Inverse problems for elliptic equations

Matti Lassas

HELSINKI UNIVERSITY OF TECHNOLOGY
Institute of Mathematics

Finnish Centre of Excellence
in Inverse Problems Research

1 Inverse problem in applications

Calderón's inverse problem: Measure electric resistance between all boundary points of a body. Can the conductivity be determined in the body?

Inverse problem for the wave equation: Let us send waves from the boundary of a body and measure the waves at the boundary. Can the wave speed be determined in the body?
Question: What happens if boundary is not well known?

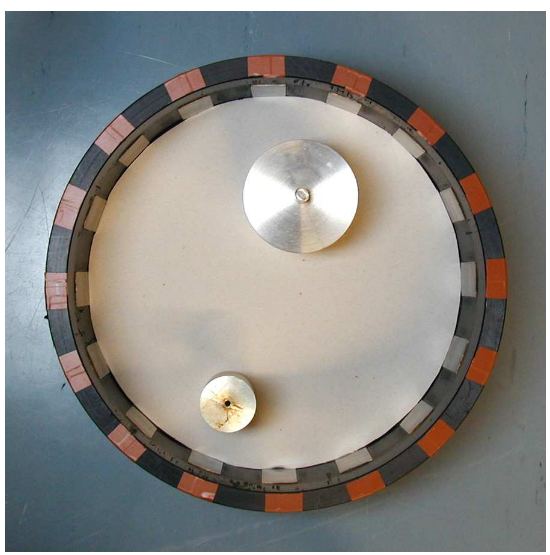
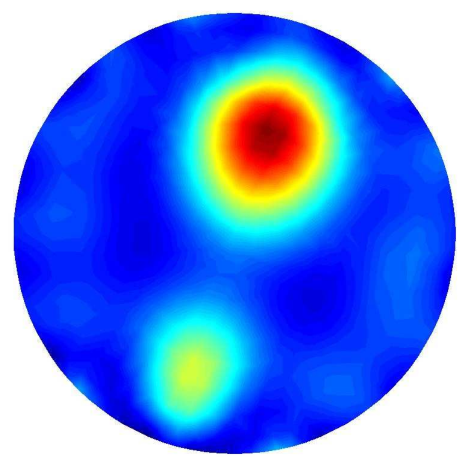



Figure: University of Kuopio.

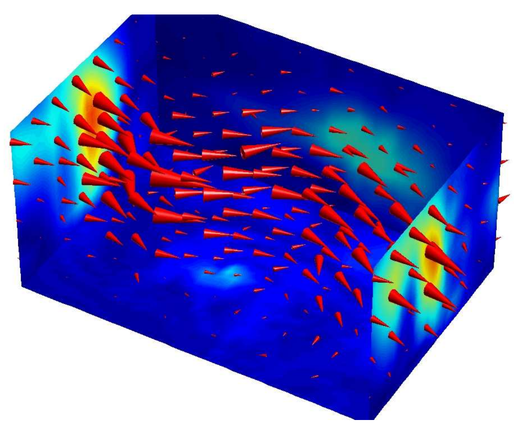
2 Inverse conductivity problem

Consider a body $\Omega \subset \mathbb{R}^n$. An electric potential $u(x)$ causes the current

$$J(x) = \sigma(x)\nabla u(x).$$

Here the conductivity $\sigma(x)$ can be an isotropic, that is, scalar, or an anisotropic, that is, matrix valued function.
If the current has no sources inside the body, we have

$$\nabla \cdot \sigma(x)\nabla u(x) = 0.$$



Conductivity equation

$$\nabla\cdot\sigma(x)\nabla u(x) = 0 \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = f.$$

Calderón's inverse problem: Do the measurements made on the boundary determine the conductivity, that is, does $\partial\Omega$ and the Dirichlet-to-Neumann map Λ_σ,

$$\Lambda_\sigma(f) = \nu\cdot\sigma\nabla u|_{\partial\Omega}$$

determine the conductivity $\sigma(x)$ in Ω?

Some previous results for inverse conductivity problem:

- Calderón 1980: Solution of the linearized inverse problem.
- Sylvester-Uhlmann 1987: Uniqueness of inverse problem in $\mathbb{R}^n$, $n \geq 3$
- Nachman 1996: Calderón's problem in $\mathbb{R}^2$
- Astala-Päivärinta 2003: Uniqueness of Calderón's problem in $\mathbb{R}^2$ with L^∞-conductivity
- Sylvester 1990: Inverse problem for an anisotropic conductivity near constant in $\mathbb{R}^2$.
- Siltanen-Mueller-Isaacson 2000: Explicit numerical solution for the 2D-inverse problem.
- Kenig-Sjöstrand-Uhlmann 2006: Reconstructions with limited data.

What happens when the following standard assumptions are not valid?

- The boundary $\partial\Omega$ is known.
- Topology of Ω is known.
- Conductivity satisfies

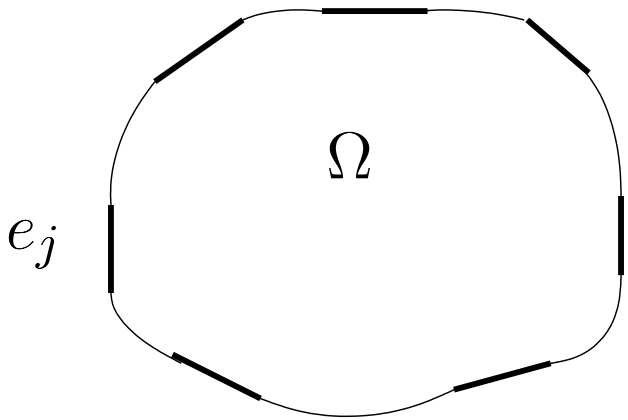
$$C_0 \leq \gamma(x) \leq C_1, \quad C_0, C_1 > 0.$$

3 Electrical Impedance Tomography with an unknown boundary

Practical task: In medical imaging one often wants to find an image of the conductivity, when the domain Ω is poorly known.

Figure: Rensselaer Polytechnic Institute.

Complete electrode model. Let $e_j \subset \partial\Omega$, $j = 1, \ldots, J$ be disjoint open sets (electrodes) and



$$\nabla\cdot \gamma \nabla v = 0 \quad \text{in } \Omega,$$
$$z_j \nu\cdot \gamma \nabla v + v|_{e_j} = V_j,$$
$$\nu\cdot \gamma \nabla v|_{\partial\Omega \setminus \cup_{j=1}^{J} e_j} = 0.$$

Here z_j are the contact impedance of electrodes and $V_j \in \mathbb{R}$. The boundary measurements are the currents

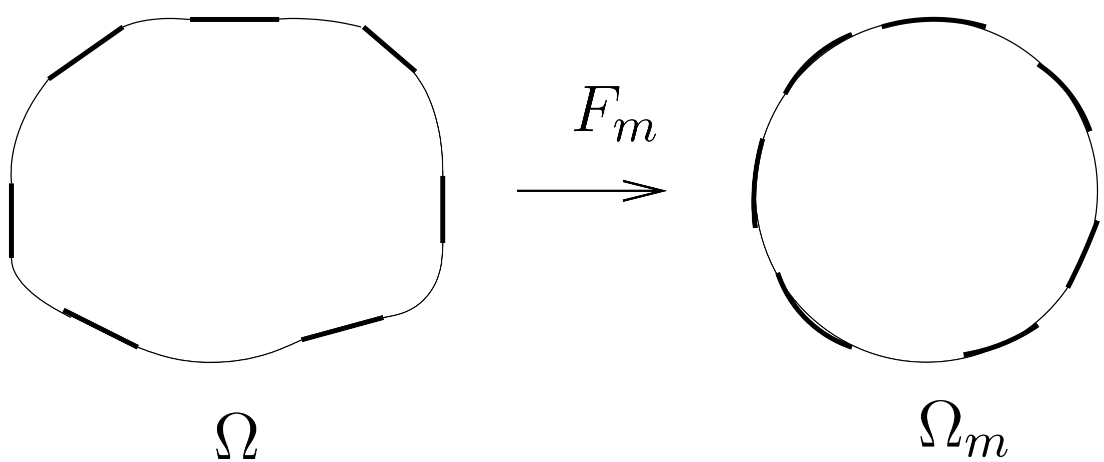
$$I_j = \frac{1}{|e_j|} \int_{e_j} \nu\cdot \gamma \nabla v(x)\, ds(x), \quad j = 1, \ldots, J.$$

The matrix $E : (V_j)_{j=1}^{J} \to (I_j)_{j=1}^{J}$ is the electrode measurement matrix.

