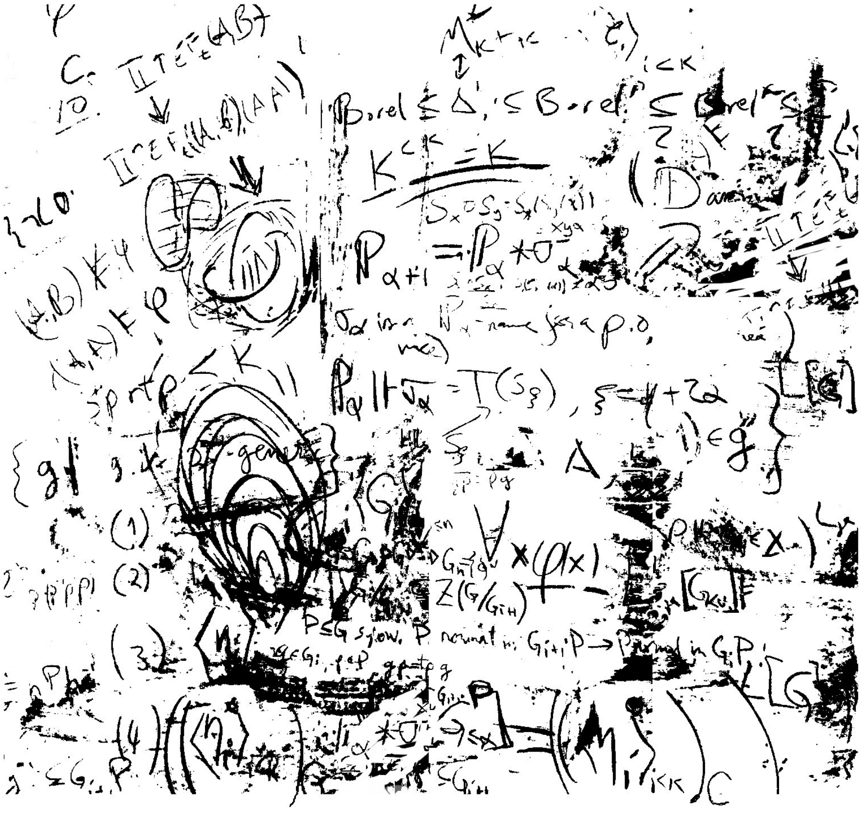
Generalized Descriptive Set Theory And Classification Theory

Vadim Kulikov

Licentiate's thesis
Supervisor: Tapani Hyttinen
Joint work with Tapani Hyttinen and Sy-David Friedman
December 1, 2010
University of Helsinki
Department of Mathematics and Statistics



Acknowledgement

This work is joint with prof. Sy-David Friedman and Tapani Hyttinen who supervised this thesis.

I wish to express my most sincere gratitude to Tapani for extraodinarily careful attention towards this work and towards my developement as a mathematician. During these months I learned from him a lot.

I am greatly indebted to Sy-David Friedman for the collaboration and for the great amount of careful comments he made on the drafts of this work.

I wish to thank Jouko Väänänen for his support in many respects, Philipp Schlicht, Daisuke Ikegami and many other collegues for interesting and fruitful discussions on this and related topics.

I thank my parents, grandparents and dear friends for their support and understanding during this journey.

This thesis would not have been possible without the financial support by the Research Foundation of the University of Helsinki and the Finnish National Graduate School in Mathematics and its Applications. All the authors are indebted to Mittag-Leffler Institute (Royal Swedish Academy of Sciences) and the John Templeton Foundation for its support through its project Myriad Aspects of Infinity (ID #13152) as well as the Academy of Finland for its support through its grant number 1123110.

Contents

I History And Motivation

There is a long tradition in studying connections between Borel structure of Polish spaces (descriptive set theory) and model theory. The connection arises from the fact that any class of countable structures can be coded into a subset of the space 2^ω provided all structures in the class have domain ω. A survey on this topic is given in [**7**]. Suppose X and Y are subsets of 2^ω and let E_1 and E_2 be equivalence relations on X and Y respectively. If $f\colon X \to Y$ is a map such that $E_1(x,y) \iff E_2(f(x),f(y))$, we say that f is a *reduction of* E_1 *to* E_2. If there exists a Borel or continuous reduction, we say that E_1 is Borel or continuously *reducible* to E_2, denoted $E_1 \leqslant_B E_2$ or $E_1 \leqslant_c E_2$. The mathematical meaning of this is that f *classifies* E_1*-equivalence in terms of* E_2*-equivalence.*

A reducibility result, say $E_1 \leqslant_B E_2$, tells us that E_1 is at most as complicated as E_2; once E_2 is understood, so is E_1 (modulo the reduction). An irreducibility result, $E_1 \not\leqslant_B E_2$ tells that there is no hope in trying to classify E_1 in terms of E_2, at least in a "Borel way". From the model theoretic point of view, the isomorphism relation, and the elementary equivalence relation (in some language) on some class of structures are the equivalence relations of main interest. But model theory in general does not restrict itself to countable structures. Most of stability theory and Shelah's classification theory characterizes first-order theories in terms of their uncountable models. This leads to the generalization adopted in this paper. We consider the space 2^κ for an uncountable cardinal κ with the idea that the models of size κ are coded into the elements of that space.

This approach, to connect such uncountable descriptive set theory with model theory, began in the early 1990's. One of the pioneering papers was by Mekler and Väänänen [**22**]. A survey on the research done in 1990's can be found in [**34**] and a discussion of the motivational background for this work in [**33**]. Let us explain how our approach differs from the earlier ones and why is it useful. For a first-order complete countable theory in a countable vocabulary T and a cardinal $\kappa \geqslant \omega$, define

$$S_T^\kappa = \{\eta \in 2^\kappa \mid \mathcal{A}_\eta \models T\} \text{ and } \cong_T^\kappa = \{(\eta,\xi) \in (S_T^\kappa)^2 \mid \mathcal{A}_\eta \cong \mathcal{A}_\xi\}.$$

where $\eta \mapsto \mathcal{A}_\eta$ is some fixed coding of (all) structures of size κ. We can now define the partial order on the set of all theories as above by

$$T \leqslant^\kappa T' \iff \cong^\kappa_T \leqslant_B \cong^\kappa_{T'} .$$

As pointed out above, $T \leqslant^\kappa T'$ says that $\cong^\kappa_T$ is at most as difficult to classify as $\cong^\kappa_{T'}$. But does this tell us whether T is a simpler theory than T'? Rough answer: *If $\kappa = \omega$, then no but if $\kappa > \omega$, then yes.*

To illustrate this, let $T = \mathrm{Th}(\mathbb{Q}, \leqslant)$ be the theory of the order of the rational numbers (DLO) and let T' be the theory of a vector space over the field of rational numbers. Without loss of generality we may assume that they are models of the same vocabulary.

It is easy to argue that the model class defined by T' is strictly simpler than that of T. (For instance there are many questions about T, unlike T', that cannot be answered in ZFC; say existence of a saturated model.) On the other hand $\cong^\omega_T \leqslant_B \cong^\omega_{T'}$ and $\cong^\omega_{T'} \nleqslant_B \cong^\omega_T$ because there is only one countable model of T and there are infinitely many countable models of T'. But for $\kappa > \omega$ we have $\cong^\kappa_T \nleqslant_B \cong^\kappa_{T'}$ and $\cong^\kappa_{T'} \leqslant_B \cong^\kappa_T$, since there are 2^κ equivalence classes of $\cong^\kappa_T$ and only one equivalence class of $\cong^\kappa_T$. Another example, introduced in Martin Koerwien's Ph.D. thesis and his article [**18**] shows that there exists an ω-stable theory without DOP and without OTOP with depth 2 for which $\cong^\omega_T$ is not Borel, while we show here that for $\kappa > \omega$, $\cong^\kappa_T$ is Borel for all classifiable shallow theories.

The results suggest that the order $\leqslant^\kappa$ for $\kappa > \omega$ corresponds naturally to the classification of theories in stability theory: the more complex a theory is from the viewpoint of stability theory, the higher it seems to sit in the ordering $\leqslant^\kappa$ and vice versa. Since dealing with uncountable cardinals often implies the need for various cardinality or set theoretic assumptions beyond ZFC, the results are not always as simple as in the case $\kappa = \omega$, but they tell us a lot. For example, our results easily imply the following (modulo some mild cardinality assumptions on κ):

- If T is deep and T' is shallow, then $\cong_T \nleqslant_B \cong_{T'}$.
- If T is unstable and T' is classifiable, then $\cong_T \nleqslant_B \cong_{T'}$.

Jouko Väänänen writes in [**33**]:

> *When we look into the deep eyes of the uncountable structures, we are perhaps starting to see there some compassion for our modest advances, our budding infinite trees, our courageous appeals to stability and our resolve to play the game to the end.*

II Introduction

II.1. Notations and Conventions

II.1.1. Set Theory

We use standard set theoretical notation:

- $A \subset B$ means that A is a subset of B or is equal to B.
- $A \subsetneq B$ means proper subset.
- Union, intersection and set theoretical difference are denoted respectively by $A \cup B$, $A \cap B$ and $A \setminus B$. For larger unions and intersections $\bigcup_{i \in I} A_i$ etc..

Usually the Greek letters κ, λ and μ will stand for cardinals and α, β and γ for ordinals, but this is not strict. Also η, ξ, ν are usually elements of κ^κ or 2^κ and p, q, r are elements of $\kappa^{<\kappa}$ or $2^{<\kappa}$. $\operatorname{cf}(\alpha)$ is the cofinality of α (the least ordinal β for which there exists an increasing unbounded function $f\colon \beta \to \alpha$).

By S^κ_λ we mean $\{\alpha < \kappa \mid \operatorname{cf}(\alpha) = \lambda\}$. A λ-*cub set* is a subset of a limit ordinal (usually of cofinality $> \lambda$) which is unbounded and contains suprema of all bounded increasing sequences of length λ. A set is *cub* if it is λ-cub for all λ. A set is *stationary* if it intersects all cub sets and λ-*stationary* if it intersects all λ-cub sets. Note that $C \subset \kappa$ is λ-cub if and only if $C \cap S^\kappa_\lambda$ is λ-cub and $S \subset \kappa$ is λ-stationary if and only if $S \cap S^\kappa_\lambda$ is (just) stationary.

If $(\mathbb{P}, \leqslant)$ is a forcing notion, we write $p \leqslant q$ if p and q are in $\mathbb{P}$ and q forces more than p. Usually $\mathbb{P}$ is a set of functions equipped with inclusion and $p \leqslant q \iff p \subset q$. In that case $\varnothing$ is the weakest condition and we write $\mathbb{P} \Vdash \varphi$ to mean $\varnothing \Vdash_{\mathbb{P}} \varphi$.

II.1.2. Functions

We denote by $f(x)$ the value of x under the mapping f and by $f[A]$ or just fA the image of the set A under f. Similarly $f^{-1}[A]$ or just $f^{-1}A$ indicates the inverse image of A. Domain and range are denoted respectively by $\operatorname{dom} f$ and $\operatorname{ran} f$.

If it is clear from the context that f has an inverse, then f^{-1} denotes that inverse. For a map $f\colon X \to Y$ *injective* means the same as *one-to-one* and *surjective* the same as *onto*

Suppose $f\colon X \to Y^\alpha$ is a function with range consisting of sequences of elements of Y of length α. The projection pr_β is a function $Y^\alpha \to Y$ defined by $\mathrm{pr}_\beta((y_i)_{i<\alpha}) = y_\beta$. For the coordinate functions of f we use the notation $f_\beta = \mathrm{pr}_\beta \circ f$ for all $\beta < \alpha$.

By support of a function f we mean the subset of $\operatorname{dom} f$ in which f takes non-zero values, whatever "zero" means depending on the context (hopefully never unclear). The support of f is denoted by $\operatorname{sprt} f$.

II.1.3. Model Theory

In Section II.2.3 we fix a countable vocabulary and assume that all theories are theories in this vocabulary. Moreover we assume that they are first-order, complete and countable. By $\mathrm{tp}(\bar a/A)$ we denote the complete type of $\bar a = (a_1, \dots, a_{\mathrm{length}\,\bar a})$ over A.

We think of models as tuples $\mathcal{A} = \langle \operatorname{dom} \mathcal{A}, P_n^{\mathcal{A}} \rangle_{n<\omega}$ where the P_n are relation symbols in the vocabulary and the $P_n^{\mathcal{A}}$ are their interpretations. If a relation R has arity n (a property of the vocabulary), then for its interpretation it holds that $R^{\mathcal{A}} \subset (\operatorname{dom} \mathcal{A})^n$. In Section II.2.3 we adopt more conventions concerning this.

II.2. Ground Work

II.2.1. Trees And Topologies

Throughout the paper κ is assumed to be an uncountable regular cardinal which satisfies

$$\kappa^{<\kappa} = \kappa \tag{$*$}$$

(For justification of this, see below.) We look at the space κ^κ, i.e. the functions from κ to κ and the space formed by the initial segments $\kappa^{<\kappa}$. It is useful to think of $\kappa^{<\kappa}$ as a tree ordered by inclusion and of κ^κ as a topological space of the branches of $\kappa^{<\kappa}$; the topology is defined below. Occasionally we work in 2^κ and $2^{<\kappa}$ instead of κ^κ and $\kappa^{<\kappa}$.

1. Definition. A *tree* t is a partial order with a root in which the sets $\{x \in t \mid x < y\}$ are well ordered for each $y \in t$. A *branch* in a tree is a maximal linear suborder.

A tree is called *a* $\kappa\lambda$*-tree*, if there are no branches of length λ or higher and no element has $\geqslant \kappa$ immediate successors. If t and t' are trees, we write $t \leqslant t'$ to mean that there exists an order preserving map $f\colon t \to t'$, $a <_t b \Rightarrow f(a) <_{t'} f(b)$.

Convention. *Unless otherwise said, by a tree* $t \subset (\kappa^{<\kappa})^n$ *we mean a tree with domain being a downward closed subset of*

$$(\kappa^{<\kappa})^n \cap \{(p_0, \dots, p_{n-1}) \mid \operatorname{dom} p_0 = \dots = \operatorname{dom} p_{n-1}\}$$

ordered as follows: $(p_0, \dots, p_{n-1}) < (q_0, \dots, q_{n1})$ *if* $p_i \subset q_i$ *for all* $i \in \{0, \dots, n-1\}$. *It is always a* $\kappa^+, \kappa+1$*-tree.*

2. Example. Let $\alpha < \kappa^+$ be an ordinal and let t_α be the tree of descending sequences in α ordered by end extension. The root is the empty sequence. It is a $\kappa^+\omega$-tree. Such t_α can be embedded into $\kappa^{<\omega}$, but note that not all subtrees of $\kappa^{<\omega}$ are $\kappa^+\omega$-trees (there are also $\kappa^+, \omega+1$-trees).

In fact the trees $\kappa^{<\beta}$, $\beta \leqslant \kappa$ and t_α are universal in the following sense:

Fact ($\kappa^{<\kappa} = \kappa$). *Assume that* t *is a* $\kappa^+, \beta+1$*-tree,* $\beta \leqslant \kappa$ *and* t' *is* $\kappa^+\omega$*-tree. Then*

(1) there is an embedding $f\colon t \to \kappa^{<\beta}$,

(2) and an embedding $f\colon t' \to t_\alpha$ *for some* $\alpha < \kappa^+$. $\square$

Define the topology on κ^κ as follows. For each $p \in \kappa^{<\kappa}$ define the basic open set

$$N_p = \{\eta \in \kappa^\kappa \mid \eta \restriction \operatorname{dom}(p) = p\}.$$

Open sets are precisely the empty set and the sets of the form $\bigcup X$, where X is a collection of basic open sets. Similarly for 2^κ.

There are many justifications for the assumption $(*)$ which will be most apparent after seeing the proofs of our theorems. The crucial points can be summarized as follows: if $(*)$ does not hold, then

- the space κ^κ does not have a dense subset of size κ,
- there are open subsets of κ^κ that are not κ-unions of basic open sets which makes controlling Borel sets difficult (see Definition 14 on page 15).
- Vaught's generalization of the Lopez-Escobar theorem (Theorem 22) fails, see Remark 23 on page 23.

- The model theoretic machinery we are using often needs this cardinality assumption (see e.g. Theorem 27 and proof of Theorem 63).

Initially the motivation to assume $(*)$ was simplicity. Many statements concerning the space $\kappa^{<\kappa}$ are independent of ZFC and using $(*)$ we wanted to make the scope of such statements neater. In the statements of (important) theorems we mention the assumption explicitly.

Because the intersection of less than κ basic open sets is either empty or a basic open set, we get the following.

FACT $(\kappa^{<\kappa} = \kappa)$. *The following hold for a topological space* $P \in \{2^\kappa, \kappa^\kappa\}$*:*

(1) The intersection of less than κ *basic open sets is either empty or a basic open set,*
(2) The intersection of less than κ *open sets is open,*
(3) Basic open sets are closed,
(4) $|\{A \subset P \mid A \text{ is basic open}\}| = \kappa$*,*
(5) $|\{A \subset P \mid A \text{ is open}\}| = 2^\kappa$*.*

In the space $\kappa^\kappa \times \kappa^\kappa = (\kappa^\kappa)^2$ we define the ordinary product topology.

3. DEFINITION. A set $Z \subset \kappa^\kappa$ is Σ^1_1 if it is a projection of a closed set $C \subset (\kappa^\kappa)^2$. A set is Π^1_1 if it is the complement of a Σ^1_1 set. A set is Δ^1_1 if it is both Σ^1_1 and Π^1_1.

As in standard descriptive set theory $(\kappa = \omega)$, we have the following:

4. THEOREM. *For* $n < \omega$ *the spaces* $(\kappa^\kappa)^n$ *and* κ^κ *are homeomorphic.* $\square$

REMARK. This standard theorem can be found for example in Jech's book [**15**]. Applying this theorem we can extend the concepts of Definition 3 to subsets of $(\kappa^\kappa)^n$. For instance a subset A of $(\kappa^\kappa)^n$ is Σ^1_1 if for a homeomorphism $h\colon (\kappa^\kappa)^n \to \kappa^\kappa$, $h[A]$ is Σ^1_1 according to Definition 3.

II.2.2. Ehrenfeucht-Fraïssé Games

We will need Ehrenfeucht-Fraïssé games in various connections. It serves also as a way of coding isomorphisms.

5. DEFINITION (Ehrenfeucht-Fraïssé games). For a tree t, a cardinal κ and structures $\mathcal{A}$ and $\mathcal{B}$, the game $\mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B})$ is played by players **I** and **II** as follows. Note that t might be an ordinal. Player **I** chooses subsets of $A \cup B$

and climbs up the tree t and player **II** chooses partial functions $A \to B$ as follows. Suppose a sequence

$$(X_i, p_i, f_i)_{i<\gamma}$$

has been played (if $\gamma = 0$, then the sequence is empty). Player **I** picks a set $X_\gamma \subset A \cup B$ of cardinality strictly less than κ such that $X_\delta \subset X_\gamma$ for all ordinals $\delta < \gamma$. Then player **I** picks a $p_\gamma \in t$ which is $<_t$-minimal subject to $p_\delta <_t p_\gamma$ for all $\delta < \gamma$. Then player **II** chooses a partial function $f_\gamma \colon A \to B$ such that $X_\gamma \cap A \subset \operatorname{dom} f_\gamma$, $X_\gamma \cap B \subset \operatorname{ran} f_\gamma$, $|\operatorname{dom} f_\gamma| < \kappa$ and $f_\delta \subset f_\gamma$ for all ordinals $\delta < \gamma$. The game ends when player **I** cannot go up the tree anymore, i.e. $(p_i)_{i<\gamma}$ is a branch. Player **II** wins if

$$f = \bigcup_{i<\gamma} f_i$$

is a partial isomorphism. Otherwise player **I** wins.

A *strategy* of player **II** in $\mathrm{EF}_t^\kappa(\mathcal{A}.\mathcal{B})$ is a function

$$\sigma \colon ([A \cup B]^{<\kappa} \times t)^{<\mathrm{ht}(t)} \to \bigcup_{I \in [A]^{<\kappa}} B^I,$$

where $[R]^{<\kappa} = \{\text{subsets of } R \text{ of size } < \kappa\}$ and $\mathrm{ht}(t)$ is the *height* of the tree, i.e.

$$\mathrm{ht}(t) = \sup\{\alpha \mid \alpha \text{ is an ordinal and there is an order preserving embedding } \alpha \to t\}.$$

A strategy of **I** is similarly a function

$$\tau \colon \Big(\bigcup_{I \in [A]^{<\kappa}} B^I \Big)^{<\mathrm{ht}(t)} \to [A \cup B]^{<\kappa} \times t.$$

We say that a strategy τ of player **I** *beats* strategy σ of player **II** if the play $\tau * \sigma$ is a win for **I**. The play $\tau * \sigma$ is just the play where **I** uses τ and **II** uses σ. Similarly σ beats τ if $\tau * \sigma$ is a win for **II**. We say that a strategy is a *winning strategy* if it beats all opponents strategies.

The notation $X \uparrow \mathrm{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$ means that player X has a winning strategy in $\mathrm{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$

REMARK. By our convention $\operatorname{dom} \mathcal{A} = \operatorname{dom} \mathcal{B} = \kappa$, so while player **I** picks a subset of $\operatorname{dom} \mathcal{A} \cup \operatorname{dom} \mathcal{B}$ he actually just picks a subset of κ, but as a small analysis shows, this does not alter the game.

Consider the game $\mathrm{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$, where $|\mathcal{A}| = |\mathcal{B}| = \kappa$, $|t| \leqslant \kappa$ and $\mathrm{ht}(t) \leqslant \kappa$. The set of strategies can be identified with κ^κ, for example as follows. The moves of player **I** are members of $[A \cup B]^{<\kappa} \times t$ and the moves of player **II** are members of $\bigcup_{I\in[A]^{<\kappa}} B^I$. By our convention $\operatorname{dom}\mathcal{A} = \operatorname{dom}\mathcal{B} = A = B = \kappa$, so these become $V = [\kappa]^{<\kappa} \times t$ and $U = \bigcup_{I\in[\kappa]^{<\kappa}} \kappa^I$. By our cardinality assumption $\kappa^{<\kappa} = \kappa$, these sets are of cardinality κ.

Let

$$\begin{aligned} f\colon U &\to \kappa \\ g\colon U^{<\kappa} &\to \kappa \\ h\colon V &\to \kappa \\ k\colon V^{<\kappa} &\to \kappa \end{aligned}$$

be bijections. Let us assume that $\tau\colon U^{<\kappa} \to V$ is a strategy of player **I** (there cannot be more than κ moves in the game because we assumed $\operatorname{ht}(t) \leqslant \kappa$). Let $\nu_\tau\colon \kappa \to \kappa$ be defined by

$$\nu_\tau = h \circ \tau \circ g^{-1}$$

and if $\sigma\colon V^{<\kappa} \to U$ is a strategy of player **II**, let ν_σ be defined by

$$\nu_\sigma = f \circ \sigma \circ k^{-1}.$$

We say that ν_τ *codes* τ.

6. Theorem ($\kappa^{<\kappa} = \kappa$). *Let $\lambda \leqslant \kappa$ be a cardinal. The set*

$$C = \{(\nu,\eta,\xi) \in (\kappa^\kappa)^3 \mid \nu \textit{ codes a w.s. of } \mathbf{II} \textit{ in } \mathrm{EF}^\kappa_\lambda(\mathcal{A}_\eta,\mathcal{A}_\xi)\} \subset (\kappa^\kappa)^3$$

is closed. If $\lambda < \kappa$, then also the corresponding set for player **I**

$$D = \{(\nu,\eta,\xi) \in (\kappa^\kappa)^3 \mid \nu \textit{ codes a w.s. of } \mathbf{I} \textit{ in } \mathrm{EF}^\kappa_\lambda(\mathcal{A}_\eta,\mathcal{A}_\xi)\} \subset (\kappa^\kappa)^3$$

is closed.

Remark. Compare to Theorem 12.

Proof. Assuming $(\nu_0,\eta_0,\xi_0) \notin C$, we will show that there is an open neighbourhood U of (ν_0,η_0,ξ_0) such that $U \subset (\kappa^\kappa)^3 \setminus C$. Denote the strategy that ν_0 codes by σ_0. By the assumption there is a strategy τ of **I** which beats σ_0. Consider the game in which **I** uses τ and **II** uses σ_0.

