Anisotropic conductivities that cannot be detected by EIT

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Abstract. We construct anisotropic conductivities in dimension three that give rise to the same voltage and current measurements at the boundary of a body as a homogeneous isotropic conductivity. These conductivities are non-zero but very small close to some surfaces inside the body.

1 Introduction

There has been much progress in recent years in understanding the *isotropic* electrical impedance tomography (EIT) problem. However, in the case of *anisotropic* conductivities, particularly in three dimensions, very little is understood. One of the difficulties is that one cannot determine an anisotropic conductor of a medium uniquely by making current and voltage measurements at the boundary of the medium. Namely, any smooth change of variable (diffeomorphism) which fixes the boundary gives rise to the same electrical measurements; we explain this below.

Let $D \subset \mathbf{R}^n$ be a domain; we will mainly be interested in $n = 2$ or $n = 3$. An anisotropic conductivity is defined by a symmetric, positive definite matrix-valued function, $\sigma = (\sigma^{ij}(x))$. In the absence of sources or sinks, the potential u satisfies

$$\begin{aligned} (\nabla\cdot \sigma\nabla)u = \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\sigma^{jk}(x)\frac{\partial}{\partial x^j}u &= 0 \text{ on } D \qquad (1)\\ u|_{\partial D} &= f, \end{aligned}$$

where f is the prescribed voltage on the boundary. The resulting voltage-to-current (or Dirichlet-to-Neumann) map is then defined by

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$$\Lambda_\sigma(f) = Bu|_{\partial D} \tag{2}$$

where

$$Bu = \sum_{j,k=1}^{n} \nu_j \sigma^{jk} \frac{\partial u}{\partial x^k}, \tag{3}$$

u is the solution of (1) and $\nu = (\nu_1, \ldots, \nu_n)$ is the unit normal vector of ∂D.

The operator Λ_σ encodes all possible current and voltage measurements made at the boundary of the body. The inverse problem is to determine σ from Λ_σ. Applying the divergence theorem, we have

$$Q_\sigma(f) =: \int_D \sum_{j,k=1}^{n} \sigma^{jk}(x) \frac{\partial u}{\partial x^j} \frac{\partial u}{\partial x^k} dx = \int_{\partial D} \Lambda_\sigma(f) f dS, \tag{4}$$

where u solves (1) and dS denotes surface measure on ∂D. $Q_\sigma(f)$ represents the power needed to maintain the potential f on ∂D. Note that, by (4), knowing Q_σ is equivalent with knowing Λ_σ. Now, if $F : D \to D$, $F = (F^1, \ldots, F^n)$, is a diffeomorphism with $F|_{\partial D} =$ Identity, then by making the change of variables $y = F(x)$ and denoting $u = v \circ F^{-1}$ in the first integral in (4), we obtain

$$\Lambda_{F_*\sigma} = \Lambda_\sigma,$$

where

$$(F_*\sigma)^{jk}(y) = \left. \frac{1}{\det[\frac{\partial F^j}{\partial x^k}(x)]} \sum_{p,q=1}^{n} \frac{\partial F^j}{\partial x^p}(x) \frac{\partial F^k}{\partial x^q}(x) \sigma^{pq}(y) \right|_{x=F^{-1}(y)} \tag{5}$$

is the "push-forward" of the conductivity σ by F. Thus, there is a large (infinite-dimensional) class of conductivities which give rise to the same electrical measurements at the boundary.

Notice that if σ is an isotropic conductivity, i.e., for some scalar function $\gamma(x) > 0$ we have

$$\sigma^{ij}(x) = \gamma(x)\delta^{ij}, \tag{6}$$

and if F is a change of variables which is the identity at the boundary such that $F_*\sigma$ is also isotropic (which need not be the case in general), then F is the identity. Therefore this obstruction is not present for isotropic conductivities. In fact for isotropic conductivities, one has the following global uniqueness result, proven in dimensions greater than or equal to three in [SyU] and in dimension two in [Na]:

Theorem 1 *Let $D \subset \mathbf{R}^n$ be a bounded domain with smooth boundary. If σ and $\widetilde{\sigma}$ are two isotropic C^2-conductivities in $D \subset \mathbf{R}^n$, which are strictly positive and for which $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$, then $\widetilde{\sigma} =.\sigma$*

For further progress on this problem, see the surveys [CnIN], [U1],[U2], the book [Is] and the recent article [GLU].

