Some comments on magic squares and Survo puzzles

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Outline

Magic Squares

There is a huge number of literature related to magic squares, see the new magic bibliography by Styan and Trenkler (2007b).

Here we consider the 640 essentially different, **classic** 4×4 magic squares of Dudeney Types I–VI[1] having rank equal to 3.

Their **eigenvalues** $\lambda_1 = 34$ (the **magic sum**) and $\lambda_4 = 0$. We focus on the rest of the eigenvalues as well as the **magic key**, denoted by κ (Styan and Trenkler, 2007a). Now, we have simply $\kappa = |\lambda_2|^2 = |\lambda_3|^2$, since $|\lambda_2| = |\lambda_3|$.

Variations of each magic square matrix can be constructed by using a **flip matrix**, see Chu and Styan (2007). The two main groups are called **sweet** and **sour** by Styan and Trenkler (2007a).

[1] Thus we omit the 240 non-singular squares of Dudeney Types VII–XII here. All the 880 4×4 magic squares are listed, e.g., on Harvey D. Heinz's website `http://www.geocities.com/~harveyh/magicsquare.htm`.

Examples of Magic Squares

In the following, **F** refers to Frénicle index:

F294 (Type I)

$$\begin{bmatrix} 2 & 7 & 12 & 13 \\ 16 & 9 & 6 & 3 \\ 5 & 4 & 15 & 10 \\ 11 & 14 & 1 & 8 \end{bmatrix}$$

F298 (Type VI)

$$\begin{bmatrix} 2 & 7 & 13 & 12 \\ 14 & 11 & 1 & 8 \\ 3 & 6 & 16 & 9 \\ 15 & 10 & 4 & 5 \end{bmatrix}$$

F299 (Type III)

$$\begin{bmatrix} 2 & 7 & 13 & 12 \\ 16 & 9 & 3 & 6 \\ 11 & 14 & 8 & 1 \\ 5 & 4 & 10 & 15 \end{bmatrix}$$

F294	F298	F299	
$\lambda_2 = \lambda_3 = 0$ $\kappa = 0$	$\lambda_2 = +8, \lambda_3 = -8$ $\kappa = -64$	$\lambda_2 = \lambda_3 = 0$ $\kappa = 0$	(**sweet**)
$\lambda_2 = +8i, \lambda_3 = -8i$ $\kappa = 64$	$\lambda_2 = -8, \lambda_3 = +8$ $\kappa = -64$	$\lambda_2 = \lambda_3 = 0$ $\kappa = 0$	(**sour**)

Note that κ is equal to the sum of the principal minors, e.g. **F294**:

$$\kappa = \begin{vmatrix} 2 & 7 \\ 16 & 9 \end{vmatrix} + \begin{vmatrix} 9 & 6 \\ 4 & 15 \end{vmatrix} + \begin{vmatrix} 15 & 10 \\ 1 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 12 \\ 5 & 15 \end{vmatrix} + \begin{vmatrix} 2 & 13 \\ 11 & 8 \end{vmatrix} + \begin{vmatrix} 9 & 3 \\ 14 & 8 \end{vmatrix} = 0.$$