Denote the γ^{th} move in this game by (X_γ, h_γ) where $X_\gamma \subset A_{\eta_0} \cup A_{\xi_0}$ and $h_\gamma\colon A_{\eta_0} \to A_{\xi_0}$ are the moves of the players. Since player **I** wins this game, there is $\alpha < \lambda$ for which h_α is not a partial isomorphism between $\mathcal{A}_{\eta_0}$ and $\mathcal{A}_{\xi_0}$. Let

$$\varepsilon = \sup(X_\alpha \cup \operatorname{dom} h_\alpha \cup \operatorname{ran} h_\alpha)$$

(Recall $\operatorname{dom}\mathcal{A}_\eta = A_\eta = \kappa$ for any η by convention.) Let π be the coding function defined in Definition 11 on page 14. Let

$$\beta_1 = \pi[\varepsilon^{<\omega}] + 1.$$

The idea is that $\eta_0 \restriction \beta_1$ and $\xi_0 \restriction \beta_1$ decide the models $\mathcal{A}_{\eta_0}$ and $\mathcal{A}_{\xi_0}$ as far as the game has been played. Clearly $\beta_1 < \kappa$.

Up to this point, player **II** has applied her strategy σ_0 precisely to the sequences of the moves made by her opponent, namely to $S = \{(X_\gamma)_{\gamma<\beta} \mid \beta < \alpha\} \subset \operatorname{dom}\sigma_0$. We can translate this set to represent a subset of the domain of ν_0: $S' = k[S]$, where k is as defined before the statement of the present theorem. Let $\beta_2 = (\sup S') + 1$ and let

$$\beta = \max\{\beta_1, \beta_2\}.$$

Thus $\eta_0 \restriction \beta$, $\xi_0 \restriction \beta$ and $\nu_0 \restriction \beta$ decide the moves $(h_\gamma)_{\gamma<\alpha}$ and the winner.

Now

$$\begin{aligned} U &= \{(\nu,\eta,\xi) \mid \nu \restriction \beta = \nu_0 \restriction \beta \wedge \eta \restriction \beta = \eta_0 \restriction \beta \wedge \xi \restriction \beta = \xi_0 \restriction \beta\} \\ &= N_{\nu_0 \restriction \beta} \times N_{\eta_0 \restriction \beta} \times N_{\xi_0 \restriction \beta}. \end{aligned}$$

is the desired neighbourhood. Indeed, if $(\nu,\eta,\xi) \in U$ and ν codes a strategy σ, then τ beats σ on the structures $\mathcal{A}_\eta, \mathcal{A}_\xi$, since the first α moves are exactly as in the corresponding game of the triple (ν_0, η_0, ξ_0).

Let us now turn to D. The proof is similar. Assume that $(\nu_0, \eta_0, \xi_0) \notin D$ and ν_0 codes strategy τ_0 of player **I**. Then there is a strategy of **II**, which beats τ_0. Let $\beta < \kappa$ be, as before, an ordinal such that all moves have occurred before β and the relations of the substructures generated by the moves are decided by $\eta_0 \restriction \beta, \xi_0 \restriction \beta$ as well as the strategy τ_0. Unlike for player **I**, the win of **II** is determined always only in the end of the game, so β can be $\geqslant \lambda$. This is why we made the assumption $\lambda < \kappa$, by which we can always have $\beta < \kappa$ and so

$$\begin{aligned} U &= \{(\nu,\eta,\xi) \mid \nu \restriction \beta = \nu_0 \restriction \beta \wedge \eta \restriction \beta = \eta_0 \restriction \beta \wedge \xi \restriction \beta = \xi_0 \restriction \beta\} \\ &= N_{\nu_0 \restriction \beta} \times N_{\eta_0 \restriction \beta} \times N_{\xi_0 \restriction \beta}. \end{aligned}$$

is an open neighbourhood of (ν_0, η_0, ξ_0) in the complement of D. $\square$

Let us list some theorems concerning Ehrenfeucht-Fraïssé games which we will use in the proofs.

7. Definition. Let T be a theory and $\mathcal{A}$ a model of T of size κ. The $L_{\infty\kappa}$-*Scott height* of $\mathcal{A}$ is

$$\sup\{\alpha \mid \exists\mathcal{B} \models T(\mathcal{A} \not\cong \mathcal{B} \wedge \mathbf{II} \uparrow \mathrm{EF}^{\kappa}_{t_\alpha}(\mathcal{A},\mathcal{B}))\},$$

where t_α is as in Example 2 and the subsequent Fact.

Remark. Sometimes the Scott height is defined in terms of quantifier ranks, but this gives an equivalent definition by Theorem 9 below.

8. Definition. The *quantifier rank* $R(\varphi)$ of a formula $\varphi \in L_{\infty\infty}$ is an ordinal defined by induction on the length of φ as follows. If φ quantifier free, then $R(\varphi) = 0$. If $\varphi = \exists\bar{x}\psi(\bar{x})$, then $R(\varphi) = R(\psi(\bar{x})) + 1$. If $\varphi = \neg\psi$, then $R(\varphi) = R(\psi)$. If $\varphi = \bigwedge_{\alpha<\lambda}\psi_\alpha$, then $R(\varphi) = \sup\{R(\psi_\alpha \mid \alpha < \lambda)\}$

9. Theorem. *Models $\mathcal{A}$ and $\mathcal{B}$ satisfy the same $L_{\infty\kappa}$-sentences of quantifier rank $< \alpha$ if and only if $\mathbf{II} \uparrow \mathrm{EF}^{\kappa}_{t_\alpha}(\mathcal{A},\mathcal{B})$.* □

The following theorem is a well known generalization of a theorem of Karp [**16**]:

10. Theorem. *Models $\mathcal{A}$ and $\mathcal{B}$ are $L_{\infty\kappa}$-equivalent if and only if $\mathbf{II} \uparrow \mathrm{EF}^{\kappa}_{\omega}(\mathcal{A},\mathcal{B})$.* □

II.2.3. Coding Models

There are various degrees of generality to which the content of this text is applicable. Many of the results generalize to vocabularies with infinitary relations or to uncountable vocabularies, but not all. We find it reasonable though to fix the used vocabulary to make the presentation clearer.

Models can be coded to models with just one binary predicate. Function symbols often make situations unnecessarily complicated from the point of view of this paper.

Thus our approach is, without great loss of generality, to fix our attention to models with finitary relation symbols of all finite arities.

Let us fix L to be the countable relational vocabulary consisting of the relations P_n, $n < \omega$, $L = \{P_n \mid n < \omega\}$, where each P_n is an n-ary relation: the interpretation of P_n is a set consisting of n-tuples. We can assume without loss of generality that the domain of each L-structure of size κ is κ, i.e. $\operatorname{dom}\mathcal{A} = \kappa$. If we restrict our attention to these models, then the set of all L-models has the same cardinality as κ^κ.

We will next present the way we code the structures and the isomorphisms between them into the elements of κ^κ (or equivalently – as will be seen – to 2^κ).

11. Definition. Let π be a bijection $\pi\colon \kappa^{<\omega} \to \kappa$. If $\eta \in \kappa^\kappa$, define the structure $\mathcal{A}_\eta$ to have $\operatorname{dom}(\mathcal{A}_\eta) = \kappa$ and if $(a_1, \dots a_n) \in \operatorname{dom}(\mathcal{A}_\eta)^n$, then

$$(a_1, \dots, a_n) \in P_n^{\mathcal{A}_\eta} \iff \eta(\pi(a_1, \dots, a_n)) > 0.$$

In that way the rule $\eta \mapsto \mathcal{A}_\eta$ defines a surjective (onto) function from κ^κ to the set of all L-structures with domain κ. We say that η *codes* $\mathcal{A}_\eta$.

Remark. Define the equivalence relation on κ^κ by $\eta \sim \xi \iff \operatorname{sprt}\eta = \operatorname{sprt}\xi$, where sprt means support, see Section II.1.2 on page 6. Now we have $\eta \sim \xi \iff \mathcal{A}_\eta = \mathcal{A}_\xi$, i.e. the identity map $\kappa \to \kappa$ is an isomorphism between $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ when $\eta \sim \xi$ and vice versa. On the other hand $\kappa^\kappa/\sim\ \cong 2^\kappa$, so the coding can be seen also as a bijection between models and the space 2^κ.

The distinction will make little difference, but it is convenient to work with both spaces depending on context. To illustrate the insignificance of the choice between κ^κ and 2^κ, note that $\sim$ is a closed equivalence relation and identity on 2^κ is bireducible with $\sim$ on κ^κ (see Definition 67).

II.2.4. Coding Partial Isomorphisms

Let $\xi, \eta \in \kappa^\kappa$ and let p be a bijection $\kappa \to \kappa \times \kappa$. Let $\nu \in \kappa^\alpha$, $\alpha \leqslant \kappa$. The idea is that for $\beta < \alpha$, $p_1(\nu(\beta))$ is the image of β under a partial isomorphism and $p_2(\nu(\beta))$ is the inverse image of β. That is, for a $\nu \in \kappa^\alpha$, define a relation $F_\nu \subset \kappa \times \kappa$:

$$(\beta, \gamma) \in F_\nu \iff \big(\beta < \alpha \wedge p_1(\nu(\beta)) = \gamma\big) \vee \big(\gamma < \alpha \wedge p_2(\nu(\gamma)) = \beta\big)$$

If ν happens to be such that F_ν is a partial isomorphism $\mathcal{A}_\xi \to \mathcal{A}_\eta$, then we say that ν *codes a partial isomorphism between* $\mathcal{A}_\xi$ and $\mathcal{A}_\eta$, this isomorphism being determined by F_ν. If $\alpha = \kappa$ and ν codes a partial isomorphism, then F_ν is an isomorphism and we say that ν *codes an isomorphism.*

12. Theorem. *The set*

$$C = \{(\nu, \eta, \xi) \in (\kappa^\kappa)^3 \mid \nu \textit{ codes an isomorphism between } \mathcal{A}_\eta \textit{ and } \mathcal{A}_\xi\}$$

is a closed set.

Proof. Suppose that $(\nu,\eta,\xi)\notin C$ i.e. ν does not code an isomorphism $\mathcal{A}_\eta\cong\mathcal{A}_\xi$. Then (at least) one of the following holds:

(1) F_ν is not a function,
(2) F_ν is not one-to-one,
(3) F_ν does not preserve relations of $\mathcal{A}_\eta$, $\mathcal{A}_\xi$.

(Note that F_ν is always onto if it is a function and $\operatorname{dom}\nu=\kappa$.) If (1), (2) or (3) holds for ν, then respectively (1), (2) or (3) holds for any triple (ν',η',ξ') where $\nu'\in N_{\nu\restriction\gamma}$, $\eta'\in N_{\eta\restriction\gamma}$ and $\xi'\in N_{\xi\restriction\gamma}$, so it is sufficient to check that (1), (2) or (3) holds for $\nu\restriction\gamma$ for some $\gamma<\kappa$, because

Let us check the above in the case that (3) holds. The other cases are left to the reader. Suppose (3) holds. There is $(a_0,\dots,a_{n-1})\in(\operatorname{dom}\mathcal{A}_\eta)^n=\kappa^n$ such that $(a_0,\dots,a_{n-1})\in P_n$ and $(a_0,\dots,a_{n-1})\in P_n^{\mathcal{A}_\eta}$ and $(F_\nu(a_0),\dots,F_\nu(a_{n-1}))\notin P_n^{\mathcal{A}_\xi}$. Let β be greater than

$$\max(\{\pi(a_0,\dots,a_{n-1}),\pi(F_\nu(a_0),\dots,F_\nu(a_{n-1}))\}\cup\{a_0,\dots a_{n-1},F_\nu(a_0),\dots,F_\nu(a_{n-1})\})$$

Then it is easy to verify that any $(\eta',\xi',\nu')\in N_{\eta\restriction\beta}\times N_{\xi\restriction\beta}\times N_{\nu\restriction\beta}$ satisfies (3) as well. □

13. Corollary. *The set* $\{(\eta,\xi)\in(\kappa^\kappa)^2\mid\mathcal{A}_\eta\cong\mathcal{A}_\xi\}$ *is* Σ^1_1.

Proof. It is the projection of the set C of Theorem 12. □

II.3. Generalized Borel Sets

14. Definition. We have already discussed Δ^1_1 sets which generalize Borel subsets of Polish space in one way. Let us see how else can we generalize usual Borel sets to our setting.

- [**3, 22**] The collection of λ-*Borel* subsets of κ^κ is the smallest set, which contains the basic open sets of κ^κ and is closed under complementation and under taking intersections of size λ. Since we consider only κ-Borel sets, we write Borel = κ-Borel.
- The collection $\Delta^1_1=\Sigma^1_1\cap\Pi^1_1$.
- [**3, 22**] The collection of *Borel** subsets of κ^κ. A set A is Borel* if there exists a $\kappa^+\kappa$-tree t in which each increasing sequence of limit order type has a unique supremum and a function

$$h\colon\{\text{branches of } t\}\to\{\text{basic open sets of } \kappa^\kappa\}$$

such that $\eta \in A \iff$ player **II** has a winning strategy in the game $G(t, h, \eta)$. The game $G(t, h, \eta)$ is defined as follows. At the first round player **I** picks a minimal element of the tree, on successive rounds he picks an immediate successor of the last move played by player **II** and if there is no last move, he chooses an immediate successor of the supremum of all previous moves. Player **II** always picks an immediate successor of the Player **I**'s choice. The game ends when the players cannot go up the tree anymore, i.e. have chosen a branch b. Player **II** wins, if $\eta \in h(b)$. Otherwise **I** wins.

A *dual* of a Borel* set B is the set

$$B^d = \{\xi \mid \mathbf{I} \uparrow G(t, h, \xi)\}$$

where t and h satisfy the equation $B = \{\xi \mid \mathbf{II} \uparrow G(t, h, \xi)\}$. The dual is not unique.

REMARK. Suppose that t is a $\kappa^+\kappa$ tree and $h\colon \{\text{branches of } t\} \to \text{Borel}^*$ is a labeling function taking values in Borel* sets instead of basic open sets. Then $\{\eta \mid \mathbf{II} \uparrow G(t, h, \eta)\}$ is a Borel* set.

Thus if we change the basic open sets to Borel* sets in the definition of Borel*, we get Borel*.

15. REMARK. Blackwell [**2**] defined Borel* sets in the case $\kappa = \omega$ and showed that in fact Borel=Borel*. When κ is uncountable it is not the case. But it is easily seen that if t is a $\kappa^+\omega$-tree, then the Borel* set coded by t (with some labeling h) is a Borel set, and vice versa: each Borel set is a Borel* set coded by a $\kappa^+\omega$-tree. We will use this characterization of Borel.

It was first explicitly proved in [**22**] that these are indeed generalizations:

16. THEOREM ([**22**], $\kappa^{<\kappa} = \kappa$). *Borel* $\subset \Delta^1_1 \subset$ *Borel** $\subset \Sigma^1_1$.

Proof. (Sketch) If A is Borel*, then it is Σ^1_1, intuitively, because $\eta \in A$ if and only if *there exists* a winning strategy of player **II** in $G(t, h, \eta)$ where (t, h) is a tree that codes A (here one needs the assumption $\kappa^{<\kappa} = \kappa$ to be able to code the strategies into the elements of κ^κ). By Remark 15 above if A is Borel, then there is also such a tree. Since Borel $\subset$ Borel* by Remark 15 and Borel is closed under taking complements, Borel sets are Δ^1_1.

The fact that Δ^1_1 sets are Borel* is a more complicated issue; it follows from a separation theorem proved in [**22**]. The separation theorem says that any two disjoint Σ^1_1 sets can be separated by Borel* sets. It is proved in [**22**] for $\kappa = \omega_1$, but the proof generalizes to any κ (with $\kappa^{<\kappa} = \kappa$). $\square$

Remark. There are sets that are Δ^1_1 but not Borel. One example being the universal Borel set

$$B = \{(\eta,\xi) \in 2^\kappa \times 2^\kappa \mid \eta \text{ is in the set coded by } (t_\xi, h_\xi)\},$$

where $\xi \mapsto (t_\xi, h_\xi)$ is a continuous coding of ($\kappa^+\omega$-tree, labeling)-pairs in such a way that for all $\kappa^+\omega$-trees $t \subset \kappa^{<\omega}$ and labelings h there is ξ with $(t_\xi, h_\xi) = (t, h)$. It is not Borel since if it were, then the diagonal's complement

$$D = \{\eta \mid (\eta,\eta) \notin B\}$$

would be a Borel set which it is not, since it cannot be coded by any (t_ξ, h_ξ). On the other hand its complement $C = (2^\kappa)^2 \setminus B$ is Σ^1_1, because $(\eta,\xi) \in C$ if and only if *there exists* a winning strategy of player **I** in the Borel-game $G(t_\xi, h_\xi, \eta)$ and the latter can be coded to a Borel set. It is left to the reader to verify that when $\kappa > \omega$, then the set

$$F = \{(\eta,\xi,\nu) \mid \nu \text{ codes a w.s. for } \mathbf{I} \text{ in } G(t_\xi, h_\xi, \eta)\}$$

is closed.

The existence of an isomorphism relation which is Δ^1_1 but not Borel follows from Theorems 61 and 62.

In the next section we show that it is consistent that $\Delta^1_1 \subsetneq \text{Borel}^*$.

Open Problem. Is it consistent that $\Delta^1_1 = \text{Borel}^*$? Is it consistent (or even provable in ZFC) that $\text{Borel}^* = \Sigma^1_1$?

17. Theorem. *For a set $S \subset \kappa^\kappa$ the following are equivalent.*

- *(1) S is Σ^1_1,*
- *(2) S is a projection of a Borel set,*
- *(3) S is a projection of a Σ^1_1 set,*
- *(4) S is a continuous image of a closed set.*

Proof. Let us go in the order.

(1) $\Rightarrow$ (2)**:** Closed sets are Borel.

(2) $\Rightarrow$ (3)**:** The same proof as in the standard case $\kappa = \omega$ gives that Borel sets are Σ^1_1 (see for instance [**15**]).

(3) $\Rightarrow$ (4)**:** Let $A \subset \kappa^\kappa \times \kappa^\kappa$ be a Σ^1_1 set which is the projection of A, $S = \text{pr}_0 A$. Then let $C \subset \kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa$ be a closed set such that $\text{pr}_1 C = A$. Here $\text{pr}_0 \colon \kappa^\kappa \times \kappa^\kappa \to \kappa^\kappa$ and $\text{pr}_1 \colon \kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa \to \kappa^\kappa \times \kappa^\kappa$ are the obvious projections. Let $f \colon \kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa \to \kappa^\kappa$ be a homeomorphism. Then S is the image of the closed set $f[C]$ under the continuous map $\text{pr}_0 \circ \text{pr}_1 \circ f^{-1}$.

(4) $\Rightarrow$ (1)**:** The image of a closed set under a continuous map f is the projection of the graph of f restricted to that closed set. It is a basic topological fact that a graph of a continuous partial function with closed domain is closed (provided the range is Hausdorff).

$\square$

18. THEOREM ([**22**]). *Borel* sets are closed under unions and intersections of size κ.* $\square$

19. DEFINITION. A Borel* set B is *determined* if there exists a tree t and a labeling function h such that the corresponding game $G(t, h, \eta)$ is determined for all $\eta \in \kappa^\kappa$ and

$$B = \{\eta \mid \mathbf{II} \text{ has a winning strategy in } G(t, h, \eta)\}.$$

20. THEOREM ([**22**]). *Δ^1_1 sets are exactly the determined Borel* sets.* $\square$

III Borel Sets, Δ_1^1 Sets and Infinitary Logic

III.1. The Language $L_{\kappa^+\kappa}$ and Borel Sets

The interest in the class of Borel sets is explained by the fact that the Borel sets are relatively simple yet at the same time this class includes many interesting definable sets. We prove Vaught's theorem (Theorem 22), which equates "invariant" Borel sets with those definable in the infinitary language $L_{\kappa^+\kappa}$. Note that on models of size κ, the languages $L_{\infty\kappa}$ and $L_{\kappa^+\kappa}$ are equivalent. Vaught proved his theorem for the case $\kappa = \omega_1$ in [**35**], but the proof works for arbitrary kappa (assuming as always $\kappa^{<\kappa} = \kappa$).

21. Definition. Denote by S_κ the set of all permutations of κ. If $u \in \kappa^{<\kappa}$, denote

$$\bar{u} = \{p \in S_\kappa \mid p^{-1} \restriction \operatorname{dom} u = u\}.$$

Note that $\bar{\varnothing} = S_\kappa$ and if $u \in \kappa^\alpha$ is not injective, then $\bar{u} = \varnothing$.

A permutation $p\colon \kappa \to \kappa$ acts on 2^κ by

$$p\eta = \xi \iff p\colon \mathcal{A}_\eta \to \mathcal{A}_\xi \text{ is an isomorphism.}$$

The map $\eta \mapsto p\eta$ is well defined for every p and it is easy to check that it defines an action of the permutation group S_κ on the space 2^κ. We say that a set $A \subset 2^\kappa$ is *closed under permutations* if it is a union of orbits of this action.

22. Theorem ([**35**], $\kappa^{<\kappa} = \kappa$). *A set $B \subset \kappa^\kappa$ is Borel and closed under permutations if and only if there is a sentence φ in $L_{\kappa^+\kappa}$ such that $B = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$.*

Proof. Let φ be a sentence in $L_{\kappa^+\kappa}$. Then $\{\eta \in 2^\kappa \mid \mathcal{A}_\eta \models \varphi\}$ is closed under permutations, because if $\eta = p\xi$, then $\mathcal{A}_\eta \cong \mathcal{A}_\xi$ and $\mathcal{A}_\eta \models \varphi \iff \mathcal{A}_\xi \models \varphi$ for every sentence φ. If φ is a formula with parameters $(a_i)_{i<\alpha} \in \kappa^\alpha$, one easily verifies by induction on the complexity of φ that the set

$$\{\eta \in 2^\kappa \mid \mathcal{A}_\eta \models \varphi((a_i)_{i<\alpha})\}$$

is Borel. This of course implies that for every sentence φ the set $\{\eta \mid \mathcal{A}_\eta \models \varphi\}$ is Borel.

The converse is less trivial. Note that the set of permutations $S_\kappa \subset \kappa^\kappa$ is Borel, since

$$S_\kappa = \bigcap_{\beta<\kappa} \bigcup_{\alpha<\kappa} \underbrace{\{\eta \mid \eta(\alpha) = \beta\}}_{\text{open}} \cap \bigcap_{\alpha<\beta<\kappa} \underbrace{\{\eta \mid \eta(\alpha) \neq \eta(\beta)\}}_{\text{open}}. \qquad (\cdot)$$

For a set $A \subset \kappa^\kappa$ and $u \in \kappa^{<\kappa}$, define

$$A^{*u} = \big\{\eta \in 2^\kappa \mid \{p \in \bar{u} \mid p\eta \in A\} \text{ is co-meager in } \bar{u}\big\}.$$

From now on in this section we will write "$\{p \in \bar{u} \mid p\eta \in A\}$ is co-meager", when we really mean "co-meager in $\bar{u}$".

Let us show that the set

$$Z = \{A \subset 2^\kappa \mid A^{*u} \text{ is } L_{\kappa^+\kappa}\text{-definable for all } u \in \kappa^{<\kappa}\}$$

contains basic open sets, is closed under intersections of size κ and under complementation in the three steps (a),(b) and (c) below. This implies that all Borel sets are in Z. We will additionally keep track of the fact that the formula, which defines A^{*u} depends only on A and $\operatorname{dom} u$, i.e. for each $\beta < \kappa$ and Borel set A there exists $\varphi = \varphi_\beta^A$ such that for all $u \in \kappa^\beta$ we have $A^{*u} = \{\eta \mid \mathcal{A}_\eta \models \varphi((u_i)_{i<\beta})\}$. Setting $u = \varnothing$, we have the intended result, because $A^{*\varnothing} = A$ for all A which are closed under permutations and φ is a sentence (with no parameters).

If A is fixed we denote $\varphi_\beta^A = \varphi_\beta$.