Mathematical formulation of EIT with unknown boundary:

1. Assume that γ is an isotropic conductivity in Ω.
2. Assume that we are given a set Ω_m that is our best guess for Ω. Let $F_m : \Omega \to \Omega_m$ be a map corresponding to the modeling error.
3. The given data is the electrode measurement matrix $E \in \mathbb{R}^{J\times J}$.

Fact: The deformation $F_m : \Omega \to \Omega_m$ can change an isotropic conductivity to an anisotropic conductivity.



4 Anisotropic inverse problems

- Non-uniqueness.
- Invariant formulation. Uniqueness and non-uniqueness results
- Applications to Euclidean space: non-uniqueness results.

Deformation of the domain. Assume that $\gamma(x) = (\gamma^{jk}(x)) \in \mathbb{R}^{n\times n}$,

$$\nabla\cdot \gamma \nabla u = 0 \quad \text{in } \Omega.$$

Let F be diffeomorphism

$$F : \Omega \to \Omega, \quad F|_{\partial\Omega} = Id.$$

Then

$$\nabla\cdot \widetilde{\gamma} \nabla v = 0 \quad \text{in } \Omega,$$

where

$$v(x) = u(F^{-1}(x)), \quad \widetilde{\gamma}(y) = F_*\gamma(y) = \left.\frac{(DF)\cdot \gamma\cdot (DF)^t}{\det (DF)}\right|_{x=F^{-1}(y)}$$

Then $\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma$.

Invariant formulation.
Assume $n \geq 3$. Consider Ω as a Riemannian manifold with

$$g^{jk}(x) = (\text{det}\,\gamma(x))^{-1/(n-2)}\gamma^{jk}(x).$$

Then conductivity equation is the Laplace-Beltrami equation

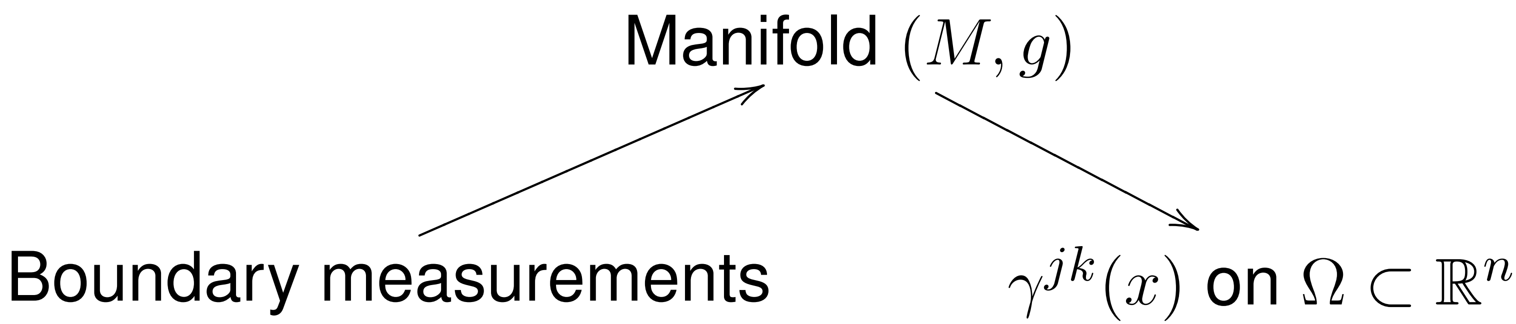
$$\Delta_g u = 0 \quad \text{in } \Omega, \quad \text{where}$$

$$\Delta_g u = \sum_{j,k=1}^{n} g^{-1/2} \frac{\partial}{\partial x^j}(g^{1/2} g^{jk} \frac{\partial}{\partial x^k} u)$$

and $g = \text{det}\,(g_{ij})$, $[g_{ij}] = [g^{jk}]^{-1}$.
Inverse problem: Can we determine the Riemannian manifold (M, g) by knowing ∂M and

$$\Lambda_{M,g} : u|_{\partial M} \mapsto \partial_\nu u|_{\partial M}.$$

Generally, solutions of anisotropic inverse problems are not unique. However, if we have enough a priori knowledge of the form of the conductivity, we can sometimes solve the inverse problem uniquely.



Uniqueness results

Theorem 1 (L.-Taylor-Uhlmann 2003) *Assume that* (M, g) *is a complete* n*-dimensional real-analytic Riemannian manifold and* $n \geq 3$. *Then* ∂M *and*

$$\Lambda_{M,g} : u|_{\partial M} \mapsto \partial_\nu u|_{\partial M}$$

determine (M, g) *uniquely.*

Theorem 2 (L.-Uhlmann 2001) *Assume that* (M, g) *is a compact* 2*-dimensional Riemannian manifold. Then* ∂M *and*

$$\Lambda_{M,g} : u|_{\partial M} \mapsto \partial_\nu u|_{\partial M}$$

determine conformal class

$$\{(M, \alpha g) : \quad \alpha \in C^\infty(M),\ \alpha(x) > 0\}$$

uniquely.

5 Anisotropic problem in $\Omega \subset \mathbb{R}^2$.

Isotropic case:

Theorem 3 (Astala-Päivärinta 2003) *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected bounded domain and* $\sigma \in L^\infty(\Omega; \mathbb{R}_+)$ *an isotropic conductivity function. Then the Dirichlet-to-Neumann map* Λ_σ *for the equation*

$$\nabla \cdot \sigma \nabla u = 0$$

determines uniquely the conductivity σ*.*

Next we denote $\sigma \in \Sigma(\Omega)$ if $\sigma(x) \in \mathbb{R}^{2\times 2}$ is symmetric, measurable, and

$$C_1\, I \leq \sigma(x) \leq C_2\, I, \quad \text{for a.e. } x \in \Omega$$

with some $C_1, C_2 > 0$.

Sylvester 1990, Sun-Uhlmann 2003, Astala-L.-Päivärinta 2005

Theorem 4 *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected bounded domain and* $\sigma_1, \sigma_2 \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$ *conductivity tensors. If* $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ *then there is a* $W^{1,2}$*-diffeomorphism*

$$F : \Omega \to \Omega, \quad F|_{\partial\Omega} = Id$$

such that

$$\sigma_1 = F_* \sigma_2.$$

Recall that if $F : \Omega \to \widetilde{\Omega}$ is a diffeomorphism, it transforms the conductivity σ in Ω to $\widetilde{\sigma} = F_* \sigma$ in $\widetilde{\Omega}$,

$$\widetilde{\sigma}(x) = \left. \frac{DF(y)\, \sigma(y)\, (DF(y))^t}{|\det DF(y)|} \right|_{y=F^{-1}(x)}$$

Proof. Identify $\mathbb{R}^2 = \mathbb{C}$. Let σ be an anisotropic conductivity, $\sigma(x) = I$ for $x \in \mathbb{C} \setminus \Omega$. There is $F : \mathbb{C} \to \mathbb{C}$ such that

$$\gamma = F_* \sigma$$

is isotropic. There are $w(x, k)$ such that

$$\nabla \cdot \gamma \nabla w = 0 \text{ in } \mathbb{C}$$

and

$$\lim_{x \to \infty} w(x, k) e^{-ikx} = 1, \qquad \lim_{k \to \infty} \frac{1}{k} \log(w(x, k) e^{-ikx}) = 0.$$