Magic Survo

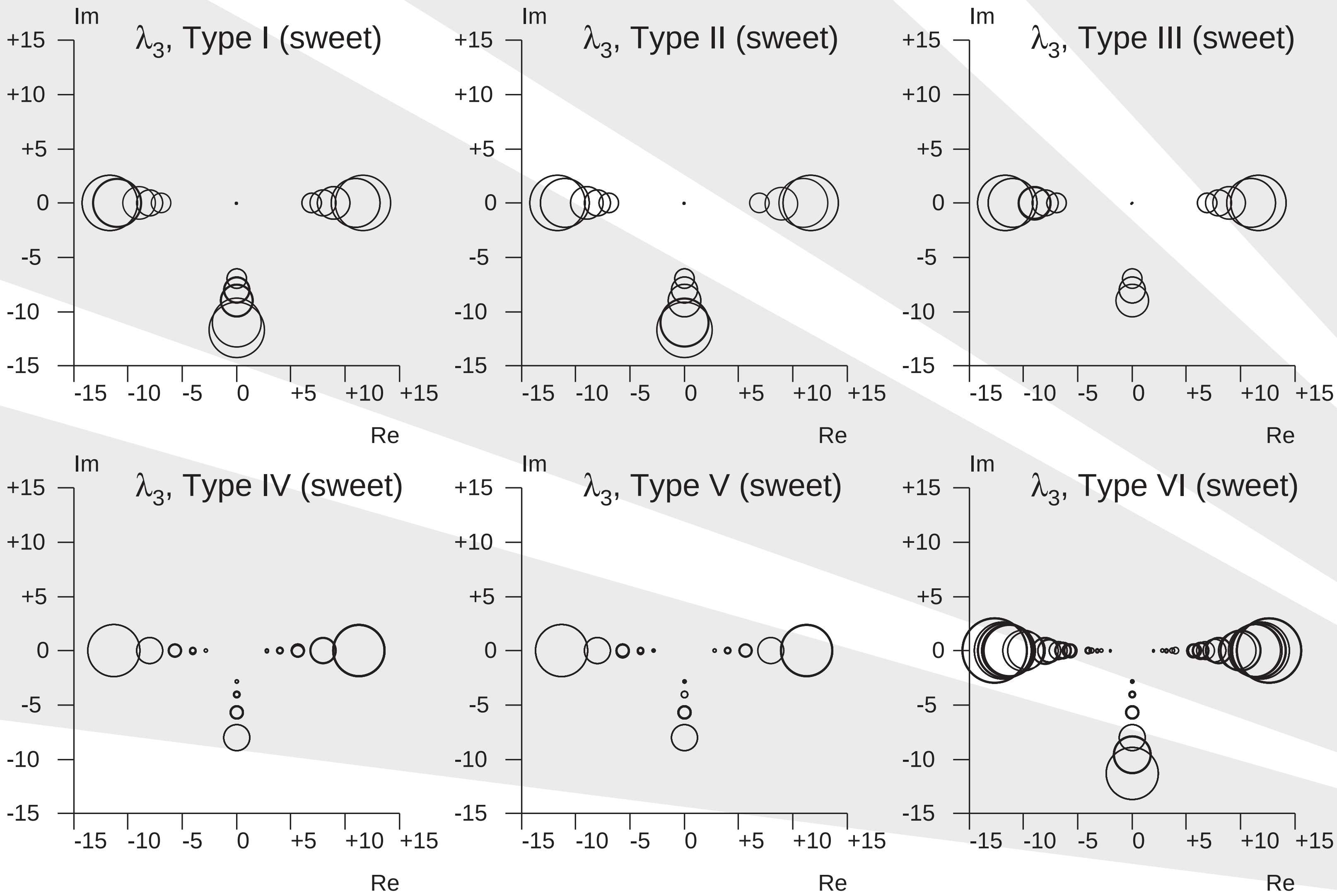
I computed[2] the numerical results of the sweet-and-sours of the 640 magic square matrices using **SURVO MM** (Mustonen, 2001), especially its matrix interpreter and *sucro* (*Su*rvo mac*ro*) language. Below, a **Survo data file** is created for the results:

```
FILE CREATE DUDENEY3
FIELDS:
  1 N 2 Frenicle Frenicle index                    (###)
  2 N 1 Dudeney  Dudeney Type {1,6}                  (#)   Here, I would like
  3 N 8 eig2Re   eigenval #2 real        (###.##########)
  4 N 8 eig2Im   eigenval #2 imaginary   (###.##########)     to demonstrate
  5 N 8 eig3Re   eigenval #3 real        (###.##########)
  6 N 8 eig3Im   eigenval #3 imaginary   (###.##########)        some computations
  7 N 8 pm1      principal minor 1                (####)
  8 N 8 pm2      principal minor 2                (####)           using Survo (SURVO MM).
  9 N 8 pm3      principal minor 3                (####)
 10 N 8 pm4      principal minor 4                (####)              Let us see...
 11 N 8 pm5      principal minor 5                (####)
 12 N 8 pm6      principal minor 6                (####)
 13 N 8 kappa    kappa (sum of pm's)              (####)
END
```

[2] I am grateful to **Seppo Mustonen** for his advice on the complex eigenvalues. I am also grateful to **George P. H. Styan** for his advice on the principal minors.