(a) Assume $q \in 2^{<\kappa}$ and let N_q be the corresponding basic open set. Let us show that $N_q \in Z$. Let $u \in \kappa^\beta$ be arbitrary. We have to find $\varphi_\beta^{N_q}$. Let θ be a quantifier free formula with α parameters such that:

$$N_q = \{\eta \in 2^\kappa \mid \mathcal{A}_\eta \models \theta((\gamma)_{\gamma<\alpha})\}.$$

Here $(\gamma)_{\gamma<\alpha}$ denotes both an initial segment of κ as well as an α-tuple of the structure. Suppose $\alpha \leqslant \beta$. We have $p \in \bar{u} \Rightarrow u \subset p^{-1}$, so

$$\begin{aligned}
\eta \in N_q^{*u} \quad &\iff \quad \{p \in \bar{u} \mid p\eta \in N_q\} \text{ is co-meager} \\
&\iff \quad \{p \in \bar{u} \mid \mathcal{A}_{p\eta} \models \theta((\gamma)_{\gamma<\alpha})\} \text{ is co-meager} \\
&\iff \quad \{p \in \bar{u} \mid \mathcal{A}_{\eta} \models \theta((p^{-1}(\gamma))_{\gamma<\alpha})\} \text{ is co-meager} \\
&\iff \quad \{p \in \bar{u} \mid \underbrace{\mathcal{A}_{\eta} \models \theta((u_\gamma)_{\gamma<\alpha})}_{\text{independent of } p}\} \text{ is co-meager} \\
&\iff \quad \mathcal{A}_{\eta} \models \theta((u_\gamma)_{\gamma<\alpha}).
\end{aligned}$$

Then $\varphi_\beta = \theta$.

Assume then that $\alpha > \beta$. By the above, we still have

$$\eta \in N_q^{*u} \iff E = \big\{p \in \bar{u} \mid \mathcal{A}_\eta \models \theta\big((p^{-1}(\gamma))_{\gamma<\alpha}\big)\big\} \text{ is co-meager}$$

Assume that $w = (w_\gamma)_{\gamma<\alpha} \in \kappa^\alpha$ is an arbitrary sequence with no repetition and such that $u \subset w$. Since $\bar{w}$ is an open subset of $\bar{u}$ and E is co-meager, there is $p \in \bar{w} \cap E$. Because $p \in E$, we have $\mathcal{A}_\eta \models \theta\big((p^{-1}(\gamma))_{\gamma<\alpha}\big)$. On the other hand $p \in \bar{w}$, so we have $w \subset p^{-1}$, i.e. $w_\gamma = w(\gamma) = p^{-1}(\gamma)$ for $\gamma < \alpha$. Hence

$$\mathcal{A}_\eta \models \theta((w_\gamma)_{\gamma<\alpha}). \tag{$\star$}$$

On the other hand, if for every injective $w \in \kappa^\alpha$, $w \supset u$, we have $(\star)$, then in fact $E = \bar{u}$ and is trivially co-meager. Therefore we have an equivalence:

$$\eta \in N_q^{*u} \iff (\forall w \supset u)(w \in \kappa^\alpha \wedge w \text{ inj.} \Rightarrow \mathcal{A}_\eta \models \theta((w_\gamma)_{\gamma<\alpha})).$$

But the latter can be expressed in the language L_{κ^+k} by the formula $\varphi_\beta((w_i)_{i<\beta})$:

$$\bigwedge_{i<j<\beta}(w_i \neq w_j) \wedge \Big(\bigforall_{\beta\leqslant i<\alpha} w_i\Big)\Big(\bigwedge_{i<j<\alpha}(w_i \neq w_j) \to \theta((w_i)_{i<\alpha})\Big)$$

θ was defined to be a formula defining N_q with parameters. It is clear thus that θ is independent of u. Furthermore the formulas constructed above from θ depend only on $\beta = \operatorname{dom} u$ and on θ. Hence the formulas defining N_q^{*u} and N_q^{*v} for $\operatorname{dom} u = \operatorname{dom} v$ are the same modulo parameters.

(b) For each $i < \kappa$ let $A_i \in Z$. We want to show that $\bigcap_{i<\kappa} A_i \in Z$. Assume that $u \in \kappa^{<\kappa}$ is arbitrary. It suffices to show that

$$\bigcap_{i<\kappa}(A_i^{*u}) = \Big(\bigcap_{i<\kappa} A_i\Big)^{*u},$$

because then $\varphi_\beta^{\bigcap_i A_i}$ is just the κ-conjunction of the formulas $\varphi_\beta^{A_i}$ which exist by the induction hypothesis. Clearly the resulting formula depends again only on $\operatorname{dom} u$ if the previous did. Note that a κ-intersection of co-meager sets is co-meager. Now

$$\begin{aligned}
&\eta \in \bigcap_{i<\kappa}(A_i^{*u}) \\
\iff\ & (\forall i<\kappa)(\{p\in\bar u \mid p\eta\in A_i\} \text{ is co-meager}) \\
\iff\ & (\forall i<\kappa)(\forall i<\kappa)(\{p\in\bar u \mid p\eta\in A_i\} \text{ is co-meager}) \\
\iff\ & \bigcap_{i<\kappa}\{p\in\bar u \mid p\eta\in A_i\} \text{ is co-meager} \\
\iff\ & \{p\in\bar u \mid p\eta\in \bigcap_{i<\kappa} A_i\} \text{ is co-meager} \\
\iff\ & \eta\in\Big(\bigcap_{i<\kappa} A_i\Big)^{*u}.
\end{aligned}$$

(c) Assume that $A\in Z$ i.e. that A^{*u} is definable for any u. Let $\varphi_{\operatorname{dom} u}$ be the formula, which defines A^{*u}. Let now $u\in\kappa^{<\kappa}$ be arbitrary and let us show that $(A^c)^{*u}$ is definable. We will show that

$$(A^c)^{*u} = \bigcap_{v\supset u}(A^{*v})^c$$

i.e. for all η

$$\eta\in(A^c)^{*u} \iff \forall v\supset u(\eta\notin A^{*v}). \tag{1}$$

Granted this, one can write the formula "$\forall v\supset u\neg\varphi_{\operatorname{dom} u}((v_i)_{i<\operatorname{dom} v})$", which is not of course the real $\varphi^{A^c}_\beta$ which we will write in the end of the proof.

To prove (1) we have to show first that for all $\eta\in\kappa^\kappa$ the set $B=\{p\in\bar u\mid p\eta\in A\}$ has Property of Baire.

The set of all permutations $S_\kappa\subset\kappa^\kappa$ is Borel by $(\cdot)$ on page 20. The set $\bar u$ is an intersection of S_κ with an open set. Again the set $\{p\in\bar u\mid p\eta\in A\}$ is the intersection of $\bar u$ and the inverse image of A under the continuous map $(p\mapsto p\eta)$, so is Borel and so has Property of Baire (P.B.), see Section IV.3.

We can now turn to proving the equivalence (1). First "$\Leftarrow$":

$$\begin{aligned}
\eta\notin(A^c)^{*u} \ &\Rightarrow\ B=\{p\in\bar u\mid p\eta\in A\} \text{ is not meager in } \bar u \\
&\Rightarrow\ \text{By P.B. of } B \text{ there is a non-empty open } U \text{ such that } U\setminus B \text{ is meager} \\
&\Rightarrow\ \text{There is non-empty } \bar v\subset\bar u \text{ such that } \bar v\setminus B \text{ is meager.} \\
&\Rightarrow\ \text{There exists } \bar v\subset\bar u \text{ such that } \{p\in\bar v\mid p\eta\in A\}=\bar v\cap B \text{ is co-meager} \\
&\Rightarrow\ \exists v\supset u(\eta\in A^{*v}).
\end{aligned}$$

And then the other direction "⇒":

$$\begin{aligned}\eta\in(A^c)^{*u} &\Rightarrow \{p\in\bar u\mid p\eta\in A\}\text{ is meager}\\ &\Rightarrow \text{ for all }\bar v\subset\bar u\text{ the set }\{p\in\bar v\mid p\eta\in A\}\text{ is meager.}\\ &\Rightarrow \forall\bar v\subset\bar u(\eta\notin A^{*v}).\end{aligned}$$

Let us now write the formula $\psi=\varphi^{A^c}_\beta$ such that

$$\forall\bar v\subset\bar u(\eta\notin A^{*v})\iff\mathcal{A}_\eta\models\psi((u_i)_{i<\beta}),$$

where $\beta=\operatorname{dom}u$: let $\psi((u_i)_{i<\beta})$ be

$$\bigwedge_{\beta\leqslant\gamma<\kappa}\ \forall_{i<\gamma}x_i\left(\Big[\bigwedge_{j<\beta}(x_j=u_j)\wedge\bigwedge_{i<j<\gamma}(x_i\neq x_j)\Big]\to\neg\varphi_\gamma((x_i)_{i<\gamma})\right)$$

One can easily see, that this is equivalent to $\forall v\supset u\big(\neg\varphi_{\operatorname{dom}v}((v_i)_{i<\operatorname{dom}v})\big)$ and that ψ depends only on $\operatorname{dom}u$ modulo parameters. □

23. Remark. If $\kappa^{<\kappa}>\kappa$, then the direction from right to left of the above theorem does not in general hold. Let $\langle\kappa,\lessdot,A\rangle$ be a model with domain κ, $A\subset\kappa$ and $\lessdot$ a well ordering of κ of order type κ. Väänänen and Shelah have shown in [**29**] (Corollary 17) that if $\kappa=\lambda^+$, $\kappa^{<\kappa}>\kappa$, $\lambda^{<\lambda}=\lambda$ and a forcing axiom holds (and $\omega_1^L=\omega_1$ if $\lambda=\omega$) then there is a sentence of $L_{\kappa\kappa}$ defining the set

$$\mathrm{STAT}=\{\langle\kappa,\lessdot,A\rangle\mid A\text{ is stationary}\}.$$

If now STAT is Borel, then so would be the set CUB defined in Section IV.3, but by Theorem 42 this set cannot be Borel since Borel sets have Property of Baire.

Open Problem. Does the direction left to right of Theorem 22 hold without the assumption $\kappa^{<\kappa}=\kappa$?

III.2. The Language $M_{\kappa^+\kappa}$ And Δ^1_1-Sets

In this section we will present a theorem similar to Theorem 22. The idea of the proof is due to Sam Coskey and Philipp Schlicht:

24. Theorem ($\kappa^{<\kappa}=\kappa$). *A set $D\subset\kappa^\kappa$ is Δ^1_1 and closed under permutations if and only if there is a determined sentence φ in $M_{\kappa^+\kappa}$ such that $D=\{\eta\mid\mathcal{A}_\eta\models\varphi\}$.*

We have to define these concepts before the proof.

25. Definition (Karttunen [**17**]). Let λ and κ be cardinals. The language $M_{\lambda\kappa}$ is then defined to be the set of pairs (t, l) of a tree t and a labeling function $\mathscr{L}$. The tree t is a $\lambda\kappa$-tree where the limits of increasing sequences of t exist and are unique. The labeling $\mathscr{L}$ is a function satisfying the following conditions:

(1) $\mathscr{L}\colon t \to a \cup \bar{a} \cup \{\bigwedge, \bigvee\} \cup \{\exists x_i \mid i < \kappa\} \cup \{\forall x_i \mid i < \kappa\}$ where a is the set of atomic formulas and $\bar{a}$ is the set of negated atomic formulas.
(2) If $x \in t$ has no successors, then $\mathscr{L}(t) \in a \cup \bar{a}$.
(3) If $x \in t$ has exactly one immediate successor then $\mathscr{L}(t)$ is either $\exists x_i$ or $\forall x_i$ for some $i < \kappa$.
(4) Otherwise $\mathscr{L}(t) \in \{\bigvee, \bigwedge\}$.
(5) If $x < y$, $\mathscr{L}(x) \in \{\exists x_i, \forall x_i\}$ and $\mathscr{L}(y) \in \{\exists x_j, \forall x_j\}$, then $i \neq j$.

26. Definition. Truth for $M_{\lambda\kappa}$ is defined in terms of a semantic game. Let $(t, \mathscr{L})$ be the pair which corresponds to a particular sentence φ and let $\mathcal{A}$ be a model. The semantic game $S(\varphi, \mathcal{A}) = S(t, \mathscr{L}, \mathcal{A})$ for $M_{\lambda\kappa}$ is played by players **I** and **II** as follows. At the first move the players are at the root and later in the game at some other element of t. Let us suppose that they are at the element $x \in t$. If $\mathscr{L}(x) = \bigvee$, then Player **II** chooses a successor of x and the players move to that chosen element. If $\mathscr{L}(x) = \bigwedge$, then player **I** chooses a successor of x and the players move to that chosen element. If $\mathscr{L}(x) = \forall x_i$ then player **I** picks an element $a_i \in \mathcal{A}$ and if $\mathscr{L}(x) = \exists x_i$ then player **II** picks an element a_i and they move to the immediate successor of x. If they come to a limit, they move to the unique supremum. If x is a maximal element of t, then they plug the elements a_i in place of the corresponding free variables in the atomic formula $\mathscr{L}(x)$. Player **II** wins if this atomic formula is true in $\mathcal{A}$ with these interpretations. Otherwise player **I** wins.

We define $\mathcal{A} \models \varphi$ if and only if **II** has a winning strategy in the semantic game.

Given a sentence φ, the sentence $\sim \varphi$ is defined by modifying the labeling function as follows. The atomic formulas are replaced by their negations, the symbols $\bigvee$ and $\bigwedge$ switch places and the quantifiers $\forall$ and $\exists$ switch places. A sentence $\varphi \in M_{\lambda\kappa}$ is *determined* if for all models $\mathcal{A}$ either $\mathcal{A} \models \varphi$ or $\mathcal{A} \models \sim \varphi$.

Now the statement of Theorem 24 makes sense. Before the proof let us recall a separation theorem for $M_{\kappa^+\kappa}$, Theorem 3.9 from [**32**]:

27. THEOREM. *Assume* $\kappa^{<\kappa} = \lambda$ *and let* $\exists R\varphi$ *and* $\exists S\psi$ *be two* Σ^1_1 *sentences where* φ *and* ψ *are in* $M_{\kappa^+\kappa}$ *and* $\exists R$ *and* $\exists S$ *are second order quantifiers. If* $\exists R\varphi \wedge \exists S\psi$ *does not have a model, then there is a sentence* $\theta \in M_{\lambda^+\lambda}$ *such that for all models* $\mathcal{A}$

$$\mathcal{A} \models \exists R\varphi \Rightarrow \mathcal{A} \models \theta \text{ and } \mathcal{A} \models \exists S\psi \Rightarrow \mathcal{A} \models \sim \theta \qquad \square$$

28. DEFINITION. For a tree t, let σt be the tree of downward closed linear subsets of t ordered by inclusion.

Proof of Theorem 24. Let us first show that if φ is an arbitrary sentence of $M_{\kappa^+\kappa}$, then $D_\varphi = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$ is Σ^1_1. The proof has the same idea as the proof of Theorem 16 that Borel* $\subset \Sigma^1_1$. Note that this implies that if φ is determined, then D_φ is Δ^1_1.

A strategy in the semantic game $S(\varphi, \mathcal{A}_\eta) = S(t, \mathscr{L}, \mathcal{A}_\eta)$ is a function

$$\upsilon \colon \sigma t \times (\operatorname{dom} \mathcal{A}_\eta)^{<\kappa} \to t \cup (t \times \operatorname{dom} \mathcal{A}_\eta).$$

This is because the previous moves always form an initial segment of a branch of the tree together with the sequence of constants picked by the players from $\operatorname{dom} \mathcal{A}_\eta$ at the quantifier moves, and a move consists either of going to some node of the tree or going to a node of the tree together with choosing an element from $\operatorname{dom} \mathcal{A}_\eta$. By the convention that $\operatorname{dom} \mathcal{A}_\eta = \kappa$, a strategy becomes a function

$$\upsilon \colon \sigma t \times \kappa^{<\kappa} \to t \cup (t \times \kappa),$$

Because t is a $\kappa^+\kappa$-tree, there are fewer than κ moves in a play (there are no branches of length κ and the players go up the tree on each move). Let

$$f \colon \sigma t \times \kappa^{<\kappa} \to \kappa$$

be any bijection and let

$$g \colon t \cup (t \times \kappa) \to \kappa$$

be another bijection. Let F be the bijection

$$F \colon (t \cup (t \times \kappa))^{\sigma t \times \kappa^{<\kappa}} \to \kappa^\kappa$$

defined by $F(\upsilon) = g \circ \upsilon \circ f^{-1}$. Let

$$C = \{(\eta, \xi) \mid F^{-1}(\xi) \text{ is a winning strategy of } \mathbf{II} \text{ in } S(t, \mathscr{L}, \mathcal{A}_\eta)\}.$$

Clearly D_φ is the projection of C. Let us show that C is closed. Consider an element (η,ξ) in the complement of C. We shall show that there is an open neighbourhood of (η,ξ) outside C. Denote $\upsilon = F^{-1}(\xi)$. Since υ is not a winning strategy there is a strategy τ of **I** that beats υ. There are $\alpha+1<\kappa$ moves in the play $\tau * \upsilon$ (by definition all branches have successor order type). Assume that $b=(x_i)_{i\leqslant\alpha}$ is the chosen branch of the tree and $(c_i)_{i<\alpha}$ the constants picked by the players. Let $\beta<\kappa$ be an ordinal with the properties $\{f((x_i)_{i<\gamma},(c_i)_{i<\gamma}) \mid \gamma \leqslant \alpha+1\} \subset \beta$ and

$$\eta' \in N_{\eta\restriction\beta} \to \mathcal{A}_{\eta'} \not\models \mathscr{L}(x_\alpha)((c_i)_{i<\alpha}). \qquad (\star)$$

Such β exists, since $|\{f((x_i)_{i<\gamma},(c_i)_{i<\gamma}) \mid \gamma \leqslant \alpha+1\}| < \kappa$ and $\mathscr{L}(x_\alpha)$ is a (possibly negated) atomic formula which is not true in $\mathcal{A}_\eta$, because **II** lost the game $\tau * \upsilon$ and because already a fragment of size $<\kappa$ of $\mathcal{A}_\eta$ decides this. Now if $(\eta',\xi') \in N_{\eta\restriction\beta} \times N_{\xi\restriction\beta}$ and $\upsilon' = F^{-1}(\xi')$, then $\upsilon * \tau$ is the same play as $\tau * \upsilon'$. So $\mathcal{A}_{\eta'} \not\models \mathscr{L}(x_\alpha)((c_i)_{i<\alpha})$ by $(\star)$ and (η',ξ') is not in C and

$$N_{\eta\restriction\beta} \times N_{\xi\restriction\beta}$$

is the intended open neighbourhood of (η,ξ) outside C. This completes the "if"-part of the proof.

Now for a given $A \in \Delta_1^1$ which is closed under permutations we want to find a determined sentence $\varphi \in M_{\kappa^+\kappa}$ such that $A = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$. By our assumption $\kappa^{<\kappa}=\kappa$ and Theorems 20 and 27, it is enough to show that for a given Borel* set B which is closed under permutations, there is a sentence $\exists R\psi$ which is Σ_1^1 over $M_{\kappa^+\kappa}$, such that $B = \{\eta \mid \mathcal{A}_\eta \models \exists R\psi\}$.

The sentence "R is a well ordering of the universe of order type κ", is definable by the formula $\theta=\theta(R)$ of $L_{\kappa^+\kappa} \subset M_{\kappa^+\kappa}$:

$$\begin{aligned} &\text{"}R\text{ is a linear ordering on the universe"}\\ \wedge\quad &\Big(\forall_{i<\omega} x_i\Big)\Big(\bigvee_{i<\omega} \neg R(x_{i+1},x_i)\Big)\\ \wedge\quad &\forall x \bigvee_{\alpha<\kappa} \exists_{i<\alpha} y_i \left[\Big(\forall y (R(y,x) \to \bigvee_{i<\alpha} y_i = y)\Big)\right] \end{aligned} \qquad (2)$$

(We assume $\kappa>\omega$, so the infinite quantification is allowed. The second row says that there are no descending sequences of length ω and the third row says that the initial segments are of size less than κ. This ensures that $\theta(R)$ says that R is a well ordering of order type κ).

Let t and h be the tree and the labeling function corresponding to B. Define the tree $t^\star$ as follows.

(1) Assume that b is a branch of t with $h(b) = N_{\xi\restriction\alpha}$ for some $\xi \in \kappa^\kappa$ and $\alpha < \kappa$. Then attach a sequence of order type α^* on top of b where
$$\alpha^* = \bigcup_{s\in\pi^{-1}[\alpha]} \operatorname{ran} s,$$
where π is the bijection $\kappa^{<\omega} \to \kappa$ used in the coding, see Definition 11 on page 14.

(2) Do this to each branch of t and add a root r to the resulting tree.

After doing this, the resulting tree is $t^\star$. Clearly it is a $\kappa^+\kappa$-tree, because t is. Next, define the labeling function $\mathscr{L}$. If $x \in t$ then either $\mathscr{L}(x) = \bigwedge$ or $\mathscr{L}(x) = \bigvee$ depending on whether it is player **I**'s move or player **II**'s move: formally let $n < \omega$ be such that $\mathrm{OTP}(\{y \in t^\star \mid y \leqslant x\}) = \alpha + n$ where α is a limit ordinal or 0; then if n is odd, put $\mathscr{L}(x) = \bigwedge$ and otherwise $\mathscr{L}(x) = \bigvee$. If $x = r$ is the root, then $\mathscr{L}(x) = \bigwedge$. Otherwise, if x is not maximal, define
$$\beta = \mathrm{OTP}\{y \in t^\star \setminus (t \cup \{r\}) \mid y \leqslant x\}$$
and set $\mathscr{L}(x) = \exists x_\beta$.

Next we will define the labeling of the maximal nodes of $t^\star$. By definition these should be atomic formulas or negated atomic formulas, but it is clear that they can be replaced without loss of generality by any formula of $M_{\kappa^+\kappa}$; this fact will make the proof simpler. Assume that x is maximal in $t^\star$. $\mathscr{L}(x)$ will depend only on $h(b)$ where b is the unique branch of t leading to x. Let us define $\mathscr{L}(x)$ to be the formula of the form $\theta \wedge \Theta_b((x_i)_{i<\alpha^*})$, where θ is defined above and Θ_b is defined below. The idea is that
$$\mathcal{A}_\eta \models \Theta_b((a_\gamma)_{\gamma<\alpha^*})\} \iff \eta \in h(b) \text{ and } \forall\gamma < \alpha^*(a_\gamma = \gamma).$$
Let us define such a Θ_b. Suppose that ξ and α are such that $h(b) = N_{\xi\restriction\alpha}$. Define for $s \in \pi^{-1}[\alpha]$ the formula A_b^s as follows:
$$A_b^s = \begin{cases} P_{\operatorname{dom} s}, & \text{if } \mathcal{A}_\xi \models P_{\operatorname{dom} s}((s(i))_{i\in\operatorname{dom} s}) \\ \neg P_{\operatorname{dom} s}, & \text{if } \mathcal{A}_\xi \not\models P_{\operatorname{dom} s}((s(i))_{i\in\operatorname{dom} s}) \end{cases}$$
Then define
$$\begin{aligned} \psi_0((x_i)_{i<\alpha^*}) &= \bigwedge_{i<\alpha^*} \left[\forall y(R(y, x_i) \leftrightarrow \bigvee_{j<i}(y = x_j))\right] \\ \psi_1((x_i)_{i<\alpha^*}) &= \bigwedge_{s\in\pi^{-1}[\alpha]} A_b^s((x_{s(i)})_{i\in\operatorname{dom} s}), \\ \Theta_b &= \psi_0 \wedge \psi_1. \end{aligned}$$
The disjunction over the empty set is considered false.

