}_A \in p$ and $\bar{a}'_A \in p'$, $\text{tp}^w(\bar{a}_A/A) = \text{tp}^w(\bar{a}'_A/A)$. Also when $B' \subset B$, the restriction $p \upharpoonright B'$ is defined as

$$p \upharpoonright B' = \{\bar{a}_A \in p : A \subset B'\}.$$

We also say that p is realized if there is \bar{a} such that $\text{tp}^w(\bar{a}/A) = \text{tp}^w(\bar{a}_A/A)$ for each finite $A \subset B$ and $\bar{a}_A \in p$. We recall the following result, which is Lemma 3.13 of [7].

Lemma 4.21 *Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an \aleph_0 -stable finitary AEC. Assume that \mathcal{A} is a model and $|\mathcal{A}| \leq \aleph_1$. Let p be a finitely realized type over \mathcal{A} . Then p is realized.*

The next lemma shows that for a finitely realized weak type p over a model \mathcal{A} there is a finite $E \subset \mathcal{A}$ such that p does not split over E . We define that a finitely realized weak type p over a model \mathcal{A} does not split over E , if for every finite $B \subset \mathcal{A}$ and \bar{a} realizing $p \upharpoonright (E \cup B)$, the type $\text{tp}^w(\bar{a}/E \cup B)$ does not split over E . By finite character of splitting, this is a reasonable definition.

Lemma 4.22 *Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an \aleph_0 -stable finitary AEC. Let p be a finitely realized weak type over a model \mathcal{A} . Then there is finite $E \subset \mathcal{A}$ such that p does not split over E .*

Proof: We assume the contrary. By finite character of splitting we can construct a sequence of countable models $\mathcal{A}_n \preceq_{\mathbb{K}} \mathcal{A}$, $n < \omega$ such that the type $p \upharpoonright \mathcal{A}_{n+1}$ splits over each finite $E \subset \mathcal{A}_n$. Denote $\mathcal{A}' = \bigcup_{n < \omega} \mathcal{A}_n \preceq_{\mathbb{K}} \mathcal{A}$. But now, the type $p \upharpoonright \mathcal{A}'$ is realized and splits over any finite $E \subset \mathcal{A}'$, a contradiction. \square

Finally we represent the three equivalent conditions.

Proposition 4.23 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable finitary AEC. The following are equivalent:*

1. $(\mathbb{K}, \preceq_{\mathbb{K}})$ has the extension property for non-splitting.
2. There exists an infinite cardinal κ such that every finitely realized weak type p over a weakly κ -saturated model \mathcal{A} is realized.
3. For any \aleph_0 -saturated model \mathcal{A} and any \bar{a} there is finite $E \subset \mathcal{A}$ such that the following holds: For any $B \supseteq \mathcal{A}$ there is \bar{b} realizing $\text{tp}^w(\bar{a}/\mathcal{A})$ such that $\text{tp}^w(\bar{b}/B)$ does not split over E .

Proof: Clearly 1 implies 3 by local character of splitting. First we show that 2 implies 1. Assume that \mathcal{A} is \aleph_0 -saturated and $\text{tp}(\bar{a}/\mathcal{A})$ does not split over finite $E \subset \mathcal{A}$. We would like to find a free extension of $\text{tp}^w(\bar{a}/\mathcal{A})$ to a set B containing \mathcal{A} . By increasing the size of B if necessary, we may assume that $B = \mathcal{B}$ is a weakly κ -saturated model for the κ from condition 2. Let $\mathcal{A}' \preceq_{\mathbb{K}} \mathcal{A}$ be a countable \aleph_0 -saturated model containing E . By countable extension, for any finite $A \subset \mathcal{B}$ there is \bar{a}_A realizing $\text{tp}^w(\bar{a}/\mathcal{A}')$ such that $\text{tp}^w(\bar{a}_A/\mathcal{A}' \cup A)$ does not split over E . By stationarity, when $A \subset A'$ are finite subsets of \mathcal{B} , $\text{tp}^w(\bar{a}_{A'}/\mathcal{A}' \cup A) = \text{tp}^w(\bar{a}_A/\mathcal{A}' \cup A)$. Then by 2, there is \bar{b} realizing the type $\text{tp}^w(\bar{a}_A/A)$ for each finite $A \subset \mathcal{B}$. By finite character of splitting, the type $\text{tp}^w(\bar{b}/\mathcal{B})$ does not split over E , and again by stationarity, $\text{tp}^w(\bar{b}/\mathcal{A}) = \text{tp}^w(\bar{a}/\mathcal{A})$.

Finally we show that 3 implies 2, where we take $\kappa = \aleph_0$. Let p be a finitely realized type over a \aleph_0 -saturated model \mathcal{A} . By Lemma 4.22 there is finite $A \subset \mathcal{A}$ such that p does not split over A . Let $\mathcal{A}' \preceq_{\mathbb{K}} \mathcal{A}$ be countable \aleph_0 -saturated model containing A and let \bar{a} realize $p \upharpoonright \mathcal{A}'$. By 3, there is finite $E \subset \mathcal{A}'$ and \bar{b} such that \bar{b} realizes $\text{tp}^w(\bar{a}/\mathcal{A}')$ and $\text{tp}^w(\bar{b}/\mathcal{A})$ does not split over E . But now by stationarity, \bar{b} realizes p . \square

We say that (K, \preceq_K) is *strongly* \aleph_0 -stable there are only countably many different Lascar strong types over a countable set E . Strong \aleph_0 -stability implies \aleph_0 -stability, since having the same weak type over a countable set E is an E -invariant equivalence relation with a bounded number of classes. In [7], we showed that an \aleph_0 -stable finitary AEC with the extension property is strongly \aleph_0 -stable. We show that also without the extension

property, strong \aleph_0 -stability is implied by weak categoricity in some uncountable κ . We recall from [7], that whenever $(\bar{a}_i)_{i < \alpha}$ is a strongly E -indiscernible sequence, we have that $\text{Lstp}(\bar{a}_i/E) = \text{Lstp}(\bar{a}_j/E)$ for each $i < j < \alpha$. This is based only on the definitions of strong indiscernibility and the Lascar strong type.

Proposition 4.24 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a finitary AEC and weakly categorical in an uncountable cardinal κ . Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ is strongly \aleph_0 -stable.*

Proof: Let E be a countable set and let $(\bar{b}_i)_{i < \omega_1}$ be a sequence of distinct tuples. We want to find $i < j < \omega_1$ such that $\text{Lstp}(\bar{b}_i/E) = \text{Lstp}(\bar{b}_j/E)$. Let $EM(\kappa \cup E)$ be a model as in Proposition 2.13, where κ is a wellorder of length κ . By weak κ -categoricity, the model $EM(\kappa \cup E)$ is weakly \aleph_1 -saturated, and then also \aleph_1 -saturated respect to Galois types. We are able to construct an automorphism mapping the sequence $(\bar{b}_i)_{i < \omega_1}$ into $EM(\kappa \cup E)$ and fixing E pointwise. Since having the same Lascar strong type over E is invariant under automorphisms fixing E , we may assume that $(\bar{b}_i)_{i < \omega_1} \subset EM(\kappa \cup E)$.

Since τ^* and E are countable, we may assume that each \bar{b}_j is generated similarly and with the same sequence of terms of τ^* from a fixed finite subset of E and some $(a_{\alpha_0^j}, \dots, a_{\alpha_n^j})$ for $\alpha_0^j < \dots < \alpha_n^j < \kappa$. By Lemma 2.19 there is a subsequence $(\bar{b}_{j_k})_{k < \omega_1}$ such that it is a strongly E -indiscernible sequence. Thus $\text{Lstp}(\bar{b}_{i_k}/E) = \text{Lstp}(\bar{b}_{i_p}/E)$ for all $k < p < \omega_1$. \square

We recall the group $\text{Saut}(\mathfrak{M}/E)$, which is a normal subgroup of $\text{Aut}(\mathfrak{M}/E)$ such that $f \in \text{Saut}(\mathfrak{M}/E)$ if for all \bar{a} , $\text{Lstp}(\bar{a}/E) = \text{Lstp}(f(\bar{a})/E)$. We say that an $f \in \text{Saut}(\mathfrak{M}/E)$ is a strong automorphism. We also recall that a model \mathcal{A} is said to be a -saturated, if for any finite $E \subset \mathcal{A}$ and \bar{a} , there is $\bar{b} \in \mathcal{A}$ such that $\text{Lstp}(\bar{b}/E) = \text{Lstp}(\bar{a}/E)$. Imitating the proofs of similar results in [7], we are able to prove the following.

Lemma 4.25 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a strongly \aleph_0 -stable finitary AEC. Then we have that*

1. *Let E be countable. Then $\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E)$ if and only if there is $f \in \text{Saut}(\mathfrak{M}/E)$ such that $f(\bar{a}) = \bar{b}$.*
2. *Each \aleph_0 -saturated model is also a -saturated.*
3. *For countable E and $\bar{a} \neq \bar{b}$, $\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E)$ if and only if there exists $n < \omega$, \bar{a}_i and strongly E -indiscernible sequences I_i for $i \leq n$ such that $\bar{a}_0 = \bar{a}$, $\bar{a}_n = \bar{b}$ and $\bar{a}_i, \bar{a}_{i+1} \in I_i$ for $i < n$.²*
4. *Let \mathcal{A} be an \aleph_0 -saturated model and $E \subset \mathcal{A}$ finite. Then $\text{tp}^w(\bar{a}/\mathcal{A})$ Lascar-splits over E if and only if there are $\bar{c}, \bar{d} \in \mathcal{A}$ such that $\text{Lstp}(\bar{c}/E) = \text{Lstp}(\bar{d}/E)$ and $\text{tp}^w(\bar{c}/E \cup \bar{a}) \neq \text{tp}^w(\bar{d}/E \cup \bar{a})$.*
5. *Assume that $\mathcal{A} \subset \mathcal{B}$ are \aleph_0 -saturated. Let \bar{a} and \bar{b} be such that $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$, the type $\text{tp}^w(\bar{a}/\mathcal{B})$ does not Lascar-split over finite $E \subset \mathcal{A}$ and the type $\text{tp}^w(\bar{b}/\mathcal{B})$ does not split over finite $E' \subset \mathcal{A}$. Then we have that $\text{tp}^w(\bar{a}/\mathcal{B}) = \text{tp}^w(\bar{b}/\mathcal{B})$.*

²In the proof of item 3 we should use Corollary 2.16 instead of Corollary 6.10 of [7].

Finally we are able to show the following.

Proposition 4.26 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple finitary AEC and weakly categorical in some uncountable κ . Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ has the extension property.*

Proof: By Proposition 4.24, the class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is strongly \aleph_0 -stable and hence also \aleph_0 -stable. We will show that condition 2 of Proposition 4.23 holds. For this, let p be a finitely realized type over an \aleph_0 -saturated model \mathcal{A} .