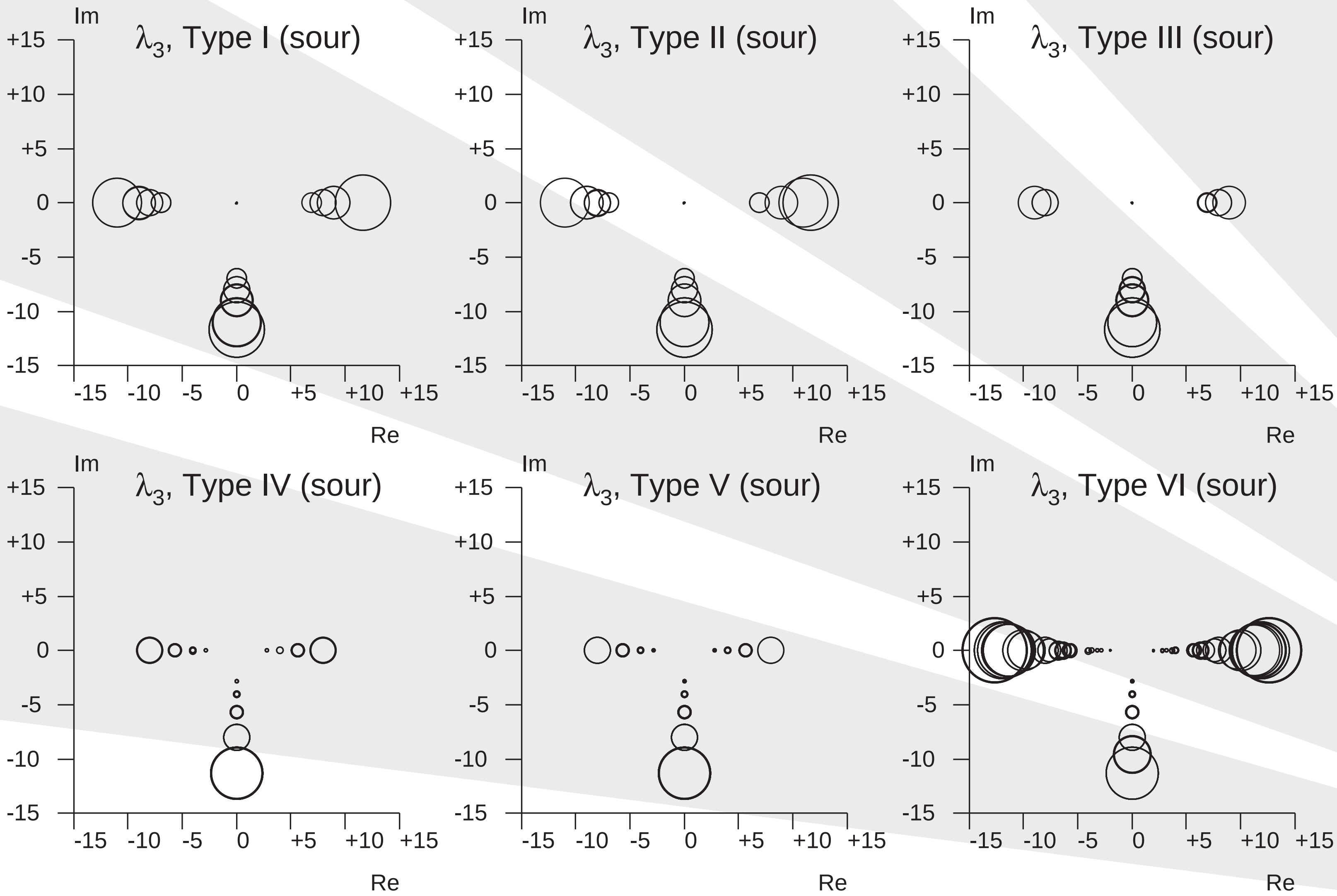
Results as Survo graphs (1)