Claim 1. Suppose for all η, R is the standard order relation on κ. Then

$$(\mathcal{A}_\eta, R) \models \Theta_b((a_\gamma)_{\gamma<\alpha^*}) \iff \eta \in h(b) \wedge \forall\gamma < \alpha^*(\alpha_\gamma = \gamma).$$

Proof of Claim 1. Suppose $\mathcal{A}_\eta \models \Theta((a_\gamma)_{\gamma<\alpha^*})$. Then by $\mathcal{A}_\eta \models \psi_0((a_\gamma)_{\gamma<\alpha^*})$ we have that $(a_\gamma)_{\gamma<\alpha^*}$ is an initial segment of $\operatorname{dom}\mathcal{A}_\eta$ with respect to R. But $(\operatorname{dom}\mathcal{A}_\eta, R) = (\kappa, <)$, so $\forall\gamma < \alpha^*(\alpha_\gamma = \gamma)$. Assume that $\beta < \alpha$ and $\eta(\beta) = 1$ and denote $s = \pi^{-1}(\beta)$. Then $\mathcal{A}_\eta \models P_{\operatorname{dom} s}((s(i))_{i\in\operatorname{dom} s})$. Since Θ is true in $\mathcal{A}_\eta$ as well, we must have $A_b^s = P_{\operatorname{dom} s}$ which by definition means that $\mathcal{A}_\xi \models P_{\operatorname{dom} s}((s(i))_{i\in\operatorname{dom} s})$ and hence $\xi(\beta) = \xi(\pi(s)) = 1$. In the same way one shows that if $\eta(\beta) = 0$, then $\xi(\beta) = 0$ for all $\beta < \alpha$. Hence $\eta\restriction\alpha = \xi\restriction\alpha$.

Assume then that $a_\gamma = \gamma$ for all $\gamma < \alpha^*$ and that $\eta \in N_{\xi\restriction\alpha}$. Then $\mathcal{A}_\eta$ trivially satisfies ψ_0. Suppose that $s \in \pi^{-1}[\alpha]$ is such that $\mathcal{A}_\xi \models P_{\operatorname{dom} s}((s(i))_{i\in\operatorname{dom} s})$. Then $\xi(\pi(s)) = 1$ and since $\pi(s) < \alpha$, also $\eta(\pi(s)) = 1$, so $\mathcal{A}_\eta \models P_{\operatorname{dom} s}((s(i))_{i\in\operatorname{dom} s})$. Similarly one shows that if

$$\mathcal{A}_\xi \not\models P_{\operatorname{dom} s}((s(i))_{i\in\operatorname{dom} s}),$$

then $\mathcal{A}_\eta \not\models P_{\operatorname{dom} s}((s(i))_{i\in\operatorname{dom} s})$. This shows that $\mathcal{A}_\eta \models A_b^s((s(i))_{i\in\operatorname{dom} s})$ for all s. Hence $\mathcal{A}_\eta$ satisfies ψ_1, so we have $\mathcal{A}_\eta \models \Theta$. $\square_{\text{Claim 1}}$

Claim 2. t, h, $t^\star$ and $\mathscr{L}$ are such that for all $\eta \in \kappa^\kappa$

$$\mathbf{II} \uparrow G(t, h, \eta) \iff \exists R \subset (\operatorname{dom}\mathcal{A}_\eta)^2 \ \ \mathbf{II} \uparrow S(t^\star, \mathscr{L}, \mathcal{A}_\eta).$$

Proof of Claim 2. Suppose σ is a winning strategy of **II** in $G(t, h, \eta)$. Let R be the well ordering of $\operatorname{dom}\mathcal{A}_\eta$ such that $(\operatorname{dom}\mathcal{A}_\eta, R) = (\kappa, <)$. Consider the game $S(t^\star, \mathscr{L}, \mathcal{A}_\eta)$. On the first move the players are at the root and player **I** chooses where to go next. They go to to a minimal element of t. From here on **II** uses σ as long as they are in t. Let us see what happens if they got to a maximal element of t, i.e. they picked a branch b from t. Since σ is a winning strategy of **II** in $G(t, h, \eta)$, we have $\eta \in h(b)$ and $h(b) = N_{\xi\restriction\alpha}$ for some ξ and α. For the next α moves the players climb up the tower defined in item (1) of the definition of $t^\star$. All labels are of the form $\exists x_\beta$, so player **II** has to pick constants from $\mathcal{A}_\eta$. She picks them as follows: for the variable x_β she picks $\beta \in \kappa = \operatorname{dom}\mathcal{A}_\eta$. She wins now if $\mathcal{A}_\eta \models \Theta((\beta)_{\beta<\alpha^*})$ and $\mathcal{A}_\eta \models \theta$. But $\eta \in h(b)$, so by Claim 1 the former holds and the latter holds because we chose R to be a well ordering of order type κ.

Let us assume that there is no winning strategy of **II** in $G(t,h,\eta)$. Let R be an arbitrary relation on $\operatorname{dom}\mathcal{A}_\eta$. Here we shall finally use the fact that B is closed under permutations. Suppose R is not a well ordering of the universe of order type κ. Then after the players reached the final node of $t^\star$, player **I** chooses to go to θ and player **II** loses. So we can assume that R is a well ordering of the universe of order type κ. Let $p\colon \kappa\to\kappa$ be a bijection such that $p(\alpha)$ is the α^{th} element of κ with respect to R. Now p is a permutation and $\{\eta \mid \mathcal{A}_{p\eta}\in B\} = B$ since B is closed under permutations. So by our assumption that $\eta\notin B$ (i.e. $\mathbf{II} \not\uparrow G(t,h,\eta)$), we also have $p\eta\notin B$, i.e. player **II** has no winning strategy in $G(t,h,p\eta)$ either.

Suppose σ is any strategy of **II** in $S(t^\star,\mathscr{L},\mathcal{A}_\eta)$. Player **I** imagines that σ is a strategy in $G(t,h,p\eta)$ and picks a strategy τ that beats it. In the game $S(t^\star,\mathscr{L},\mathcal{A}_\eta)$, as long as the players are still in t, player **I** uses τ that would beat σ if they were playing $G(t,h,p\eta)$ instead of $S(t^\star,\mathscr{L},\eta)$. Let us assume that they got to a maximal element of t and picked a branch b of t. We have that $p\eta\notin h(b)$. If **II** wants to satisfy ψ_0 of the definition of Θ_b, she is forced to pick the constants $(a_i)_{i<\alpha^*}$ such that a_i is the i^{th} element of $\operatorname{dom}\mathcal{A}_\eta$ with respect to R. Suppose that $\mathcal{A}_\eta \models \psi_1((a_i)_{i<\alpha^*})$ (recall $\Theta_b = \psi_0\wedge\psi_1$). But then $\mathcal{A}_{p\eta}\models \psi_1((\gamma)_{\gamma<\alpha^*})$ and also $\mathcal{A}_{p\eta}\models\psi_0((\gamma)_{\gamma<\alpha^*})$, so by Claim 1 we should have $p\eta\in h(b)$ which is a contradiction. $\square_{\text{Claim 2}}$

$\square_{\text{Theorem 24}}$

IV Generalizations From Classical Descriptive Set Theory

IV.1. Simple Generalizations

IV.1.1. The Identity Relation

Denote by id the equivalence relation $\{(\eta,\xi)\in(2^\kappa)^2\mid \eta=\xi\}$. With respect to our choice of topology, the natural generalization of the equivalence relation

$$E_0=\{(\eta,\xi)\in 2^\omega\times 2^\omega\mid \exists n<\omega\forall m>n(\eta(m)=\xi(m))\}$$

is equivalence modulo sets of size $<\kappa$:

$$E_0^{<\kappa}=\{(\eta,\xi)\in 2^\kappa\times 2^\kappa\mid \exists\alpha<\kappa\forall\beta>\alpha(\eta(\beta)=\xi(\beta))\},$$

although the equivalences modulo sets of size $<\lambda$ for $\lambda<\kappa$ can also be studied:

$$E_0^{<\lambda}=\{(\eta,\xi)\in 2^\kappa\times 2^\kappa\mid \exists A\subset\kappa[|A|<\lambda\wedge\forall\beta\notin A(\eta(\beta)=\xi(\beta))]\},$$

but for $\lambda<\kappa$ these turn out to be bireducible with id (see below). Similarly one can define $E_0^{<\lambda}$ on κ^κ instead of 2^κ.

It makes no difference whether we define these relations on 2^κ or κ^κ since they become bireducible to each other:

29. Theorem. *Let $\lambda\leqslant\kappa$ be a cardinal and let $E_0^{<\lambda}(P)$ denote the equivalence relation $E_0^{<\lambda}$ on $P\in\{2^\kappa,\kappa^\kappa\}$ (notation defined above). Then*

$$E_0^{<\lambda}(2^\kappa)\leqslant_c E_0^{<\lambda}(\kappa^\kappa)\text{ and }E_0^{<\lambda}(\kappa^\kappa)\leqslant_c E_0^{<\lambda}(2^\kappa).$$

Note that when $\lambda=1$, we have $E_0^{<1}(P)=\mathrm{id}_P$.

Proof. In this proof we think of functions $\eta,\xi\in\kappa^\kappa$ as graphs $\eta=\{(\alpha,\eta(\alpha))\mid\alpha<\kappa\}$. Fix a bijection $h\colon\kappa\to\kappa\times\kappa$. Let $f\colon 2^\kappa\to\kappa^\kappa$ be the inclusion, $f(\eta)(\alpha)=\eta(\alpha)$. Then f is easily seen to be a continuous reduction $E_0^{<\lambda}(2^\kappa)\leqslant_c E_0^{<\lambda}(\kappa^\kappa)$. Define $g\colon\kappa^\kappa\to 2^\kappa$ as follows. For $\eta\in\kappa^\kappa$ let $g(\eta)(\alpha)=1$ if $h(\alpha)\in\eta$ and $g(\eta)(\alpha)=0$ otherwise. Let us show that

g is a continuous reduction $E_0^{<\lambda}(\kappa^\kappa) \leqslant_c E_0^{<\lambda}(2^\kappa)$. Suppose $\eta, \xi \in \kappa$ are $E_0^{<\lambda}(\kappa^\kappa)$-equivalent. Then clearly $|\eta \bigtriangleup \xi| < \lambda$. On the other hand

$$I = \{\alpha \mid g(\eta)(\alpha) \neq g(\xi)(\alpha)\} = \{\alpha \mid h(\alpha) \in \eta \bigtriangleup \xi\}$$

and because h is a bijection, we have that $|I| < \lambda$.

Suppose η and ξ are not $E_0^{<\lambda}(\kappa^\kappa)$-equivalent. But then $|\eta \bigtriangleup \xi| \geqslant \lambda$ and the argument above shows that also $|I| \geqslant \lambda$, so $g(\eta)(\alpha)$ is not $E_0^{<\lambda}(2^\kappa)$-equivalent to $g(\xi)(\alpha)$.

g is easily seen to be continuous. $\square$

30. THEOREM ($\kappa^{<\kappa} = \kappa$). *$E_0^{<\lambda}$ is an equivalence relation on 2^κ for all $\lambda \leqslant \kappa$ and*

(1) $E_0^{<\lambda}$ is Borel.
(2) $E_0^{<\kappa} \not\leqslant_B \text{id}$.
(3) If $\lambda \leqslant \kappa$, then $\text{id} \leqslant_c E_0^{<\lambda}$.
(4) If $\lambda < \kappa$, then $E_0^{<\lambda} \leqslant_c \text{id}$.

Proof. $E_0^{<\lambda}$ is clearly reflexive and symmetric. Suppose $\eta E_0^{<\lambda} \xi$ and $\xi E_0^{<\lambda} \zeta$. Denote $\eta = \eta^{-1}\{1\}$ and similarly for η, ζ. Then $|\eta \bigtriangleup \xi| < \lambda$ and $|\xi \bigtriangleup \zeta| < \lambda$; but $\eta \bigtriangleup \zeta \subset (\eta \bigtriangleup \xi) \cup (\xi \bigtriangleup \zeta)$. Thus $E_0^{<\lambda}$ is indeed an equivalence relation.

(1) $E_0^{<\lambda} = \displaystyle\bigcup_{A \in [\kappa]^{<\lambda}} \bigcap_{\alpha \notin A} \underbrace{\{(\eta, \xi) \mid \eta(\alpha) = \xi(\alpha)\}}_{\text{open}}$.

(2) Assume there were a Borel reduction $f\colon 2^\kappa \to 2^\kappa$ witnessing $E_0 \leqslant_B \text{id}$. By Theorem 44 there are dense open sets $(D_i)_{i<\kappa}$ such that $f \restriction \bigcap_{i<\kappa} D_i$ is continuous. If $p, q \in 2^\alpha$ for some α and $\xi \in N_p$, let us denote $\xi^{(p/q)} = q^\frown(\xi \restriction (\kappa \setminus \alpha))$, and if $A \subset N_p$, denote

$$A^{(p/q)} = \{\eta^{(p/q)} \mid \eta \in A\}.$$

Let C is be the collection of sets, each of which is of the form

$$\bigcup_{q \in 2^\alpha} [D_i \cap N_p]^{(p/q)}$$

for some $\alpha < \kappa$ and some $p \in 2^\alpha$. It is easy to see that each such set is dense and open, so C is a collection of dense open sets. By the assumption $\kappa^{<\kappa} = \kappa$, C has size κ. Also C contains the sets D_i for all $i < \kappa$, (taking $\alpha = 0$). Denote $D = \bigcap_{i<\kappa} D_i$. Let $\eta \in \bigcap C$, $\xi = f(\eta)$ and $\xi' \neq \xi$, $\xi' \in \operatorname{ran}(f \restriction D)$. Now ξ and ξ' have disjoint open neighbourhoods V and V' respectively. Let α and $p, q \in 2^\alpha$ be such that $\eta \in N_p$ and such that $D \cap N_p \subset f^{-1}[V]$ and $D \cap N_q \subset f^{-1}[V']$.

These p and q exist by the continuity of f on D. Since $\eta \in \bigcap C$ and $\eta \in N_p$, we have

$$\eta \in [D_i \cap N_q]^{(q/p)}$$

for all $i < \kappa$, which is equivalent to

$$\eta^{(p/q)} \in [D_i \cap N_q]$$

for all $i < \kappa$, i.e. $\eta^{(p/q)}$ is in $D \cap N_q$. On the other hand (since $D_i \in C$ for all $i < \kappa$ and because $\eta \in N_p$), we have $\eta \in D \cap N_p$. This implies that $f(\eta) \in V$ and $f(\eta^{(p/q)}) \in V'$ which is a contradiction, because V and V' are disjoint and $(\eta, \eta^{p/q}) \in E_0$.

(3) Let $(A_i)_{i<\kappa}$ be a partition of κ into pieces of size κ: if $i \neq j$ then $A_i \cap A_j = \varnothing$, $\bigcup_{i<\kappa} A_i = \kappa$ and $|A_i| = \kappa$. Obtain such a collection for instance by taking a bijection $h\colon \kappa \to \kappa \times \kappa$ and defining $A_i = h^{-1}[\kappa \times \{i\}]$. Let $f\colon 2^\kappa \to 2^\kappa$ be defined by $f(\eta)(\alpha) = \eta(i) \iff \alpha \in A_i$. Now if $\eta = \xi$, then clearly $f(\eta) = f(\xi)$ and so $f(\eta) E_0^{<\lambda} f(\xi)$. If $\eta \neq \xi$, then there exists i such that $\eta(i) \neq \xi(i)$ and we have that

$$A_i \subset \{\alpha \mid f(\eta)(\alpha) \neq f(\xi)(\alpha)\}$$

and A_i is of size $\kappa \geqslant \lambda$.

(4) By Theorem 29 it is sufficient to prove this for these relations on κ^κ. Fix $\lambda < \kappa$ and let us define the reduction $f\colon E_0^{<\lambda} \to \mathrm{id}$. Let us enumerate $\kappa^{<\kappa} = \{p_i \mid i < \kappa\}$. (Recall the cardinality assumption $\kappa^{<\kappa} = \kappa$). We will define a function $f'\colon \kappa^{<\kappa} \to \kappa^{<\kappa}$ with the following properties:

(i) $\forall p(\operatorname{dom} f'(p) = \operatorname{dom} p)$

(ii) if $\gamma \leqslant \min\{\operatorname{dom} p_i, \operatorname{dom} p_j\}$, then $|\{\alpha < \gamma \mid p_i(\alpha) \neq p_j(\alpha)\}| < \lambda$ if and only if $f'(p_i) \restriction \gamma = f'(p_j) \restriction \gamma$.

Once we have that, letting $f(\eta) = \bigcup_{\alpha<\kappa} f'(\eta \restriction \alpha)$, f is the intended reduction. Let us define $f'(p_i)$ by induction on i.

To begin with let $f'(p_0) = p_0$. Suppose that for all $i < j$, $f'(p_i)$ is defined and conditions (i) and (ii) hold so far. Consider p_j. Let us deal with two separate cases:

(a) There is $i < j$ with the properties $\operatorname{dom} p_i \geqslant \operatorname{dom} p_j$ and

$$|\{\alpha < \operatorname{dom} p_j \mid p_j(\alpha) \neq p_i(\alpha)\}| < \lambda.$$

(b) There is no such $i < j$.

If (a), then let $f'(p_j) = f'(p_i) \restriction \operatorname{dom} p_j$. If (b), let R be the set

$$R = \{f'(p_i) \restriction \gamma \mid (i < j) \wedge (\gamma < \kappa) \wedge (|\{\alpha < \gamma \mid p_i(\alpha) \neq p_j(\alpha)\}| < \lambda)\},$$

By the induction hypothesis, item (ii), for any $p, q \in R$ we have $p \subset q$ or $q \subset p$. Let $\alpha < \operatorname{dom} p_j$. If $\alpha < \sup\{\operatorname{dom} p \mid p \in R\}$, then let $f'(p_j)(\alpha) = (\cup R)(\alpha)$. Suppose now that $\alpha \geqslant \sup\{\operatorname{dom} p \mid p \in R\}$. Let

$$J = \{i < j \mid \alpha \in \operatorname{dom} p_i\}$$

and let $f'(p_j)(\alpha)$ be such that $f'(p_j)(\alpha) \notin \{f'(p_i)(\alpha) \mid i \in J\}$. This is possible since $|J| < \kappa$. Let us check that the set $\{p_i \mid i \leqslant j\}$ satisfies (i) and (ii). Clearly $\operatorname{dom} f'(p_j) = \operatorname{dom} p_j$ and by the induction hypothesis $\operatorname{dom} f'(p_i) = \operatorname{dom} p_i$ for all $i < j$.

Suppose $i, k \leqslant j$. Let us show that (ii) holds. If $i, k < j$, then (ii) holds by the induction hypothesis, so suppose $k = j$, and suppose that

$$|\{\alpha < \gamma \mid p_i(\alpha) \neq p_j(\alpha)\}| < \lambda. \tag{$*$}$$

If we mapped p_j according to the case (a), then $f'(p_j) = f'(p_l) \restriction \operatorname{dom} p_j$ for some $l < j$ and $f'(p_j) \restriction \gamma = f'(p_l) \restriction \gamma$. By $(*)$ and the definition of (a) we have $|\{\alpha < \gamma \mid p_i(\alpha) \neq p_l(\alpha)\}| < \lambda$ which implies by the induction hypothesis that

$$f'(p_i) \restriction \gamma = f'(p_l) \restriction \gamma = f'(p_j) \restriction \gamma.$$

Assume that we mapped p_j according to (b). But then i and γ are as in the definition of R, so $f'(p_i) \restriction \gamma \in R$ and hence $f'(p_i) \restriction \gamma \subset f'(p_j)$ by the definition of $f'(p_j)$ in the case (b). Suppose then that

$$|\{\alpha < \gamma \mid p_i(\alpha) \neq p_j(\alpha)\}| \geqslant \lambda. \tag{$**$}$$

If p_j is mapped according to (a), then there exists $l < j$ such that

$$|\{\alpha < \gamma \mid p_j(\alpha) \neq p_l(\alpha)\}| < \lambda \tag{$***$}$$

which together with $(**)$ implies that

$$|\{\alpha < \gamma \mid p_i(\alpha) \neq p_l(\alpha)\}| \geqslant \lambda. \qquad (****).$$

Now by the induction hypothesis it follows from $(***)$ and $(****)$ that

$$f'(p_i) \restriction \gamma \neq f'(p_l) \restriction \gamma = f'(p_j) \restriction \gamma.$$

Suppose that p_j is mapped according to (b). Then $f'(p_i) \restriction \gamma \notin R$. If $\sup\{\operatorname{dom} p \mid p \in R\} \geqslant \gamma$, then it means that there is an l such that $f'(p_l) \restriction \gamma \neq f'(p_i) \restriction \gamma$ and $f'(p_l) \restriction \gamma \subset f'(p_j)$ and we are done. If $\sup\{\operatorname{dom} p \mid p \in R\} < \gamma$, then by the choice of $f'(p_j)(\alpha)$ for $\sup\{\operatorname{dom} p \mid p \in R\} \leqslant \alpha < \gamma$ we have $f'(p_j)(\alpha) \neq f'(p_i)(\alpha)$ which again gives $f'(p_j) \restriction \gamma \neq f'(p_i) \restriction \gamma$.

Let us define $f(\eta) = \bigcup_{\alpha<\kappa} f'(\eta\restriction\alpha)$ and let us show that f is the needed reduction. First, let us check that f is well defined. Suppose $\alpha_0 < \alpha_1 < \kappa$. Since

$$|\{\beta < \alpha_0 \mid (\eta\restriction\alpha_0)(\alpha) \neq (\eta\restriction\alpha_1)(\alpha_0)\}| = 0,$$

by (ii), we have that $f'(\eta\restriction\alpha_0) = f'(\eta\restriction\alpha_1)\restriction\alpha_0$. Thus the sequence $\langle f'(\eta\restriction\alpha)\rangle_{\alpha<\kappa}$ is a chain and f is well defined.

Let us show next that f is continuous. Let $p \in 2^{<\kappa}$. Then

$$f^{-1}N_p = \{\eta \mid p \subset f(\eta)\} = \{\eta \mid p \subset \bigcup_{\alpha<\kappa} f'(\eta\restriction\alpha)\}$$

Since f' has the property that $\operatorname{dom} f'(\eta\restriction\alpha) = \alpha$, we get further

$$= \{\eta \mid p = f'(\eta\restriction\operatorname{dom} p)\}$$

which is clearly open.

Let us show that f is a reduction. Suppose $\eta E_0^{<\lambda}\xi$. Then $f(\eta) = \bigcup_{\alpha<\kappa} f'(\eta\restriction\alpha)$ and $f(\xi) = \bigcup_{\alpha<\kappa} f'(\xi\restriction\alpha)$, so it is sufficient to show that for all $\alpha_0 < \alpha_1$

$$f'(\eta\restriction\alpha_0) \subset f'(\xi\restriction\alpha_1).$$

This follows immediately from the construction of f': if $\eta\restriction\alpha_0$ comes before $\xi\restriction\alpha_1$ in the enumeration of $2^{<\kappa}$, then apply case (b), namely $f'(\eta\restriction\alpha_0)$ will be in R and if they come in the reverse order, apply (a) to get $f'(\eta\restriction\alpha_0) \subset f'(\xi\restriction\alpha_1)$. Suppose then that $\neg\eta E_0^{<\lambda}\xi$. Since $\lambda < \kappa$, there is an $\alpha < \kappa$ such that

$$|\{\beta < \alpha+1 \mid (\eta\restriction\alpha+1)(\beta) \neq (\xi\restriction\alpha+1)(\beta)\}| \geqslant \lambda.$$

Suppose that $\eta\restriction(\alpha+1) = p_i$ and $\xi\restriction(\alpha+1) = p_j$ and assume without loss of generality that $i < j$. Now as in the proof of (ii) in the induction step of the construction of f', we get that $f'(p_i)(\alpha) \neq f'(p_j)(\alpha)$. $\square$

IV.2. On Silver Dichotomy

IV.2.1. Silver Dichotomy for Isomorphism Relations

Although Silver's dichotomy for Borel sets is not provable from ZFC for $\kappa > \omega$ (see Theorem 37 on page 42), it holds when the equivalence relation is an isomorphism relation, if $\kappa > \omega$ is an inaccessible cardinal:

31. Theorem. *Assume that κ is inaccessible and $\kappa^{<\kappa} = \kappa$. If the number of equivalence classes of $\cong_T$ is greater than κ, then* $\mathrm{id} \leqslant_c \cong_T$.