Returning to the case of anisotropic conductivities, we define $\sigma, \widetilde{\sigma}$ to be equivalent if, for some diffeomorphism $F : D \to D$ fixing the boundary, $\widetilde{\sigma} = F_*\sigma$ as in (5). The relevant inverse problem is then whether two inequivalent conductivities can have the same Dirichlet-to-Neumann map. This problem has a geometric formulation that we will review in the next section. In particular, given the invariance of the problem under changes of coordinates, it is natural, for $n \geq 3$, to consider it on a Riemannian manifold (M, g), where there is a direct correspondence between the Riemannian metric g and the conductivity σ. In section 2 we reformulate the EIT problem in geometric terms and also describe some positive results for the case that the conductivity is bounded from below, that is,

$$c_0 I \leq [\sigma^{jk}(x)]_{j,k=1}^n \leq c_1 I, \qquad \text{for some} \quad c_0, c_1 > 0, \tag{7}$$

where I denotes the $n \times n$ identity matrix. In section 3, we construct examples of inequivalent anisotropic conductivities that have the same Dirichlet-to-Neumann map as isotropic conductivities. These conductivities do not satisfy (7); in particular, they are arbitrarily small near a surface contained in the body. We conclude the paper with a brief discussion in section 4 of the implications of these examples.

2 Geometrical interpretation of EIT

Let us assume now that (M, g) is an n-dimensional Riemannian manifold with smooth boundary ∂M. The metric g is assumed to be symmetric and positive definite. The invariant object analogous to the conductivity equation (1) is the Laplace-Beltrami operator, which is given by

$$\Delta_g u = \sum_{j,k=1}^n G^{-1/2} \partial_j (G^{1/2} g^{jk} \partial_k u) \tag{8}$$

where $G = \det(g_{jk})$, $[g_{jk}] = [g^{jk}]^{-1}$. The Dirichlet-to-Neumann map is defined by solving the Dirichlet problem

$$\Delta_g u = 0 \quad \text{in} \quad M, \tag{9}$$
$$u|_{\partial M} = f.$$

The operator analogous to Λ_σ is then

$$\Lambda_g(f) = G^{1/2} \sum_{j,k=1}^{n} \nu_j g^{jk} \frac{\partial u}{\partial x_k}|_{\partial M}, \tag{10}$$

with $\nu = (\nu_1, \ldots, \nu_n)$ the outward unit normal to ∂M. In dimension three or higher, the conductivity equation and the Riemannian metric are related by

$$\sigma^{jk} = \det(g)^{1/2} g^{jk}, \quad \text{or} \quad g^{jk} = \det(\sigma)^{(2/(n-2)} \sigma^{jk}. \tag{11}$$

Moreover, $\Lambda_g = \Lambda_\sigma$, and $\Lambda_{\psi^* g} = \Lambda_g$, where $\psi^* g$ denotes the pullback of the metric g by a diffeomorphism of M fixing ∂M [LeU].

In dimension two, (11) is not valid; in this case, the conductivity equation can be reformulated as

$$\text{Div}_g(\beta\, \text{Grad}_g u) = 0 \quad \text{in} \quad M, \tag{12}$$
$$u|_{\partial M} = f$$

where β is the scalar function $\beta = |\det \sigma|^{1/2}$, $g = (g_{jk})$ is equal to (σ_{jk}), and Div_g and Grad_g are the divergence and gradient operators with respect to the Riemannian metric g. Thus we see that in two dimensions, Laplace-Beltrami operators correspond only to those conductivity equations for which $\det(\sigma) = 1$.

After one has solved the EIT problem for an abstract Riemannian manifold, then one tries to embed the manifold (M, g) into a Euclidean space $F : (M, g) \to D \subset \mathbf{R}^n$ with a diffeomorphism F.

For domains in two dimensions, Sylvester[Sy] showed, using special coordinates called isothermal coordinates, that one can reduce the anisotropic problem to the isotropic one; combining this with the result of Nachman[Na], one obtains

Theorem 2 *If σ and $\widetilde{\sigma}$ are two C^3-smooth anisotropic conductivities in $D \subset \mathbf{R}^2$ for which $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$, then there is a diffeomorphism $F : D \to D$, $F|_{\partial D} = Id$ such that $\widetilde{\sigma} = F_* \sigma$.*