By Lemma 4.22 there is finite $E \subset \mathcal{A}$ such that p does not split over E . Then p does not Lascar-split over E neither. Let $\mathcal{A}_0 \preceq_{\mathbb{K}} \mathcal{A}$ be countable and \aleph_0 -saturated such that $E \subset \mathcal{A}_0$. Then by Lemma 4.21 there is \bar{a} realizing $p \upharpoonright_{\mathcal{A}_0}$. But now by simplicity, $\bar{a} \downarrow_{\mathcal{A}_0} \mathcal{A}_0$ and thus there is finite $E' \subset \mathcal{A}_0$ and \bar{b} such that $\text{tp}^w(\bar{b}/\mathcal{A}_0) = \text{tp}^w(\bar{a}/\mathcal{A}_0)$ and $\text{tp}^w(\bar{b}/\mathcal{A})$ does not Lascar-split over E' . But now by 5 of Lemma 4.25, \bar{b} realizes $p \upharpoonright_{\mathcal{B}}$ for each countable \aleph_0 -saturated $\mathcal{A}_0 \preceq_{\mathbb{K}} \mathcal{B} \preceq_{\mathbb{K}} \mathcal{A}$, and thus realizes p . \square

5 F-primary models

In this section we introduce f-primary models. The notion imitates the notion of F^f -primary in [20]. The existence of f-primary models is implied by simplicity, and they can be used to show that weak categoricity implies every \aleph_0 -saturated model being weakly saturated. Again we will assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable finitary AEC with the extension property.

To construct F-primary models we need to assume both simplicity and the extension property for non-splitting, mainly to use Corollary 4.20. In [9] we construct a-primary models where the use of splitting and U -rank is replaced with the use of Lascar strong types. There we have a different notion of isolation and the use of Corollary 4.20 is replaced with a different result (see Proposition 3.11 of [9]).

Definition 5.1 (F-isolated) *Let \bar{a} be a tuple and A a set. A weak type $\text{tp}^w(\bar{a}/A)$ is f-isolated over $E \subset A$ if whenever \bar{b} realizes $\text{tp}^w(\bar{a}/E)$, then $\bar{b} \downarrow_E A$.*

We remark that if $\text{tp}^w(\bar{a}/A)$ is f-isolated over $E \subset A$ and $E \subset E' \subset A$, then $\text{tp}^w(\bar{a}/A)$ is also f-isolated over E' .

Definition 5.2 (F-primary) *We say that \mathcal{A} is f-primary over a set A for some ordinal ξ there are tuples \bar{a}_i and finite sets A_i for $i < \xi$ such that*

1. *the weak type $\text{tp}^w(\bar{a}_i/A \cup \bigcup_{j < i} \bar{a}_j)$ is f-isolated over $A_i \subset A \cup \bigcup_{j < i} \bar{a}_j$ and*
2. *$\mathcal{A} = A \cup \bigcup_{i < \xi} \bar{a}_i$ has the \aleph_0 -saturation property.*

Lemma 5.3 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. For every tuple \bar{a} , set A and finite $B \subset A$ there is \bar{b} and finite $A_0 \subset A$ such that $\text{tp}^w(\bar{b}/B) = \text{tp}^w(\bar{a}/B)$ and $\text{tp}^w(\bar{b}/A)$ is f-isolated over A_0 .*

Proof: Assume that \bar{a} , A and finite $B \subset A$ would witness the contrary. We define tuples \bar{a}_i and finite sets A_i for $i < \omega$ to contradict Corollary 4.20. First let $\bar{a}_0 = \bar{a}$ and $A_0 = B$. Then assume we have defined \bar{a}_n and A_n for $i \leq n$ such that

1. $\text{tp}^w(\bar{a}_i/B) = \text{tp}^w(\bar{a}/B)$,
2. sets A_i are finite and $A_i \subset A_{i+1} \subset A$,
3. $\text{tp}^w(\bar{a}_{i+1}/A_i) = \text{tp}^w(\bar{a}_i/A_i)$ and
4. $\bar{a}_{i+1} \not\downarrow_{A_i} A_{i+1}$.

Since we have 1, the type $\text{tp}^w(\bar{a}_n/A)$ can't be f-isolated over finite $A_n \subset A$. Thus there is a tuple \bar{a}_{n+1} such that $\text{tp}^w(\bar{a}_{n+1}/A_n) = \text{tp}^w(\bar{a}_n/A_n)$ but $\bar{a}_{n+1} \not\downarrow_{A_n} A$. Furthermore, by finite character of independence, there is finite $A_{n+1} \subset A$ such that $\bar{a}_{n+1} \not\downarrow_{A_n} A_{n+1}$. We may assume that $A_n \subset A_{n+1}$. This construction contradicts Corollary 4.20. \square

Lemma 5.4 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. For every set A there is an \aleph_0 -saturated model \mathcal{B} of size $|A|$ such that it is f-primary over A . Furthermore, if \mathcal{B}' is \aleph_0 -saturated model containing A , we can choose such \mathcal{B} that $\mathcal{B} \preceq_{\mathbb{K}} \mathcal{B}'$.*

Proof: We prove the last claim. Denote $|A| = \lambda$. By induction on $n < \omega$ we define sets $B_n \subset \mathcal{B}'$ of size λ , tuples $\bar{a}_i^n \in \mathcal{B}'$ and finite sets $A_i^n \subset \mathcal{B}'$ for $i < \lambda$. First let $B_0 = A$.

Assume we have defined B_n . Let $(\text{tp}^w(\bar{b}_i/D_i))_{i < \lambda}$ enumerate all weak types over finite subsets of B_n . Such enumeration exists by \aleph_0 -stability. Let $i < \lambda$ and assume we have defined \bar{a}_j^n for $j < i$. We use the previous lemma to find a tuple \bar{d}_i^n realizing $\text{tp}^w(\bar{b}_i/D_i)$ and a finite subset $A_i^n \subset B_n \cup \bigcup_{j < i} \bar{a}_j^n$ such that $\text{tp}^w(\bar{d}_i^n/B_n \cup \bigcup_{j < i} \bar{a}_j^n)$ is f-isolated over A_i^n . By \aleph_0 -saturation, there is $\bar{a}_i^n \in \mathcal{B}'$ realizing $\text{tp}^w(\bar{d}_i^n/D_i \cup A_i^n)$. Then also $\text{tp}^w(\bar{a}_i^n/B_n \cup \bigcup_{j < i} \bar{a}_j^n)$ is f-isolated over A_i^n . Finally let $B_{n+1} = B_n \cup \bigcup_{i < \lambda} \bar{a}_i^n$.

Each weak type over a finite subset of B_n is realized in B_{n+1} . Thus $\mathcal{B} = \bigcup_{n < \omega} B_n = A \cup \bigcup_{(n,i) \in \omega \times \lambda} \bar{a}_i^n$ has the \aleph_0 -saturation property. We have that \mathcal{B} is an f-primary model over A and is of size λ . \square

Lemma 5.5 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Let \mathcal{A} be \aleph_0 -saturated, $A_0, A_1 \subset \mathcal{A}$ finite and*

1. $\text{tp}^w(\bar{a}_0/\mathcal{A} \cup \bar{b})$ is f-isolated over $A_0 \cup \bar{b}$ and
2. $\text{tp}^w(\bar{a}_1/\mathcal{A} \cup \bar{b} \cup \bar{a}_0)$ f-isolated over $A_1 \cup \bar{b} \cup \bar{a}_0$.

Then there is finite $A \subset \mathcal{A}$ such that $\text{tp}^w(\bar{a}_0 \cup \bar{a}_1/\mathcal{A} \cup \bar{b})$ is f-isolated over $A \cup \bar{b}$.

Proof: Let $A \subset \mathcal{A}$ be finite such that $A_0 \cup A_1 \subset A$ and $\bar{b} \downarrow_A \mathcal{A}$. We claim that this is the set as wanted. To prove the claim, we assume the contrary. Let \bar{c}_0 and \bar{c}_1 be such that $\text{tp}^w(\bar{c}_0 \cup \bar{c}_1/A \cup \bar{b}) = \text{tp}^w(\bar{a}_0 \cup \bar{a}_1/A \cup \bar{b})$ but $\bar{c}_0 \cup \bar{c}_1 \not\downarrow_{A \cup \bar{b}} \mathcal{A}$. By finite character there is finite $\bar{d} \in \mathcal{A}$ such that $\bar{c}_0 \cup \bar{c}_1 \not\downarrow_{A \cup \bar{b}} \bar{d}$. From 1 we get that $\bar{c}_0 \downarrow_{A \cup \bar{b}} \mathcal{A}$. Since $\bar{b} \downarrow_A \mathcal{A}$, we get from symmetry and transitivity that $\mathcal{A} \downarrow_A \bar{b} \cup \bar{c}_0$. Similarly $\mathcal{A} \downarrow_A \bar{b} \cup \bar{a}_0$.

Let $f \in \text{Aut}(\mathfrak{M}/A \cup \bar{b})$ be such that $f(\bar{c}_0) = \bar{a}_0$. By Proposition 4.8, there is $\bar{d}^* \in \mathcal{A}$ such that $\text{Lstp}(\bar{d}^*/A) = \text{Lstp}(f(\bar{d})/A)$. Since $\bar{d}^* \in \mathcal{A}$, we have $\bar{d}^* \downarrow_A \bar{b} \cup \bar{a}_0$. Furthermore $\bar{d} \downarrow_A \bar{b} \cup \bar{c}_0$ implies that $f(\bar{d}) \downarrow_A \bar{b} \cup \bar{a}_0$. From stationarity for Lascar strong types we get

that $\text{tp}^w(\bar{d}^*/A \cup \bar{b} \cup \bar{a}_0) = \text{tp}^w(f(\bar{d})/A \cup \bar{b} \cup \bar{a}_0)$ and furthermore $\text{tp}^w(\bar{d}^* \cup \bar{a}_0/A \cup \bar{b}) = \text{tp}^w(\bar{d} \cup \bar{c}_0/A \cup \bar{b})$. Let \bar{a}'_1 be such that

$$\text{tp}^w(\bar{d}^* \cup \bar{a}_0 \cup \bar{a}'_1/A \cup \bar{b}) = \text{tp}^w(\bar{d} \cup \bar{c}_0 \cup \bar{c}_1/A \cup \bar{b}).$$

Since $\bar{a}'_1 \cup \bar{a}_0 \not\downarrow_{A \cup \bar{b}} \bar{d}^*$ and $\bar{a}_0 \downarrow_{A \cup \bar{b}} \bar{d}^*$, Pairs Lemma implies that $\bar{a}'_1 \not\downarrow_{A \cup \bar{b} \cup \bar{a}_0} \bar{d}^*$, and thus by monotonicity $\bar{a}'_1 \not\downarrow_{A_1 \cup \bar{b} \cup \bar{a}_0} \bar{d}^* \cup A$. Now we have that $\text{tp}^w(\bar{a}_1/A_1 \cup \bar{b} \cup \bar{a}_0) = \text{tp}^w(\bar{a}'_1/A_1 \cup \bar{b} \cup \bar{a}_0)$ but $\bar{a}'_1 \not\downarrow_{A_1 \cup \bar{b} \cup \bar{a}_0} \mathcal{A}$, a contradiction with 2. \square

Lemma 5.6 *Assume that $\mathcal{A} = A \cup \bigcup_{i < \xi} \bar{a}_i$ is f -primary over A and $\bar{d} \in \mathcal{A}$. Then there is finite $A' \subset A$ and $i_0 < \dots < i_n < \xi$ such that $\bar{d} \subset A' \cup \bar{a}_0, \dots, \bar{a}_n$ and for each $0 \leq p \leq n$ there is finite $A_p \subset A' \cup \bar{a}_0, \dots, \bar{a}_{p-1}$ such that $\text{tp}^w(\bar{a}_p/A \cup \bar{a}_0, \dots, \bar{a}_{p-1})$ is f -isolated over A_p .*