Proof. Suppose that there is more than κ equivalence classes of $\cong_T$. We will show that then $\mathrm{id}_{2^\kappa} \leqslant_c \cong_T$. If T is not classifiable, then as was done in [**26**], we can construct a tree $t(S)$ for each $S \subset S^\kappa_\omega$ and Ehrenfeucht-Mostowski-type models $M(t(S))$ over these trees such that if $S \triangle S'$ is stationary, then $M(t(S)) \ncong M(t(S'))$. Now it is easy to construct a reduction $f\colon \mathrm{id}_{2^\kappa} \leqslant_c E_{S^\kappa_\omega}$ (see notation 69), so then $\eta \mapsto M(t(f(\eta)))$ is a reduction $\mathrm{id} \leqslant_c \cong_T$.

Assume now that T is classifiable. By [**25**] Theorem XIII.4.8 (this is also mentioned in [**6**] Theorem 2.5), assuming that $\cong_T$ has more than κ equivalence classes, it has depth at least 2 and so there are: a $\lambda(T)^+$-saturated model $\mathcal{B} \models T$, $|\mathcal{B}| = \lambda(T)$, and a $\lambda(T)^+$-saturated elementary submodel $\mathcal{A} \preccurlyeq \mathcal{B}$ and $a \notin \mathcal{B}$ such that $\mathrm{tp}(a/\mathcal{B})$ is orthogonal to $\mathcal{A}$. Let $f\colon \kappa \to \kappa$ be strictly increasing and such that for all $\alpha < \kappa$, $f(\alpha) = \mu^+$, for some μ with the properties $\lambda(T) < \mu < \kappa$, $\mathrm{cf}(\mu) = \mu$ and $\mu^{2^\omega} = \mu$. For each $\eta \in 2^\kappa$ with $\eta^{-1}\{1\}$ is unbounded we will construct a model $\mathcal{A}_\eta$. As above, it will be enough to show that $\mathcal{A}_\eta \ncong \mathcal{A}_\xi$ whenever $\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}$ is λ-stationary where $\lambda = \lambda(T)^+$. Fix $\eta \in 2^\kappa$ and let $\lambda = \lambda(T)^+$.

For each $\alpha \in \eta^{-1}\{1\}$ choose $\mathcal{B}_\alpha \supset \mathcal{A}$ such that

(1) $\exists \pi_\alpha\colon \mathcal{B} \cong \mathcal{B}_\alpha$, $\pi_\alpha \restriction \mathcal{A} = \mathrm{id}_{\mathcal{A}}$.
(2) $\mathcal{B}_\alpha \downarrow_{\mathcal{A}} \bigcup\{\mathcal{B}_\beta \mid \beta \in \eta^{-1}\{1\}, \beta \neq \alpha\}$

Note that 2 implies that if $\alpha \neq \beta$, then $\mathcal{B}_\alpha \cap \mathcal{B}_\beta = \mathcal{A}$. For each $\alpha \in \eta^{-1}\{1\}$ and $i < f(\alpha)$ choose tuples a^α_i with the properties

(3) $\mathrm{tp}(a^\alpha_i/\mathcal{B}_\alpha) = \pi_\alpha(\mathrm{tp}(a/\mathcal{B}))$
(4) $a^\alpha_i \downarrow_{\mathcal{B}_\alpha} \bigcup\{a^\alpha_j \mid j < f(\alpha), j \neq i\}$

Let $\mathcal{A}_\eta$ be F^s_λ-primary over

$$S_\eta = \bigcup\{B_\alpha \mid a < \eta^{-1}\{1\}\} \cup \bigcup\{a^\alpha_i \mid \alpha < \eta^{-1}\{1\}, i < f(\alpha)\}.$$

It remains to show that if $S^\kappa_\lambda \cap \eta^{-1}\{1\} \triangle \xi^{-1}\{1\}$ is stationary, then $\mathcal{A}_\eta \ncong \mathcal{A}_\xi$. Without loss of generality we may assume that $S^\kappa_\lambda \cap \eta^{-1}\{1\} \setminus \xi^{-1}\{1\}$ is stationary. Let us make a counter assumption, namely that there is an isomorphism $F\colon \mathcal{A}_\eta \to \mathcal{A}_\xi$.

Without loss of generality there exist singletons b^η_i and sets B^η_i, $i < \kappa$ of size $< \lambda$ such that $\mathcal{A}_\eta = S_\eta \cup \bigcup_{i<\kappa} b^\eta_i$ and $(S_\eta, (b^\eta_i, B^\eta_i)_{i<\kappa})$ is an F^s_λ-construction.

Let us find an ordinal $\alpha < \kappa$ and sets $C \subset \mathcal{A}_\eta$ and $D \subset \mathcal{A}_\xi$ with the properties listed below:

(a) $\alpha \in \eta^{-1}\{1\} \setminus \xi^{-1}\{1\}$
(b) $D = F[C]$

(c) $\forall\beta \in (\alpha+1) \cap \eta^{-1}\{1\}(\mathcal{B}_\beta \subset C)$ and $\forall\beta \in (\alpha+1) \cap \xi^{-1}\{1\}(\mathcal{B}_\beta \subset D)$,
(d) for all $i < f(\alpha)$, $\forall\beta \in \alpha\cap\eta^{-1}\{1\}(a_i^\beta \in C)$ and $\forall\beta \in \alpha\cap\xi^{-1}\{1\}(a_i^\beta \in D)$,
(e) $|C| = |D| < f(\alpha)$,
(f) For all β, if $\mathcal{B}_\beta \cap C \setminus \mathcal{A} \neq \varnothing$, then $\mathcal{B}_\beta \subset C$ and if $\mathcal{B}_\beta \cap D \setminus \mathcal{A} \neq \varnothing$, then $\mathcal{B}_\beta \subset D$,
(g) C and D are λ-saturated,
(h) if $b_i^\eta \in C$, then $B_i^\eta \subset [S_\eta \cup \bigcup\{b_i^\eta \mid j < i\}] \cap C$ and if $b_i^\xi \in D$, then $B_i^\xi \subset [S_\xi \cup \bigcup\{b_i^\xi \mid j < i\}] \cap D$.

This is possible, because $\eta^{-1}\{1\} \setminus \xi^{-1}\{1\}$ is stationary and we can close under the properties (b)–(h).

Now $\mathcal{A}_\eta$ is F^s_λ-primary over $C \cup S_\eta$ and $\mathcal{A}_\xi$ is F^s_λ-primary over $D \cup S_\eta$ and thus $\mathcal{A}_\eta$ is F^s_λ-atomic over $C \cup S_\eta$ and $\mathcal{A}_\xi$ is F^s_λ-atomic over $D \cup S_\xi$. Let

$$I_\alpha = \{a_i^\alpha \mid i < f(\alpha)\}.$$

Now $|I_\alpha \setminus C| = f(\alpha)$, because $|C| < f(\alpha)$, and so $I_\alpha \setminus C \neq \varnothing$. Let $c \in I_\alpha \setminus C$ and let $A \subset S_\xi \setminus D$ and $B \subset D$ be such that $\mathrm{tp}(F(c)/A \cup B) \vdash \mathrm{tp}(F(c)/D \cup S_\xi)$ and $|A \cup B| < \lambda$. Since $\alpha \notin \xi^{-1}\{1\}$, we can find (just take disjoint copies) a sequence $(A_i)_{i<f(\alpha)^+}$ such that $A_i \subset I_\alpha \cap \mathcal{A}_\xi$, $\mathrm{tp}(A_i/D) = \mathrm{tp}(A/D)$ and $A_i \downarrow_D \bigcup\{A_j \mid j \neq i, j < f(\alpha)^+\}$

Now we can find $(d_i)_{i<f(\alpha)^+}$, such that

$$\mathrm{tp}(d_i{}^\frown A_i{}^\frown B_i/\varnothing) = \mathrm{tp}(F(c){}^\frown A^\frown B/\varnothing).$$

Then it is a Morley sequence over D and for all $i < f(\alpha)^+$,

$$\mathrm{tp}(d_i/D) = \mathrm{tp}(F(c)/D),$$

which implies

$$\mathrm{tp}(F^{-1}(d_i)/C) = \mathrm{tp}(c/C),$$

for some i, since for some i we have $c = a_i^\alpha$. Since by (c), $\mathcal{B}_\alpha \subset C$, the above implies that

$$\mathrm{tp}(F^{-1}(d_i)/\mathcal{B}_\alpha) = \mathrm{tp}(a_i^\alpha/\mathcal{B}_\alpha)$$

which by the definition of a_i^α, item 3 implies

$$\mathrm{tp}(F^{-1}(d_i)/\mathcal{B}_\alpha) = \pi_\alpha(\mathrm{tp}(a/\mathcal{B})).$$

Thus the sequence $(F^{-1}(d_i))_{i<f(\alpha)^+}$ witnesses that the dimension of $\pi_\alpha(\mathrm{tp}(a/\mathcal{B}))$ in $\mathcal{A}_\eta$ is greater than $f(\alpha)$. Denote that sequence by J. Since $\pi_\alpha(\mathrm{tp}(a/\mathcal{B}))$ is orthogonal to $\mathcal{A}$, we can find $J' \subset J$ such that $|J'| = f(\alpha)^+$ and J' is a Morley sequence over S_η. Since $f(\alpha)^+ > \lambda$, this contradicts Theorem 4.9(2) of Chapter IV of [**25**]. $\square$

IV.2.2. Theories Bireducible With id

32. Theorem. *Suppose κ is inaccessible and $\kappa^{<\kappa} = \kappa$. Then there is a theory T such that $\cong_T$ is bireducible with id_{2^κ}.*

Proof. Let $\mathcal{M}$ be the model with domain $M = \operatorname{dom}\mathcal{M} = \omega \cup (\omega \times \omega)$ and a binary relation R which is interpreted

$$R^{\mathcal{M}} = \{(a,(b,c)) \in M^2 \mid a \in \omega, (b,c) \in \omega \times \omega, a = b\}.$$

Then our intended theory is the complete first-order theory of this structure $T = \mathrm{Th}(\mathcal{M})$.

Let $\hat{C} = \{\aleph_\beta \mid \beta \leqslant \kappa\}$ and $C = \omega \cup \hat{C}$.

Let $\mathcal{A}$ be a model of T of size κ and let $f_{\mathcal{A}} \colon \hat{C} \to C$ be a function such that

$$f_{\mathcal{A}}(\aleph_\beta) = \operatorname{card}(\{x \in A \mid \operatorname{card}(\{(a,b) \in A \mid R(x,(a,b))\}) = \aleph_\beta\}), \qquad (*)$$

i.e. $f_{\mathcal{A}}(\aleph_\beta)$ equals the number of elements which are R-related to exactly $\aleph_\beta$ elements. Clearly $\mathcal{A} \cong \mathcal{B}$ is equivalent to $f_{\mathcal{A}} = f_{\mathcal{B}}$.

Let $g_0 \colon \hat{\mu} \to \hat{C}$ and $g_1 \colon \mu \to C$ be bijections. Let us define the function F by

$$F(\xi) = g_1^{-1} \circ f_{\mathcal{A}_\xi} \circ g_0.$$

Now F is a reduction $\cong_T \leqslant \mathrm{id}_{\kappa^\kappa}$. By Theorem 29 $\mathrm{id}_{\kappa^\kappa}$ is continuously bireducible with id_{2^κ}. Let us show that F is Borel. In order to do it, we will use the easy direction (right to left) of Theorem 22. Because every basic open set in κ^κ is an intersection of the sets of the form

$$U_{\gamma\delta} = \{\eta \in \kappa^\kappa \mid \eta(\gamma) = \delta\},$$

it is enough to show that $F^{-1}[U_{\gamma\delta}]$ is Borel for any $\gamma, \delta \in \kappa$.

$\eta \in F^{-1}[U_{\gamma\delta}]$ is equivalent to

($\star$) *there exists exactly $g_1(\delta)$ elements in $F^{-1}(\eta)$ which are R-related to exactly $g_0(\gamma)$ elements.*

We can express ($\star$) in $L_{\kappa^+\kappa}$. First, let us define the formula φ_λ for $\lambda < \kappa$ which says that the variable x is R-related to exactly λ elements:

$$\varphi_\lambda(x) : \exists_{i<\lambda} y_i \Big[\Big(\bigwedge_{j_0<j_1<\lambda} \neg y_{j_0} = y_{j_1}\Big) \wedge \bigwedge_{i<\lambda} R(x,y_i) \wedge \forall z \Big(R(x,z) \to \bigvee_{i<\lambda} z = y_i\Big)\Big].$$

Then one can write the formula which says that there are exactly $\nu < \kappa$ such x_k that satisfy φ_λ:

$$\psi_{\lambda\nu} : \mathop{\exists}\limits_{k<\nu} x_k \Big[\Big(\bigwedge_{i<j<\nu} \neg x_i = x_j \Big) \wedge \bigwedge_{k<\nu} \varphi_\lambda(x_k) \wedge \forall z \Big(\varphi_\lambda(z) \to \bigvee_{k<\nu} (z = x_k) \Big) \Big]$$

For the cases $\gamma = \kappa$, $\delta = \kappa$, define

$$\varphi_\kappa(x_k) : \bigwedge_{\beta<\kappa} \mathop{\forall}\limits_{i<\beta} y_i \Big[\exists y_\beta \Big[\Big(\bigwedge_{i<\beta} (y_\beta \neq y_i) \Big) \wedge R(x_k, y_\beta) \Big] \Big]$$

and

$$\psi_{\kappa\lambda} : \bigwedge_{\beta<\kappa} \mathop{\forall}\limits_{k<\beta} x_k \Big[\exists x_\beta \Big[\Big(\bigwedge_{k<\beta} (x_\beta \neq x_k) \Big) \wedge \varphi_\lambda(x_\beta) \Big] \Big]$$

Note that the last formulas say “for all $\beta < \kappa$ there exist more than β...”, but it is equivalent to “there exist exactly κ...” in our class of models, because the models are all of size κ.

Thus $\psi_{g_0(\gamma), g_1(\delta)}$ is defined for all $\gamma \leqslant \kappa$ and $\delta \leqslant \kappa$. By the direction right to left of Theorem 22 this implies that the sets $F^{-1}U_{\gamma\delta}$ are Borel. This proves $\cong_T \leqslant_B \mathrm{id}_{2^\kappa}$.

Since κ is inaccessible, the other direction follows from Theorem 31. On the other hand one easily constructs such a reduction from scratch. Let us do it for the sake of completeness.

Let us show that $\mathrm{id} \leqslant_c \cong_T$. Let us modify the setting a little; let $C_{<\kappa} = \{\lambda < \kappa \mid \lambda \text{ is a cardinal}\}$ and $C^\omega_{<\kappa} = C_{<\kappa} \setminus \omega$ and let

$$h_0 \colon \kappa \to C^\omega_{<\kappa}$$

and

$$h_1 \colon \kappa \to C_{<\kappa}$$

be increasing bijections. Suppose $\eta \in \kappa^\kappa$ and define $f_\eta \colon C^\omega_{<\kappa} \to C_{<\kappa}$ by

$$f_\eta(\lambda) = [(h_1 \circ \eta \circ h_0^{-1})(\lambda)]^+$$

(recall that κ is inaccessible). Let us now build the model $\mathcal{M}_\eta$:

$$\operatorname{dom} \mathcal{M}_\eta = \bigcup_{\lambda \in C^\omega_{<\kappa}} \{(\lambda, f_\eta(\lambda))\} \times [f_\eta(\lambda) \cup f_\eta(\lambda) \times \lambda]$$

(that is, formally $\operatorname{dom} \mathcal{M}_\eta$ consists of pairs and triples the first projection being a pair of the form $(\lambda, f_\eta(\lambda))$) and for all $x, y \in \operatorname{dom} \mathcal{M}_\eta$:

$$R(x,y) \iff \exists\lambda\exists\alpha\exists\beta\big(x = ((\lambda, f_\eta(\lambda)), \alpha) \wedge y = ((\lambda, f_\eta(\lambda)), \alpha, \beta)\big).$$

Denote the mapping $\eta \mapsto \mathcal{M}_\eta$ by G, i.e. $G(\eta) = \mathcal{M}_\eta$. Clearly $\mathcal{M}_\eta \models T$. Let us show that

$$\mathcal{M}_\eta \cong \mathcal{M}_\xi \iff \mathcal{M}_\eta = \mathcal{M}_\xi \iff \eta = \xi.$$

The implications from right to left are evident. Suppose $h\colon \mathcal{M}_\eta \to \mathcal{M}_\xi$ is an isomorphism. Since it preserves relations, the restrictions send bijectively the λ-levels to some other λ'-levels:

$$h \restriction \{(\lambda, f_\eta(\lambda))\} \times [\{\alpha\} \cup \{\beta\} \times \lambda] \to \{(\lambda', f_\eta(\lambda'))\} \times [\{\alpha'\} \cup \{\beta'\} \times \lambda']$$

is a bijection which implies $\lambda = \lambda'$. Further, by bijectivity, the map $\alpha \mapsto \alpha'$ induced by these restrictions is also bijective (by preservation of relations, pairs are sent to pairs), so this map $\alpha \mapsto \alpha'$ is a bijection between $f_\eta(\lambda)$ and $f_\xi(\lambda)$, thus they are the same cardinal for all λ, i.e. $f_\eta = f_\xi$.

For a model of the form $\mathcal{M}_\eta$ and $\alpha < \kappa$, let

$$\mathcal{M}_{\eta\restriction\alpha} = \bigcup_{\substack{\lambda \in C^\omega_{<\kappa} \\ \lambda < h_0(\alpha)}} \{(\lambda, f_\eta(\lambda))\} \times [f_\eta(\lambda) \cup f_\eta(\lambda) \times \lambda]$$

equipped with the relation $R^{\mathcal{M}_{\eta\restriction\alpha}} = R^{\mathcal{M}} \cap (\operatorname{dom} \mathcal{M}_{\eta\restriction\alpha})^2$.

Let us fix a well ordering of $\operatorname{dom} \mathcal{A}$ for each model $\mathcal{A} \in \operatorname{ran} G$ as follows. If $x, y \in \operatorname{dom} \mathcal{M}_\eta$, then

$$\begin{aligned} x \lessdot y \iff & \quad \operatorname{pr}_1(x) < \operatorname{pr}_1(y) \\ \text{or} & \quad \operatorname{pr}_1(x) = \operatorname{pr}_1(y) \wedge \operatorname{pr}_2(x) < \operatorname{pr}_2(y) \\ \text{or} & \quad \operatorname{pr}_1(x) = \operatorname{pr}_1(y) \wedge \operatorname{pr}_2(x) = \operatorname{pr}_2(y) \wedge \operatorname{pr}_3(x) < \operatorname{pr}_3(y) \end{aligned}$$

Note that in the last case it might happen that there is no third projection of x, in that case define $\operatorname{pr}_3(x)$ to be -1. (If $\operatorname{pr}_3(y)$ were also undefined, then we had $x = y$.) The initial segments with respect to $\lessdot$ are of size less than κ, because $f_\eta(\lambda)$ and λ are elements of $C_{<\kappa}$ and $\lessdot$ is clearly a well ordering. Moreover, since we added the + in the definition of $f_\eta(\lambda)$, we have that $\forall\lambda\forall\eta(f_\eta(\lambda) > 0)$, so we get the following:

($\star\star$) Suppose x is the γ^{th} element of the model with respect to $\lessdot$. Then $\operatorname{pr}_1(x) \leqslant \gamma$. Hence for any η

$$\begin{aligned} & \mathcal{M}_\eta \cap \{x \in \operatorname{dom} \mathcal{M}_\eta \mid \operatorname{OTP}_{\lessdot}(x) < \gamma\} \\ \subset \; & \mathcal{M}_{\eta\restriction(\gamma+1)} \end{aligned}$$

Note also that if $\mathcal{M}_{\eta\restriction\alpha} = \mathcal{M}_{\xi\restriction\alpha}$, then the identity map $\mathrm{id}\colon \mathcal{M}_{\eta\restriction\alpha} = \mathcal{M}_{\xi\restriction\alpha}$ preserves $\lessdot$.

Recall the coding $\eta \mapsto \mathcal{A}_\eta$ of the Definition 11. In the definition it is assumed that $\operatorname{dom}\mathcal{A} = \kappa$, but instead of that we can use the well-ordering $\lessdot$. More precisely, for a given model $\mathcal{A}$, let $c(\mathcal{A})$ denote some η such that there is an isomorphism $f\colon \mathcal{A}_\eta \cong \mathcal{A}$ which preserves the ordering of the domain: $f(\alpha)$ is the α^{th} element of $\operatorname{dom}\mathcal{A}$ with respect to $\lessdot$. In our present case, $c\colon \operatorname{ran} G \to \kappa^\kappa$.

Let us show that the map $F = c \circ G\colon \eta \mapsto c(\mathcal{M}_\eta)$ is continuous and therefore is the intended bijection. For that purpose let us equip $\operatorname{ran} G$ with a topology τ. We will then show that G is continuous with respect to that topology and then show that also c is continuous.

Let τ be the topology on $\operatorname{ran} G$ generated by

$$U_p = \{\mathcal{M}_\eta \mid p \subset \eta\}$$

for $p \in \kappa^{<\kappa}$. In fact τ is the topology co-induced by G, so it trivially makes G continuous:

$$G^{-1}U_p = N_p.$$

Let us show that

$$U_p = \{\mathcal{M} \in \operatorname{ran} G \mid \mathcal{M}_p \subset \mathcal{M}\}. \tag{$\star\star\star$}$$

Suppose $\mathcal{M}_\eta \in U_p$ for some η. This is equivalent to that there is ξ with $p \subset \xi$ such that $\mathcal{M}_\eta = \mathcal{M}_\xi$. This in turn is equivalent with $p \subset \eta$, since necessarily $\eta = \xi$. So $\mathcal{M}_\eta \in U_p$ implies

$$\begin{aligned}
\mathcal{M}_p &= \mathcal{M}_{\eta\restriction\operatorname{dom} p}\\
&= \mathcal{M}_\eta \cap \bigcup_{\substack{\lambda\in C^\omega_{<\kappa}\\ \lambda<h_0(\operatorname{dom} p)}} \{\lambda\}\times[f_\eta(\lambda)\cup f_\eta(\lambda)\times\lambda]\\
&\subset \mathcal{M}_\eta.
\end{aligned}$$

Assume that $\mathcal{M} \in \operatorname{ran} G$, $\mathcal{M}_p \subset \mathcal{M}$ and that η is such that $\mathcal{M} = \mathcal{M}_\eta$. Let us assume that ξ is such that $p \subset \xi$ and let us show that $\xi\restriction\operatorname{dom} p \subset \eta$. Let $\lambda < h_0(\operatorname{dom} p)$. Then because $f_\xi(\lambda) > 0$, we have

$$(\lambda, f_\xi(\lambda), 0) \in \mathcal{M}_p.$$

By the assumption $\mathcal{M}_p \subset \mathcal{M}_\eta$, this implies $(\lambda, f_\xi(\lambda), 0) \in \mathcal{M}_\eta$. By definition, this can only happen if $f_\eta(\lambda) = f_\xi(\lambda)$. Thus for all $\lambda < h_0(\operatorname{dom} p)$, we have

$f_\eta(\lambda) = f_\xi(\lambda)$. Recall that h_1 and h_0 are an increasing bijections, so

$$\begin{aligned} & [\forall\lambda < h_0(\operatorname{dom} p)](f_\eta(\lambda) = f_\xi(\lambda)) \\ \iff & [\forall\lambda < h_0(\operatorname{dom} p)]((h_1 \circ \eta \circ h_0^{-1})(\lambda) = (h_1 \circ \xi \circ h_0^{-1})(\lambda)) \\ \iff & [\forall\alpha < \operatorname{dom} p]((h_1 \circ \eta)(\alpha) = (h_1 \circ \xi)(\alpha)) \\ \iff & [\forall\alpha < \operatorname{dom} p](\eta(\alpha) = \xi(\alpha)) \\ \iff & [\forall\alpha < \operatorname{dom} p](\eta(\alpha) = p(\alpha)) \end{aligned}$$

$\Rightarrow p \subset \eta$.