Let $u(x, k) = w(F^{-1}(x), k)$. The Λ_σ determines $u(x, k)$ for $x \in \mathbb{C} \setminus \Omega$ and

$$F^{-1}(x) = \lim_{k \to \infty} \frac{\log u(x, k)}{ik}, \qquad x \in \mathbb{C} \setminus \Omega.$$

Corollaries:

1. Inverse problem in the half space.

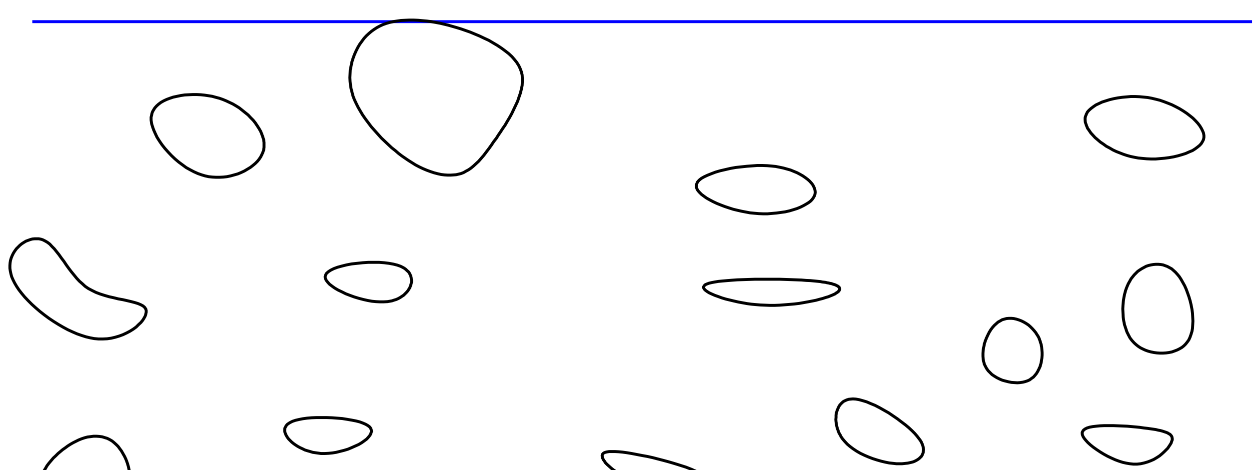
Let $\sigma \in C^\infty(\mathbb{R}^2_-)$ satisfy $0 < C_1 \le \sigma \le C_2$ and

$$\nabla\cdot \sigma \nabla u = 0 \qquad \text{in } \mathbb{R}^2_- = \{(x^1, x^2) \,|\, x^2 < 0\}, \tag{1}$$

$$u|_{\partial \mathbb{R}^2_-} = f, \quad u \in L^\infty(\mathbb{R}^2_-). \tag{2}$$

Notice that here the radiation condition at infinity (2) is quite simple. Let

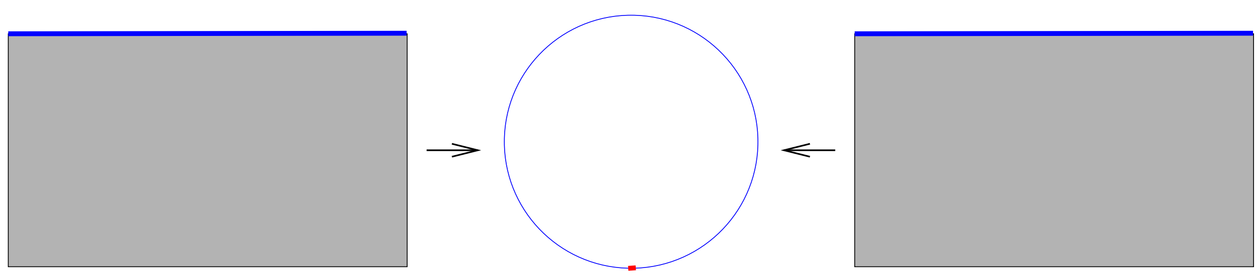
$$\Lambda_\sigma : H^{1/2}_{comp}(\partial \mathbb{R}^2_-) \to H^{-1/2}(\partial \mathbb{R}^2_-), \quad f \mapsto \nu\cdot \sigma \nabla u|_{\partial \mathbb{R}^2_-}.$$



Corollary 5.1 (Astala-L.-Päivärinta 2005) *The map Λ_σ determines the equivalence class*

$$E_\sigma = \{\sigma_1 \in \Sigma(\mathbb{R}^2_-) \quad | \quad \sigma_1 = F_*\sigma,\ F : \mathbb{R}^2_- \to \mathbb{R}^2_- \text{ is } W^{1,2}\text{-diffeo}, \\ F|_{\partial\mathbb{R}^2_-} = I\}.$$

Moreover, each class E_σ contains at most one isotropic conductivity.
Thus, if σ is known to be isotropic, it is determined uniquely by Λ_σ.



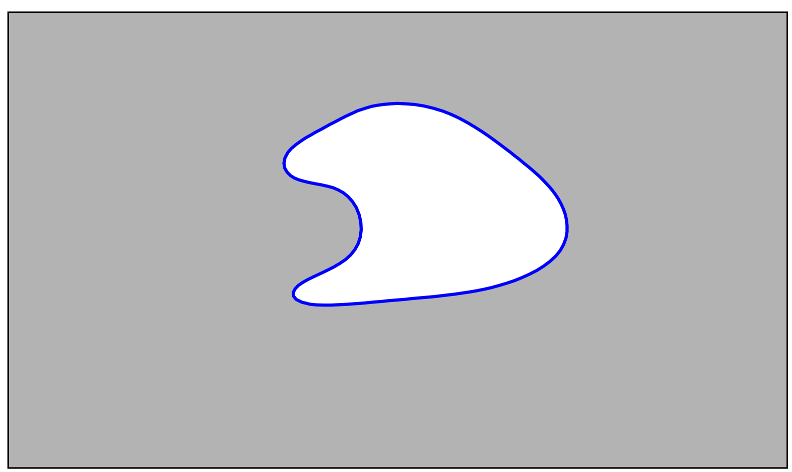
Open problem: Inverse problem in $\mathbb{R}^3_+$.

2. Inverse problem in the exterior domain. Let $S = \mathbb{R}^2 \setminus \overline{D}$, where D is a bounded Jordan domain. Let

$$\begin{aligned} \nabla\cdot \sigma\nabla u &= 0 \quad \text{in } S, \\ u|_{\partial S} &= f \in H^{1/2}(\partial S), \\ u &\in L^\infty(S). \end{aligned}$$

We define

$$\Lambda_\sigma : H^{1/2}(\partial S) \to H^{-1/2}(\partial S), \quad f \mapsto \nu\cdot \sigma\nabla u|_{\partial S}.$$



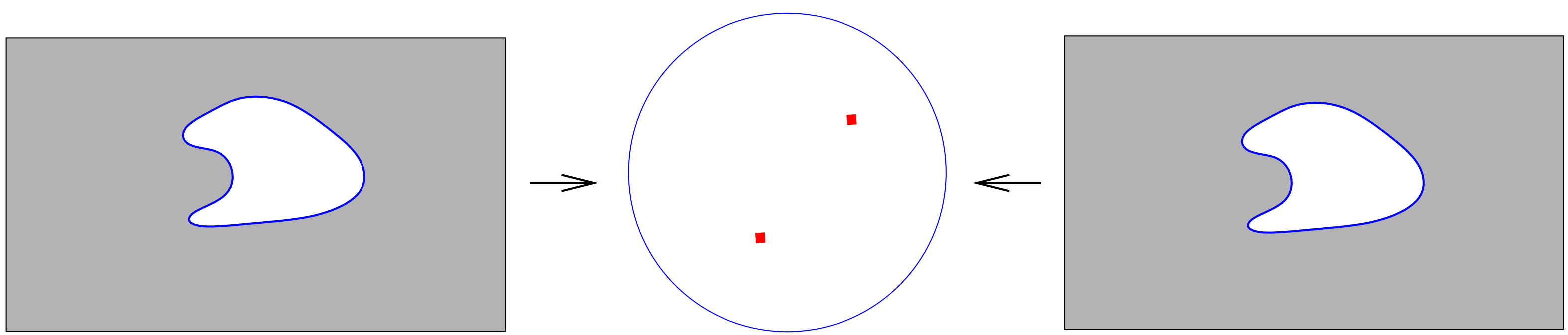
Let $S \subset \mathbb{R}^2$, $\mathbb{R}^2 \setminus S$ compact, and denote $\overline{S} = S \cup \{\infty\}$.