λ_3 on the complex plane, size of the point proportional to $|\kappa|$ (sweet):



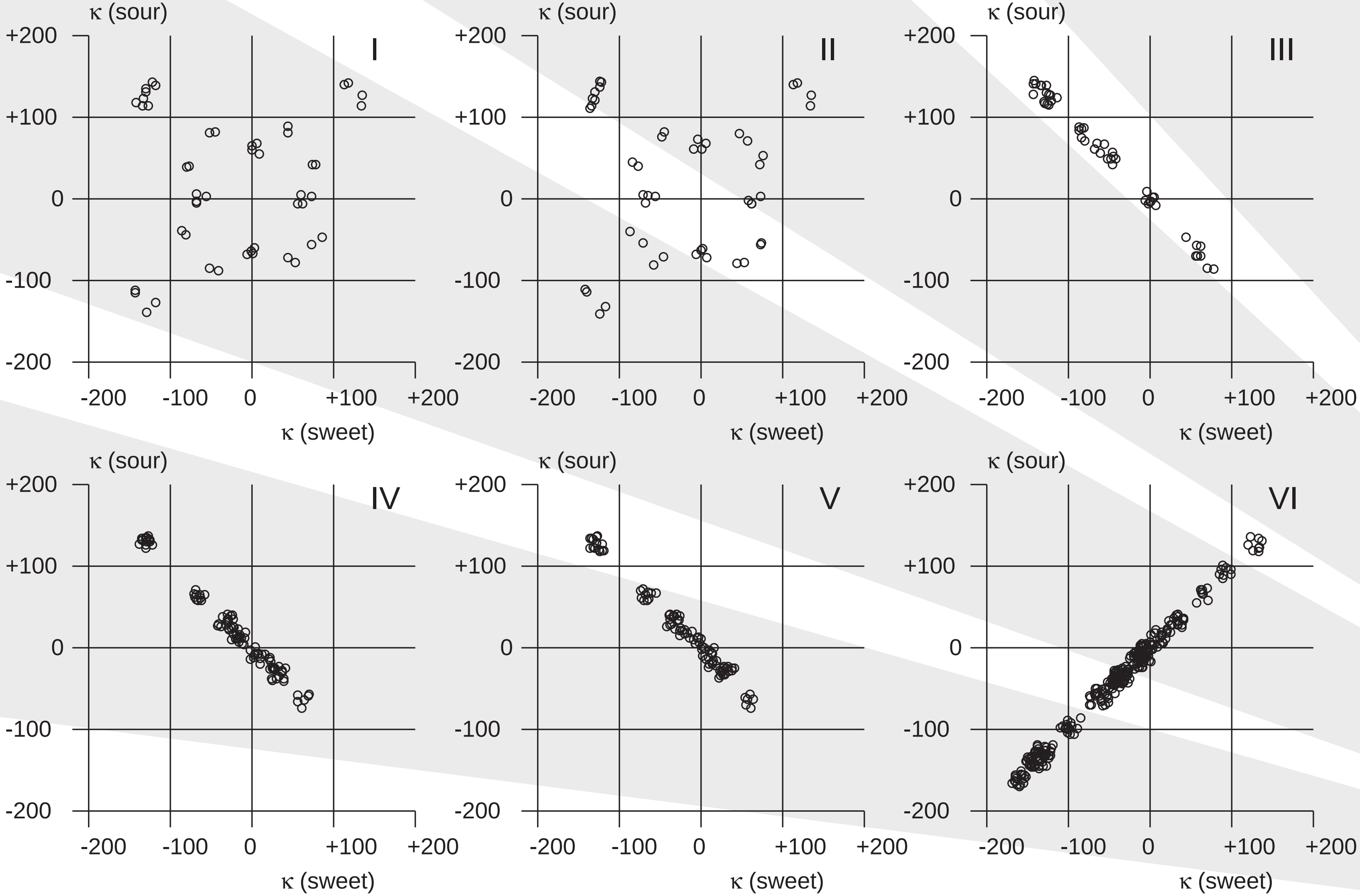
Results as Survo graphs (2)