In a two-dimensional Riemannian manifold (M,g), the Laplace-Beltrami operator is conformally invariant, i.e.,

$$\Delta_{\beta g} = \frac{1}{\beta}\Delta_g \tag{13}$$

for $\beta(x) > 0$ any smooth conformal factor. Thus, one can expect to recover at most the conformal class of (M,g) from Λ_g. In fact, it was proven in [LU]

Theorem 3 *Assume that (M,g) is a compact two-dimensional Riemannian manifold with boundary. Then Λ_g determines the conformal class of (M,g).*

In dimensions $n \geq 3$, it was shown in [LU] and [LTU] that the following holds:

Theorem 4 *Assume that (M,g) is a complete, real-analytic Riemannian manifold with boundary. Then Λ_g determines (M,g) up to isometry.*

3 Counterexamples

In this section we do not assume that the uniforms bounds (7) are valid, as we allow the conductivity to be degenerate on a surface. In the previous section we showed that inverse conductivity problem can be solved in two steps: First one constructs a Riemannian manifold and then embeds this manifold to Euclidean space. To motivate our counterexample, let us consider the two-dimensional manifold shown in Figure 3. When the bridge connecting the two parts of the manifold gets smaller, the boundary measurements give less information about the isolated part and in the limit no information can be obtained inside the isolated area. When these manifolds are embedded in $\mathbf{R}^2$ (or similar 3-dimensional manifolds to $\mathbf{R}^3$) we should obtain conductivities whose boundary measurements give no information about conductivity on some parts of the domain.

Example 1a. To build a conductivity corresponding to the manifold shown in Figure 3, let us consider set $\Omega = B(0,2)$ that is a ball in $\mathbf{R}^3$ centered at the origin and having radius R. Let $D = B(0,1)$, the ball of radius 1.. The set Ω has two parts D and $\Omega \setminus D$.

Consider the map $F : \Omega \setminus \{0\} \to \Omega \setminus \overline{D}$ given by

$$F : x \mapsto (\frac{|x|}{2} + 1)\frac{x}{|x|}. \tag{14}$$

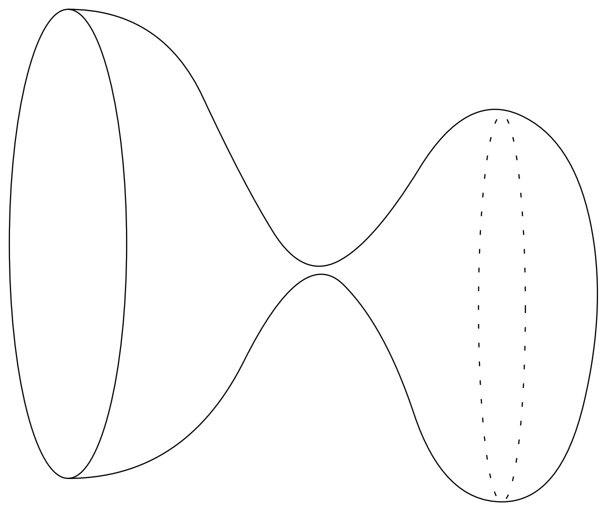

Figure 1: A manifold that collapses to two parts when the width of the bridge connecting two parts goes to zero.

This map takes points in $B(0,1)\setminus 0$ and enlarges them to points in the ball $B(0,\frac{3}{2})$. Let $\gamma = 1$ be the homogeneous conductivity in Ω and define $\sigma = F_*\gamma$. In this way we obtain a conductivity in $\Omega \setminus D$. Then we extend the metric σ smoothly as a Riemannian metric to $\overline{D}$.

Next we consider these conductivities in the standard polar coordinates of $\Omega\setminus\{0\}$ that are denoted by $(r,\phi,\theta)\in R_+\times(-\pi,\pi)\times(0,\pi)$. Now the the metric tensor g and the corresponding conductivity σ_g are related by $\sigma_g = |\det g|^{1/2}g^{jk}$. Let g be the metric corresponding to γ and $\widetilde{g}$ be the metric corresponding to σ. Then in polar coordinates we see that g and γ correspond to the matrices

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix},$$

$$\gamma = \begin{pmatrix} r^2\sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & (\sin\theta)^{-1} \end{pmatrix}$$

and the metrics $\widetilde{g}$ and σ corresponding in the domain $\{1<r<2\}$ to the matrices

$$\widetilde{g} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4(r-1)^2 & 0 \\ 0 & 0 & 4(r-1)^2 \sin^2\theta \end{pmatrix},$$

$$\sigma = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}.$$

It can be shown that the equation

$$\begin{aligned} &\nabla\cdot \sigma \nabla v(x) = 0 \text{ in } \Omega \\ &v|_{\partial\Omega} = f_0, \\ &v \in L^\infty(\Omega) \end{aligned}$$

has a unique solution defined *in the sense of distributions* and that the Dirichlet-to-Neumann maps of σ and γ coincide. So, the conductivity σ looks like a homogeneous media no matter what the conductivity is inside D!