Corollary 5.2 (Astala-L.-Päivärinta 2005) *Let $\sigma \in \Sigma(S)$. Then the map Λ_σ determines the equivalence class*

$$E_{\sigma,S} = \{\sigma_1 \in \Sigma(S) \quad | \quad \sigma_1 = F_*\sigma,\ F : \overline{S} \to \overline{S} \text{ is a } W^{1,2}\text{-diffeo}, \\ F|_{\partial S} = I\ \}.$$

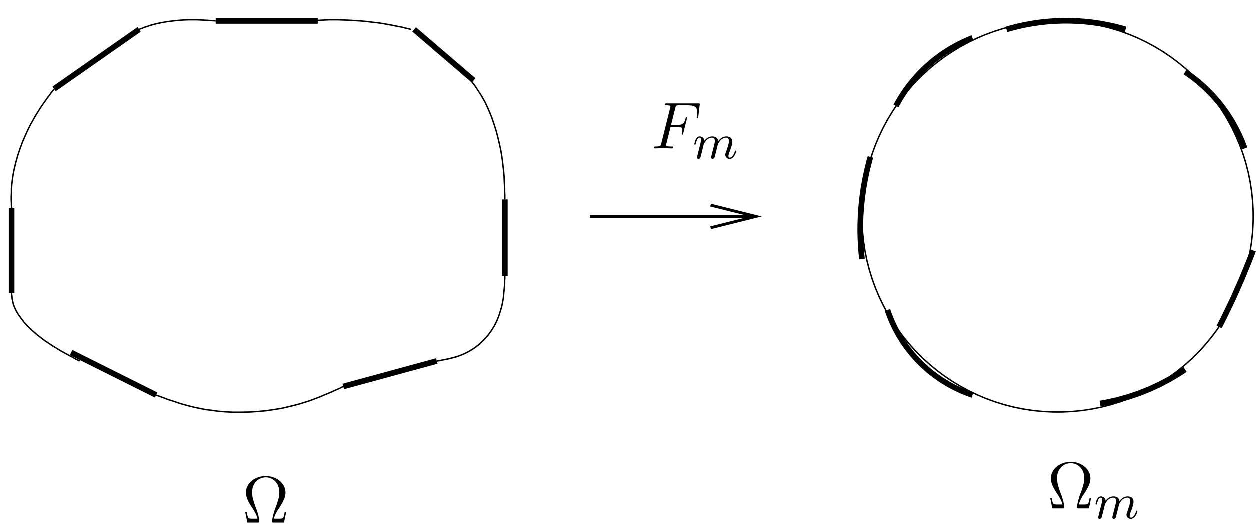
Moreover, if σ is known to be isotropic, it is determined uniquely by Λ_σ.

The group of diffeomorphisms preserving the data do not map $S \to S$.

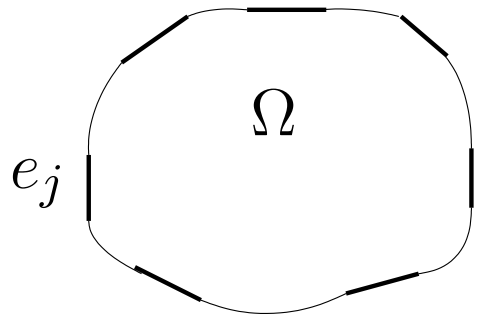


6 Unknown boundary problem in $\mathbb{R}^2$.

1. Assume that γ is an isotropic conductivity in Ω.
2. Assume that we are given a set Ω_m that is our best guess for Ω. Let $F_m : \Omega \to \Omega_m$ be a map corresponding to the modeling error.
3. We are given the electrode measurement matrix $E \in \mathbb{R}^{J \times J}$.



Complete electrode model Let $e_j \subset \partial\Omega$, $j = 1, \ldots, J$ be disjoint open sets (electrodes) and



$$
\begin{aligned}
\nabla\cdot \gamma \nabla v &= 0 \quad \text{in } \Omega, \\
z_j \nu\cdot \gamma \nabla v + v|_{e_j} &= V_j, \\
\nu\cdot \gamma \nabla v|_{\partial\Omega \setminus \cup_{j=1}^{J} e_j} &= 0,
\end{aligned}
$$

where z_j are the contact impedances and V_j are the potentials on electrode e_j. Measure currents

$$
I_j = \frac{1}{|e_j|} \int_{e_j} \nu\cdot \gamma \nabla v(x)\, ds(x), \quad j = 1, \ldots, J.
$$

This give us electrode measurements matrix $E : \mathbb{R}^J \to \mathbb{R}^J$, $E(V_1, \ldots, V_J) = (I_1, \ldots, I_J)$.

Continuous model. The electrical potential u satisfy

$$\begin{aligned} \nabla \cdot \gamma \nabla u = 0, \qquad & x \in \Omega, \\ (z\nu \cdot \gamma \nabla u + u)|_{\partial\Omega} = h, & \end{aligned}$$

where γ is an isotropic conductivity and z is the contact impedance on the boundary.
Boundary measurements are modeled by the Robin-to-Neumann map $R = R_{\gamma,z}$ given by

$$R_{\gamma,z} : h \mapsto \nu \cdot \gamma \nabla u|_{\partial\Omega}$$

The power needed to maintain the given voltage $(V_1, \ldots, V_J)$ or h at boundary are given by

$$p(V) = E[V, V], \qquad p(h) = R[h, h],$$

where we have quadratic forms

$$E[V, \widetilde{V}] = \sum_{j=1}^{J} (EV)_j \widetilde{V}_j |e_j|, \quad R[h, \widetilde{h}] = \int_{\partial\Omega} (Rh)\, \widetilde{h}\, ds.$$

The form $E[\,\cdot\,,\cdot\,]$ can be viewed as a discretization of $R[\,\cdot\,,\cdot\,]$.

