Factor analysis and the reliability of measurement scales*

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*joint work with **Simo Puntanen** and **Lauri Tarkkonen**

Outline

Sources of uncertainty

Main sources of **uncertainty** in statistical research:

- sampling (well known)
- **measurement** (too often neglected!)
  1. **validity:** *are we measuring the right thing?*
     - closely connected to the substantial theory
     - only partially a statistical question
     - within the *measurement framework* we can assess:
       (a) structural validity of the measurement model
       (b) predictive validity of the measurement scale
  2. **reliability:** *are we measuring accurately enough?*
     - relevant: only if validity acceptable
     - definition: ratio of true variance to total variance
     - required: estimate of measurement error variance

Estimation of reliability

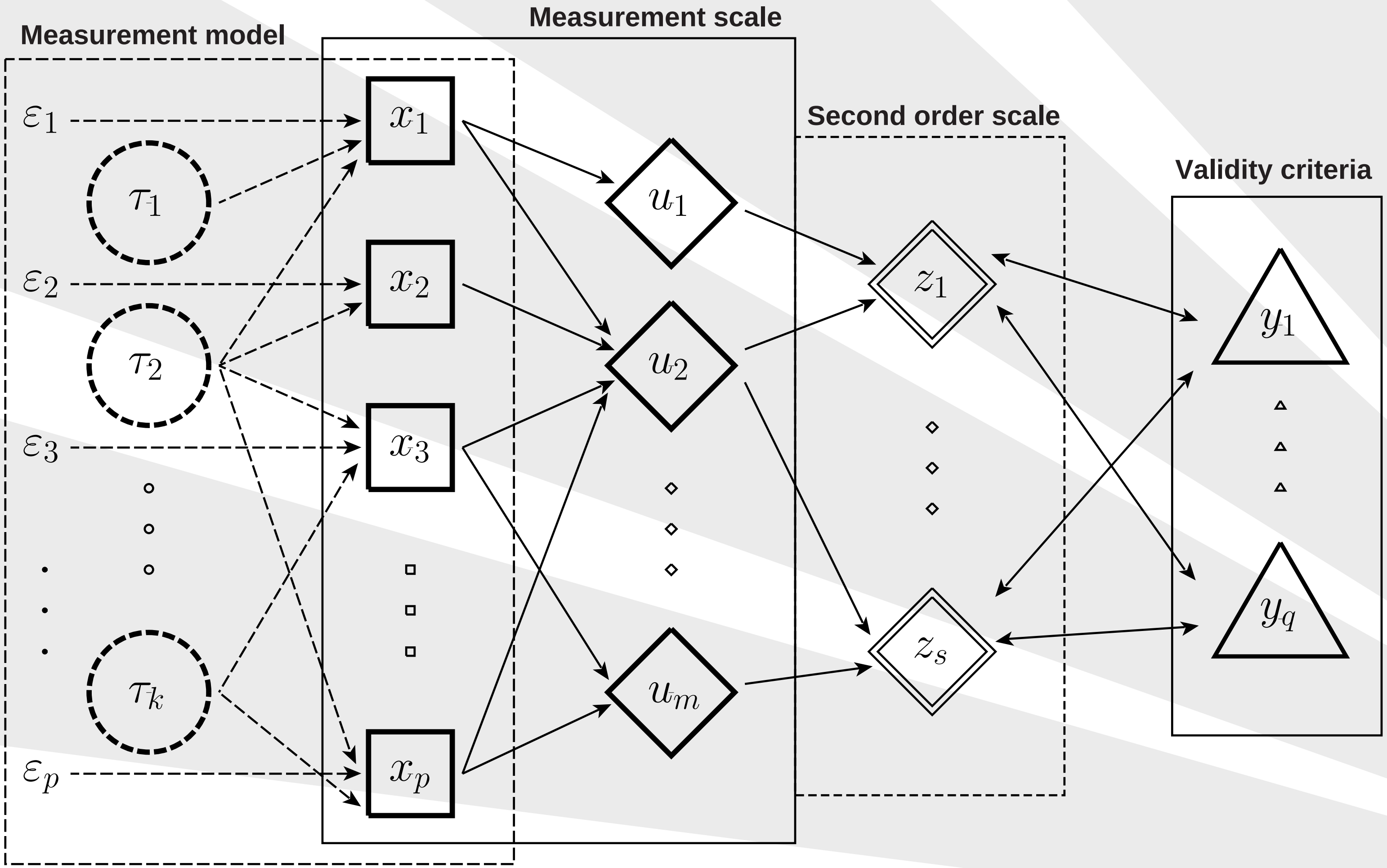
Estimation of reliability depends on the assumptions made about the **measurement model** and the **measurement scale**.