Proof: First let $B_0 \cup \bar{a}_{i_{(0,0)}} < \dots < \bar{a}_{i_{(0,k_0)}}$ be such that $B_0 \subset A$ and the finite tuple \bar{d} is included in these. For each $0 \leq p \leq k_0$ there is finite $A_{(0,p)} \subset A \cup \bigcup_{j < i_{(0,p)}} \bar{a}_j$ such that $\text{tp}^w(\bar{a}_{i_{(0,p)}}/A \cup \bigcup_{j < i_{(0,p)}} \bar{a}_j)$ is f -isolated over $A_{(0,p)}$. We let $B_1 \cup \bar{a}_{i_{(1,0)}} < \dots < \bar{a}_{i_{(1,k_1)}}$, where $B_1 \subset A$, contain all tuples included in the sets $A_{(0,p)}$. Then the sets $A_{(1,p)} \subset A \cup \bigcup_{j < i_{(1,p)}} \bar{a}_j$ are defined similarly. We continue this and define a tree of ordinals such that $i_{(n+1,p)} <_T i_{(n,p')}$ if $\bar{a}_{i_{(n+1,p)}}$ is included in $A_{(n,p')}$. We get tree whose branches go down the ordinals and each level is finite. Since there can't be an infinite branch, the tree must be finite. Let A' contain all sets B_m for those m appearing in the tree and let $\bar{a}_0, \dots, \bar{a}_n$ contain all $\bar{a}_{i_{(m,p)}}$ similarly. \square

Proposition 5.7 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Let \mathcal{A} be \aleph_0 -saturated and B a set. Let $\mathcal{A}^* = \mathcal{A} \cup B \cup \bigcup_{i < \xi} \bar{a}_i$ be f -primary over $\mathcal{A} \cup B$ and let \bar{d} be a tuple in \mathcal{A}^* . Then there are $\bar{a} = \bar{a}_{i_0}, \dots, \bar{a}_{i_n}$ for $i_0 < \dots < i_n < \xi$, finite $A \subset \mathcal{A}$ and $\bar{b} \subset B$ such that*

1. $\bar{d} \subset A \cup \bar{b} \cup \bar{a}$,
2. $\text{tp}^w(\bar{a}/\mathcal{A} \cup \bar{b})$ is f -isolated over $A \cup \bar{b}$ and
3. the tuple \bar{b} dominates $\bar{a} \cup \bar{b}$ over \mathcal{A} .

Proof: Let $A' \cup \bar{b} \subset \mathcal{A} \cup B$, A_p for $0 \leq p \leq n$ and $i_0 < \dots < i_n < \xi$ be as in the previous Lemma. We show by induction on $m \leq n$ that there is finite $A'_m \subset \mathcal{A}' \cup \bar{b} \cup \bar{a}_{i_0}, \dots, \bar{a}_{i_{n-m-1}}$ such that

$$\text{tp}^w(\bar{a}_{i_{n-m}} \cup \dots \cup \bar{a}_{i_n}/\mathcal{A}' \cup \bar{b} \cup \bar{a}_{i_0}, \dots, \bar{a}_{i_{n-m-1}}) \text{ is } f\text{-isolated over } A'_m.$$

When $m = 0$, $n - m = n$ and the claim is clear. Assume the claim holds for m . Then

1. $\text{tp}^w(\bar{a}_{i_{n-m}} \cup \dots \cup \bar{a}_{i_n}/\mathcal{A}' \cup \bar{b} \cup \bar{a}_{i_0}, \dots, \bar{a}_{i_{n-(m+1)}})$ is f -isolated over A'_m and
2. $\text{tp}^w(\bar{a}_{i_{n-(m+1)}}/\mathcal{A}' \cup \bar{b} \cup \bar{a}_{i_0}, \dots, \bar{a}_{i_{n-(m+1)-1}})$ is f -isolated over $A_{n-(m+1)}$.

Item 1 holds by induction and item 2 by the choice of $A_{n-(m+1)}$. We find $A'_{m+1} \subset \mathcal{A} \cup \bar{b} \cup \bar{a}_{i_0}, \dots, \bar{a}_{i_{n-(m+1)-1}}$ by Lemma 5.5. We can take $A = A'_n$. This proves 1 and 2.

To prove 3, assume to the contrary, that $\bar{c} \downarrow_{\mathcal{A}} \bar{b}$ but $\bar{c} \not\downarrow_{\mathcal{A}} \bar{b} \cup \bar{a}$ for some tuple \bar{c} . Let $A' \subset \mathcal{A}$ be finite such that $A \subset A'$ and $\bar{c} \downarrow_{A'} \mathcal{A} \cup \bar{b}$. Since $A \subset A'$, the type $\text{tp}^w(\bar{a}/\mathcal{A} \cup \bar{b})$ is f-isolated over $A' \cup \bar{b}$. By finite character, there is finite B' such that $A' \subset B' \subset \mathcal{A}$ and $\bar{c} \not\downarrow_{A'} B' \cup \bar{a} \cup \bar{c}$. By symmetry and Lemma 4.6, $\bar{b} \downarrow_{\mathcal{A}}^s \bar{c}$. We may assume that $\text{tp}^w(\bar{b}/\mathcal{A} \cup \bar{c})$ does not split over B' .

Since \mathcal{A} is \aleph_0 -saturated, there is $f \in \text{Aut}(\mathfrak{M}/B')$ such that $f(\bar{c}) \in \mathcal{A}$. Since $\text{tp}^w(\bar{b}/\mathcal{A} \cup \bar{c})$ does not split over B' , we may assume that $f(\bar{b}) = \bar{b}$. By f-isolation we get that $f(\bar{a}) \downarrow_{A' \cup \bar{b}} \mathcal{A}$, and furthermore $f(\bar{a}) \downarrow_{A' \cup \bar{b}} f(\bar{c}) \cup B'$. On the other hand, by monotonicity and invariance $f(\bar{c}) \downarrow_{A'} B' \cup \bar{b}$, and by symmetry, $\bar{b} \cup B' \downarrow_{A'} f(\bar{c})$. Now by the Pairs Lemma, $f(\bar{a}) \cup \bar{b} \cup B' \downarrow_{A'} f(\bar{c})$.

But since $\bar{c} \not\downarrow_{A'} \bar{a} \cup \bar{b} \cup B'$, by invariance $f(\bar{c}) \not\downarrow_{A'} f(\bar{a}) \cup \bar{b} \cup B'$, and by symmetry, $f(\bar{a}) \cup \bar{b} \cup B' \not\downarrow_{A'} f(\bar{c})$, a contradiction. \square

By finite character we gain the following corollary.

Corollary 5.8 *Let $\mathcal{B} = \mathcal{A} \cup B \cup (\bar{a}_i)_{i < \xi}$ be an f-primary model over $\mathcal{A} \cup B$, where \mathcal{A} is an \aleph_0 -saturated model. Then B dominates \mathcal{B} over the model \mathcal{A} .*

We prove here also an easy remark using Pairs Lemma.

Remark 5.9 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Assume that $\bar{a}_p \downarrow_{A \cup \bigcup_{0 \leq k < p} \bar{a}_k} B$ for all $0 \leq p \leq n$. Then $\bar{a}_0 \cup \dots \cup \bar{a}_n \downarrow_A B$.*

Proof: By induction on p we prove that

$$\bar{a}_{n-p} \cup \dots \cup \bar{a}_n \downarrow_{A \cup \bigcup_{0 \leq k < n-p} \bar{a}_k} B.$$

First when $p = 0$, $n - p = n$, and the claim follows from the assumption. Then assume that the claim holds for p . In addition we have that $\bar{a}_{n-(p+1)} \downarrow_{A \cup \bigcup_{0 \leq k < n-(p+1)} \bar{a}_k} B$, and thus the claim for $p + 1$ follows from Pairs Lemma. \square

5.1 Categoricity transfer

In the next proposition we prove the ‘weak categoricity transfer’ for \aleph_0 -saturated models.

Proposition 5.10 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ a simple finitary AEC with the extension property, weakly categorical in some uncountable cardinal κ . Assume that \mathcal{A} is an uncountable \aleph_0 -saturated model. Let $B \subset \mathcal{A}$ be such that $|B| < |\mathcal{A}|$ and let $\bar{d} \in \mathfrak{M}$. Then the type $\text{tp}^w(\bar{d}/B)$ is realized in \mathcal{A} .*

Proof: It is enough to prove the claim for all such \mathcal{A} that $|\mathcal{A}|$ is a successor cardinal. If $|\mathcal{A}|$ is a limit, there is an \aleph_0 -saturated model $B \subset \mathcal{A}' \preceq_{\mathbb{K}} \mathcal{A}$ of size $|B|^+$, and $\text{tp}^w(\bar{d}/B)$ is realized in \mathcal{A}' .

By Lemma 3.19 there is a Morley sequence $(b_i)_{i < \aleph_1} \subset \mathcal{A}$ over an \aleph_0 -saturated model $\mathcal{B} \subset \mathcal{A}$ containing B . Also there is finite $E \subset \mathcal{B}$ such that $\text{tp}^w(b_i/\mathcal{B} \cup \bigcup_{j < i} b_j)$ does not

split over E for all $i < \aleph_1$. By local character there is finite $E' \subset \mathcal{B}$ such that $\bar{d} \downarrow_{E'}^s \mathcal{B}$. Let $\mathcal{C} \preceq_{\mathbb{K}} \mathcal{B}$ be countable and \aleph_0 -saturated model containing $E \cup E'$. Then $\bar{d} \downarrow_{\mathcal{C}}^s \mathcal{B}$ and $b_i \downarrow_{\mathcal{C}}^s \mathcal{B} \cup \bigcup_{j < i} b_j$ for all $i < \aleph_1$. We show that $\text{tp}^w(\bar{d}/\mathcal{B})$ is realized in \mathcal{A} .

Using extension we continue the Morley sequence to $(b_i)_{i < \kappa}$, where κ is the (weak) categoricity cardinal. Let

$$\mathcal{C}^* = \mathcal{C} \cup \bigcup_{i < \kappa} b_i \cup \bigcup_{j < \xi} \bar{a}_j$$

be f-primary over $\mathcal{C} \cup \bigcup_{i < \kappa} b_i$. By weak categoricity the model \mathcal{C}^* is \aleph_1 -saturated and thus the weak type $\text{tp}^w(\bar{d}/\mathcal{C})$ is realized in \mathcal{C}^* . We find finite $A \subset \mathcal{C}$, $\bar{b} = b_{i_0} \cup \dots \cup b_{i_m}$, $i_0 < \dots < i_m < \kappa$ and $\bar{a} = \bar{a}_{j_0}, \dots, \bar{a}_{j_n}$, $j_0 < \dots < j_n < \xi$ as in Proposition 5.7. Denote $\bar{b}^* = b_0, \dots, b_m \in \mathcal{A}$. By Lemma 3.20 and since \mathcal{C} is a countable model, there is an automorphism $f \in \text{Aut}(\mathfrak{M}/\mathcal{C})$ mapping b_{i_k} to b_k for each $0 \leq k \leq m$. Proposition 4.8 gives $\bar{a}^* \in \mathcal{A}$ such that

$$\text{Lstp}(\bar{a}^*/A \cup \bar{b}^*) = \text{Lstp}(f(\bar{a})/A \cup \bar{b}^*).$$

By 2 of Proposition 5.7 and invariance, $\text{tp}^w(f(\bar{a})/\mathcal{C} \cup \bar{b}^*)$ is f-isolated over $A \cup \bar{b}^*$ and thus $\bar{a}^* \downarrow_{A \cup \bar{b}^*} \mathcal{C}$. By stationarity of Lascar strong types \bar{a}^* realizes $\text{tp}^w(f(\bar{a})/\mathcal{C} \cup \bar{b}^*)$. Then 3 of Proposition 5.7 gives that \bar{b}^* dominates $\bar{a}^* \cup \bar{b}^*$ over \mathcal{C} . But we had that $b_i \downarrow_{\mathcal{C}}^s \mathcal{B} \cup \bigcup_{j < i} b_j$ for all $i < \aleph_1$, and thus by Remark 5.9, $\bar{b}^* \downarrow_{\mathcal{C}}^s \mathcal{B}$. Using symmetry we gain that $\bar{a}^* \cup \bar{b}^* \downarrow_{\mathcal{C}} \mathcal{B}$.