Let $F_m : \Omega \to \Omega_m$ be deformation of the domain and $f_m = F_m|_{\partial\Omega}$. On $\partial\Omega_m$ we define

$$\widetilde{R} = (f_m)_* R_{\gamma,z}.$$

Then the quadratic form R corresponding to the power needed to have the given voltage on the boundary satisfies

$$\widetilde{R}[h,h] = R[h \circ f_m, h \circ f_m], \quad h \in H^{-1/2}(\partial\Omega_m).$$

Thus the electrode measurement matrix on $\partial\Omega_m$ corresponds in the continuous model to the map

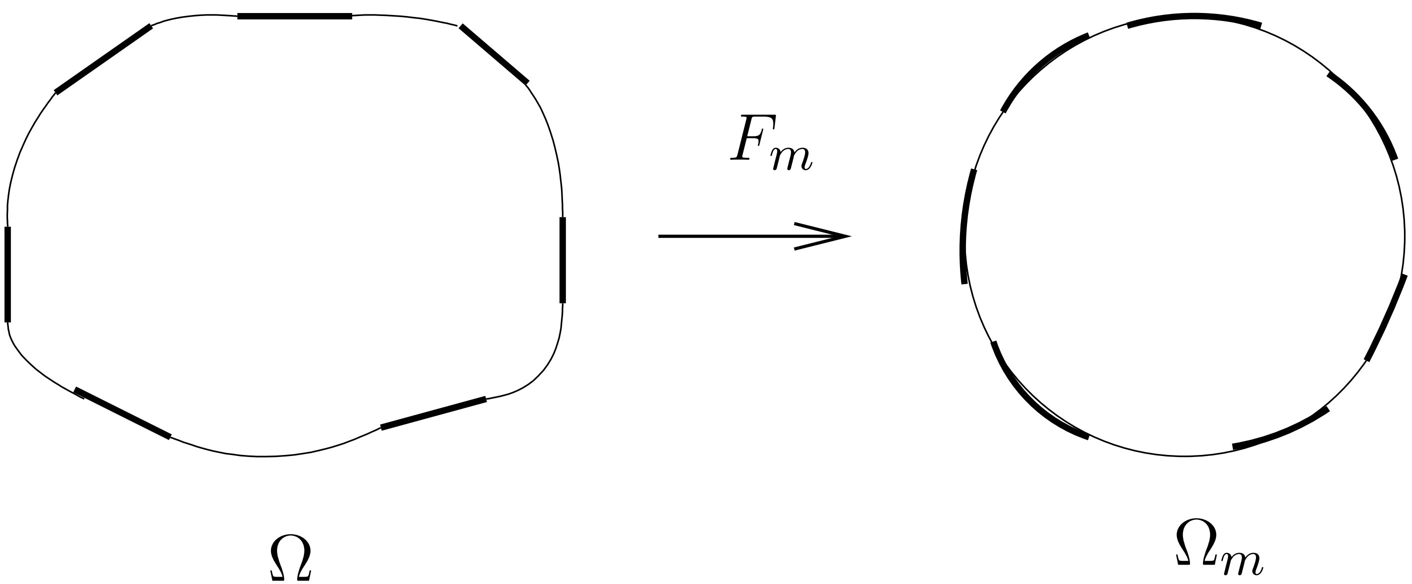
$$\widetilde{R} = (f_m)_* R_{\gamma,z}.$$

Fact: $\widetilde{R} = R_{\widetilde{\gamma},\widetilde{z}}$ where

$$\widetilde{\gamma} = (F_m)_* \gamma, \quad \widetilde{z} = (F_m)_* z.$$

Thus the boundary map $\widetilde{R}$ on $\partial\Omega_m$ is equal to $R_{\widetilde{\gamma},\widetilde{z}}$ that corresponds to boundary measurements made with an anisotropic conductivity $\widetilde{\gamma} = (F_m)_*\gamma$ in Ω_m and $\widetilde{z} = z \circ f_m^{-1}$.

Assume we are given Ω_m and $\widetilde{R}$. Our aim is to find a conductivity tensor in Ω_m that is as close as possible to an isotropic conductivity and has the Robin-to-Neumann map $\widetilde{R}$.

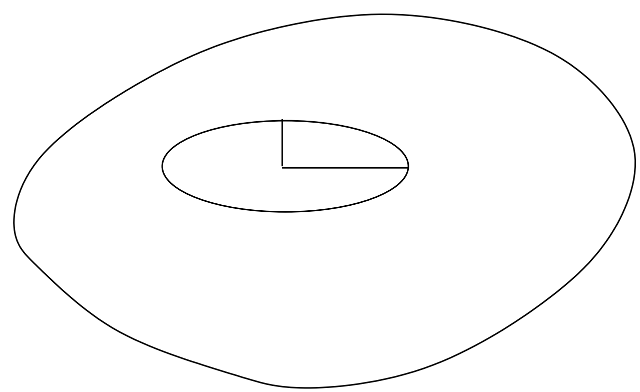


Definition 6.1 *Let $\gamma = \gamma^{jk}(x)$ be a matrix valued conductivity. Let $\lambda_1(x)$ and $\lambda_2(x)$, $\lambda_1(x) \geq \lambda_2(x)$ be its eigenvalues. Anisotropy of γ at x is*

$$K(\gamma, x) = \left(\frac{\lambda_1(x) - \lambda_2(x)}{\lambda_1(x) + \lambda_2(x)} \right)^{1/2}.$$

The maximal anisotropy of γ in Ω is

$$K(\gamma) = \sup_{x \in \Omega} K(\gamma, x).$$



The anisotropy function $K(\widehat{\gamma}, x)$ is constant for

$$\widehat{\gamma}(x) = \eta(x) R_{\theta(x)} \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix} R_{\theta(x)}^{-1}$$

where

$$\begin{aligned} &\lambda \geq 1, \\ &\eta(x) \in \mathbb{R}_+, \\ &R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \end{aligned}$$

We say that $\widehat{\gamma} = \widehat{\gamma}_{\lambda,\theta,\eta}$ is a uniformly anisotropic conductivity.

Theorem 6.2 (Kolehmainen-L.-Ola 2005) *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected $C^{1,\alpha}$–domain, $\gamma \in L^\infty(\overline{\Omega}, \mathbb{R})$ be isotropic conductivity, and $z \in C^1(\partial\Omega)$ be the contact impedance.*

Let Ω_m be a model domain and $f_m : \partial\Omega \to \partial\Omega_m$ be a $C^{1,\alpha}$–diffeomorphism.

Assume that we know $\partial\Omega_m$ and $\widetilde{R} = (f_m)_* R_{\gamma,z}$. These data determine $\widetilde{z} = z \circ f_m^{-1}$ and an anisotropic conductivity σ on Ω_m such that

1. $R_{\sigma,\widetilde{z}} = \widetilde{R}$.
3. *If σ_1 satisfies $R_{\sigma_1,\widetilde{z}} = \widetilde{R}$ then $K(\sigma_1) \geq K(\sigma)$.*

Moreover, the conductivity σ is uniformly anisotropic.

Algorithm:
In following, we assume that $z = 0$ and denote $R_\sigma = R_{\sigma,z}$. The conductivity $\sigma = \widehat{\gamma}_{\lambda,\eta,\theta}$ can obtained by solving the minimization problem

$$\min_{(\lambda,\theta,\eta)\in S} \lambda, \quad \text{where } S = \{(\lambda,\theta,\eta) : \; R_{\widehat{\gamma}(\lambda,\theta,\eta)} = \widetilde{R}\}.$$

In implementation of the algorithm we approximate this by

$$\min_{(\lambda,\theta,\eta)} \|R_{\widehat{\gamma}(\lambda,\theta,\eta)} - \widetilde{R}\|^2 + \varepsilon_1|\lambda - 1|^2 + \varepsilon_2(\|\theta\|^2 + \|\eta\|^2).$$