We note that in Example 1a. the conductivity is bounded from above and the solution of (1) is unique and the Dirichlet-to-Neumann map is well defined.

We construct a second counterexample corresponding to Figure 3. In

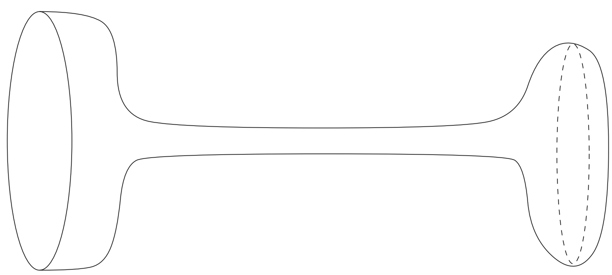

Figure 2: A manifold that collapses to two parts when the length of the bridge connecting the two parts goes to infinity.

this situation, when the bridge connecting the two parts of the manifold goes to infinity, the boundary measurements give less information about the isolated part and in the limit no information can be obtained inside the isolated area. Also similar 3-dimensional manifolds can be embedded into $\mathbf{R}^3$ to obtain counterexamples.

Example 1b. To build a conductivity corresponding to the manifold shown in Figure 3, let us consider again the sets $\Omega = B(0,2)$ and $D = B(0,1)$.

In polar coordinates we define the metric $\widehat{g}$ and the conductivity $\widehat{\sigma}$ in the domain $\{1 < r < 2\}$ by the matrices

$$\widehat{g} = \begin{pmatrix} (r-1)^{-2} & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin^2 \theta \end{pmatrix},$$

$$\widehat{\sigma} = \begin{pmatrix} (r-1)\rho^2 \sin\theta & 0 & 0 \\ 0 & (r-1)^{-1} \sin\theta & 0 \\ 0 & 0 & (r-1)^{-1} \sin^{-1}\theta \end{pmatrix}$$

where $\rho > 0$ is a constant Thus, in $\Omega \setminus D$ the metric is the product metric on $\mathbf{R}_+ \times S^2_\rho$ where S^2_ρ is a 2-sphere with radius ρ. Now we extend the metric $\widehat{g}$ arbitrarily to Ω. It can be shown that in the domain $\Omega \setminus \overline{D}$ the equation

$$\begin{aligned} &\nabla \cdot \sigma \nabla v(x) = 0 \text{ in } \Omega \setminus \overline{D} \\ &v|_{\partial\Omega} = f_0, \\ &v \in L^\infty(\Omega \setminus \overline{D}) \end{aligned}$$

has a unique solution.

Also, when ρ is small enough, we can extend the definition of $v(x)$ to the whole domain Ω by defining it as a constant $c_0 = \lim_{x \to \partial D} v(x)$ in D. Then the obtained function $v(x)$ is a solution of the boundary value problem

$$\begin{aligned} &\nabla \cdot \sigma \nabla v(x) = 0 \text{ in } \Omega \\ &v|_{\partial\Omega} = f_0, \\ &v \in L^\infty(\Omega) \end{aligned}$$

in the sense of distributions. In this case case the boundary measurements would not give information about the metric inside D.

4 Discussion

We emphasize that in the above counterexamples the conductivity tensor is not bounded below. However, in practice many objects are modeled to be perfectly insulating, which correspond in mathematical terms to zero conductivity. This means of course that the conductivity is so small that it can be modeled to be zero when compared to real objects with measurement precision. In particular, in impedance tomography, an ill-posed, the measurement precision is quite poor. Thus even reasonably low conductivity materials, in particular those that are located far from measurement electrodes, may give rise to measurements similar to perfectly insulating materials. The counterexamples presented in this paper might give an explanation of effects that may be seen in practical measurement configurations. In particular, for applications of EIT, very roughly speaking, we can consider a case where in a patient's body there is a cancerous tumor that is covered with low conductivity, anisotropic tissue. In this case, it is possible that the tumor may appear in measurements to be healthy tissue.

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