We had that $\text{tp}^w(\bar{d}/\mathcal{C})$ was realized in $\bar{a} \cup \bar{b} \cup A$, where $A \subset \mathcal{C}$, and $\text{tp}^w(\bar{a} \cup \bar{b}/\mathcal{C}) = \text{tp}^w(f(\bar{a}) \cup \bar{b}^*/\mathcal{C}) = \text{tp}^w(\bar{a}^* \cup \bar{b}^*/\mathcal{C})$. We claim that $\text{tp}^w(\bar{d}/\mathcal{B})$ is realized in $\bar{a}^* \cup \bar{b}^* \cup A \subset \mathcal{A}$. But this follows from stationarity over \aleph_0 -saturated models, since $\bar{d} \downarrow_{\mathcal{C}} \mathcal{B}$ and $\bar{a}^* \cup \bar{b}^* \downarrow_{\mathcal{C}} \mathcal{B}$. \square

Finally we state the main theorem. For an abstract elementary class $(\mathbb{K}, \preceq_{\mathbb{K}})$, we define

$$((\mathbb{K})^\omega, \preceq_{\mathbb{K}}) = \{\mathcal{A} \in \mathbb{K} : \mathcal{A} \text{ is } \aleph_0\text{-saturated}\}.$$

If $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable finitary AEC with the extension property, then clearly also the class $((\mathbb{K})^\omega, \preceq_{\mathbb{K}})$ is. We formulate the ‘weak categoricity transfer’ as follows.

Theorem 5.11 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple finitary AEC and weakly categorical in some uncountable cardinal κ . Then*

1. $((\mathbb{K})^\omega, \preceq_{\mathbb{K}})$ is weakly categorical in each uncountable κ and
2. $(\mathbb{K}, \preceq_{\mathbb{K}})$ is weakly categorical in each λ such that $\lambda \geq \min\{\kappa, \mathfrak{H}\}$.

Proof: By Theorem 2.24 and Proposition 4.26, the class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is also \aleph_0 -stable and has the extension property. The first claim follows from Proposition 5.10. For the second claim, we need to show that each model of size $\geq \min\{\kappa, \mathfrak{H}\}$ is \aleph_0 -saturated. If $\kappa \leq \mathfrak{H}$, this is clear, since the model of size κ is \aleph_0 -saturated. But also each model of size $\geq \mathfrak{H}$ is \aleph_0 -saturated by Lemma 2.26. \square

We recall the definition of tameness. The class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is tame (or LS(\mathbb{K})-tame), if whenever \mathcal{A} is a model and

$$\text{tp}^g(\bar{a}/\mathcal{A}) \neq \text{tp}^g(\bar{b}/\mathcal{A}),$$

there is a submodel $\mathcal{B} \preceq_{\mathbb{K}} \mathcal{A}$ of size $\text{LS}(\mathbb{K})$ such that

$$\text{tp}^g(\bar{a}/\mathcal{B}) \neq \text{tp}^g(\bar{b}/\mathcal{B}).$$

With tameness the result of Theorem 2.21 generalizes as follows.

Theorem 5.12 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable tame finitary AEC and \mathcal{A} is a model. Then $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$ if and only if $\text{tp}^g(\bar{a}/\mathcal{A}) = \text{tp}^g(\bar{b}/\mathcal{A})$.*

We recall that an \aleph_0 -stable and tame finitary AEC always has the extension property. The previous theorem guarantees that unions of types over models behave well, and thus the condition 2 of Proposition 4.23 holds. This was stated in [7] as Theorem 4.13.

Theorem 5.13 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable tame finitary AEC. Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ has the extension property.*

Let κ be an uncountable cardinal. By Corollary 3.8, κ -categoricity always implies weak κ -categoricity for an \aleph_0 -stable finitary AEC. Now Theorem 5.12 together with Lemma 2.22 imply that if $(\mathbb{K}, \preceq_{\mathbb{K}})$ is also tame, weak κ -categoricity is equivalent with κ -categoricity. As a corollary we get the following.

Corollary 5.14 *Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple tame finitary AEC categorical in some uncountable κ . Then*

1. $(\mathbb{K}^\omega, \preceq_{\mathbb{K}})$ is categorical in each uncountable κ and
2. $(\mathbb{K}, \preceq_{\mathbb{K}})$ is categorical in each λ such that $\lambda \geq \min\{\kappa, \mathfrak{H}\}$.

We remark that if in the previous theorem $(\mathbb{K}, \preceq_{\mathbb{K}})$ happens to be \aleph_0 -categorical (equivalently *atomic*, see Definition 6.1), then we get the full Morley theorem for $(\mathbb{K}, \preceq_{\mathbb{K}})$. That is, categoricity in any uncountable cardinal transfers to total categoricity. In this case, the class is also definable in $L_{\omega_1\omega}$ by Kueker [14]. By earlier work by Shelah and Keisler, the full Morley theorem holds also for $L_{\omega_1\omega}$ -definable classes which are either *excellent* or *homogeneous*. Such classes are tame finitary AECs but they are not necessarily simple. Also neither excellence nor homogeneity is known to follow from the assumptions of Corollary 5.14, except as in Corollary 5.15. Hence our theorem gives a new Morley theorem for sentences in $L_{\omega_1\omega}$ which is incomparable to the previous results. We will study the relation between the three $L_{\omega_1\omega}$ -definable frameworks in the following section.

We should remark that the assumptions of 5.14 *do* imply excellence using the corollary 5.14 itself. This is based on the famous result by Shelah that under certain set-theoretic assumptions, if a sentence in $L_{\omega_1\omega}$ is categorical in each \aleph_n for $n < \omega$, then it is excellent. See the book [1] for the definition of excellence and proof of the result mentioned above.

Corollary 5.15 *Assume that $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for each $n < \omega$. Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple tame finitary AEC categorical in \aleph_0 and in some uncountable κ . Then it is excellent.*

6 Examples

In this section we present several examples of finitary AECs. The examples are meant to study the relations between such concepts as excellence, homogeneity, tameness, \aleph_0 -stability, atomicity, simplicity etc in such classes. However, we will not present any examples motivated by some ‘natural’ mathematical examples arising outside model theory, although we would be very interested to find such examples. These examples are motivated by the study of the relations between different non-elementary frameworks, mostly definable in $L_{\omega_1\omega}$, which have been studied in the literature.

We state the following definition.

Definition 6.1 *We say that a finitary abstract elementary class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is almost atomic, if*

$$\mathbb{K} = (\mathbb{K})^{\omega},$$

that is, all models in \mathbb{K} are \aleph_0 -saturated.

By Kueker’s results in [14], if $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable finitary class, it is almost atomic if and only if it is definable with a complete sentence ϕ in $L_{\omega_1\omega}$ and $\preceq_{\mathbb{K}}$ is the syntactic \mathcal{F} -elementary equivalence relation a countable fragment $\mathcal{F} \subset L_{\omega_1\omega}$ containing ϕ . Also a non- almost atomic finitary class might be defined with a sentence in $L_{\omega_1\omega}$, but then the sentence cannot be complete.

We write *almost* atomic, since the class will become an atomic AEC after adding countable many predicates to the language as in section 2.3. However, since *excellence* is only defined for atomic AECs in [1], we will have to be careful with what we mean by excellence in an almost atomic class. See the discussion after Question 6.28.

The following easy example demonstrates that there are finitary classes, which are not definable with a single sentence in $L_{\omega_1\omega}$. For more examples, see [14].

Example 6.2 *Define the language L to be countable many unary predicates $P_n, n < \omega$. Denote by $S(L)$ the set of all (complete) basic L -types i.e. types consisting of atomic formulas and negations of atomic formulas. We have that $|S(L)| = 2^{\aleph_0}$.*

For any subset $\Phi \subsetneq S(L)$ denote by $(\mathbb{K}_{\Phi}, \preceq_{\mathbb{K}})$ the class of all L -structures omitting Φ with $\preceq_{\mathbb{K}}$ as substructure. Then $(\mathbb{K}_{\Phi}, \preceq_{\mathbb{K}})$ is a finitary abstract elementary class.

For different $\Phi_1, \Phi_2 \subset S(L)$, the classes $\mathbb{K}_{\Phi_1}, \mathbb{K}_{\Phi_2}$ are not equivalent in $L_{\omega_1\omega}$. Also there are $2^{(2^{\aleph_0})}$ different subsets of $S(L)$. Since there are only 2^{\aleph_0} different sentences in $L_{\omega_1\omega}$, all such classes cannot be definable with a sentence in $L_{\omega_1\omega}$.

The classes are also simple and tame, and also superstable with the Tarski-Vaught property, see [9] for the definitions of the latter two. These properties are inherited from the first-order empty theory of L , since the monster model of each class \mathbb{K}_{Φ} is a submodel of the first-order monster model.

However, all the classes are closed under $L_{\omega_1\omega}$ -equivalence and weak type equals $L_{\omega_1\omega}$ -type. Also, each class is definable with a sentence in $L_{\kappa\omega}$ for some $\kappa \leq (2^{\aleph_0})^+$.

On the other hand, if such class \mathbb{K}_{Φ} is \aleph_0 -stable, the set $S(T) \setminus \Phi$ must be countable. Then the class is definable in $L_{\omega_1\omega}$, since we can list all possible types in one sentence.

Another easy example shows that the class $((\mathbb{K})^{\omega}, \preceq_{\mathbb{K}})$ of \aleph_0 -saturated models of an AEC $(\mathbb{K}, \preceq_{\mathbb{K}})$ does not determine the original class $(\mathbb{K}, \preceq_{\mathbb{K}})$. We define two different finitary

classes $(\mathbb{K}_1, \preceq_{\mathbb{K}_1})$ and $(\mathbb{K}_2, \preceq_{\mathbb{K}_2})$ where the classes $((\mathbb{K}_1)^\omega, \preceq_{\mathbb{K}_1})$ and $((\mathbb{K}_2)^\omega, \preceq_{\mathbb{K}_2})$ are equal. This example is due to Kueker.

Example 6.3 Let $L = \{E\}$, where E is a binary relation symbol. We let \mathbb{K} be the L -structures such that E is an equivalence relation with infinitely many infinite classes and infinitely many 2-element classes but no finite classes of size n for $n > 2$. One-element classes are allowed. We let $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$.

For $\mathcal{A}, \mathcal{B} \in \mathbb{K}$, we define that $A \preceq_{\mathbb{K}_1} B$ if A is a substructure of B and all finite E -classes in A stay finite in B .

For $\mathcal{A}, \mathcal{B} \in \mathbb{K}$, we define that $A \preceq_{\mathbb{K}_2} B$ if A is a substructure of B and all two-element classes in A are not increased in B .