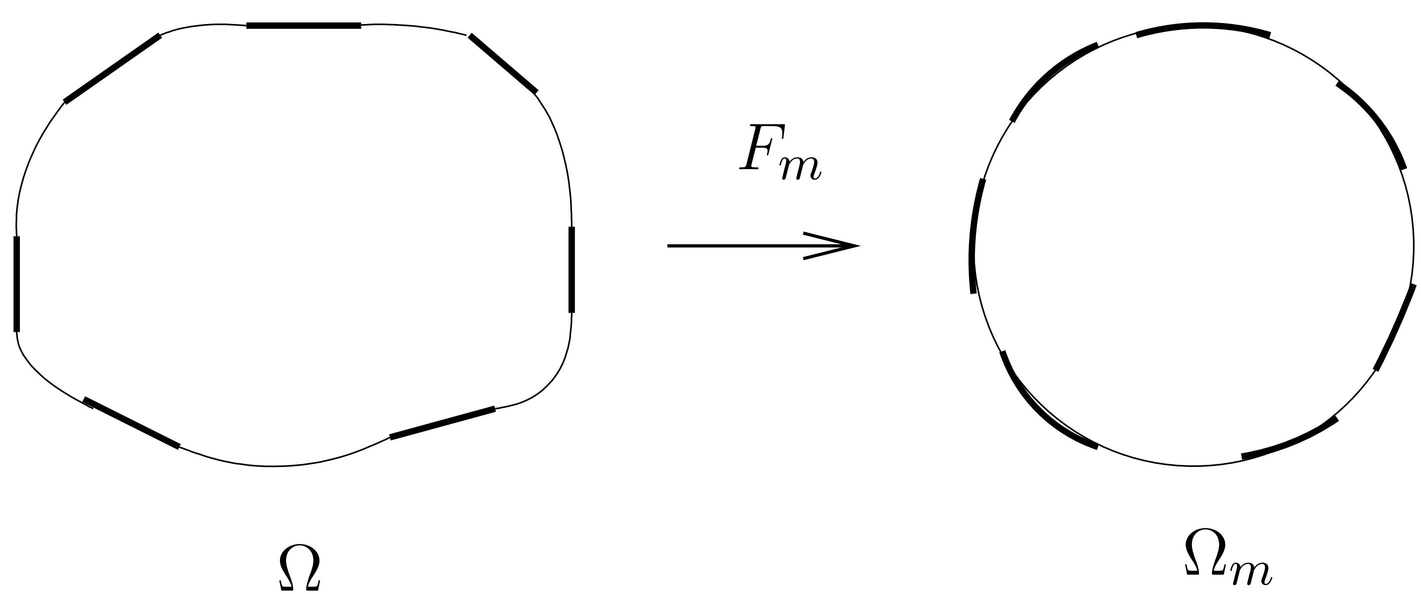
Let $f_m : \partial\Omega \to \partial\Omega_m$ be the boundary modeling map and σ be the conductivity with the smallest possible anisotropy such that $R_\sigma = \widetilde{R}$. Then

Corollary 6.3 *Then there is a unique map*

$$F_e : \Omega \to \Omega_m, \qquad F_e|_{\partial\Omega} = f_m$$

depending only on $f_m : \partial\Omega \to \partial\Omega_m$ *such that*

$$\textit{det}\,(\sigma(x))^{1/2} = \gamma(F_e^{-1}(x)).$$



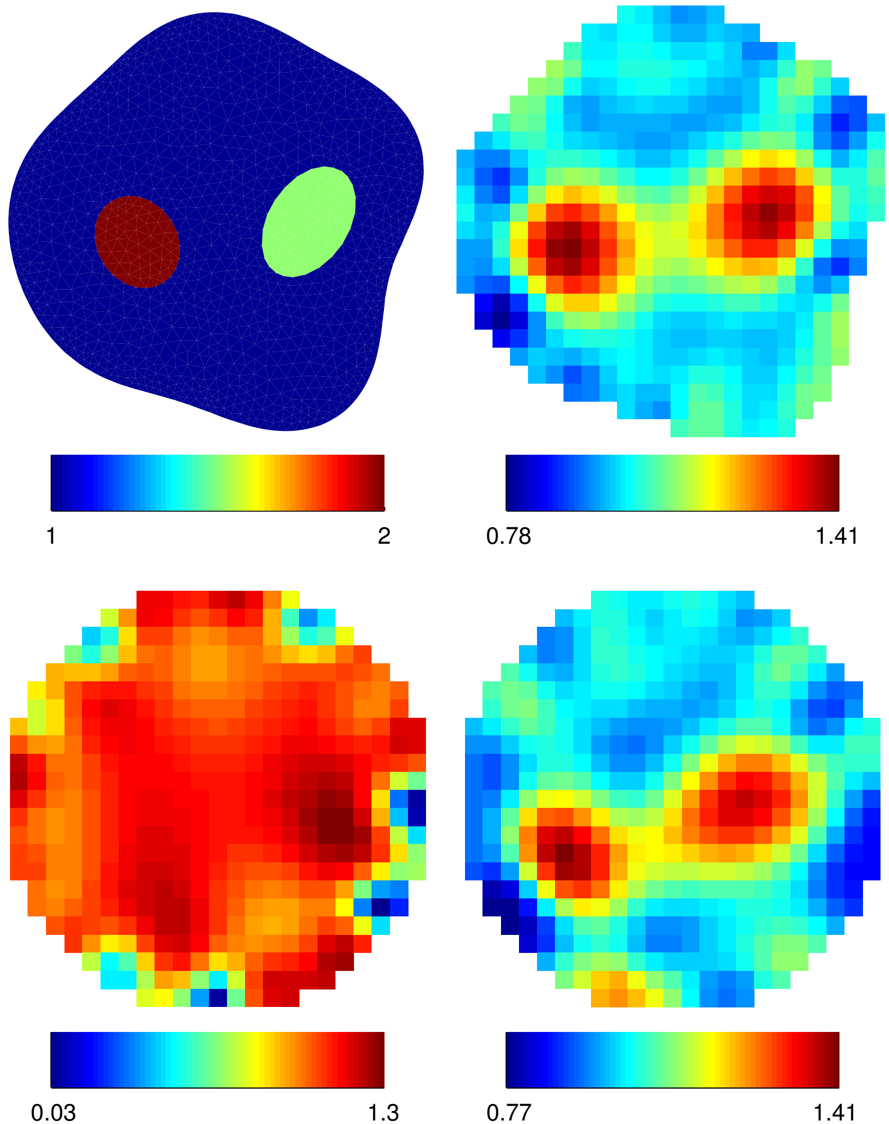
Idea of the proof. If $F:\Omega \to \Omega$ is a diffeomorphism and γ_1 is an isotropic conductivity, then