Several estimators suggested, we focus on two of them:

- new alternative: **Tarkkonen's rho**
  - based on measurement framework approach [**1**, **2**, **3**]
  - realistic assumptions, well applicable in practice
  - multidimensionality now stressed in psychology [**4**, **5**]
- most widely used: **Cronbach's alpha**
  - based on Spearman's one-factor model (>100 years ago)
  - routinely used for >50 years (despite of criticism)
  - problem: underestimation (too strict assumptions)

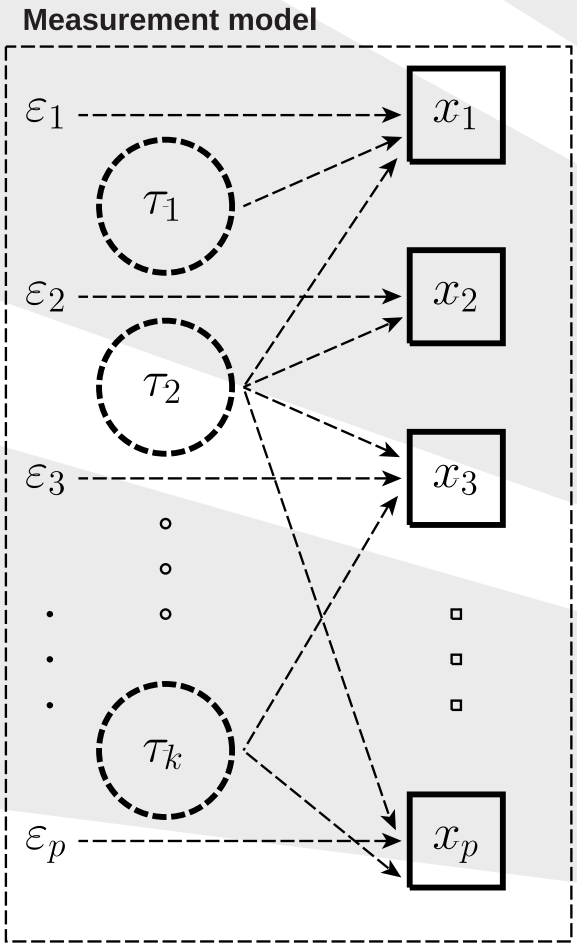
Measurement framework

- *guidelines of the study from the plans to the analyses*
- basis for a consistent assessment of **measurement quality**



Measurement model

1. **What** is to be studied? **How many** dimensions are there?
2. **How** to measure it – as well as possible?



Measurement model

Let $\boldsymbol{x} = (x_1, \ldots, x_p)'$ measure k **(important here:** $k < p$**)** unobservable **true scores** $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k)'$ with unobservable **measurement errors** $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_p)'$.

Assume $\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$, $\mathrm{cov}(\boldsymbol{\tau}, \boldsymbol{\varepsilon}) = \mathbf{0}$. The measurement model is

$$\boldsymbol{x} = \boldsymbol{\mu} + \boldsymbol{B}\boldsymbol{\tau} + \boldsymbol{\varepsilon}, \tag{1}$$

where $\boldsymbol{B} \in \mathbb{R}^{p \times k}$ specifies the relationship between $\boldsymbol{x}$ and $\boldsymbol{\tau}$.

Denoting $\mathrm{cov}(\boldsymbol{\tau}) = \boldsymbol{\Phi}$ and $\mathrm{cov}(\boldsymbol{\varepsilon}) = \boldsymbol{\Psi}$ we have

$$\mathrm{cov}(\boldsymbol{x}) = \boldsymbol{\Sigma} = \boldsymbol{B}\boldsymbol{\Phi}\boldsymbol{B}' + \boldsymbol{\Psi}, \tag{2}$$

where it is assumed that $\boldsymbol{\Sigma} > \mathbf{0}$ and $\boldsymbol{B}$ has full column rank.