For both cases, the class of \aleph_0 -saturated models is the class of models with no one-element classes. In this class the notions $\preceq_{\mathbb{K}_1}$ and $\preceq_{\mathbb{K}_2}$ agree. The notion of a Galois 1-type differs in the classes $(\mathbb{K}_1, \preceq_{\mathbb{K}_1})$ and $(\mathbb{K}_2, \preceq_{\mathbb{K}_2})$. In \mathbb{K}_1 , there are two 1-types, one saying that ‘ x is an element in a finite class’ and ‘ x is an element in an infinite class’. In \mathbb{K}_2 , the 1-types are ‘ x is an element in a two-element class’ and ‘ x is not an element in a two-element class’. Both classes are finitary and \aleph_0 -stable and definable in $L_{\omega_1\omega}$, but the notions $\preceq_{\mathbb{K}_1}$ and $\preceq_{\mathbb{K}_2}$ are not given by $L_{\omega_1\omega}$.

6.1 A non-atomic example

We give another example, where the class of $((\mathbb{K})^\omega, \preceq_{\mathbb{K}})$ is very different than $(\mathbb{K}, \preceq_{\mathbb{K}})$. We define a simple, \aleph_0 -stable, \aleph_0 -tame, finitary class, which has the maximal number of models in each cardinality. Also it is not excellent in the sense that we don’t have prime models over independent diagrams. But, the class $(\mathbb{K})^\omega$ is excellent and categorical in all powers. We thank David Kueker and John Baldwin for comments on this example.

Example 6.4 Let $L = \{\pi_1, \pi_2\} \cup \{Q\} \cup \{P_\eta \mid \eta \in 2^{<\omega}\}$, where π_1 and π_2 are unary function symbols, Q is a unary predicate symbol and the others are ternary relation symbols. Let T be a first-order theory that says:

- (i) $\forall z(Q(z) \vee \neg Q(z) \vee \exists x \exists y P_\emptyset(x, y, z))$
- (ii) $\forall x \forall y \forall z (P_\emptyset(x, y, z) \rightarrow (Q(x) \wedge Q(y) \wedge \neg Q(z)))$
- (iii) $\forall x \forall y \forall x' \forall y' \forall z (P_\emptyset(x, y, z) \wedge P_\emptyset(x', y', z) \rightarrow (x = x' \wedge y = y'))$
- (iv) Q is infinite
- (v) for all $\eta \in 2^{<\omega}$,
 - (a) $\forall x \forall y ((Q(x) \wedge Q(y)) \rightarrow \exists z P_\eta(x, y, z))$
 - (b) $\forall x \forall y ((Q(x) \wedge Q(y)) \rightarrow \forall z (P_\eta(x, y, z) \leftrightarrow (P_{\eta \frown (0)}(x, y, z) \vee P_{\eta \frown (1)}(x, y, z))))$
 - (c) $\forall x \forall y \neg \forall z (P_{\eta \frown (0)}(x, y, z) \wedge P_{\eta \frown (1)}(x, y, z))$.
- (vi) (a) $\forall x (\pi_1(x) = x \leftrightarrow Q(x)) \wedge \forall x \forall z (\pi_1(z) = x \leftrightarrow \exists y P_\emptyset(x, y, z))$.
(b) $\forall y (\pi_2(y) = y \leftrightarrow Q(y)) \wedge \forall y \forall z (\pi_2(z) = y \leftrightarrow \exists x P_\emptyset(x, y, z))$.

For all $\eta \in 2^\omega$, let

$$q_\eta = \{z \neq z'\} \cup \{P_{\eta \upharpoonright n}(x, y, z) \wedge P_{\eta \upharpoonright n}(x, y, z') \mid n \in \omega\}$$

and

$$p_\eta = \{P_{\eta \upharpoonright n}(x, y, z) \mid n \in \omega\}.$$

Let \mathbb{K} be the class of all models of T which omit every type q_η and those p_η such that both $\eta^{-1}(0)$ and $\eta^{-1}(1)$ are infinite. Let $\preceq_{\mathbb{K}}$ be the first-order relation $\preceq_{\omega\omega}$.

Items (i) – (iv) say that the predicate P_\emptyset divides the elements outside the predicate Q to infinitely many disjoint sets indexed by the pairs (x, y) in the predicate. We call such a set a tree indexed by (x, y) . Furthermore, (v) says that each element in the tree satisfies a type p_η for some branch η in the binary tree.

Functions π_1 and π_2 are added for convenience, to make the theory T to have quantifier elimination. Axiom (vi) gives the coordinates $(\pi_1(z), \pi_2(z))$ for the tree of each z not in Q . Hence the ‘columns’ and ‘rows’ of trees are definable with quantifier-free formulas $\pi_1(z) = x$ and $\pi_2(z) = y$.

We omit the types q_η to make each type p_η to have at most one representative, and we omit all but countably many types p_η to make all trees countable. The class $(\mathbb{K}, \preceq_{\mathbb{K}})$ becomes \aleph_0 -stable.

For any model \mathcal{A}_0 in \mathbb{K} , we can define two models \mathcal{A}_1 and \mathcal{A}_2 extending \mathcal{A}_0 such that $Q(\mathcal{A}_1) \setminus Q(\mathcal{A}_0) = a_1$ and $Q(\mathcal{A}_2) \setminus Q(\mathcal{A}_0) = a_2$ for some distinct a_1 and a_2 . Then the models $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ form an independent 2-diagram. However, there are \mathcal{B} and $\mathcal{B}' \preceq_{\mathbb{K}}$ -extending both \mathcal{A}_1 and \mathcal{A}_2 such that no substructure of \mathcal{B} can be embedded to \mathcal{B}' over $\mathcal{A}_1 \cup \mathcal{A}_2$ or vice versa. Namely, we can choose the (a_1, a_2) -tree in \mathcal{B} so that all branches are eventually one and in \mathcal{B}' so that all branches are eventually zero. Hence there cannot be a prime model over $\mathcal{A}_1 \cup \mathcal{A}_2$.

In an \aleph_0 -saturated model, all the trees will be isomorphic. Hence the class $(\mathbb{K})^\omega$ is categorical in every infinite cardinal.

6.2 Non-simple examples

All the remaining examples in this section will be some further developments of the construction introduced by Hart and Shelah in [6] and further studied by Baldwin and Kolesnikov in [2]. We will refer to the paper [2] for the properties of the construction, but we will also recall the necessary definitions here. Unlike we claimed in the introduction of [9], the constructed AEC is not simple. However, we get a non-tame \aleph_0 -stable finitary AEC with the extension property for non-splitting.

To see that the original example is not simple, one can use a similar proof than we will show in the following. However, here we present a much more elementary example of non-simplicity. This example will be even totally categorical, homogeneous and excellent. It will serve as an introduction to the original example. This version of the example was suggested by Kolesnikov.

The first \aleph_0 -stable and non-simple example was suggested by Shelah and is presented in Hyttinen, Lessmann [10]. There the cause for non-simplicity is similar. However, that example is not categorical.

We want to emphasize that although we only prove that the examples in this section are not simple according to our definition of simplicity, by Theorem 4.9 there could not be any other notion of independence which would give a sufficient independence calculus.

Example 6.5 *The language L contains unary predicates I, G, H and G^* , a binary function e from $G \times I$ to H and a 3-ary relation t_G on $G \times G^* \times G^*$.*

The universe of a structure M in the class contains disjoint copies of I, G, G^ and H , where*

1. I is an infinite set,

2. H is \mathbb{Z}_2 ,
3. G is the set of finite support functions from I to H and
4. G^* is an affine copy of G .

Furthermore, $e(g, i) \in H$ gives the value of the finite support function $g \in G$ at a point $i \in I$. The affine action of G on G^* is coded by t_G , we write $t_G(g, x, y)$ as $y = x + g$.

The class of structures is a finitary AEC when we take as the relation $\preceq_{\mathbb{K}}$ the \mathcal{F} -elementary substructure relation given by a suitable fragment $\mathcal{F} \subseteq L_{\omega_1\omega}$.

The construction can be defined in $L_{\omega_1\omega}$ and the models are determined up to isomorphism by the size of I . Let \mathfrak{M} again be the monster model of the class and we refer as I, G, G^*, H to the corresponding parts of the monster model.

We denote by $g_1 + g_2$ the coordinate-wise addition of two functions $g_1, g_2 \in G$. We denote by S_g the support of $g \in G$. When we write $g \in M$ for some submodel $M \subseteq \mathfrak{M}$ and $g \in G$ we mean that $S_g \subseteq M$.

We recall that $\text{tp}^w(\bar{b}/B)$ Lascar-splits over $A \subset B$ if there are $\bar{a}_0, \bar{a}_1 \in B$ which belong to the same strongly A -indiscernible sequence such that $\text{tp}^g(\bar{a}_0/A \cup \bar{b}) \neq \text{tp}^g(\bar{a}_1/A \cup \bar{b})$.

We recall that a finitary AEC is *simple*, if for every finite A and \bar{a} and arbitrary B extending A there is \bar{b} realizing $\text{tp}^w(\bar{a}/A)$ such that $\text{tp}^w(\bar{b}/B)$ does not Lascar-split over A .

Lemma 6.6 *The finitary class $(\mathbb{K}, \preceq_{\mathbb{K}})$ described in Example 6.5 is not simple.*

Proof: Let M be an elementary submodel of \mathfrak{M} such that $c, d \in M \cap G^*$ belong to a strongly indiscernible sequence. (And they are not the same element.) We can find such M and c, d , since G^* is unbounded. We claim that the type of any element $x \in G^*$ over the empty set cannot be extended to a free extension over M .

Assume (to the contrary) that $a \in G^* \setminus M$ would be free of M over \emptyset . That is, $\text{tp}^w(a/M)$ does not Lascar-split over \emptyset .

Since t_G codes a regular action, for each $a, b \in G^*$ there is a unique $g \in G$ such that $a = b + g$. If both a and b belong to the model M , we have that this g is in $M \cap G$. That is, the support S_g is in $M \cap G$. Hence we define:

- Let $g \in G$ be such that $a = c + g$.
- Let $h \in G \cap M$ be such that $c = d + h$.
- Let $h_0 \in G \cap M$ be such that $S_g \cap S_h = S_{h_0}$.

Now since h_0 is its own inverse, the functions $g + h_0$ and h have disjoint support. Let F be an automorphism of the monster such that F fixes I, \mathbb{Z}_2 and G pointwise and for all $x \in G^*$,

$$F(x) = x + h_0.$$

Then since $h_0 \in M$, F fixes M setwise.

By invariance, the element $F(a)$ is also free of M over the empty set. We can write

$$F(a) = c + (g + h_0)$$

and similarly

$$F(a) = d + (g + h_0 + h).$$

But since $g + h_0$ and h have disjoint support and h is not the zero function,

$$|S_{(g+h_0)}| \neq |S_{(g+h_0+h)}|,$$

and hence there is no automorphism of the monster model mapping $g + h_0$ to $g + h_0 + h$. Hence c and d have different Galois type over $F(a)$. This is a contradiction, since now c and d witness that $\text{tp}(F(a)/M)$ Lascar-splits over \emptyset . \square

Now we recall the construction as defined in [2]. This construction gives for each natural number $k \geq 2$ an example of a finitary AEC which is categorical in $\aleph_0, \dots, \aleph_{k-2}$ but not in \aleph_{k-1} . The examples will also be almost atomic, \aleph_0 -stable and have the extension property for non-splitting for $k \geq 3$ but non-tame and non-simple for any k .

Example 6.7 *Let $k > 2$ be a natural number. Let L again denote the vocabulary, where the L -structures contain*

1. unary predicates $I, K, G, H, G^*, H^*, ,$,
2. binary function e_G taking $G \times H$ to H ,
3. a function π_G mapping G^* to K ,
4. a function π_H mapping H^* to K ,
5. a 4-ary relation t_G on $K \times G \times G^* \times G^*$,
6. a 4-ary relation t_H on $K \times G \times G^* \times G^*$,
7. other projection functions and
8. A $(k+1)$ -ary relation symbol Q on $(G^*)^k \times H$.