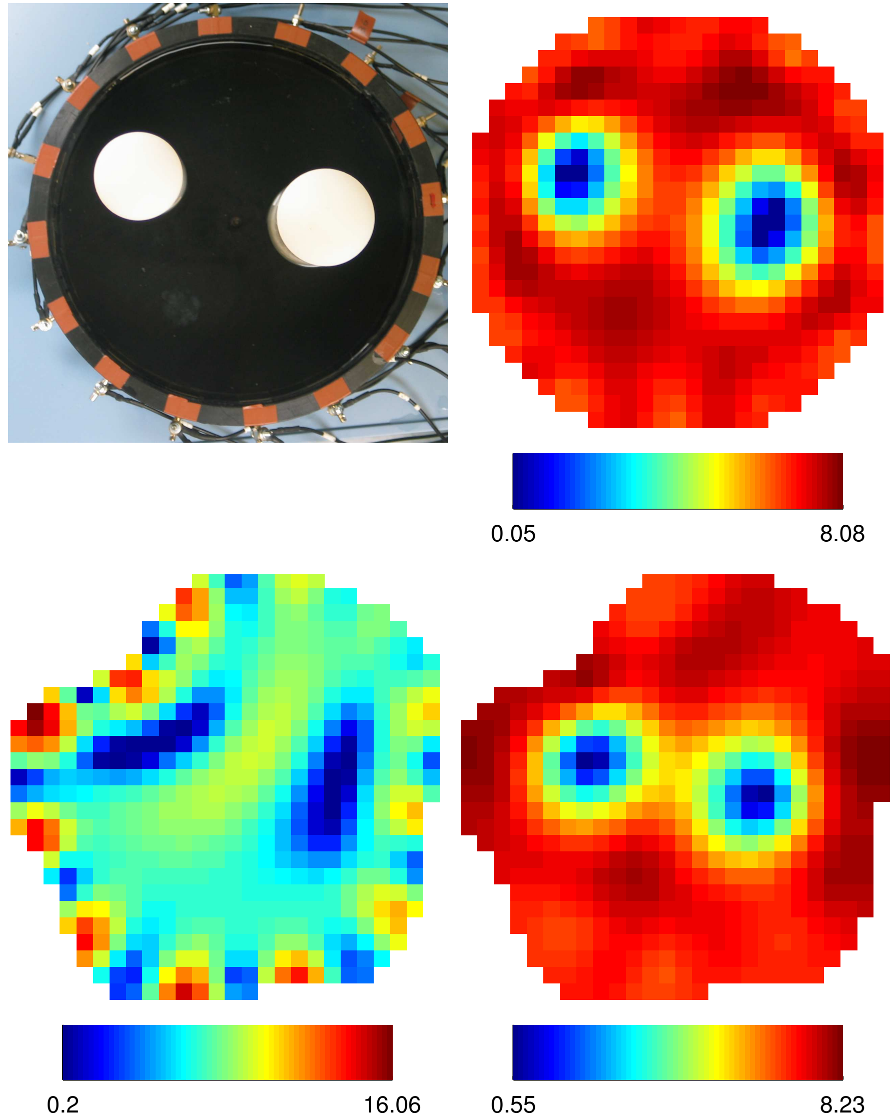
$$K(F_*\gamma_1, x) = |\mu_F(x)|$$

where

$$\mu_F = \frac{\overline{\partial} F}{\partial F}, \quad \partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}).$$

To find the minimally anisotropic conductivity we need to find a quasiconformal map with the smallest possible dilatation and the given boundary values. This is called the Teichmüller map.


1
2
0.78
1.41
0.03
1.3
0.77
1.41


0.05
8.08
0.2
16.06
0.55
8.23

7 Unknown boundary problem in $\mathbb{R}^3$.

The electrical potential u satisfies

$$\nabla \cdot \gamma \nabla u = 0, \qquad x \in \Omega \subset \mathbb{R}^3,$$
$$(z\nu \cdot \gamma \nabla u + u)|_{\partial\Omega} = h,$$

where γ is an isotropic conductivity and z is the contact impedance on the boundary.
The boundary measurements are modeled by the Robin-to-Neumann map $R = R_{z,\gamma}$ given by

$$R_{\gamma,z} : h \mapsto \nu \cdot \gamma \nabla u|_{\partial\Omega}$$

Again, let $f_m : \partial\Omega \to \partial\Omega_m$ be the modeling of boundary and $\widetilde{R} = (f_m)_* R_{\gamma,z}$.

Theorem 5 (Kolehmainen-L.-Ola 2006) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded, strictly convex, C^∞–domain. Assume that $\gamma \in C^\infty(\overline{\Omega})$ is an isotropic conductivity, $z \in C^\infty(\partial\Omega)$, $z > 0$ is the contact impedance, and $R_{\gamma,z}$ is the Robin-to-Neumann map.*

Let Ω_m be a model domain and $f_m : \partial\Omega \to \partial\Omega_m$ be a diffeomorphism.

Assume that we are given $\partial\Omega_m$, the values of the contact impedance $z(f_m^{-1}(x))$, and the map $\widetilde{R} = (f_m)_* R_{\gamma,z}$.

Then we can determine Ω upto a rigid motion T and the conductivity $\gamma \circ T^{-1}$ on the reconstructed domain $T(\Omega)$.

Idea of the proof: Let γ be the isotropic conductivity on Ω, $\widetilde{\gamma} = (F_m)_*\gamma$, $F_m|_{\partial\Omega} = f_m$. Let $\widetilde{g}$ be the metric in Ω_m corresponding to the conductivity $\widetilde{\gamma}$.

- $\widetilde{R} = R_{\widetilde{\gamma},\widetilde{z}}$ determine the contact impedance $\widetilde{z}$ and the metric $\widetilde{g}$ on boundary $\partial\Omega_m$.
- $\widetilde{z}(x)$ and $z(f_m^{-1}(x))$ determine $\beta = \det(Df_m^{-1})$.
- $\widetilde{g}$ and β determine $\gamma \circ f_m^{-1}$ on boundary $\partial\Omega_m$.
- On $\partial\Omega_m$ we find the metric corresponding to the Euclidean metric of $\partial\Omega$. This determines by the Cohn-Vossen rigidity theorem the strictly convex set Ω up to a rigid motion T.
- In $T(\Omega)$ we solve an isotropic inverse problem.

Consider now the following algorithm:

Data: Assume that we are given $\partial\Omega_m$, $\widetilde{R} = (f_m)_* R_{\gamma,z}$ and $z \circ f_m^{-1}$ on $\partial\Omega_m$.

Aim: We look for a metric $\widetilde{g}$ corresponding to the conductivity $\widetilde{\gamma}$ and $\widetilde{z}$ such that $\widetilde{R} = R_{\widetilde{\gamma},\widetilde{z}}$ and $\widetilde{z} = z \circ f_m^{-1}$.

Idea: We look for a metric $\widetilde{g}$ in Ω_m and $\rho \in C^\infty(\Omega_m)$ such that

$$\overline{g}_{ij}(x) = e^{2\rho(x)}\widetilde{g}_{ij}(x) \quad \text{is flat.}$$

Algorithm:

1. Determine the two leading terms in the symbolic expansion of $\widetilde{R}$. They determine a contact impedance $\widehat{z}$ and a metric $\widehat{g}$ on $\partial\Omega_m$ such that if $\widetilde{R} = R_{\widetilde{\gamma},\widetilde{z}}$ then $\widetilde{z} = \widehat{z}$ and $\widetilde{g}|_{\partial\Omega_m} = \widehat{g}$.

2. Compute the ratio of reconstructed i.e. $\widehat{z}$, and measured contact impedances

$$\beta(x) := \frac{z(f_m^{-1}(x))}{\widehat{z}(x)}, \quad x \in \partial\Omega_m.$$

Then $\beta = \frac{dS_{\partial\Omega_m}}{(f_m)_* dS_{\partial\Omega}}$.

3. Let $dS_{\widehat{g}}$ be the volume form of $\widehat{g}$ on $\partial\Omega_m$ and dS_E the Euclidean volume on $\partial\Omega_m$. Then

$$dS_{\widehat{g}} = (\text{det}\,\widehat{g})^{1/2}\, dS_E.$$

Define

$$\widehat{\gamma} = (\text{det}\,\widehat{g})^{1/2}\, \beta.$$

With this choice $\widehat{\gamma}$ will satisfy $\widehat{\gamma}(x) = \gamma(f_m^{-1}(x))$ for $x \in \partial\Omega_m$.

4. Define the boundary value $\widehat{\rho}$ for the function ρ by

$$\widehat{\rho}(x) = \frac{1}{2-n} \log\left(\widehat{\gamma}(x)\right), \quad x \in \partial\Omega_m.$$

5. Solve the minimization problem

$$\min F_\tau(z,\rho,\gamma)$$

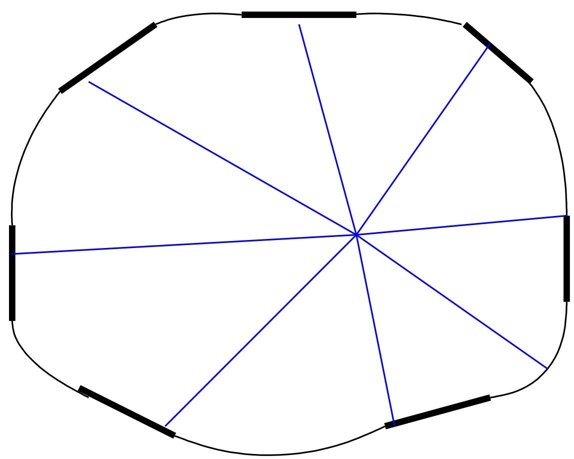
$$F_\tau(z,\rho,\gamma) = \|\widetilde{R} - R_{\gamma,z}\|^2_{L(H^{-1/2}(\partial\Omega_m))}$$
$$+\|\frac{z(x)}{z(f_m^{-1}(x))} - \beta(x)\|_{L^2(\partial\Omega_m)} + \|\,\rho|_{\partial\Omega_m} - \widehat{\rho}\,\|^2_{L^2(\partial\Omega_m)}$$
$$+\tau\|\mathsf{C}\|^2_{L^2(\Omega_m)}$$
$$+\sum_{i,j=1}^{n} \|\rho_{,ij} - \left(-\mathsf{Ric}_{ij} + \frac{1}{4}g_{ij}\mathsf{R} - \frac{1}{2}g_{ij}g^{lm}\rho_{,l}\rho_{,k}\right)\|^2_{L^2(\Omega_m)}$$

where $\tau \geq 0$, g is the metric corresponding to γ, Ric and R are the Ricci curvature and scalar curvature of g, and $\mathsf{C}_{ij} = g^{kp}g^{lq}\nabla_k(\mathsf{Ric}_{li} - \frac{1}{4}R\,g_{li})\epsilon_{pqj}$ is Cotton-York tensor.