Model: Estimation of parameters

The **parameters** are the $pk + k(k+1)/2 + p(p+1)/2$ (unique) elements of the matrices $\boldsymbol{B}$, $\boldsymbol{\Phi}$, and $\boldsymbol{\Psi}$. In general, there are too many, since $\boldsymbol{\Sigma}$ has only $p(p+1)/2$ elements.

- Identifiability is obtained by imposing assumptions on the true scores and the measurement errors.
- **Typical:** assume that $\mathrm{cov}(\boldsymbol{\tau}) = \boldsymbol{I}_k$, an identity matrix of order k, and $\mathrm{cov}(\boldsymbol{\varepsilon}) = \boldsymbol{\Psi}_d = \mathrm{diag}(\psi_1^2, \ldots, \psi_p^2)$.
- With these the model conforms with the orthogonal factor analysis model where the *common factors are directly associated with the true scores* and the *specific factors are interpreted as measurement errors*.

Assuming multinormality the parameters can be estimated using e.g., **the maximum likelihood** method of factor analysis.

Model: Structural validity

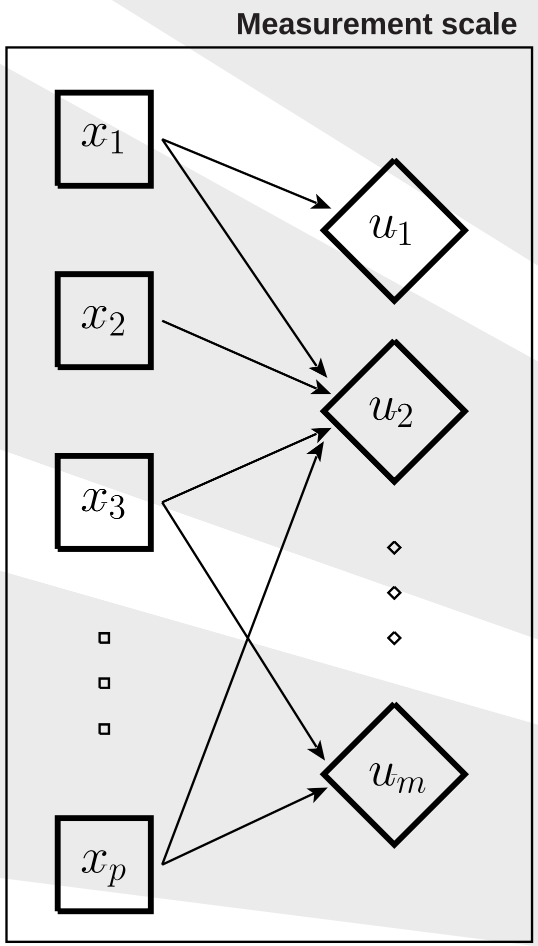
Structural validity is a property of the measurement model.

- Important, as the model forms the core of the framework and hence affects the quality of all scales created.
- Lack of structural validity can be revealed by testing
  - hypotheses on the dimension of $\boldsymbol{\tau}$
  - hypotheses on the effects of $\boldsymbol{\tau}$ on $\boldsymbol{x}$ (matrix $\boldsymbol{B}$)
- The whole approach could be called *semi-confirmatory*.
- Residuals of the model obtained by estimation of $\mathrm{var}(\boldsymbol{\varepsilon})$.
- Dimension of $\boldsymbol{\tau}$ will make the reliabilities identified.
- Appropriate (e.g. **graphical**) factor rotation is essential.

Similarly with other questions of validity, knowledge of the theory and practice of the application needed.

Measurement scale

- Measurement scale is a combination of the observed items.
- Examples: factor scores, psychological test scales, . . .