A structure $M(I)$ is as follows.

1. I is a set
2. K is $[I]^k$, the set of k -tuples of elements in I ,
3. H is \mathbb{Z}_2 , and G is the set of finite support functions from K to H .

Note that now the functions in G are from K , not I . The function $e_G(g, u) \in H$ gives the value of the finite support function on $g \in G$ on $u \in K$. Furthermore

4. G^* is the set of affine copies of G indexed by K . We write the elements of G^* as pairs (u, x) , where $u \in K$.
5. H^* is the set of affine copies of H indexed by K . We write the elements of H^* as pairs (u, i) , where $u \in K$.
6. π_G is a projection function from G^* to K . For $u \in K$, we say that $\pi_G^{-1}(u)$ is the G^* -fiber over u and write $\pi_G^{-1}(u) = G_u^*$,

7. and similarly, the pre-images of the projection $\pi_H : H^* \rightarrow K$ give the H^* -fibers H_u^* .
8. The regular (free and transitive) relation $t_G(u, g, y, x) \subset K \times G \times G^* \times G^*$ codes the affine action of G on each fiber $u \in K$. When x and y are on the same fiber u , we write $t_G(u, g, y, x)$ as $y = x + g$.
9. Similarly, $t_H(u, h, x, y)$ codes the affine action of \mathbb{Z}_2 on each fiber of H^* . On the same fiber u we may write $y = x + h$.
10. The $(k + 1)$ -ary relation Q satisfies the following:
- (a) Q is symmetric with respect to all permutations of the k first elements from G^*
 - (b) $Q((u_1, x_1), \dots, (u_k, x_k), (u_{k+1}, x_{k+1}))$ implies that u_1, \dots, u_{k+1} form a compatible $k + 1$ -tuple, i.e. all k element subsets of a $k + 1$ element subset of I .
 - (c) $((u_1, g_1), \dots, (u_k, g_k), (u_{k+1}, h)) \in Q$ iff $((u_1, g_1), \dots, (u_k, g_k), (u_{k+1}, h + 1)) \notin Q$.
 - (d) If $g \in G$ is such that $g(u_{k+1}) = 1$, then $((u_1, g_1), \dots, (u_k, g_k), (u_{k+1}, h)) \in Q$ iff $((u_1, g_1 + g), \dots, (u_k, g_k), (u_{k+1}, h)) \notin Q$.

We only take as models such structures where I is infinite. We can write the above description as an $L_{\omega_1\omega}$ -sentence ϕ_k and we take as the abstract elementary class the class $(\text{Mod}(\phi_k), \preceq_{\mathcal{F}})$, where \mathcal{F} is a suitable countable fragment of $L_{\omega_1\omega}$.

We call a *standard structure* on I the structure $M(I)$ as above, where each G_u^* and H_u^* are really a copies of G and H , and we can talk of the zero function 0 of each G_u^* and the zero 0 of each H_u^* . Of course, such things are not definable in the models of ϕ_k , but we will use these concepts in the metalanguage, when we are constructing models of ϕ_k . On the standard structure we also define for each compatible $(k + 1)$ -tuple u_1, \dots, u_k, u_{k+1} that $Q((u_1, 0), \dots, (u_k, 0), (u_{k+1}, 0))$ and hence

$$Q((u_1, g_1), \dots, (u_k, g_k), (u_{k+1}, i))$$

if and only if

$$g_1(u_{k+1}) + \dots + g_k(u_{k+1}) + i = 0 \pmod{2}.$$

The following is proved in [2].

Fact 6.8 *For any $k \geq 2$, $(\text{Mod}(\phi_k), \preceq_{\mathbb{K}})$ has arbitrarily large models, the joint embedding property and the disjoint amalgamation property. Furthermore, we can amalgamate disjointly over full substructures of structures in \mathbb{K} .*

When $k \geq 3$, $(\text{Mod}(\phi_k), \preceq_{\mathbb{K}})$ is categorical in $\aleph_0, \dots, \aleph_{k-2}$ and hence also \aleph_0 -stable.

By Fact 6.8, there is a monster model \mathfrak{M}_k for ϕ_k . Furthermore, any isomorphism between so called full substructures of the monster model extends to an automorphism.

A *full substructure* of \mathfrak{M} is defined in Definition 1.5 of [2] as a substructure of \mathfrak{M} which is isomorphic to a structure $M(I') \upharpoonright L \setminus \{Q\}$ for some, possibly finite, $I' \subset I$. To describe a full substructure, we recall the Fact 4.1 from [2].

Fact 6.9 Let $\mathcal{A} \models \phi_k$. If a subset A' of the universe of \mathcal{A} is the universe of a full substructure of \mathcal{A} , then

1. A' is an L -substructure of \mathcal{A} ,
2. $G(A')$ is the set of all finite support functions in $G(\mathcal{A})$ whose support is contained in $K(A')$,
3. for all $u \in K(A')$ and for some $x \in G_u^*(\mathcal{A})$, we have $G_u^*(A') = \{x + \gamma : \gamma \in G(A)\}$ and
4. for all $u \in K(A')$ and for some $x \in H_u^*(\mathcal{A})$, we have $H_u^*(A') = \{x + \delta : \delta \in \mathbb{Z}_2\}$

Let \mathcal{A} be a model of ϕ_k . We recall from [2] the notions of a *solution* over a set $A \subseteq \mathcal{A}(I)$ (Definition 2.3), *extension property* for solutions over a set A (Definition 2.5) and *n -amalgamation property* for solutions over a set A (Definition 2.9). For completeness, we write the definitions here:

Definition 6.10 (A solution over a set) A solution (g, h) is a pair of functions

$$g : K(\mathcal{A}) \rightarrow G^*(\mathcal{A}) \quad \text{and} \quad h : K(\mathcal{A}) \rightarrow H^*(\mathcal{A}),$$

where $g(u) \in G_u^*$ and $h(u) \in H_u^*$.

We say that (g, h) is a solution for the subset $W \subset K(\mathcal{A})$ if for each $u \in W$ there are $g(u) \in G_u^*$ and $h(u) \in H_u^*$ such that if u_1, \dots, u_k, u_{k+1} are a compatible $(k+1)$ -tuple from W , then

$$Q((u_1, g(u_1)), \dots, (u_k, g(u_k)), (u_{k+1}, h(u_{k+1}))).$$

If (g, h) is a solution for a set $W = [A]^k$ for some $A \subset I(\mathcal{A})$, we say that (g, h) is a solution over A .

Definition 6.11 (n -amalgamation property for solutions) Let A be a subset of $I(\mathcal{A})$ of size λ . Let $\{b_0, \dots, b_{n-1}\}$ be an n -element subset of $I(\mathcal{A})$ such that for any $(n-1)$ -element subset w of $n = \{0, \dots, n-1\}$, we have a solution (g_w, h_w) over $A \cup \{b_l : l \in w\}$. Assume also that the solutions are compatible, i.e. $(\bigcup_w g_w, \bigcup_w h_w)$ is a function.

We say that \mathcal{A} has n -amalgamation for solutions over sets of size λ if for every such situation, there is a solution (g, h) over $A \cup \{b_0, \dots, b_{n-1}\}$ extending simultaneously each (g_w, h_w) .

We call the 1-amalgamation property for solutions over sets of size λ as the *extension property for solutions* over sets of size λ . The same definitions can be written over *finite sets*, i.e. where A is finite. We also recall the following facts from [2]. Fact 6.12 follows from Lemmas 2.11 and 2.14, Fact 6.13 follows from the proof of Lemma 2.6 and Fact 6.14 is Lemma 2.11.

Fact 6.12 (Existence of solutions over countable sets) Let $k \geq 3$. For all $\mathcal{A} \models \phi_k$, there is a solution over every countable $J \subseteq I(\mathcal{A})$.

Fact 6.13 Let $k \geq 2$. If \mathcal{A} and \mathcal{A}' are models of ϕ_k with solutions (g, h) and (g', h') , respectively and identity is an isomorphism from $\mathcal{A} \upharpoonright (L \setminus \{Q\})$ to $\mathcal{A}' \upharpoonright (L \setminus \{Q\})$, then there is an isomorphism $f : \mathcal{A} \rightarrow \mathcal{A}'$ such that if $\mathcal{B} \subseteq \mathcal{A}$ is a full substructure, $g \upharpoonright (\mathcal{B} \cap K(\mathcal{A})) = g' \upharpoonright (\mathcal{B} \cap K(\mathcal{A}'))$ and $h \upharpoonright (\mathcal{B} \cap K(\mathcal{A})) = h' \upharpoonright (\mathcal{B} \cap K(\mathcal{A}'))$, then $f \upharpoonright \mathcal{B} = \text{id}$.

Fact 6.14 (Extension property for solutions over finite sets) *Let $k \geq 2$. If $\mathcal{A} \in \mathbb{K}$ and $J, J' \subseteq I(\mathcal{A})$ are finite and with solutions (g, h) and (g', h') and they agree on $J \cap J'$, then there is a solution for $J \cup J'$ which extends both (g, h) and (g', h') .*

We recall (Lemmas 4.2 and 4.4 of [2]) that for any $\mathcal{A} \models \phi_k$ and any set $B \subset \mathcal{A}$ there is a minimal full structure M_B in \mathcal{A} containing B . Furthermore, if B is finite, also M_B is finite and unique up to isomorphism over B . Also, if $C \subset B$, we can choose M_C to be a substructure of M_B .

Definition 6.15 *Let \mathfrak{M}_k be a monster model for ϕ_k . For any subset $A \subseteq \mathfrak{M}_k$, let M_A be a minimal full substructure \mathcal{A} of M such that $A \subseteq \mathcal{A}$. Define I_A to be the least $J \subseteq I(\mathfrak{M}_k)$ such that $A \subseteq M_J$ for some M_J .*

For subsets $A, B, C \subseteq \mathfrak{M}_k$ we write $A \downarrow_B^- C$ if $I_{AB} \cap I_{BC} = I_B$.

Studying the properties of full substructures we can prove the following characterization.

Lemma 6.16 *We have that $A \downarrow_B^- C$ iff there are full structures M_{AB} , M_B and M_{BC} such that $M_{AB} \cap M_{BC} = M_B$ and that then these structures can be found inside any M_{ABC} . Also if $A \downarrow_B^- C$, then $I_{ABC} = I_{AB} \cup I_{BC}$.*

Clearly the notion \downarrow^- is invariant under automorphisms.

By 6.16, \downarrow^- is symmetric and transitive. We show that \downarrow^- has the extension property.

Lemma 6.17 *For all finite sequences \bar{a} and all $B, C \subseteq \mathfrak{M}_k$ there is \bar{a}' such that \bar{a}' and \bar{a} have the same Galois type over B and $\bar{a}' \downarrow_B^- C$.*

Proof: Let M_B , M_{BC} and $M_{B\bar{a}}$ be some minimal full structures, containing $B, B \cup C$ and $B \cup \bar{a}$ respectively, such that $M_B \subseteq M_{BC}$ and $M_B \subseteq M_{B\bar{a}}$. By Fact 6.8, there is a disjoint amalgam \mathcal{B} of M_{BC} and $M_{B\bar{a}}$ over M_B . We may assume that $M_{BC} \subseteq \mathcal{B} \preceq_{\mathbb{K}} \mathfrak{M}_k$. Hence there is an automorphism $f \in \text{Aut}(\mathfrak{M}_k/M_B)$ such that $f(M_{B\bar{a}}) \cap M_{BC} = M_B$. By Lemma 6.16, $f(\bar{a}) \downarrow_B^- C$. \square

We prove the following lemma to show that $(\text{Mod}(\phi_k), \preceq_{\mathbb{K}})$, $k \geq 3$, has the extension property for non-splitting over models. Here it is essential that $B = M_B$ is a full substructure. If we could prove the same for arbitrary sets B , we would get simplicity. But $(\text{Mod}(\phi_k), \preceq_{\mathbb{K}})$ is not simple by Theorem 6.20.