Similarly (sour). The eigenvalues are either **real** or **pure imaginary**:



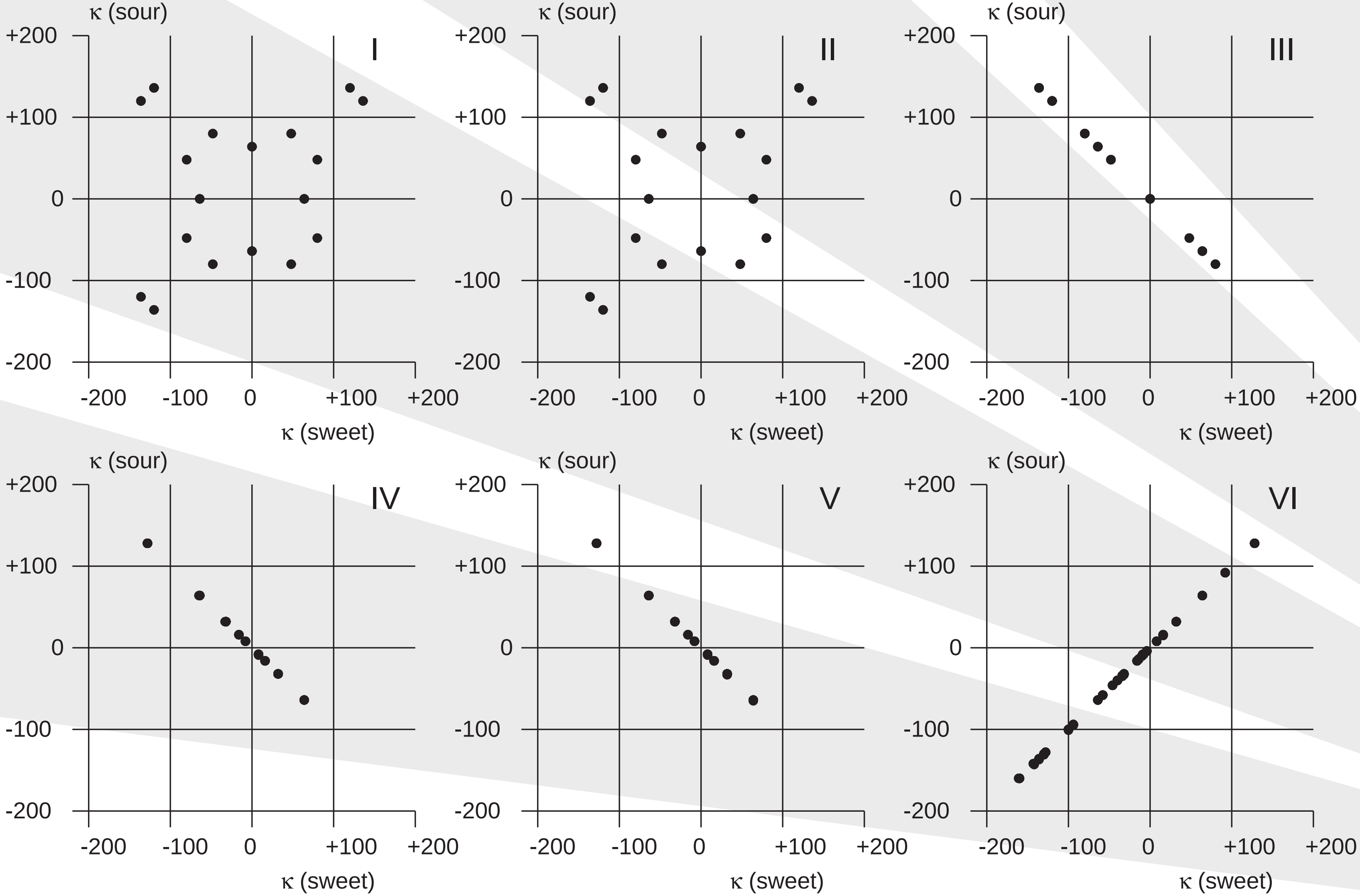
Results as Survo graphs (3)