Consider now the coding $c\colon \operatorname{ran} G \to \kappa^\kappa$. Let $N_{\xi\restriction\alpha}$ be a basic open set of κ^κ. Let $\mathcal{M}$ be a model in $c^{-1}N_{\xi\restriction\alpha}$. Let us show that there is an open τ-neighbourhood of $\mathcal{M}$ inside $c^{-1}N_{\xi\restriction\alpha}$. We know that $\xi \restriction \alpha$ decides a segment of $\mathcal{M}$ that is below γ^{th} element with respect to $\lessdot$, for some γ. Denote that segment by $S \subset \mathcal{M}$. Let η be such that $\mathcal{M} = \mathcal{M}_\eta$. From $(\star\star)$ we have:

$$\begin{aligned} S &\subset \mathcal{M}_\eta \cap \{x \in \operatorname{dom} \mathcal{M}_\eta \mid \operatorname{OTP}_\lessdot(x) < \gamma\} \\ &\subset \mathcal{M}_{\eta\restriction(\gamma+1)} \end{aligned}$$

Let us show that $U_{\eta\restriction(\gamma+1)}$ is an open neighbourhood of $\mathcal{M}$ inside $c^{-1}[N_{\xi\restriction\alpha}]$. Suppose $\mathcal{M} \in U_{\eta\restriction(\gamma+1)}$ and $c(\mathcal{M}) = \zeta$. Then by $(\star\star\star)$ we have $\mathcal{M}_{\eta\restriction(\gamma+1)} \subset \mathcal{M}$. Let $S' \subset \mathcal{M}$ be the subset of $\mathcal{M}$ decided by $\zeta \restriction \alpha$. Thus

$$\{\operatorname{OTP}_\lessdot(x) \mid x \in S'\} = \{\operatorname{OTP}_\lessdot(x) \mid x \in S\},$$

but by the note after $(\star\star)$ we have $S = S'$ and since $S \subset \mathcal{M}_{\eta\restriction(\gamma+1)}$ and $\mathcal{M}_{\eta\restriction(\gamma+1)} = \mathcal{M}_{\zeta\restriction(\gamma+1)}$, the codings must coincide and we have $\zeta \restriction \alpha = \xi \restriction \alpha$, i.e. $c(\mathcal{M}) \in N_{\xi\restriction\alpha}$. □

IV.2.3. Failure of Silver's Dichotomy

There are well-known dichotomy theorems for Borel equivalence relations on 2^ω. Two of them are:

33. THEOREM (Silver, [**30**]). *Let $E \subset 2^\omega \times 2^\omega$ be a Π^1_1 equivalence relation. If E has uncountably many equivalence classes, then $\mathrm{id}_{2^\omega} \leqslant_B E$.* □

34. THEOREM (Generalized Glimm-Effros dichotomy, [**5**]). *Let $E \subset 2^\omega \times 2^\omega$ be a Borel equivalence relation. Then either $E \leqslant_B \mathrm{id}_{2^\omega}$ or else $E_0 \leqslant_c E$.* □

NOTATION. Let $\mathcal{C} \in \{\text{Borel}, \Delta^1_1, \text{Borel}^*, \Sigma^1_1, \Pi^1_1\}$.

By *κ-SD for $\mathcal{C}$* we mean the statement that there are no equivalence relations E in the class $\mathcal{C}$ such that $E \subset 2^\kappa \times 2^\kappa$ and E has more than κ equivalence classes such that $\mathrm{id} \not\leqslant_B E$, $\mathrm{id} = \mathrm{id}_{2^\kappa}$.

Similarly *κ-PSP for $\mathcal{C}$* means the Perfect Set Property, i.e. that each member A of $\mathcal{C}$ has either size $\leqslant \kappa$ or there is a Borel injection $2^\kappa \to A$. Using Lemma 44 it is not hard to see that this definition is equivalent to the game definition given in [**22**].

As in the case $\kappa = \omega$ we have the following also for uncountable κ:

35. Theorem. *If κ-SD for Π^1_1 holds, then the κ-PSP holds for Σ^1_1-sets. More generally, if $\mathcal{C} \in \{\mathrm{Borel}, \Delta^1_1, \mathrm{Borel}^*, \Sigma^1_1, \Pi^1_1\}$, then κ-SD for $\mathcal{C}$ implies κ-PSP for $\mathcal{C}'$, where elements in $\mathcal{C}'$ are all the complements of those in $\mathcal{C}$.*

Proof. Let us prove this for $\mathcal{C} = \Pi^1_1$, the other cases are similar. Suppose we have a Σ^1_1-set A. Let

$$E = \{(\eta, \xi) \mid \eta = \xi \text{ or } ((\eta \notin A) \wedge (\xi \notin A))\}.$$

Now $E = \mathrm{id} \cup (2^\kappa \setminus A)^2$. Since A is Σ^1_1, $(2^\kappa \setminus A)^2$ is Π^1_1 and because id is Borel, also E is Π^1_1. Obviously $|A|$ is the number of equivalence classes of E provided A is infinite. Then suppose $|A| > \kappa$. Then there are more than κ equivalence classes of E, so by κ-SD for Π^1_1, there is a reduction $f \colon \mathrm{id} \leqslant E$. This reduction in fact witnesses the PSP of A. □

The idea of using Kurepa trees for this purpose arised already in the paper [**22**] by Mekler and Väänänen.

36. Definition. If $t \subset 2^{<\kappa}$ is a tree, a *path* through t is a branch of length κ. A *κ-Kurepa tree* is a tree $K \subset 2^{<\kappa}$ which satisfies the following:

(a) K has more than κ paths,
(b) K is downward closed,
(c) for all $\alpha < \kappa$, the levels are small: $|\{p \in K \mid \mathrm{dom}\, p = \alpha\}| < \kappa$.

37. Theorem. *Assume one of the following:*

(1) κ is regular but not strongly inaccessible and there exists a κ-Kurepa tree $K \subset 2^{<\kappa}$,
(2) κ is regular (might be strongly inaccessible), $2^\kappa > \kappa^+$ and there exists a tree $K \subset 2^{<\kappa}$ with less than 2^κ branches.

Then Silver dichotomy for κ does not hold. In fact there an equivalence relation $E \subset 2^\kappa \times 2^\kappa$ which is the union of a closed and an open set, has more than κ equivalence classes but $\mathrm{id}_{2^\kappa} \not\leqslant_B E$.

Proof. Let us break the proof according to the assumptions (1) and (2). So first let us consider the case where κ is not strongly inaccessible and there is a κ-Kurepa tree.

(1): Let us carry out the proof in the case $\kappa = \omega_1$. It should be obvious then how to generalize it to any κ not strongly inaccessible. So let $K \subset 2^{<\omega_1}$ be an ω_1-Kurepa tree. Let P be the collection of all paths of K. For $b \in P$, denote $b = \{b_\alpha \mid \alpha < \omega_1\}$ where b_α is an element of K with domain α. Let

$$C = \{\eta \in 2^{\omega_1} \mid \eta = \bigcup_{\alpha<\omega_1} b_\alpha, b \in P\}.$$

Clearly C is closed. Let $E = \{(\eta, \xi) \mid (\eta \notin C \land \xi \notin C) \lor (\eta \in C \land \eta = \xi)\}$. In words, E is the equivalence relation whose equivalence classes are the complement of C and the singletons formed by the elements of C. E is the union of the open set $\{(\eta, \xi) \mid \eta \notin C \land \xi \notin C\}$ and the closed set $\{(\eta, \xi) \mid \eta \in C \land \eta = \xi\} = \{(\eta, \eta) \mid \eta \in C\}$. The number of equivalence classes equals the number of paths of K, so there are more than ω_1 of them by the definition of Kurepa tree.

Let us show that $\mathrm{id}_{2^{\omega_1}}$ is not embeddable to E. Suppose that $f\colon 2^{\omega_1} \to 2^{\omega_1}$ is a Borel reduction. We will show that then K must have a level of size $\geqslant \omega_1$ which contradicts the definition of Kurepa tree. By Lemma 44 there is a co-meager set D on which $f \restriction D$ is continuous. There is at most one $\eta \in 2^{\omega_1}$ whose image $f(\eta)$ is outside C, so without loss of generality $f[D] \subset C$. Let p be an arbitrary element of K such that $f^{-1}[N_p] \neq \varnothing$. By continuity there is a $q \in 2^{<\omega_1}$ with $f[N_q \cap D] \subset N_p$. Since D is co-meager, there are η and ξ such that $\eta \neq \xi$, $q \subset \eta$ and $q \subset \xi$. Let $\alpha_1 < \omega_1$ and p_0 and p_1 be extensions of p with the properties $p_0 \subset f(\eta)$, $p_1 \subset f(\xi)$, $\alpha_1 = \operatorname{dom} p_0 = \operatorname{dom} p_1$, $f^{-1}[N_{p_0}] \neq \varnothing \neq f^{-1}[N_{p_1}]$ and $N_{p_0} \cap N_{p_1} = \varnothing$. Note that p_0 and p_1 are in K. Then, again by continuity, there are q_0 and q_1 such that $f[N_{q_0} \cap D] \subset N_{p_0}$ and $f[N_{q_1} \cap D] \subset N_{p_1}$. Continue in the same manner to obtain α_n and $p_s \in K$ for each $n < \omega$ and $s \in 2^{<\omega}$ so that $s \subset s' \iff p_s \subset p_{s'}$ and $\alpha_n = \operatorname{dom} p_s \iff n = \operatorname{dom} s$. Let $\alpha = \sup_{n<\omega} \alpha_n$. Now clearly the α's level of K contains continuum many elements: by (b) in the definition of Kurepa tree it contains all the elements of the form $\bigcup_{n<\omega} p_{\eta\restriction n}$ for $\eta \in 2^\omega$ and $2^\omega \geqslant \omega_1$.

If κ is arbitrary regular not strongly inaccessible cardinal, then the proof is the same, only instead of ω steps one has to do λ steps where λ is the least cardinal satisfying $2^\lambda \geqslant \kappa$.

(2): The argument is even simpler. Define the equivalence relation E exactly as above. Now E is again closed and has as many equivalence classes as is the number of paths in K. Thus the number of equivalence classes is $> \kappa$ but id cannot be reduced to E since there are less than 2^κ equivalence classes. $\square$

Remark. Some related results:

1. In L, the PSP fails for closed sets for all uncountable regular κ,
2. (P. Schlicht) In Silver's model where an inaccessible κ is made into ω_2 by Levy collapsing each ordinal below to ω_1 with countable conditions, every Σ_1^1 subset X of 2^{ω_1} obeys the PSP.
3. Supercompactness does not imply the PSP for closed sets.

38. Corollary. *Consistency of Silver dichotomy for Borel sets on ω_1 with CH implies the consistency of a strongly inaccessible cardinal. In fact, if there is no equivalence relation witnessing the failure of Silver's dichotomy for ω_1, then ω_2 is inaccessible in L.*

Proof. By a result of Silver, if there are no ω_1-Kurepa trees, then ω_2 is inaccessible in L, see Exercise 27.5 in Part III of [**15**]. $\square$

Remark. The article [**22**] says that the consistency of PSP for $\mathcal{C}$, follows from the consistency of an inaccessible cardinal, where $\mathcal{C}$ is the class of all sets definable with parameters in 2^κ. Unfortunately the authors of this paper could not follow the proof.

IV.3. Regularity Properties and Definability of the CUB Filter

In the standard descriptive theory ($\kappa = \omega$), the notions of Borel, Δ_1^1 and Borel* coincide and one of the most important observations in the theory is that such sets have Property of Baire and that the Σ_1^1-sets obey the Perfect Set Property. In the case $\kappa > \omega$ the situation is more complicated as the following shows. It was already pointed out in the previous section that Borel $\subsetneq \Delta_1^1$. In this section we focus on the cub filter

$$\mathrm{CUB} = \{\eta \in 2^\kappa \mid \eta^{-1}\{1\} \text{ contains a cub}\}.$$

The set CUB is easily seen to be Σ_1^1: the set

$$\{(\eta, \xi) \mid (\eta^{-1}\{1\} \subset \xi^{-1}\{1\}) \wedge (\eta^{-1}\{1\} \text{ is cub})\}$$

is Borel. CUB will serve (consistently) as a counter example to $\Delta_1^1 = \text{Borel}^*$, but we will show that it is also consistent that the cub filter is Δ_1^1. The latter implies that is consistent that Δ_1^1-sets do not have Property of Baire and we will also show that in a forcing extension of L, Δ_1^1-sets all have Property Baire.

39. Definition. A *nowhere dense set* is a subset of a set whose complement is dense and open. Let $X \subset \kappa^\kappa$. A subset $M \subset X$ is *κ-meager in X*, if $M \cap X$ is the union of no more than κ nowhere dense sets,

$$M = \bigcup_{i<\kappa} N_i.$$

We usually drop the prefix "κ-".

Clearly κ-meager sets form a κ-complete ideal. A *co-meager* set is a set whose complement is meager.

A subset $A \subset X$ *has Property of Baire* or shorter *P.B.*, if there exists an open $U \subset X$ such that the symmetric difference $U \mathbin{\triangle} A$ is meager.

Halko showed in [**3**] that Borel sets have Property of Baire (the same proof as when $\kappa = \omega$ works). This is independent of the assumption $\kappa^{<\kappa} = \kappa$. Borel* sets do not in general have Property of Baire.

40. Definition ([**21, 22, 10**]). A $\kappa^+\kappa$-tree t is a *$\kappa\lambda$-canary tree* if for all stationary $S \subset S^\kappa_\lambda$ it holds that if $\mathbb{P}$ does not add subsets of κ of size less than κ and $\mathbb{P}$ kills the stationarity of S, then $\mathbb{P}$ adds a κ-branch to t.

Remark. Hyttinen and Rautila [**10**] use notation *κ-canary tree* for our *$\kappa^+\kappa$-canary tree.*

It was shown by Mekler and Shelah [**21**] and Hyttinen and Rautila [**10**] that it is consistent with ZFC+GCH that there is a $\kappa^+\kappa$-canary tree *and* it is consistent with ZFC+GCH that there are no $\kappa^+\kappa$-canary trees. The same proof as in [**21, 10**] gives the following:

41. Theorem. *Assume GCH and assume $\lambda < \kappa$ are regular cardinals. Let $\mathbb{P}$ be the forcing which adds κ^+ Cohen subsets of κ. Then in the forcing extension there are no $\kappa\lambda$-canary trees.* □

Suppose $X \subset \kappa$ is stationary. For each such X define the set

$$\text{CUB}(X) = \{\eta \in 2^\kappa \mid X \setminus \eta^{-1}\{1\} \text{ is non-stationary}\},$$

so CUB(X) is "cub in X".

42. Theorem. *In the following κ satisfies $\kappa^{<\kappa} = \kappa > \omega$.*

(1) $\mathrm{CUB}(S^\kappa_\omega)$ *is Borel*.*
(2) For all regular $\lambda < \kappa$, $\mathrm{CUB}(S^\kappa_\lambda)$ *is not Δ^1_1 in the forcing extension after adding κ^+ Cohen subsets of κ.*
(3) Assume GCH and that κ is not a successor of a singular cardinal. For any stationary set $S \subset \kappa$ there exists a forcing notion $\mathbb{P}$ which has κ^+-c.c., does not add small subsets of κ and preserves GCH and stationary subsets of $\kappa \setminus S$ such that $\mathrm{CUB}(\kappa \setminus S)$ *is Δ^1_1 in the forcing extension.*
(4) Let the assumprions for κ be as in (3). For all regular $\lambda < \kappa$, $\mathrm{CUB}(S^\kappa_\lambda)$ *is Δ^1_1 in the forcing extension after adding κ^+ Cohen subsets of κ.*
(5) (Halko-Shelah) $\mathrm{CUB}(\kappa)$ *does not have Property of Baire.*
(6) (Independently known to P. Lücke and P. Schlicht.) It is consistent that all Δ^1_1-sets have Property of Baire.

Proof of Theorem 42.

Proof of item (1)**.** Let $t = [\kappa]^{<\omega}$ (increasing functions ordered by end extension) and for all branches $b \subset t$

$$h(b) = \{\xi \in 2^\kappa \mid \xi(\sup_{n<\omega} b(n)) \neq 0\}.$$

Now if $\kappa \setminus \xi^{-1}\{0\}$ contains an ω-cub set C, then player **II** has a winning strategy in $G(t,h,\xi)$: for her n^{th} move she picks an element $x \in t$ with domain $2n+2$ such that $x(2n+1)$ is in C. Suppose the players picked a branch b in this way. Then the condition $\xi(b(2n+1)) \neq 0$ holds for all $n < \omega$ and because C is cub outside $\xi^{-1}\{0\}$, we have $\xi(\sup_{n<\omega} b(n)) \neq 0$.

Suppose on contrary that $S = \xi^{-1}\{0\}$ is stationary. Let σ be any strategy of player **II**. Let C_σ be the set of ordinals closed under this strategy. It is a cub set, so there is an $\alpha \in C_\sigma \cap S$. Player **I** can now easily play towards this ordinal to force $\xi(b(\omega)) = 0$, so σ cannot be a winning strategy.

$\square_{\text{item (1)}}$

Proof of item (2)**.** It is not hard to see that $\mathrm{CUB}^\kappa_\lambda$ is Δ^1_1 if and only if there exists a $\kappa\lambda$-canary tree. This fact is proved in detail in [**22**] in the case $\kappa = \omega_1$, $\lambda = \omega$ and the proof generalizes easily to any regular uncountable κ along with the assumption $\kappa^{<\kappa} = \kappa$. So the statement follows from Theorem 41. $\square_{\text{item (2)}}$

Proof of item (3)**.** If $X \subset 2^\kappa$ is Δ^1_1, then $\{\eta \in X \mid \eta^{-1}\{1\} \subset \kappa \setminus S\}$ is Δ^1_1, so it is sufficient to show that we can force a set $E \subset S$ which has the claimed property. So we force a set $E \subset S$ such that E is stationary but $E \cap \alpha$ is non-stationary in α for all $\alpha < \kappa$ and $\kappa \setminus E$ is fat. A set is *fat* if its intersection with any cub set contains closed increasing sequences of all order types $< \kappa$.

This can be easily forced with

$$\mathbb{R} = \{p\colon \alpha \to 2 \mid \alpha < \kappa, p^{-1}\{1\} \cap \beta \subset S \text{ is non-stationary in } \beta \text{ for all } \beta \leqslant \alpha\}$$

ordered by end-extension. It is easy to see that for any $\mathbb{R}$-generic G the set $E = (\cup G)^{-1}\{1\}$ satisfies the requirements. Also $\mathbb{R}$ does not add small subsets of κ and has κ^+-c.c. and does not kill stationary sets.

Without loss of generality assume that such E exists in V and that $0 \in E$.

Next let $\mathbb{P}_0 = \{p\colon \alpha \to 2^{<\alpha} \mid \alpha < \kappa, p(\beta) \in 2^\beta, p(\beta)^{-1}\{1\} \subset E\}$. This forcing adds a $\diamondsuit$-sequence $\langle A_\alpha \mid \alpha \in E\rangle$ (if G is generic, set $A_\alpha = (\cup G)(\alpha)^{-1}\{1\}$) such that for all $B \subset E$ there is a stationary $S \subset E$ such that $A_\alpha = B \cap \alpha$ for all $\alpha \in S$. This forcing $\mathbb{P}_0$ is $< \kappa$-closed and clearly has κ^+-c.c., so it is easily seen that it does not add small subsets of κ and does not kill stationary sets.

Let $\psi(G, \eta, S)$ be a formula with parameters $G \in (2^{<\kappa})^\kappa$ and $\eta \in 2^\kappa$ and a free variable $S \subset \kappa$ which says:

$$\forall \alpha < \kappa(\alpha \in S \iff G(\alpha)^{-1}\{1\} = \eta^{-1}\{1\}).$$

If $\langle G(\alpha)^{-1}\{1\}\rangle_{\alpha<\kappa}$ happens to be a $\diamondsuit$-sequence, then S satisfying ψ is always stationary. Thus if G_0 is $\mathbb{P}_0$-generic over V and $\eta \in 2^E$, then $(\psi(G_0, \eta, S) \to (S \text{ is stationary}))^{V[G_0]}$.

For each $\eta \in 2^E$, let $\dot{S}_\eta$ be a nice $\mathbb{P}_0$-name for the set S such that $V[G_0] \models \psi(G_0, \eta, S)$ where G_0 is $\mathbb{P}_0$-generic over V. By the definitions, $\mathbb{P}_0 \Vdash$ "$\dot{S}_\eta \subset \check{E}$ is stationary" and if $\eta \neq \eta'$, then $\mathbb{P}_0 \Vdash$ "$\dot{S}_\eta \cap \dot{S}_{\eta'}$ is bounded".

Let us enumerate $E = \{\beta_i \mid i < \kappa\}$ such that $i < j \Rightarrow \beta_i < \beta_j$ and for $\eta \in 2^E$ and $\gamma \in \kappa$ define $\eta + \gamma$ to be the $\xi \in 2^E$ such that $\xi(\beta_i) = 1$ for all $i < \gamma$ and $\xi(\beta_{\gamma+j}) = \eta(\beta_j)$ for $j \geqslant 0$. Let

$$F_0 = \{\eta \in 2^E \mid \eta(0) = 0\}^V \qquad (*)$$

Now for all $\eta, \eta' \in F_0$ and $\alpha, \alpha' \in \kappa$, $\eta + \alpha = \eta' + \alpha'$ implies $\eta = \eta'$ and $\alpha = \alpha'$. Let us now define the formula $\varphi(G, \eta, X)$ with parameters $G \in (2^{<\kappa})^\kappa$, $\eta \in 2^\kappa$ and a free variable $X \subset \kappa \setminus E$ which says:

$$\begin{aligned}(\eta(0) = 0) \wedge \forall \alpha < \kappa[\quad & (\alpha \in X \to \exists S(\psi(G, \eta + 2\alpha, S) \wedge S \text{ is non-stationary})) \\ \wedge \quad & (\alpha \notin X \to \exists S(\psi(G, \eta+2\alpha+1, S) \wedge S \text{ is non-stationary}))].\end{aligned}$$

Now, we will construct an iterated forcing $\mathbb{P}_{\kappa^+}$, starting with $\mathbb{P}_0$, which kills the stationarity of $\dot{S}_\eta$ for suitable $\eta \in 2^E$, such that if G is $\mathbb{P}_{\kappa^+}$-generic, then for all $S \subset \kappa \setminus E$, S is stationary if and only if

$$\exists \eta \in 2^E(\varphi(G_0, \eta, S))$$

where $G_0 = G \restriction \{0\}$. In this model, for each $\eta \in F_0$, there will be a unique X such that $\varphi(G_0, \eta, X)$, so let us denote this X by X_η. It is easy to check that the mapping $\eta \mapsto X_\eta$ defined by φ is Σ^1_1 so in the result, also $\mathcal{S} = \{S \subset \kappa \setminus E \mid S \text{ is stationary}\}$ is Σ^1_1. Since cub and non-stationarity are also Σ^1_1, we get that $\mathcal{S}$ is Δ^1_1, as needed.

Let us show how to construct the iterated forcing. For $S \subset \kappa$, we denote by $T(S)$ the partial order of all closed increasing sequences contained in the complement of S. Clearly $T(S)$ is a forcing that kills the stationarity of S. If the complement of S is fat and S is non-reflecting, then $T(S)$ has all the nice properties we need, as the following claims show. Let $f\colon \kappa^+ \setminus \{0\} \to \kappa^+ \times \kappa^+$ be a bijection such that $f_1(\gamma) \leqslant \gamma$.

$\mathbb{P}_0$ is already defined and it has the κ^+-c.c. and it is $< \kappa$-closed. Suppose that $\mathbb{P}_i$ has been defined for $i < \alpha$ and σ_i has been defined for $i < \cup\alpha$ such that σ_i is a (nice) $\mathbb{P}_i$-name for a κ^+-c.c. partial order. Also suppose that for all $i < \cup\alpha$, $\{(\dot{S}_{ij}, \delta_{ij}) \mid j < \kappa^+\}$ is the list of all pairs $(\dot{S}, \delta)$ such that $\dot{S}$ is a nice $\mathbb{P}_i$-name for a subset of $\check{\kappa} \setminus \check{E}$ and $\delta < \kappa$, and suppose that

$$g_\alpha\colon \{\dot{S}_{f(i)} \mid i < \alpha\} \to F_0 \qquad (***)$$

is an injective function, where F_0 is defined at $(*)$.