Measurement scale

In further analyses, the variables $\boldsymbol{x}$ are best used by creating **multivariate measurement scales** $\boldsymbol{u} = \boldsymbol{A}'\boldsymbol{x}$, where $\boldsymbol{A} \in \mathbb{R}^{p \times m}$ is a matrix of the weights. Using (2) we obtain

$$\mathrm{cov}(\boldsymbol{u}) = \boldsymbol{A}'\boldsymbol{\Sigma}\boldsymbol{A} = \boldsymbol{A}'\boldsymbol{B}\boldsymbol{\Phi}\boldsymbol{B}'\boldsymbol{A} + \boldsymbol{A}'\boldsymbol{\Psi}\boldsymbol{A}, \tag{3}$$

the (co)variances generated by the **true scores** and
the (co)variances generated by the **measurement errors**.

Some examples of measurement scales: factor scores, psychological test scales, or any other linear combinations of the observed variables. The weights of the scale may also be predetermined values according to a theory.

Scale: Predictive validity

Predictive validity is a property of the measurement scale.

- Assessed by the correlation(s) between the (second order) scale and an *external criterion*.
- In general, a second order scale is denoted by $\boldsymbol{z} = \boldsymbol{W}'\boldsymbol{u} = \boldsymbol{W}'\boldsymbol{A}'\boldsymbol{x}$, where $\boldsymbol{W} \in \mathbb{R}^{m \times s}$ is a weight matrix and a criterion is denoted by $\boldsymbol{y} = (y_1, \ldots, y_q)'$.
- Often, these scales are produced by regression analysis, discriminant analysis, or other multivariate statistical methods.

In the most general case, the predictive validity would be assessed by the **canonical correlations** between $\boldsymbol{z}$ and $\boldsymbol{y}$.

Scale: Predictive validity, example

Example: consider the regression model $y = \beta_0 + \boldsymbol{\beta}'\boldsymbol{u} + \delta$, where y is the response variable, β_0 is the intercept, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)'$ is the vector of the regression coefficients, $\boldsymbol{u}$ is the vector of the predictors (e.g., factor scores), and δ is a model error.

Now, the criterion y is a scalar, and the second order scale is given by the prediction scale $z = \hat{\boldsymbol{\beta}}'\boldsymbol{u}$, where $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \ldots, \hat{\beta}_m)'$. Hence the predictive validity is equal to ρ_{zy}, the multiple correlation of the regression model.

Monte Carlo simulations carried out using **SURVO MM** [**6**] indicate that the factor scores offer the most stable method for predictor selection in the regression model. See [**1**] for details.

Tarkkonen's rho

According to the definition of reliability, Tarkkonen's rho is obtained as a **ratio of the variances**, i.e., the diagonal elements of the matrices in (3). Hence we have [**1**, **2**, **3**]

$$\boldsymbol{\rho}_{\boldsymbol{u}} = \operatorname{diag}\left(\frac{\boldsymbol{a}_1'\boldsymbol{B}\boldsymbol{\Phi}\boldsymbol{B}'\boldsymbol{a}_1}{\boldsymbol{a}_1'\boldsymbol{\Sigma}\boldsymbol{a}_1}, \ldots, \frac{\boldsymbol{a}_m'\boldsymbol{B}\boldsymbol{\Phi}\boldsymbol{B}'\boldsymbol{a}_m}{\boldsymbol{a}_m'\boldsymbol{\Sigma}\boldsymbol{a}_m}\right)$$

$$= (\boldsymbol{A}'\boldsymbol{B}\boldsymbol{\Phi}\boldsymbol{B}'\boldsymbol{A})_d \times [(\boldsymbol{A}'\boldsymbol{\Sigma}\boldsymbol{A})_d]^{-1}$$

or, in a form where the matrix $\boldsymbol{\Psi}$ is explicitly present:

$$\boldsymbol{\rho}_{\boldsymbol{u}} = \operatorname{diag}\left(\left[1 + \frac{\boldsymbol{a}_1'\boldsymbol{\Psi}\boldsymbol{a}_1}{\boldsymbol{a}_1'\boldsymbol{B}\boldsymbol{\Phi}\boldsymbol{B}'\boldsymbol{a}_1}\right]^{-1}, \ldots, \left[1 + \frac{\boldsymbol{a}_m'\boldsymbol{\Psi}\boldsymbol{a}_m}{\boldsymbol{a}_m'\boldsymbol{B}\boldsymbol{\Phi}\boldsymbol{B}'\boldsymbol{a}_m}\right]^{-1}\right)$$

$$= \{\boldsymbol{I}_m + (\boldsymbol{A}'\boldsymbol{\Psi}\boldsymbol{A})_d \times [(\boldsymbol{A}'\boldsymbol{B}\boldsymbol{\Phi}\boldsymbol{B}'\boldsymbol{A})_d]^{-1}\}^{-1}$$

Special cases

Many models, scales, and reliability coefficients established in the test theory of psychometrics are special cases of the framework.