κ (sweet) vs κ (sour), *points jittered:*



Results as Survo graphs (4)

κ (sweet) vs κ (sour), *points NOT jittered:*



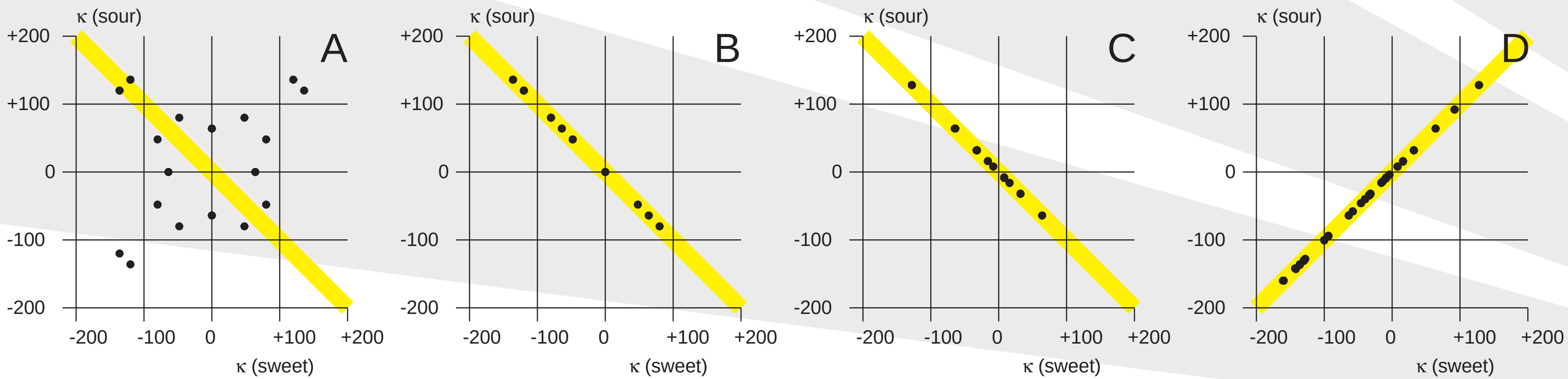
Conclusions based on the graphs and the data

Type III includes **8 cases** where $\kappa = 0$ for both sweet and sour.
These are the non-diagonable cases (Styan and Trenkler, 2007a).

Further, we have only **4 groups**, since some Types are identical:

- Group A: 96 squares of Types I & II (48+48)
- Group B: 48 squares of Type III
- Group C: 192 squares of Types IV & V (96+96)
- Group D: 304 squares of Type VI

The graphs below illustrate these groups:



Survo Puzzles

Survo Puzzles (Mustonen, 2006) offer new challenges for the players of *Sudoku* or *Kakuro*, but also give rise to interesting research questions, e.g., in combinatorics and linear algebra.

Properties compared to magic squares:

- dimensions $m \times n$ (i.e., not necessarily squares)
- elements consequtive integers $1, 2, \ldots, mn$
- row sums not necessarily equal
- column sums not necessarily equal
- no conditions for the diagonal sums

Hence, magic squares are *(rare)* special cases of Survo Puzzles. However, Survo Puzzles are more interesting when some elements are **missing**. In **open** Survo Puzzles, **all** elements are missing.

Examples of Survo Puzzles

The task is to **fill in the missing integers**, given the sums of the rows and columns (some of the integers may be given, too).

Easy (Mustonen, 2007, p. 25):

	A	B	C	D	
1		1			23
2	10		2		31
3		4		6	24
	21	12	22	23	

Quite difficult:[3]

	A	B	C	D	
1					27
2					13
3					53
4					43
	47	39	29	21	

[3] Degree of difficulty (Mustonen, 2006) equal to 1100. Originally published in `http://www.survo.fi/puzzles/index.html#090407`. The solution is given in `http://www.survo.fi/puzzles/solutions.html#230407`.