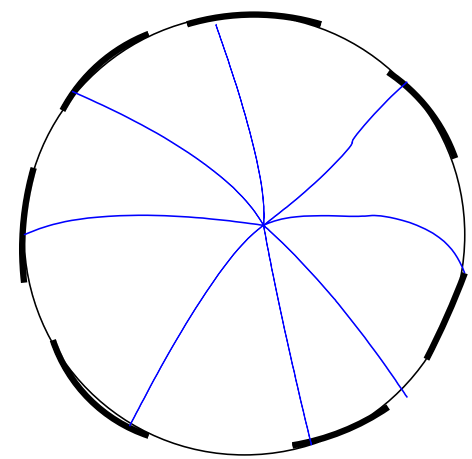
6. Find the flat metric

$$\overline{g}_{ij}(x) = e^{2\rho(x)} g_{ij}(x) = (F_m)_*(\delta_{ij})$$

on Ω_m and determine the geodesics with respect to the metric $\overline{g}$.
These give us the the embedding $F_m^{-1} : \Omega_m \to \Omega$. This gives us Ω upto a rigid motion and the conductivity γ on it.



Ω



Ω_m

Theorem 6 (Kolehmainen-L.-Ola 2006) *Let $\Omega \subset \mathbb{R}^3$ be a bounded, strictly convex, C^∞–domain. Let $\gamma \in C^\infty(\overline{\Omega})$ is an isotropic conductivity, $z \in C^\infty(\partial\Omega)$, $z > 0$ be a contact impedance.*

Let Ω_m be a model domain and $f_m : \partial\Omega \to \partial\Omega_m$ be a C^∞–smooth diffeomorphism.

Assume that we are given $\partial\Omega_m$, the values of the contact impedance $z(f_m^{-1}(x))$, $x \in \partial\Omega_m$, and the map $\widetilde{R} = (f_m)_* R_{\gamma,z}$.

Let $\tau \geq 0$. Then the minimizers $\widetilde{z}$, $\widetilde{\rho}$ and $\widetilde{\gamma}$ of $F_\tau(\widetilde{z}, \widetilde{\rho}, \widetilde{\gamma})$ determine Ω, z, and γ up to a rigid motion.

Inverse problems for conformally Euclidean metric.
We say that metric g is conformally flat if

$$g_{ij}(x) = \alpha(x)\overline{g}_{ij}(x), \qquad \text{where metric } \overline{g}_{ij}(x) \text{ is flat.}$$

Open problem: Can we determine a conformally flat metric in Ω_m from its Robin-to-Neumann map?

If this is true, then one can solve the inverse problem with an unknown boundary also for non-convex domains.

8 Maxwell's equations.

In $\Omega \subset \mathbb{R}^3$ Maxwell's equations are

$$\nabla \times E = -B_t,\ \nabla \times H = D_t,$$
$$D = \epsilon(x)E,\ B = \mu(x)H \quad \text{in } \Omega \times \mathbb{R}.$$

Let M be a 3-dimensional manifold and $\epsilon(x)$ and $\mu(x)$ metric tensors that are conformal to each other. Maxwell equations in time-domain are

$$dE = -B_t,\ dH = D_t, \quad D = *_\epsilon E,\ B = *_\mu H \quad \text{in } M \times \mathbb{R},$$
$$E|_{t<0} = 0, \quad H|_{t<0} = 0,$$

E, H are 1-forms, D, B are 2-forms, $*_\epsilon,\ *_\mu$ are Hodge-operators.

Boundary measurements:
Assume we are given $\partial\Omega$ and

$$Z : n \times E|_{\partial\Omega\times\mathbb{R}_+} \to n \times H|_{\partial\Omega\times\mathbb{R}_+},$$

Invariant formulation: Assume we are given ∂M and

$$Z : i^* E|_{\partial M\times\mathbb{R}_+} \to i^* H|_{\partial M\times\mathbb{R}_+},$$

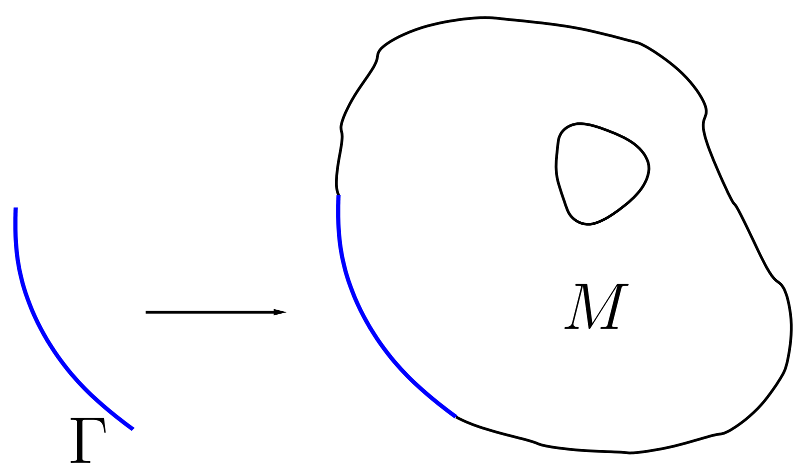
where i is the imbedding $i : \partial M \to M$.

Theorem 8.1 *[Kurylev-L.-Somersalo 2005] Let M be a compact connected 3-manifold and ϵ and μ be metric tensors conformal to each others. Assume that we are given $\Gamma \subset \partial M$ and restriction of*

$$Z_\Gamma : i^* E|_{\partial M \times \mathbb{R}_+} \to i^* H|_{\Gamma \times \mathbb{R}_+}$$

for $i^* E|_{\partial M \times \mathbb{R}_+} \in C_0^\infty(\Gamma \times \mathbb{R}_+)$. Then we can find M and ϵ, μ on M.

Corollary 8.2 *Assume that $M \subset \mathbb{R}^3$ and ϵ and μ are scalar functions. Then Γ and Z_Γ determine uniquely (M, ϵ, μ).*



Proof. We can focus the B-field to a single point:

Lemma 8.3 *Let $T > 0$ be a sufficiently large time. Then by using ∂M and map $Z_{\partial M}$ we can find all sequences of boundary values $i^* E_k|_{\partial M \times \mathbb{R}_+}$, $k = 1, 2 \ldots$ such that for some $y \in M$ and $A \in T_y^* M$*

$$\lim_{k\to\infty} B_k(x,T) = d(A\delta_y) \quad \textit{in } \mathcal{D}'(M). \tag{3}$$

The set of focusing sequences

$$\{(i^* E_k)_{k=1}^{\infty} : \text{ the limit (3) exists}\} \subset (L^2(\partial M))^{\mathbb{Z}_+}$$

can be identified with the tangent bundle TM of M,

$TM = \{(y, A) : \ y \in M, \quad A$ is a tangent vector of M at $y\}$.