Lemma 6.18 *Assume that finite tuples \bar{a} and \bar{b} have the same Galois-type over finite B with $B = M_B$, $C \supseteq B$ is finite, $\bar{a} \downarrow_B^- C$ and $\bar{b} \downarrow_B^- C$. Then \bar{a} and \bar{b} have the same Galois-type over C .*

Proof: We may assume that $C = M_C$. Fix some minimal full substructures M_{Cab} and $M_{Ba}, M_{Bb} \subseteq M_{Cab}$ such that $M_{Ba} \cap M_C = M_B$ and $M_{Bb} \cap M_C = M_B$. Since \bar{a} and \bar{b} have the same Galois-type over B , there is an isomorphism $f : M_{Ba} \rightarrow M_{Bb}$ such that $f \upharpoonright B = \text{id}$ and $f(\bar{a}) = \bar{b}$. By Facts 6.12 and 6.14 there are solutions (g, h) over I_C and (g', h') over I_{Ba} such that they agree on I_B and the values are in M_C and in M_{Ba} , respectively. Let $g'' = f \circ g' \circ f^{-1}$ and $h'' = f \circ h' \circ f^{-1}$. Then (g'', h'') is a solution over I_{Bb} and it agrees with (g, h) on I_B . By Fact 6.14 there is a solution (g_a, h_a) over I_{Ca} extending (g, h) and (g', h') and a solution (g_b, h_b) over I_{Cb} extending (g, h) and (g'', h'') . Let $J \subseteq I(\mathfrak{M})$ be

countable such that $J \cap (I_{C_a} \cup I_{C_b}) = \emptyset$. By Fact 6.14, we can extend the solutions (g_a, h_a) and (g_b, h_b) to solutions over $I_{C_a} \cup J$ and $I_{C_b} \cup J$ respectively. By Fact reffaktakaksi there is an automorphism $f^* : M_{C_a} \rightarrow M_{C_b}$ such that $M_C, M_{B_a} \subseteq M_{C_a}$, $M_C, M_{B_b} \subseteq M_{C_b}$, $f^* \upharpoonright M_C = id$ and $f^* \upharpoonright M_{B_a} = f$. Hence $f^*(a) = b$. \square

The class of models of ϕ_k has the extension property for non-splitting for $k \geq 3$. The proof uses \aleph_0 -stability.

Theorem 6.19 *For any $k \geq 3$ the class $(Mod(\phi_k), \preceq_{\mathbb{K}})$ has the extension property for non-splitting.*

Proof: We show that for any \aleph_0 -saturated model M and any \bar{a} there is finite $E \subset M$ such that for any $B \supseteq \mathcal{A}$ there is \bar{b} realizing $tp^w(\bar{a}/M)$ such that $tp^w(\bar{b}/B)$ does not split over E . This is an equivalent formulation of the extension property for non-splitting by Proposition 4.23.

Let M be an \aleph_0 -saturated model and \bar{a} a tuple. Since $(\mathbb{K}_k, \preceq_{\mathbb{K}})$ is an \aleph_0 -stable finitary AEC, splitting has local character and hence there is finite $E \subset M$ such that $tp^g(\bar{a}/M)$ does not split over E . Furthermore, there is a minimal full substructure M_E containing E , which is finite, such that $M_E \subset M$. Then the type $tp^w(\bar{a}/M)$ does not split over M_E either. We claim that this finite M_E is as required.

First we claim that $\bar{a} \downarrow_{M_E}^- M$. If not, there is some element $\bar{c} \in I_{M_E \bar{a}} \cap I_M \subset M$ such that $\bar{c} \notin I_{M_E}$. Since I_M is infinite, there is $d \in I_M \setminus I_{M_E \bar{a}} \subset M$. Clearly $tp^g(c/M_E \cup \bar{a}) \neq tp^g(d/M_E \cup \bar{a})$. But since the models of ϕ_2 have the extension property for solutions over finite sets, there is an automorphism fixing M_E pointwise and mapping c to d . Hence $tp^w(\bar{a}/M)$ splits over M_E , a contradiction.

Let B be a set containing M . We need to find \bar{b} realizing $tp^w(\bar{a}/M)$ such that $tp^w(\bar{b}/B)$ does not split over M_E . By Lemma 6.17 there is $\bar{b} \models tp^g(\bar{a}/M)$ such that $\bar{b} \downarrow_{M_E}^- B$. By invariance, $\bar{b} \downarrow_{M_E}^- M$ and by transitivity, $\bar{b} \downarrow_{M_E}^- B$.

Finally we claim that $tp^w(\bar{b}/B)$ does not split over M_E .

Assume that $tp^g(\bar{c}/M_E) = tp^g(\bar{d}/M_E)$ for some $\bar{c}, \bar{d} \in B$. By monotonicity and symmetry of \downarrow^- ,

$$\bar{c} \downarrow_{M_E}^- \bar{b} \text{ and } \bar{d} \downarrow_{M_E}^- \bar{b}.$$

Hence by Lemma 6.18, $tp^g(\bar{c}/M_E \cup \bar{b}) = tp^g(\bar{d}/M_E \cup \bar{b})$. This proves the claim. \square

Finally we see that $(Mod(\phi_k), \preceq_{\mathbb{K}})$ is not simple for any k .

Theorem 6.20 *Let $k \geq 2$. The class $(Mod(\phi_k), \preceq_{\mathcal{F}})$ is not simple.*

Proof: The proof is the same as the proof of Lemma 6.6. Only that we pick an element $u \in K$, an choose an elementary submodel M with $c, d \in G_u^* \cap M$, where c and d are elements in a strongly $\{u\}$ -indiscernible sequence. Also we do not study a type over the empty set but over the element $u \in K$, namely the type of an element on the fiber G_u^* . \square

6.3 A tame, non-excellent example

The last example is an example of a tame, simple, finitary, almost atomic class which is not excellent nor homogeneous. It is also definable in $L_{\omega_1\omega}$. This example takes advantage of the properties of the construction in Example 6.7, namely from the case $k = 4$. However, this example is not \aleph_0 -stable.

This example is not categorical, but it is *a-categorical* in all cardinals strictly above the continuum and superstable in the sense of [9].

Example 6.21 *We assume the continuum hypothesis. Let the vocabulary L^* be the vocabulary L of the previous example together with*

$$\{P, E, f\} \cup \{E_n : n < \omega\},$$

where P is a unary predicate, f a function symbol and $E, E_n, n < \omega$ binary relation symbols. We define the class \mathbb{K} to consists of L^* -structures \mathcal{A} such that

1. The relations $E, E_n, n < \omega$ are equivalence relations on \mathcal{A} .
2. The relations $E_n, n < \omega$ form a binary tree structure on $P(\mathcal{A})$: E_0 has two infinite classes on P and E_{n+1} divides each class of E_n on P into exactly two infinite classes. We call $P(\mathcal{A})$ the ‘tree of \mathcal{A} ’.

3. We omit the type

$$p := \{z \neq z'\} \cup \{E_n(z, z') : n < \omega\}$$

to make the size of the tree $P(\mathcal{A})$ at most continuum i.e. at most \aleph_1 .

4. The relation E divides $\mathcal{A} \setminus P(\mathcal{A})$ to infinitely many infinite classes, which we call ‘boxes’.
5. Each box X is a model of ϕ_4 from Example 6.7 with the L -structure respectively.
6. For the predicate $I \in L$ and any box X the function $f \upharpoonright X(I)$ is a bijection from the I -part of X to the tree $P(\mathcal{A})$.
7. We define E on $P(\mathcal{A})$ and each E_n on $\mathcal{A} \setminus P(\mathcal{A})$ to consist of singleton classes and f be the identity on $P(\mathcal{A})$.

We choose $\preceq_{\mathbb{K}}$ to be the substructure relation.

To summarize, each model of the class consists of a tree (which is some subtree of the full binary tree, countable or uncountable) and an infinite number of models of ϕ_4 , each of equal size than the tree. Since ϕ_4 is categorical in \aleph_0 and \aleph_1 , all the boxes of a model are isomorphic. We leave as an exercise to show that the class is a finitary abstract elementary class, except we prove the amalgamation property in lemma 6.25. We also leave as an exercise to show that each model of \mathbb{K} is \aleph_0 -saturated.

The monster model \mathfrak{M} of \mathbb{K} consists of a full binary tree and a big number of models of ϕ_4 of size \aleph_1 . We see that \mathbb{K} is simple: Two elements $a_1, a_2 \in \mathfrak{M}$ can be in the same strongly C -indiscernible sequence only if they are in different E -boxes not intersecting with C . If a type $\text{tp}^w(\bar{a}/B)$ Lascar-splits over $C \subset B$, there must be witnesses a_0, a_1 in A in different boxes of \mathfrak{M} , both boxes disjoint from C , such that \bar{a} intersects exactly one of their

boxes. Hence $A \downarrow_C B$ iff for each $a \in A$ and $b \in B$, if $E(a, b)$, there is $c \in C$ such that $E(a, c)$. This gives simplicity.

Since ϕ_4 has the 2-amalgamation property for solutions for sets of size \aleph_0 and the extension property for solutions over sets of size \aleph_1 , the class is tame, but since it does not have 2-amalgamation for solutions over sets of size \aleph_1 , it is not excellent. We prove these claims in Lemmas 6.26 and 6.27.

First we recall the following facts from [2]. Fact 6.22 is proved as Lemma 2.6 of [2], using the fact that the models of ϕ_4 have the extension property for solutions over sets of size \aleph_1 . Fact 6.23 is proved as Theorem 5.1 of [2], using the 2-amalgamation property for solutions over sets of size \aleph_0 .

Fact 6.22 *Let $M_1, M_2 \models \phi_4$ be of size \aleph_1 , $N_i \subset M_i$ full substructures for $i \in \{1, 2\}$, $h : N_1 \rightarrow N_2$ an isomorphism and $f : I(M_1) \rightarrow I(M_2)$ a bijection which agrees with h on $I(N_1)$. Then $f \cup h$ extends to an isomorphism between M_1 and M_2 .*

Fact 6.23 *Assume that $|M_1| = |M_2| = \aleph_1$, $M \preceq_{\mathbb{K}} M_1, M_2$ and $f : M_1(I) \rightarrow M_2(I)$ is a bijection fixing $M(I)$ pointwise, $\bar{a}_1 \in M_1$ and $\bar{a}_2 \in M_2$ and f extends to an isomorphism h between minimal full substructures $M_{\bar{a}_1} \subset M_1$, $M_{\bar{a}_2} \subset M_2$ containing \bar{a}_1, \bar{a}_2 respectively, such that $h \upharpoonright M = \text{id}$ and $h(\bar{a}_1) = \bar{a}_2$. Then $f \cup h$ extends to an isomorphism between M_1 and M_2 , fixing M pointwise.*

The following fact is not stated in [2], but it follows from Lemma 2.6, Lemma 2.14 and Theorem 7.1(2).