Solving F299 as a Survo Puzzle


```
    A  B  C  D        Let us look at row 1 and column A:
1   2  * 13  * 34     row 1: 34-2-13=19 and column A: 34-2-5=27
2   *  9  *  * 34     (both are missing two numbers).
3   *  *  8  * 34       o  We require that  DISTINCT=1 and MAX=16
4   5  *  * 15 34       o  We also have set OFF=2,9,8,15,13,5
   34 34 34 34        Now, we list the possible partitions:

COMB row1 END+2 / row1=PARTITIONS,19,2 (these apply the DISTINCT,
COMB colA END+2 / colA=PARTITIONS,27,2  MAX and OFF specs above)

Partitions 2 of 19: N[row1]=2
3 16      Can't be, see below!
7 12   ** Must be this, then.

Partitions 2 of 27: N[colA]=1
11 16  ** Only choice!

Thus 7,12 enter row 1 and 11,16 enter column A (in some order).
We may continue this example live in Survo...
```

Some topics for recreation and research

Usually it is quite demanding to 1) solve a Survo Puzzle **and** (simultaneously) 2) prove the **uniqueness** of the solution.

Surprisingly fascinating approaches have been developed by many people and undoubtedly more will be found. Various computational features of Survo have been helpful especially in harder cases.

Recently, solving a 4×4 puzzle, I came up with an approach based on the binary representation of the puzzle and operations of **Boolean algebra**. The matrix computations of Survo, e.g., the **Kronecker product**, have a central role in this approach.[4]

```
    A  B  C  D                A    B    C    D                 A  B  C  D
 1  o  +  +  .  27        1  [ 7] [12] [39] [ 3] 27        1 11  8  5  3  27
 2  +  o  +  .  13  -->>  2  [10] [ 8] [26] [ 2] 13  -->>  2  6  4  1  2  13
 3  +  +  o  .  53  -->>  3  [55] [44] [17] [11] 53  -->>  3 16 15 13  9  53
 4  .  .  .  *  43        4  [ 5] [ 4] [13] [ 1] 43        4 14 12 10  7  43
   47 39 29 21               47   39   29   21              47 39 29 21
```

[4] A sketch of my Survo-based implementation can be found in `http://www.survo.fi/puzzles/solutions.html#230407`.

More topics for research and recreation

"Nobody knows the number of essentially different 5 × 5, uniquely solvable open Survo Puzzles" (Mustonen, 2007, p. 25).

This is just one of the open questions related to Survo Puzzles. More can be found, e.g., from Mustonen (2006).

Still different recreational challenges are provided by **quick games** as Seppo Mustonen's *Java applets*[5], where the task is to fill in the missing numbers — **as quickly as possible**!

[5] See: `http://www.survo.fi/java/quick5x5.html`.

Thank you for your attention!

References

Ka Lok Chu and George P. H. Styan. Some comments on magic square matrices with three nonzero eigenvalues, Jun 2007. Invited talk at the IWMS 2007, Windsor, Canada.

Seppo Mustonen. On Survo cross sum puzzles, May 2007. Invited talk at the Annual Conference of Finnish Statisticians, Tampere, Finland, `http://www.survo.fi/papers/tampere07.pdf`.

Seppo Mustonen. SURVO MM: Computing environment for creative processing of text and numerical data. `http://www.survo.fi/mm/english.html`, 2001.

Seppo Mustonen. On certain cross sum puzzles. `http://www.survo.fi/papers/puzzles.pdf`, 2006.

George P. H. Styan and Götz Trenkler. An annotated bibliography on magic squares, with special emphasis on linear-algebraic results, as well as on very old and very new results, May 2007b.

George P. H. Styan and Götz Trenkler. An illustrated philatelic introduction to magic square matrices, Mar 2007a. Invited talk at the MAT-TRIAD 2007, Będlewo, Poland.