If α is a limit, let $\mathbb{P}_\alpha$ consist of those $p\colon \alpha \to \bigcup_{i<\alpha} \operatorname{dom} \sigma_i$ with $|\operatorname{sprt}(p)| < \kappa$ (support, see Section II.1.2 on page 6) such that for all $\gamma < \alpha$, $p \restriction \gamma \in \mathbb{P}_\gamma$ and let $g_\alpha = \bigcup_{i<\alpha} g_i$. Suppose α is a successor, $\alpha = \gamma+1$. Let $\{(\dot{S}_{\gamma j}, \delta_{\gamma j}) \mid j < \kappa\}$ be the the list of pairs as defined above. Let $(\dot{S}, \delta) = (\dot{S}_{f(\gamma)}, \delta_{f(\gamma)})$ where f is the bijection defined above. If there exists $i < \gamma$ such that $\dot{S}_{f(i)} = \dot{S}_{f(\gamma)}$ (i.e. $\dot{S}_i$ has been already under focus), then let $g_\alpha = g_\gamma$. Otherwise let

$$g_\alpha = g_\gamma \cup \{(\dot{S}_{f(\gamma)}, \eta)\}.$$

where η is some element in $F_0 \setminus \operatorname{ran} g_\gamma$. Doing this, we want to make sure that in the end $\operatorname{ran} g_{\kappa^+} = F_0$. We omit the technical details needed to ensure that.

Denote $\eta = g(\dot S_{f(\gamma)})$. Let σ_γ be a $\mathbb{P}_\gamma$-name such that for all $\mathbb{P}_\gamma$-generic G_γ it holds that

$$\mathbb{P}_\gamma \Vdash \begin{cases} \sigma_\gamma = T(\dot S_{\eta+2\delta}), & \text{if } V[G_\gamma] \models [(\delta_{f(\gamma)} \in \dot S_{f(\gamma)}) \wedge (\dot S_{f(\gamma)} \text{ is stationary})] \\ \sigma_\gamma = T(\dot S_{\eta+2\delta+1}), & \text{if } V[G_\gamma] \models [(\delta_{f(\gamma)} \notin \dot S_{f(\gamma)}) \wedge (\dot S_{f(\gamma)} \text{ is stationary})] \\ \sigma_\gamma = \{\check\varnothing\}, & \text{otherwise.} \end{cases}$$

Now let $\mathbb{P}_\alpha$ be the collection of sequences $p = \langle \rho_i \rangle_{i \leqslant \gamma}$ such that $p \restriction \gamma = \langle \rho_i \rangle_{i<\gamma} \in \mathbb{P}_\gamma$, $\rho_\gamma \in \operatorname{dom} \sigma_\gamma$ and $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \rho_\gamma \in \sigma_\gamma$ with the ordering defined in the usual way.

Let G be $\mathbb{P}_{\kappa^+}$-generic. Let us now show that the extension $V[G]$ satisfies what we want, namely that $S \subset \kappa \setminus E$ is stationary if and only if there exists $\eta \in 2^E$ such that $S = X_\eta$ (Claims 3 and 4 below).

Claim 1. For $\alpha \leqslant \kappa^+$ the forcing $\mathbb{P}_\alpha$ does not add small subsets of κ and the suborder

$$\mathbb{Q}_\alpha = \{p \mid p \in \mathbb{P}_\alpha, p = \langle \check\rho_i \rangle_{i<\alpha} \text{ where } \rho_i \in V \text{ for } i < \alpha\}$$

is dense in $\mathbb{P}_\alpha$.

Proof of Claim 1. Let us show this by induction on $\alpha \leqslant \kappa^+$. For $\mathbb{P}_0$ this is already proved and the limit case is left to the reader. Suppose this is proved for all $\gamma < \alpha < \kappa^+$ and $\alpha = \beta + 1$. Then suppose $p \in \mathbb{P}_\alpha$, $p = \langle \rho_i \rangle_{i<\alpha}$. Now $p \restriction \beta \Vdash \rho_\beta \in \sigma_\beta$. Since by the induction hypothesis $\mathbb{P}_\beta$ does not add small sets and $\mathbb{Q}_\beta$ is dense in $\mathbb{P}_\beta$, there exists a condition $r \in \mathbb{Q}_\beta$, $r > p \restriction \beta$ and a standard name $\check q$ such that $r \Vdash \check q = \rho_\beta$. Now $r^\frown(\check q)$ is in $\mathbb{Q}_\alpha$, so it is dense in $\mathbb{P}_\alpha$. To show that $\mathbb{P}_\alpha$ does not add small sets, it is enough to show that $\mathbb{Q}_\alpha$ does not. Let us think of $\mathbb{Q}_\alpha$ as a suborder of the product $\prod_{i<\alpha} 2^{<\kappa}$. Assume that τ is a $\mathbb{Q}_\alpha$-name and $p \in \mathbb{Q}_\alpha$ forces that $|\tau| = \check\lambda < \check\kappa$ for some cardinal λ. Then let $\langle M_\delta \rangle_{\delta<\kappa}$ be a sequence of elementary submodels of $H(\kappa^+)$ such that for all δ, β

(a) $|M_\delta| < \kappa$
(b) $\delta < \beta \Rightarrow M_\delta \preceq M_\beta$,
(c) $M_\delta \cap \kappa \subset M_\delta$,
(d) if β is a limit ordinal, then $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$,
(e) if $\kappa = \lambda^+$, then $M_\delta^{<\lambda} \subset M_\delta$ and if κ is inaccessible, then $M_\delta^{|M_\delta|} \subset M_{\delta+1}$,
(f) $M_\alpha \in M_{\alpha+1}$,
(g) $\{p, \kappa, \mathbb{Q}_\alpha, \tau, \check E\} \subset M_0$.

This (especially (e)) is possible since κ is not a successor of a singular cardinal and GCH holds. Now the set $C = \{M_\delta \cap \kappa \mid \delta < \kappa\}$ is cub, so because $\kappa \setminus E$ is fat, there is a closed sequence s of length $\lambda + 1$ in $C \setminus E$. Let $(\delta_i)_{i \leqslant \lambda}$ be the sequence such that $s = \langle M_{\delta_i} \cap \kappa \rangle_{i \leqslant \lambda}$. For $q \in \mathbb{Q}_\alpha$, let

$$m(q) = \inf_{\gamma \in \operatorname{sprt} q} \operatorname{ran} q(\gamma). \tag{$\star$}$$

Let $p_0 = p$ and for all $i < \gamma$ let $p_{i+1} \in M_{\delta_{i+1}} \setminus M_{\delta_i}$ be such that $p_i < p_{i+1}$, p_{i+1} decides $i+1$ first values of τ (think of τ as a name for a function $\lambda \to \kappa$ and that p_i decides the first i values of that function) and $m(p_{i+1}) \geqslant M_{\delta_i} \cap \kappa$. This p_{i+1} can be found because clearly $p_i \in M_{\delta_{i+1}}$ and $M_{\delta_{i+1}}$ is an elementary submodel. If i is a limit, $i < \lambda$, then let p_i be an upper bound of $\{p_j \mid j < i\}$ which can be found in $M_{\delta_{i+1}}$ by the assumptions (f), (e) and (b), and because $M_{\delta_i} \cap \kappa \notin E$. Finally let p_λ be an upper bound of $\langle p_i \rangle_{i<\lambda}$ which exists because for all $\alpha \in \bigcup_{i<\lambda} \operatorname{sprt} p_i$ $\sup_{i<\lambda} \operatorname{ran} p_i(\alpha) = M_{\delta_\lambda} \cap \kappa$ is not in E and the forcing is closed under such sequences. So p_λ decides the whole τ. This completes the proof of the claim.

$\square_{\text{Claim 1}}$

So for simplicity, instead of $\mathbb{P}_{\kappa^+}$ let us work with $\mathbb{Q}_{\kappa^+}$.

Claim 2. Let G be $\mathbb{P}_{\kappa^+}$-generic over V. Suppose $S \subset \kappa$, $S \in V[G]$ and $\dot{S}$ is a nice name for a subset of κ such that $\dot{S}_G = S$. Then let γ be the smallest ordinal with $S \in V[G_\gamma]$. If $(S \subset \kappa \setminus E \text{ is stationary})^{V[G_\gamma]}$, then S is stationary in $V[G]$. If $\dot{S} = \dot{S}_\eta$ for some $\eta \in V$ and $V[G_\gamma] \models \sigma_\gamma \neq T((\dot{S}_\eta)_{G_\gamma \restriction \{0\}})$ for all $\gamma < \kappa^+$, then S is stationary in $V[G]$.

Proof of Claim 2. Recall, σ_γ is as in the construction of $\mathbb{P}_{\kappa^+}$. Suppose first that $S \subset \kappa \setminus E$ is a stationary set in $V[G_\gamma]$ for some $\gamma < \kappa^+$. Let us show that S is stationary in $V[G]$. Note that $V[G] = V[G_\gamma][G^\gamma]$ where $G^\gamma = G \restriction \{\alpha \mid \alpha \geqslant \gamma\}$. Let us show this in the case $\gamma = 0$ and $S \in V$, the other cases being similar. Let $\dot{C}$ be a name and p a condition which forces that $\dot{C}$ is cub. Let us show that then $p \Vdash \check{S} \cap \dot{C} \neq \check{\varnothing}$. For $q \in \mathbb{Q}_{\kappa^+}$ let $m(q)$ be defined as in $(\star)$ above.

Like in the proof of Claim 1, construct a continuous increasing sequence $\langle M_\alpha \rangle_{\alpha<\kappa}$ of elementary submodels of $H(\kappa^{++})$ such that $\{p, \kappa, \mathbb{P}_{\kappa^+}, \check{S}, \dot{C}\} \subset M_0$ and $M_\alpha \cap \kappa$ is an ordinal. Since $\{M_\alpha \cap \kappa \mid \alpha < \kappa, M_\alpha \cap \kappa = \alpha\}$ is cub, there exists $\alpha \in S$ such that $M_\alpha \cap \kappa = \alpha$ and because E does not reflect to α there exists a cub sequence

$$c \subset \{M_\beta \cap \kappa \mid \beta < \alpha, M_\beta \cap \kappa = \beta\} \setminus E,$$

$c = \langle c_i \rangle_{i<\mathrm{cf}(\alpha)}$. Now, similarly as in the proof of Claim 1, we can choose an increasing $\langle p_i \rangle_{i \leqslant \mathrm{cf}(\alpha)}$ such that $p_0 = p$, $p_i \in \mathbb{Q}_{\kappa^+}$ for all i, $p_{i+1} \Vdash \check{\beta} \in \dot{C}$ for some $c_i \leqslant \beta \leqslant c_{i+1}$, $p_{i+1} \in M_{c_{i+1}} \setminus M_{c_i}$ and $m(p_{i+1}) \geqslant c_i$. If i is a limit, let p_i be again an upper bound of $\{p_j \mid j < i\}$ in M_{c_i}. Since the limits are not in E, the upper bounds exist. Finally $p_{\mathrm{cf}(\alpha)} \Vdash \alpha \in \dot{C}$, which implies $p_{\mathrm{cf}(\alpha)} \Vdash \check{S} \cap \dot{C} \neq \varnothing$, because α was chosen from S.

Assume then that $\dot{S} = \dot{S}_\eta$ for some $\eta \in V$ such that

$$V[G_\gamma] \models \sigma_\gamma \neq T((\dot{S}_\eta)_{G_\gamma \restriction \{0\}})$$

for all $\gamma < \kappa^+$. To prove that $(\dot{S}_\eta)_G$ is stationary in $V[G]$, we carry the same argument as the above, a little modified. Let us work in $V[G_0]$ and let p_0 force that

$$\forall \gamma < \kappa^+ (\sigma_\gamma \neq T(S_\eta)).$$

(This p_0 exists for example because there is at most one γ such that $\sigma_\gamma = T(S_\eta)$) Build the sequences c, $\langle M_{c_i} \rangle_{i<\mathrm{cf}(\alpha)}$ and $\langle p_i \rangle_{i<\mathrm{cf}(\alpha)}$ in the same fashion as above, except that assume additionally that the functions g_{κ^+} and f, defined along with $\mathbb{P}_{\kappa^+}$, are in M_{c_0}.

At the successor steps one has to choose p_{i+1} such that for each $\gamma \in \operatorname{sprt} p_i$, p_{i+1} decides σ_γ. This is possible, since there are only three choices for σ_γ, namely $\{\varnothing\}$, $T(S_{\xi+2\alpha+1})$ or $T(S_{\xi+2\alpha})$ where ξ and α are justified by the functions g_{κ^+} and f. For all $\gamma \in \operatorname{sprt} p_i$ let us denote by ξ_γ the function such that $p_{i+1} \restriction \gamma \Vdash \sigma_\gamma = T(S_{\xi_\gamma})$. Clearly $\eta \neq \xi_\gamma$ for all $\gamma \in \operatorname{sprt} p_i$. Further demand that $m(p_{i+1}) > \sup(S_\eta \cap S_{\xi_\gamma})$ for all $\gamma \in \operatorname{sprt} p_i$. It is possible to find such p_{i+1} from M_{i+1} because M_{i+1} is an elementary submodel and such can be found in $H(\kappa^{++})$ since $\xi_\gamma \neq \eta$ and by the definitions $S_\eta \cap S_{\xi_\gamma}$ is bounded. $\square_{\text{Claim 2}}$

Claim 3. In $V[G]$ the following holds: if $S \subset \kappa \setminus E$ is stationary, then there exists $\eta \in 2^E$ with $\eta(0) = 0$ such that $S = X_\eta$.

Proof of Claim 3. Recall the function g_{κ^+} from the construction of $\mathbb{P}_{\kappa^+}$ (defined at $(***)$ and the paragraph below that). Let $\eta = g_{\kappa^+}(\dot{S})$ where $\dot{S}$ is a nice name $\dot{S} \in V$ such that $\dot{S}_G = S$. If $\alpha \in S$, then there is the smallest γ such that $\dot{S} = S_{f(\gamma)}$ and $\alpha = \delta_{f(\gamma)}$ (where f is as in the definition of $\mathbb{P}_{\kappa^+}$). This stage γ is the only stage where it is possible that $V[G_\gamma] \models \sigma_\gamma = T(S_{\eta+2\alpha+1})$, but since $V[G_\gamma] \models \check{\alpha} \in \dot{S}$, by the definition of $\mathbb{P}_{\kappa^+}$ it is not the case, so the stationarity of $S_{\eta+2\alpha+1}$ has not been killed by Claim 2. On the other hand the stationarity of $S_{\eta+2\alpha}$ is killed at this level

γ of the construction, so $\alpha \in X_\eta$ by the definitions of φ and X_η. Similarly if $\alpha \notin S$, we conclude that $\alpha \notin X_\eta$. $\square_{\text{Claim 3}}$

Claim 4. In $V[G]$ the following holds: if $S \subset \kappa \setminus E$ is not stationary, then for all $\eta \in 2^E$ with $\eta(0) = 0$ we have $S \neq X_\eta$.

Proof of Claim 4. It is sufficient to show that X_η is stationary for all $\eta \in 2^E$ with $\eta(0) = 0$. Suppose first that $\eta \in F_0 \subset V$. Then since g_{κ^+} is a surjection onto F_0 (see $(*\,*\,*)$), there exists a name $\dot{S}$ such that $S = \dot{S}_G$ is stationary, $S \subset \kappa \setminus E$ and $g_{\kappa^+}(S) = \eta$. Now the same argument as in the proof of Claim 3 implies that $X_\eta = S$, so X_η is stationary by Claim 2.

If $\eta \notin F_0$, then by the definition of $\eta \mapsto X_\eta$ it is sufficient to show that the $\diamondsuit$-sequence added by $\mathbb{P}_0$ guesses in $V[G]$ every new set on a stationary set.

Suppose that τ and $\dot{C}$ are nice $\mathbb{P}_{\kappa^+}$-names for subsets of $\check{\kappa}$ and let p be a condition forcing that $\dot{C}$ is cub. We want to find γ and $q > p$ such that

$$q \Vdash ((\cup \dot{G}_0)(\check{\gamma})^{-1}\{1\} = \tau \cap \check{\gamma}) \wedge (\check{\gamma} \in \dot{C})$$

where $\dot{G}_0 = \dot{G} \restriction \{0\}$ is the name for the $\mathbb{P}_0$-generic. To do that let $p_0 \geqslant p$ be such that $p_0 \Vdash \tau \notin \mathcal{P}(\check{\kappa})^V$.

Similarly as in the proofs above define a suitable sequence $\langle M_i \rangle_{i<\lambda}$ of elementary submodels, of length $\lambda < \kappa$, where λ is a cofinality of a point in E, such that $\sup_{i<\lambda}(M_i \cap \kappa) = \alpha \in E$ and $M_i \cap \kappa \notin E$ for all $i < \lambda$. Assume also that $p_0 \in M_0$. Suppose $p_i \in M_i$ is defined. Let $p_{i+1} > p_i$ be an element of $M_{i+1} \setminus M_i$ satisfying the following:

1. p_{i+1} decides σ_β for all $\beta \in \operatorname{sprt} p_i$,
2. for all $\beta \in \operatorname{sprt} p_i$ there is $\beta' \in M_{i+1}$ such that $p_{i+1} \Vdash \beta' \in \tau \triangle \xi_\beta$, where ξ_β is defined as in the proof of Claim 2 and p_{i+1} decides what it is,
3. p_{i+1} decides τ up to $M_i \cap \kappa$,
4. $p_{i+1} \Vdash \delta \in \dot{C}$ for some $\delta \in M_{i+1} \setminus M_i$,
5. $m(p_{i+1}) > M_i \cap \kappa$, ($m(p)$ is defined at $(\star)$),

Item (1) is possible for the same reason as in the proof of Claim 2 and (2) is possible since $p_i \Vdash \forall \eta \in \mathcal{P}(\check{\kappa})^V (\tau \neq S_{\check{\eta}})$.

Since $M_i \cap \kappa \notin E$ for $i < \lambda$, this ensures that the sequence $p_0 \leqslant p_1 \leqslant \dots$ closes under limits $< \lambda$. Let $p_\lambda = \bigcup_{i<\lambda} p_i$ and let us define $q \supset p_\lambda$ as follows: $\operatorname{sprt} q = \operatorname{sprt} p_\lambda$, for $\delta \in \operatorname{sprt} p_\lambda \setminus \{0\}$ let $\operatorname{dom} q = \alpha + 1$, $p_\lambda(\delta) \subset q(\delta)$, $q(\alpha) = 1$

and $q(0)(\alpha) = \tau \cap \gamma$ (τ means here what have been decided by $\{p_i \mid i < \lambda\}$). Now q is a condition in the forcing notion.

Now certainly, if $q \in G$, then in the extension $\tau_G \cap \alpha = (\cup G_0)(\alpha)^{-1}\{1\}$ and $\alpha \in C$, so we finish. $\square_{\text{Claim 4}}$

$\square_{\text{item (3)}}$

Proof of item (4)**.** If $\kappa = \lambda^+$, this follows from the result of Mekler and Shelah [**21**] and Hyttinen and Rautila [**10**] that the existence of a $\kappa\lambda$-canary tree is consistent. For arbitrary $\lambda < \kappa$ the result follows from the item 3 of this theorem proved above (take $S = \kappa \setminus S^\kappa_\lambda$). $\square_{\text{item (4)}}$

Proof of item (5)**.** This was proved by Halko and Shelah in [**4**], Theorem 4.2. $\square_{\text{item (5)}}$

Proof of item (6)**.** Our proof is different from that given by Lücke and Schlicht. Suppose $\kappa^{<\kappa} = \kappa > \omega$. We will show that in a generic extension of V all Δ^1_1-sets have Property of Baire. Let

$$\mathbb{P} = \{p \mid p \text{ is a function}, |p| < \kappa, \operatorname{dom} p \subset \kappa \times \kappa^+, \operatorname{ran} p \subset \{0,1\}\}$$

with the ordering $p < q \iff p \subset q$ and let G be $\mathbb{P}$-generic over V. Suppose that $X \subset 2^\kappa$ is a Δ^1_1-set in $V[G]$. It is sufficient to show that for every $r \in 2^{<\kappa}$ there is $q \supset r$ such that either $N_q \setminus X$ or $N_q \cap X$ is co-meager. So let $r \in 2^{<\kappa}$ be arbitrary.

Now suppose that $\langle p_i \rangle_{i<\kappa}$ and $\langle q_i \rangle_{i<\kappa}$ are sequences in $V[G]$ such that $p_i, q_i \in (2^{<\kappa})^2$ for all $i < \kappa$ and X is the projection of

$$C_0 = (2^\kappa)^2 \setminus \bigcup_{i<\kappa} N_{p_i}$$

and $2^\kappa \setminus X$ is the projection of

$$C_1 = (2^\kappa)^2 \setminus \bigcup_{i<\kappa} N_{q_i}.$$

(By N_{p_i} we mean $N_{p_i^1} \times N_{p_i^2}$ where $p_i = (p_i^1, p_i^2)$.) Since these sequences have size κ, there exists $\alpha_1 < \kappa^+$ such that they are already in $V[G_{\alpha_1}]$ where $G_{\alpha_1} = \{p \in G \mid \operatorname{dom} p \subset \kappa \times \alpha_1\}$. More generally, for $E \subset \mathbb{P}$ and $A \subset \kappa^+$, we will denote $E_A = \{p \in E \mid \operatorname{dom} p \subset \kappa \times A\}$ and if $p \in \mathbb{P}$, similarly $p_A = p \restriction (\kappa \times A)$.

Let $\alpha_2 \geqslant \alpha_1$ be such that $r \in G_{\{\alpha_2\}}$ (identifying $\kappa \times \{\alpha_2\}$ with κ). This is possible since G is generic. Let $x = G_{\{\alpha_2\}}$. Since in $V[G]$, $x \in X$ or

$x \in 2^\kappa \setminus X$, there are $\alpha_3 > \alpha_2$, $p \in G_{\alpha_3}$, $p_{\{\alpha_2\}} \supset r$ and a name τ such that p forces that $(x, \tau) \notin N_{p_i}$ for all $i < \kappa$ or $(x, \tau) \notin N_{q_i}$ for all $i < \kappa$. Without loss of generality assume that p forces that $(x, \tau) \notin N_{p_i}$ for all $i < \kappa$. Also we can assume that τ is a $\mathbb{P}_{\alpha_3}$-name and that $\alpha_3 = \alpha_2 + 2$.

By working in $V[G_{\alpha_2}]$ we may assume that $\alpha_2 = 0$. For all $q \in \mathbb{P}_{\{1\}}$, $p_{\{1\}} \subseteq q$ and $i < \kappa$, let $D_{i,q}$ be the set of all $s \in \mathbb{P}_{\{0\}}$ such that $p_{\{0\}} \subseteq s$, $\mathrm{dom}(s) \geqslant \mathrm{dom}(p_i^1)$ and there is $q' \in \mathbb{P}_{\{1\}}$ such that $q \subseteq q'$ and $s \cup q'$ decides $\tau \restriction \mathrm{dom}(p_i^2)$. Clearly each $D_{i,q}$ is dense above $p_{\{0\}}$ in $\mathbb{P}_{\{0\}}$ and thus it is enough to show that if $y \in 2^\kappa$ is such that for all $i < \kappa$ and q as above there is $\alpha < \kappa$ such that $y \restriction \alpha \in D_{i,q}$, then $y \in X$.