Example: $\boldsymbol{x} = \boldsymbol{\mu} + \mathbf{1}\tau + \boldsymbol{\varepsilon}$ and $u = \mathbf{1}'\boldsymbol{x}$ (unweighted sum). Now, $\boldsymbol{\Sigma} = \sigma_\tau^2\mathbf{1}\mathbf{1}' + \boldsymbol{\Psi}_d$ and $\sigma_u^2 = \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1} = p^2\sigma_\tau^2 + \mathrm{tr}(\boldsymbol{\Psi}_d)$.

$$\rho_{uu} = \frac{p^2\sigma_\tau^2}{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}} = \frac{p}{p-1}\left(\frac{p^2\sigma_\tau^2 - p\sigma_\tau^2}{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}\right)$$

$$= \frac{p}{p-1}\left(\frac{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1} - \mathrm{tr}(\boldsymbol{\Psi}_d) - \mathrm{tr}(\boldsymbol{\Sigma}) + \mathrm{tr}(\boldsymbol{\Psi}_d)}{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}\right)$$

$$= \frac{p}{p-1}\left(1 - \frac{\mathrm{tr}(\boldsymbol{\Sigma})}{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}\right) = \frac{p}{p-1}\left(1 - \frac{\sum_{i=1}^p \sigma_{x_i}^2}{\sigma_u^2}\right),$$

which is the original form of Cronbach's alpha [**7**].

Some propositions for further research

- **developing** means for the correction for attenuation in various statistics, e.g., regression coefficients (see [**1**])
- **specifying** the connections between the measurement framework and multivariate statistical methods, such as discriminant analysis, canonical correlations, and correspondence analysis (see [**1**])
- **examining** the connections between the measurement framework and generalizability theory (see [**8**])
- **studying** the statistical properties of Tarkkonen's rho (sampling distribution etc.)
- **modifying** t-test for the measurement error variances
- **building** confidence intervals using the standard error of measurement
- **determining** the scales that maximize the reliability
- **combining** the reliability studies with multilevel models

Thank you for your attention!

1 Kimmo Vehkalahti, Simo Puntanen, and Lauri Tarkkonen. Effects of measurement errors in predictor selection of linear regression model. *Computational Statistics & Data Analysis*, 2007. `http://dx.doi.org/10.1016/j.csda.2007.05.005`.

2 Kimmo Vehkalahti, Simo Puntanen, and Lauri Tarkkonen. Estimation of reliability: a better alternative for Cronbach's alpha. Reports on Mathematics 430, Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland, 2006. `http://mathstat.helsinki.fi/reports/Preprint430.pdf`.

3 L. Tarkkonen and K. Vehkalahti. Measurement errors in multivariate measurement scales. *Journal of Multivariate Analysis*, 96:172–189, 2005.

4 Joseph F. Lucke. The α and the ω of congeneric test theory: An extension of reliability and internal consistency to heterogeneous tests. *Applied Psychological Measurement*, 29:65–81, 2005.

5 Jos M. F. Ten Berge and Gregor Soĉan. The greatest lower bound to the reliability of a test and the hypothesis of unidimensionality. *Psychometrika*, 69:611–623, 2004.

6 Seppo Mustonen. SURVO MM: Computing environment for creative processing of text and numerical data. `http://www.survo.fi/mm/english.html`, 2001.

7 L. J. Cronbach. Coefficient alpha and the internal structure of tests. *Psychometrika*, 16:297–334, 1951.

8 Maria Valaste, Kimmo Vehkalahti, and Lauri Tarkkonen. Generalizability: Reliability or Validity? Presentation, IMPS2007, CTT Session on Fri 13 July, 2007.