Fact 6.24 *Models of ϕ_4 do not have the 2-amalgamation property for solutions over all sets of size \aleph_0 , that is: There are $a, b, J \subseteq I(\mathfrak{M}_4)$ such that $a \neq b$, $|J| = \aleph_1$ and solutions (g_a, h_a) and (g_b, h_b) over $J \cup \{a\}$ and $J \cup \{b\}$ respectively, which agree on J but $(g_a, h_a) \cup (g_b, h_b)$ does not extend to a solution over $J \cup \{a, b\}$.*

Lemma 6.25 *The class \mathbb{K} has the amalgamation property and the joint embedding property.*

Proof: Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ be in \mathbb{K} such that $\mathcal{A}_0 \preceq_{\mathbb{K}} \mathcal{A}_i$ for $i \in \{1, 2\}$ or \mathcal{A}_0 is empty. First we amalgamate the trees $P(\mathcal{A}_1)$ and $P(\mathcal{A}_2)$ over $P(\mathcal{A}_0)$ and denote the amalgam by P . This will be the P -part of the final amalgam.

Both \mathcal{A}_1 and \mathcal{A}_2 may add new E -classes not intersecting \mathcal{A}_0 or extend an existing E -class in \mathcal{A}_0 . For each E -class in \mathcal{A}_1 and \mathcal{A}_2 we must construct a E -class in the amalgam together with the corresponding embedding.

First we treat the case where an E -class X_1 in \mathcal{A}_1 and an E -class X_2 in \mathcal{A}_2 extend an E -class X_0 in \mathcal{A}_0 . We need to amalgamate the ϕ_4 -structures X_1 and X_2 over X_0 . Denote by f_0, f_1 and f_2 the bijections from $I(X_i)$ to $P(\mathcal{A}_i)$ respectively. They determine how to amalgamate $I(X_1)$ and $I(X_2)$ over $I(X_0)$ so that $f_1 \cup f_2$ becomes a 1-to-1 map onto the new tree extending f_0 . We might not be able to amalgamate $I(X_1) \setminus I(X_0)$ and $I(X_2) \setminus I(X_0)$ disjointly.

Let M be the standard structure on the amalgam of $I(X_1)$ and $I(X_2)$. This will be one E -class in the amalgam of \mathcal{A}_1 and \mathcal{A}_2 . We have natural embeddings of the K, H and G -parts of X_i to M , but for each 4-element subset $u \in K(X_i)$ we need to embed the fibers $G_u^*(X_i)$ and $H_u^*(X_i)$ to M so that Q is preserved.

There are solutions (g_1, h_1) and (g_2, h_2) over $I(X_1)$ and $I(X_2)$ in X_1 and X_2 which agree over $I(X_0)$. We use these to define the embeddings $F_i : X_i \rightarrow M$ as follows: for any 4-element subset $u \in K(X_i)$, map each $g_i(u)$ to the zero function of $G_u^*(M)$ and $h_i(u)$ to the zero of $H_u^*(M)$. Then induce the rest of the map with the actions on the fibers.

The second case, where one of the structures, say \mathcal{A}_1 , adds a new E -class X to the structure \mathcal{A}_0 , is even easier. In order to construct a new E -class to the amalgam, we need to extend $I(X)$, and $K(X), H(X)$ and $G(X)$ respectively, in order to extend the bijection $f : I(X) \rightarrow P(\mathcal{A}_1)$ onto the tree P of the amalgam. This can be done just adding the right number (countable or \aleph_1 many) elements to get $I'(X)$ extending $I(X)$ and picking an arbitrary extension $f' : I'(X) \rightarrow P$ of the bijection f . Then again let M be the standard structure on $I'(X)$. M will be a new E -class of the amalgam. Embeddings of $I(X), K(X), H(X)$ and $G(X)$ to M can be taken to be the identity. A solution on $I(X)$ gives an embedding of each fiber $G_u^*(X)$ and $H_u^*(X)$ to M .

Finally the number of E -classes in the amalgam will be the maximum of numbers of E -classes in \mathcal{A}_1 and \mathcal{A}_2 . We are done with the construction. \square

Lemma 6.26 *The class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is tame.*

Proof: Let \mathfrak{M} be the monster model of \mathbb{K} , and let $\mathcal{A} \preceq_{\mathbb{K}} \mathbb{K}$ and $\bar{a}, \bar{b} \in \mathbb{K}$ be such that $\text{tp}^g(\bar{a}/\mathcal{A}') = \text{tp}^g(\bar{b}/\mathcal{A}')$ for all countable $\mathcal{A}' \preceq_{\mathbb{K}} \mathcal{A}$. We need to show that $\text{tp}^g(\bar{a}/\mathcal{A}) = \text{tp}^g(\bar{b}/\mathcal{A})$.

Let $\mathcal{A}_0 \preceq_{\mathbb{K}} \mathcal{A}$ be an arbitrary countable substructure and let $h_0 \in \text{Aut}(\mathfrak{M}/\mathcal{A}_0)$ map \bar{a} to \bar{b} . We study the automorphism h_0 .

First we see, that since $P(\mathcal{A}_0)$ is ‘dense’ in $P(\mathfrak{M})$, h_0 fixes $P(\mathfrak{M})$ pointwise: If there would be $a \in P(\mathfrak{M})$ such that $a \neq h_0(a)$, by 3. of the definition we could find some $n < \omega$ such that $\neg E_n(a, h_0(a))$. Since E_n has exactly $2^{(n+1)}$ classes, there is some $a' \in P(\mathcal{A}_0)$ such that $E_n(a, a')$. But now $\neg E_n(h_0(a), h_0(a'))$, a contradiction.

Since h_0 preserves E -classes, it maps each box X of \mathfrak{M} bijectively to some other box $h_0(X)$ of \mathfrak{M} , preserving the L -structure of X . Furthermore, on elements $i \in I(X)$, h_0 is determined by the equation

$$f(h_0(i)) = h_0(f(i)) = f(i).$$

Hence if $X \cap \mathcal{A}_0 \neq \emptyset$, h_0 must be an automorphism of X fixing $I(X)$ pointwise. On the other hand, if $M_{\bar{a}}^X \subseteq X$ is a minimal full substructure containing $\bar{a} \cap X$, h_0 maps it to a full substructure of $h_0(X)$ containing $\bar{b} \cap h_0(X)$, again fixing $\mathcal{A}_0 \cap I(X)$ pointwise.

Since this holds for arbitrary countable $\mathcal{A}_0 \preceq_{\mathbb{K}} \mathcal{A}$ (especially a countable \mathcal{A}_0 such that for some choices of minimal full structures $M_{\bar{a}}^X$ in X we have that $M_{\bar{a}}^X \cap \mathcal{A} \subset \mathcal{A}_0 \cap X$ in each box X intersecting \bar{a}), we can use Facts 6.22 and 6.26 to construct isomorphisms h_X between corresponding boxes such that the automorphism of \mathfrak{M} consisting of these and the identity on $\mathfrak{M}(P)$ fixes each $\mathcal{A} \upharpoonright X$ and maps \bar{a} to \bar{b} . \square

Lemma 6.27 *There are $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \in \mathbb{K}$ such that $\mathcal{A}_0 \subseteq \mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_1 \downarrow_{\mathcal{A}_0} \mathcal{A}_2$ but there is no primary model over $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$.*

Proof: Let $J, a, b \in I(\mathfrak{M}_4)$ and solutions $(g_a, h_a), (g_b, h_b)$ be as in Fact 6.24. Let $M_J, M_{J \cup a}, M_{J \cup b}$ be full substructures containing $J, J \cup a$ and $J \cup b$ respectively, with the images of the solutions included in the models. Then clearly $M_{J \cup a} \downarrow_{M_J}^- M_{J \cup b}$. We claim that there is no prime model over the triple. Assume the contrary, that N would be the prime model.

The solutions (g_a, h_a) and (g_b, h_b) induce \mathbb{K} -embeddings f_a, f_b from $M_{J \cup a}$ and $M_{J \cup b}$ to the standard model M containing $J \cup a \cup b$. Also $f_a \cup f_b$ is an elementary function, since the solutions agree on J . Since N is prime, $f_a \cup f_b$ extends to a \mathbb{K} -embedding $f : N \rightarrow M$. But when $\bar{0}_u$ denotes the zero function on a stalk u in $G^*(M)$ for $u \in K(M)$, the mappings $g : u \mapsto g^{-1}(\bar{0}_u)$ and $h : u \mapsto g^{-1}(0_u)$ give a solution (g, h) over $J \cup \{a, b\}$ extending (g_a, h_a) and (g_b, h_b) . This is a contradiction.

Now we can build some models $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ with $\mathcal{A}_1 \downarrow_{\mathcal{A}_0} \mathcal{A}_2$ such that $M_J, M_{J \cup a}$ and $M_{J \cup b}$ are in corresponding boxes in these structures. There cannot be a prime model over this triple, either. \square

Although this example has the amalgamation property over models, it cannot have amalgamation over sets, i.e. it is not included in homogeneous model theory. Amalgamation over countable sets would imply categoricity transfer for models of ϕ_4 by Keisler's theorem [13].

We remark also that the class is *a-categorical* in cardinals $> 2^{\aleph_0}$ in the sense of [9]. In an \aleph_1 -saturated structure of $\mathcal{A} \in \mathbb{K}_a$, the tree of \mathcal{A} is the full binary tree. For \aleph_1 -saturated \mathcal{A} and \mathcal{B} , the isomorphism between $P(\mathcal{A})$ and $P(\mathcal{B})$ induces a bijection between the I -parts of any two boxes in \mathcal{A} and \mathcal{B} , given by the function f in the language. Each such bijection extends to an isomorphism between the boxes by Fact 6.22. If $|\mathcal{A}| > 2^{\aleph_0}$, the number of boxes must be $|\mathcal{A}|$. The class is also superstable. This can be shown straightforwardly or by Theorem 3.38 of [9], which stated that superstability is implied by \aleph_1 -categoricity in a suitable cardinal.

Now we have two examples of simple, finitary classes which are even tame but not excellent, namely Examples 6.4 and 6.21. It is only that the Example 6.4 is not almost atomic, even though it is \aleph_0 -stable, and the almost atomic Example 6.21 is not \aleph_0 -stable.

By Example 6.5, simplicity does not follow from tameness even with categoricity. Since our only non-tame example (Example 6.7) is not simple, we could also ask if tameness would follow from simplicity. However, the authors believe this is very unlikely. The following questions remain.

Question 6.28 *Is there an example of a simple, tame finitary almost atomic class which is \aleph_0 -stable but not excellent?*

Question 6.29 *Is there a simple finitary AEC which is not tame?*

To ask Question 6.28 correctly, we will have to be careful what excellence means. We require that, to have excellence, we will have to have *prime* models over independent diagrams of models. This requirement is somewhat looser than to require *primary* models over independent diagrams, where primary refers to the notion of isolation available in atomic AECs. We can add countably many predicates to the language to transfer an \aleph_0 -stable almost atomic finitary class into an atomic AEC (see section 2.3) and then the definitions will become equal.

If we do not add any predicates to the language of a finitary AEC and look at the more strict definition of excellence, we will get more examples of ‘non-excellence’. For example, the \aleph_0 -saturated models of a complete, \aleph_0 -stable first-order theory with eni-dop will become an example of a simple, tame, finitary, \aleph_0 -stable and almost atomic class, which is not ‘excellent’, see Laskowski [15] for such examples. That is due to the fact that the atomic model over the diagram (in the first-order sense) is not \aleph_0 -saturated. There is a *prime* model in the class of \aleph_0 -saturated models which is not atomic, but will become atomic after adding the predicates.

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