So let y be such. Then we can find $z \in 2^\kappa$ such that for all $i < \kappa$ and q as above there are $\alpha, \beta < \kappa$ such that $\alpha \geqslant \mathrm{dom}(p_i^1)$ and $y \restriction \alpha \cup z \restriction \beta$ decides $t = \tau \restriction \mathrm{dom}(p_i^2)$. By the choise of p, $(y \restriction \mathrm{dom}(p_i^1), t) \neq p_i$. Thus letting τ^* be the function as decided by y and z, $(y, \tau^*) \in C_0$ and thus $y \in X$. $\square_{\text{item (6)}}$

$\square_{\text{Theorem 42}}$

43. Corollary. *For a regular $\lambda < \kappa$ let NS_λ denote the equivalence relation on 2^κ such that $\eta \mathrm{NS}_\lambda \xi$ if and only if $\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}$ is not λ-stationary. Then NS_λ is not Borel and it is not Δ_1^1 in the forcing extension after adding κ^+ Cohen subsets of κ.*

Proof. Define a map $f \colon 2^\kappa \to (2^\kappa)^2$ by $\eta \mapsto (\varnothing, \kappa \setminus \eta)$. Suppose for a contradiction that NS_λ is Borel. Then

$$\mathrm{NS}_\varnothing = \mathrm{NS}_\lambda \cap \underbrace{\{(\varnothing, \eta) \mid \eta \in 2^\kappa\}}_{\text{closed}}$$

is Borel, and further $f^{-1}[\mathrm{NS}_\varnothing]$ is Borel by continuity of f. But $f^{-1}[\mathrm{NS}_\varnothing]$ equals CUB which is not Borel by Theorem 42 (4). Similarly using (2) of Theorem 42 one can show that if there is no canary tree, then NS_λ is not Δ_1^1 and the statement follows from Theorem 41. $\square$

Remark. There are some more results and strengthenings of the results in Theorem 42:

(1) In L, the CUB filter on κ is not Δ_1^1.
(2) (Independently known by S. Coskey and P. Schlicht) In L there is a Δ_1^1 well order of $P(\kappa)$ and this implies that there is a Δ_1^1 set without the BP.

Finally let us state the following standard lemma which will be referred later on in the text:

44. Lemma. *Borel functions are continuous on a co-meager set.*

Proof. For each $\eta \in \kappa^{<\kappa}$ let V_η be an open subset of κ^κ such that $V_\eta \triangle f^{-1}N_\eta$ is meager. Let

$$D = \kappa^\kappa \setminus \bigcup_{\eta \in \kappa^{<\kappa}} V_\eta \triangle f^{-1}N_\eta.$$

Then D is as intended. Clearly it is co-meager, since we took away only a κ-union of meager sets. Let $\xi \in \kappa^{<\kappa}$ be arbitrary. The set $D \cap f^{-1}N_\xi$ is open in D since $D \cap f^{-1}N_\xi = D \cap V_\xi$ and so $f \restriction D$ is continuous. □

IV.4. The Partial Orders $\langle \mathcal{E}, \leqslant_{\mathcal{C}} \rangle$

Let $\mathcal{C} \in \{\text{Borel}, \Delta_1^1, \text{Borel}^*, \Sigma_1^1\}$ and define

$$\mathcal{E}_\kappa^{\mathcal{C}} = \{E \subset 2^\kappa \times 2^\kappa \mid (E \in \mathcal{C}) \wedge (E \text{ is an equivalence relation})\}.$$

Equip $\mathcal{E}_\kappa^{\mathcal{C}}$ with the partial order $\leqslant_B$. In the case $\kappa = \omega$ there are many known results that describe the order $\langle \mathcal{E}_\kappa^{\text{Borel}}, \leqslant_B \rangle$. Some results were discussed in Section IV.2.3, some other results show that this order is very complicated. To mention two:

45. Theorem (Louveau-Velickovic). *The partial order $\langle \mathcal{P}(\omega), \subset_* \rangle$ can be embedded into $\langle \mathcal{E}_\omega^{\text{Borel}}, \leqslant_B \rangle$, where $A \subset_* B$ if $A \setminus B$ is finite.*

46. Theorem (Adams-Kechris [**1**]). *The partial order $\langle \mathcal{B}, \subset \rangle$ can be embedded into $\langle \mathcal{E}_\omega^{\text{Borel}}, \leqslant_B \rangle$, where $\mathcal{B}$ is the collection of all Borel subsets of the real line $\mathbb{R}$. In fact, the embedding is into the suborder of $\langle \mathcal{E}_\omega^{\text{Borel}}, \leqslant_B \rangle$ consisting of the countable Borel equivalence relations, i.e., those Borel equivalence relations each of whose equivalence classes is countable.*

We aim to prove the following weak version of such a theorem for $\kappa > \omega$:

47. Theorem. *Suppose GCH and kappa is regular, uncountable. Then in a cofinality and GCH preserving forcing extension, there is an embedding*

$$\langle \mathcal{P}(\kappa), \subset \rangle \to \langle \mathcal{E}^{Borel^*}, \leqslant_B \rangle.$$

Proof. In this proof we identify functions $\eta \in 2^{\leqslant \kappa}$ with the sets $\eta^{-1}\{1\}$: for example we write $\eta \cap \xi$ to mean $\eta^{-1}\{1\} \cap \xi^{-1}\{1\}$.

For $X \subset S_\mu^\kappa$, we denote by E_X the relation

$$E_X = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}) \cap X \text{ is not stationary}\}.$$

This relation is easily seen to be Σ_1^1. If $\mu = \omega$, then it is in fact Borel*. To see this use the same argument as in the proof of Theorem 42 (1) that the $\mathrm{CUB}_\omega^\kappa$ set is Borel*. So, we will carry out the argument for an almost arbitrary regular cardinal $\mu < \kappa$ where "almost" means: if $\kappa = \lambda^+$ and λ is singular, then we demand that $\mu \leqslant \mathrm{cf}(\lambda)$. The statement of the theorem then follows putting $\mu = \omega$.

The embedding will look as follows. Let $(S_i)_{i<\kappa}$ be pairwise disjoint stationary subsets of

$$\lim S_\mu^\kappa = \{\alpha \in S_\mu^\kappa \mid \alpha \text{ is a limit of ordinals in } S_\mu^\kappa\}. \tag{$*$}$$

Denote

$$K(A) = E_{\bigcup\limits_{\alpha \in A} S_\alpha}. \tag{$**$}$$

We intend that $A \mapsto K(A)$ is the embedding. If $X_1 \subset X_2 \subset \kappa$, then $E_{X_1} \leqslant_B E_{X_2}$, because $f(\eta) = \eta \cap X_1$ is a reduction. This guarantees that

$$A_1 \subset A_2 \Rightarrow K(A_1) \leqslant_B K(A_2).$$

Now suppose that for all $\alpha < \kappa$ we have killed (by forcing) all reductions from $K(\alpha) = E_{S_\alpha}$ to $K(\kappa \setminus \alpha) = E_{\bigcup_{\beta \neq \alpha} S_\beta}$ for all $\alpha < \kappa$. Then if $K(A_1) \leqslant_B K(A_2)$ it follows that $A_1 \subset A_2$: Otherwise choose $\alpha \in A_1 \setminus A_2$ and we have:

$$K(\alpha) \leqslant_B K(A_1) \leqslant_B K(A_2) \leqslant_B K(\kappa \setminus \alpha),$$

contradiction. So we have:

$$A_1 \subset A_2 \iff K(A_1) \leqslant_B K(A_2)$$

and therefore K is the desired embedding.

Suppose that $f\colon E_X \leqslant_B E_Y$ is a Borel reduction. Then $g\colon 2^\kappa \to 2^\kappa$ defined by $g(\eta) = f(\eta) \triangle f(0)$ is a Borel function with the following property:

$$\eta \cap X \text{ is stationary } \iff g(\eta) \cap Y \text{ is stationary.}$$

The function g is Borel, so by Lemma 44 there are dense open sets D_i for $i < \kappa$ such that $g \restriction D$ is continuous where $D = \bigcap_{i<\kappa} D_i$. Note that D_i are open so for each i we can write $D_i = \bigcup_{j<\kappa} N_{p(i,j)}$, where $(p(i,j))_{j<\kappa}$ is a suitable collection of elements of $2^{<\kappa}$.

Next define $Q_g\colon 2^{<\kappa} \times 2^{<\kappa} \to \{0,1\}$ by $Q_g(p,q) = 1 \iff N_p \cap D \subset g^{-1}[N_q]$ and $R_g\colon \kappa \times \kappa \to 2^{<\kappa}$ by $R_g(i,j) = p(i,j)$ where $p(i,j)$ are as above.

For any $Q\colon 2^{<\kappa} \times 2^{<\kappa} \to \{0,1\}$ define $Q^*\colon 2^\kappa \to 2^\kappa$ by

$$Q^*(\eta) = \begin{cases} \xi, & \text{s.t. } \forall \alpha < \kappa \exists \beta < \kappa Q(\eta \restriction \beta, \xi \restriction \alpha) = 1 \text{ if such exists,} \\ 0, & \text{otherwise.} \end{cases}$$

And for any $R\colon \kappa \times \kappa \to 2^{<\kappa}$ define

$$R^* = \bigcap_{i<\kappa} \bigcup_{j<\kappa} N_{R(i,j)}.$$

Now clearly $R^*_g = D$ and $Q^*_g \restriction D = g \restriction D$, i.e. (Q, D) *codes* $g \restriction D$ in this sense. Thus we have shown that if there is a reduction $E_X \leqslant_B E_Y$, then there is a pair (Q, R) which satisfies the following conditions:

(1) $Q\colon (2^{<\kappa})^2 \to \{0,1\}$ is a function.
(2) $Q(\varnothing, \varnothing) = 1$,
(3) If $Q(p,q) = 1$ and $p' > p$, then $Q(p', q) = 1$,
(4) If $Q(p,q) = 1$ and $q' < q$, then $Q(p, q') = 1$
(5) Suppose $Q(p,q) = 1$ and $\alpha > \operatorname{dom} q$. There exist $q' > q$ and $p' > p$ such that $\operatorname{dom} q' = \alpha$ and $Q(p', q') = 1$,
(6) If $Q(p,q) = Q(p,q') = 1$, then $q \leqslant q'$ or $q' < q$,
(7) $R\colon \kappa \times \kappa \to 2^{<\kappa}$ is a function.
(8) For each $i \in \kappa$ the set $\bigcup_{j<\kappa} N_{R(i,j)}$ is dense.
(9) For all $\eta \in R^*$, $\eta \cap X$ is stationary if and only if $Q^*(\eta \cap X) \cap Y$ is stationary.

Let us call a pair (Q, R) which satisfies (1)–(9) *a code for a reduction (from E_X to E_Y)*. Note that it is not the same as the Borel code for the graph of a reduction function as a set. Thus we have shown that if $E_X \leqslant_B E_Y$, then there exists a code for a reduction from E_X to E_Y. We will now prove the following lemma which is stated in a general enough form so we can use it also in the next section:

48. Lemma (GCH). *Suppose μ_1 and μ_2 are regular cardinals less than κ such that if $\kappa = \lambda^+$, then $\mu_2 \leqslant \operatorname{cf}(\lambda)$, and suppose X is a stationary subset of $S^\kappa_{\mu_1}$, Y is a subset of $S^\kappa_{\mu_2}$, $X \cap Y = \varnothing$ (relevant if $\mu_1 = \mu_2$) and if $\mu_1 < \mu_2$ then $\alpha \cap X$ is not stationary in α for all $\alpha \in Y$. Suppose that (Q, R) is an arbitrary pair. Denote by φ the statement "(Q, R) is not a code for a reduction from E_X to E_Y". Then there is a κ^+-c.c. $< \kappa$-closed forcing $\mathbb{R}$ such that $\mathbb{R} \Vdash \varphi$.*

REMARK. Clearly if $\mu_1 = \mu_2 = \omega$, then the condition $\mu_2 \leqslant \mathrm{cf}(\lambda)$ is of course trivially true. We need this assumption in order to have $\nu^{<\mu_2} < \kappa$ for all $\nu < \kappa$.

Proof of Lemma 48. We will show that one of the following holds:

1. φ already holds, i.e. $\{\varnothing\} \Vdash \varphi$,
2. $\mathbb{P} = 2^{<\kappa} = \{p \colon \alpha \to 2 \mid \alpha < \kappa\} \Vdash \varphi$,
3. $\mathbb{R} \Vdash \varphi$,

where

$$\mathbb{R} = \{(p, q) \mid p, q \in 2^{\alpha}, \alpha < \kappa, X \cap p \cap q = \varnothing, q \text{ is } \mu_1\text{-closed}\}$$

Above "q is μ_1-closed" means "$q^{-1}\{1\}$ is μ_1-closed" etc., and we will use this abbreviation below. Assuming that (1) and (2) do not hold, we will show that (3) holds.

Since (2) does not hold, there is a $p \in \mathbb{P}$ which forces $\neg\varphi$ and so $\mathbb{P}_p = \{q \in \mathbb{P} \mid q > p\} \Vdash \neg\varphi$. But $\mathbb{P}_p \cong \mathbb{P}$, so in fact $\mathbb{P} \Vdash \neg\varphi$, because φ has only standard names as parameters (names for elements in V, such as Q, R, X and Y). Let G be any $\mathbb{P}$-generic and let us denote the set $G^{-1}\{1\}$ also by G. Let us show that $G \cap X$ is stationary. Suppose that $\dot{C}$ is a name and $r \in \mathbb{P}$ is a condition which forces that $\dot{C}$ is cub. For an arbitrary q_0, let us find a $q > q_0$ which forces $\dot{C} \cap \dot{G} \cap \check{X} \neq \varnothing$. Make a counter assumption: no such $q > q_0$ exists. Let $q_1 > q_0$ and $\alpha_1 > \operatorname{dom} q_0$ be such that $q_1 \Vdash \check{\alpha}_1 \in \dot{C}$, $\operatorname{dom} q_1 > \alpha_1$ is a successor and $q_1(\max \operatorname{dom} q_1) = 1$. Then by induction on $i < \kappa$ let q_{i+1} and $\alpha_{i+1} > \operatorname{dom} q_i$ be such that $q_{i+1} \Vdash \check{\alpha}_{i+1} \in \dot{C}$, $\operatorname{dom} q_{i+1} > \alpha_{i+1}$ is a successor and $q_{i+1}(\max \operatorname{dom} q_{i+1}) = 1$. If j is a limit ordinal, let $q_j = \bigcup_{i<j} q_i \cup \{(\sup_{i<j} \operatorname{dom} q_i, 1)\}$ and $\alpha_j = \sup_{i<j} \alpha_i$. We claim that for some $i < \kappa$, the condition q_i is as needed, i.e.

$$q_i \Vdash \dot{G} \cap \check{X} \cap \dot{C} \neq \varnothing.$$

Clearly for limit ordinals j, we have $\alpha_j = \max \operatorname{dom} q_j$ and $q_j(\alpha_j) = 1$ and $\{\alpha_j \mid j \text{ limit}\}$ is cub. Since X is stationary, there exists a limit j_0 such that $\alpha_{j_0} \in X$. Because q_0 forces that $\dot{C}$ is cub, $q_j > q_i > q_0$ for all $i < j$, $q_i \Vdash \check{\alpha}_i \in \dot{C}$ and $\alpha_j = \sup_{i<j} \alpha_i$, we have $q_j \Vdash \alpha_j \in \dot{C} \cap \check{X}$. On the other hand $q_j(\alpha_j) = 1$, so $q_j \Vdash \alpha_j \in G$ so we finish.

So now we have in $V[G]$ that $G \cap X$ is stationary, $G \in R^*$ (since R^* is co-meager) and Q is a code for a reduction, so Q^* has the property (9) and $Q^*(G \cap X) \cap Y$ is stationary. Denote $Z = Q^*(G \cap X) \cap Y$. We will now construct a forcing $\mathbb{Q}$ in $V[G]$ such that

$$V[G] \models (\mathbb{Q} \Vdash \text{“}G \cap X \text{ is not stationary, but } Z \text{ is stationary”}).$$

Then $V[G] \models (\mathbb{Q} \Vdash \varphi)$ and hence $\mathbb{P} * \mathbb{Q} \Vdash \varphi$. On the other hand $\mathbb{Q}$ will be chosen such that $\mathbb{P} * \mathbb{Q}$ and $\mathbb{R}$ give the same generic extensions. So let

$$\mathbb{Q} = \{q \colon \alpha \to 2 \mid X \cap G \cap q = \varnothing, q \text{ is } \mu_1 - \text{closed}\}, \qquad (***)$$

Clearly $\mathbb{Q}$ kills the stationarity of $G \cap X$. Let us show that it preserves the stationarity of Z. For that purpose it is sufficient to show that for any nice $\mathbb{Q}$-name $\dot{C}$ for a subset of κ and any $p \in \mathbb{Q}$, if $p \Vdash$ “$\dot{C}$ is μ_2-cub”, then $p \Vdash (\dot{C} \cap \check{Z} \neq \check{\varnothing})$.

So suppose $\dot{C}$ is a nice name for a subset of κ and $p \in \mathbb{Q}$ is such that

$$p \Vdash \text{“}\dot{C} \text{ is cub”}$$

Let $\lambda > \kappa$ be a sufficiently large regular cardinal and let N be an elementary submodel of $\langle H(\lambda), p, \dot{C}, \mathbb{Q}, \kappa \rangle$ which has the following properties:

- $|N| = \mu_2$
- $N^{<\mu_2} \subset N$
- $\alpha = \sup(N \cap \kappa) \in Z$ (This is possible because Z is stationary).

Now by the assumption of the theorem, $\alpha \backslash X$ contains a μ_1-closed unbounded sequence of length μ_2, $\langle \alpha_i \rangle_{i<\mu_2}$. Let $\langle D_i \rangle_{i<\mu_2}$ list all the dense subsets of $\mathbb{Q}^N$ in N. Let $q_0 \geqslant p$, $q_0 \in \mathbb{Q}^N$ be arbitrary and suppose $q_i \in \mathbb{Q}^N$ is defined for all $i < \gamma$. If $\gamma = \beta + 1$, then define q_γ to be an extension of q_β such that $q_\gamma \in D_\beta$ and $\operatorname{dom} q_\gamma = \alpha_i$ for some $\alpha_i > \operatorname{dom} q_\beta$. To do that, for instance, choose $\alpha_i > \operatorname{dom} q_\beta$ and define $q' \supset q_\beta$ by $\operatorname{dom} q' = \alpha_i$, $q(\delta) = 0$ for all $\delta \in \operatorname{dom} q' \setminus \operatorname{dom} q_\beta$ and then extend q' to q_β in D_β. If γ is a limit ordinal with $\operatorname{cf}(\gamma) \neq \mu_1$, then let $q_\gamma = \bigcup_{i<\gamma} q_i$. If $\operatorname{cf}(\gamma) = \mu_1$, let

$$q_\gamma = \Big(\bigcup_{i<\gamma} q_i\Big)^\frown \langle \sup_{i<\gamma} \operatorname{dom} q_i, 1 \rangle$$

Since N is closed under taking sequences of length less than μ_2, $q_\gamma \in N$. Since we required elements of $\mathbb{Q}$ to be μ_1-closed but not γ-closed if $\operatorname{cf}(\gamma) \neq \mu_1$, $q_\gamma \in \mathbb{Q}$ when $\operatorname{cf}(\gamma) \neq \mu_1$. When $\operatorname{cf}(\gamma) = \mu_1$, the limit $\sup_{i<\gamma} \operatorname{dom} q_i$ coincides with a limit of a subsequence of $\langle \alpha_i \rangle_{i<\mu_2}$ of length μ_1, i.e. the limit is α_β for some β since this sequence is μ_1-closed. So by definition $\sup_{i<\gamma} \operatorname{dom} q_i \notin X$ and again $q_\gamma \in \mathbb{Q}$.

Then $q = \bigcup_{\gamma<\mu} q_\gamma$ is a $\mathbb{Q}^N$-generic over N. Since $X \cap Y = \varnothing$, also $(X \cap G) \cap Z = \varnothing$ and $\alpha \notin X \cap G$. Hence $q^\frown(\alpha, 1)$ is in $\mathbb{Q}$. We claim that $q \Vdash (\dot{C} \cap \check{Z} \neq \varnothing)$.

Because $p \Vdash$ "$\dot{C}$ is unbounded", also $N \models (p \Vdash$ "$\dot{C}$ is unbounded"$)$ by elementarity. Assuming that λ is chosen large enough, we may conclude that for all $\mathbb{Q}^N$-generic g over N, $N[g] \models$ "$\dot{C}_g$ is unbounded", thus in particular $N[g] \models$ "$\dot{C}_g$ is unbounded in κ". Let G_1 be $\mathbb{Q}$-generic over $V[G]$ with $q \in G_1$. Then $\dot{C}_{G_1} \supset \dot{C}_q$ which is unbounded in α by the above, since $\sup(\kappa \cap N) = \alpha$. Because $\dot{C}_{G_1}$ is μ_2-cub, α is in $\dot{C}_{G_1}$.

Thus $\mathbb{P} * \mathbb{Q} \Vdash \varphi$. It follows straightforwardly from the definition of iterated forcing that $\mathbb{R}$ is isomorphic to a dense suborder of $\mathbb{P} * \dot{\mathbb{Q}}$ where $\dot{Q}$ is a $\mathbb{P}$-name for a partial order such that $\dot{\mathbb{Q}}_G$ equals $\mathbb{Q}$ as defined in $(***)$ for any $\mathbb{P}$-generic G.

Now it remains to show that $\mathbb{R}$ has the κ^+-c.c. and is $<\kappa$-closed. Since $\mathbb{R}$ is a suborder of $\mathbb{P}\times\mathbb{P}$, which has size κ, it trivially has the κ^+-c.c. Suppose $(p_i, q_i)_{i<\gamma}$ is an increasing sequence, $\gamma < \kappa$. Then the pair

$$(p,q) = \Big\langle \Big(\bigcup_{i<\gamma} p_i\Big)^\frown \langle \alpha, 0\rangle, \Big(\bigcup_{i<\gamma} q_i\Big)^\frown \langle \alpha, 1\rangle \Big\rangle$$

is an upper bound. $\square_{\text{Lemma 48}}$

49. Corollary (GCH). *Let $K\colon A \mapsto E_{\bigcup_{\alpha\in A} S_\alpha}$ be as in the beginning of the proof. For each pair (Q,R) and each α there is a $<\kappa$-closed, κ^+-c.c. forcing $\mathbb{R}(Q,R,\alpha)$ such that*

$\mathbb{R}(Q,R,\alpha) \Vdash$ *"(Q,R) is not a code for a reduction from $K(\{\alpha\})$ to $K(\kappa\backslash\{\alpha\})$"*

Proof. By the above lemma one of the choices $\mathbb{R} = \{\varnothing\}$, $\mathbb{R} = 2^{<\kappa}$ or

$$\mathbb{R} = \{(p,q) \mid p,q \in 2^\beta, \beta < \kappa, S_\alpha \cap p \cap q = \varnothing, q \text{ is } \mu\text{-closed}\}$$

suffices. $\square$

Start with a model satisfying GCH. Let $h\colon \kappa^+ \to \kappa^+ \times \kappa \times \kappa^+$ be a bijection such that $h_3(\alpha) < \alpha$ for $\alpha > 0$ and $h_3(0) = 0$. Let $\mathbb{P}_0 = \{\varnothing\}$. For each $\alpha < \kappa$, let $\{\sigma_{\beta\alpha 0} \mid \beta < \kappa^+\}$ be the list of all $\mathbb{P}_0$-names for codes for a reduction from $K(\{\alpha\})$ to $K(\kappa \setminus \{\alpha\})$. Suppose $\mathbb{P}_i$ and $\{\sigma_{\beta\alpha i} \mid \beta < \kappa^+\}$ are defined for all $i < \gamma$ and $\alpha < \kappa$, where $\gamma < \kappa^+$ is a successor $\gamma = \beta + 1$, $\mathbb{P}_i$ is $<\kappa$-closed and has κ^+-c.c.

Consider $\sigma_{h(\beta)}$. By the above corollary, the following holds:

$$\begin{aligned}\mathbb{P}_\beta \ \Vdash\ \ & \big[\exists \mathbb{R} \in \mathcal{P}(2^{<\kappa} \times 2^{<\kappa})(\mathbb{R} \text{ is } <\kappa\text{-closed, } \kappa^+\text{-c.c. p.o. and} \\ & \quad \mathbb{R} \Vdash \text{"}\sigma_{h(\beta)} \text{ is not a code for a reduction."})\big]\end{aligned}$$

So there is a $\mathbb{P}_\beta$-name ρ_β such that $\mathbb{P}_\beta$ forces that ρ_β is as $\mathbb{R}$ above. Define

$$\mathbb{P}_\gamma = \{(p_i)_{i<\gamma} \mid ((p_i)_{i<\beta} \in \mathbb{P}_\beta) \wedge ((p_i)_{i<\beta} \Vdash p_\beta \in \rho_\beta)\}.$$

And if $p = (p_i)_{i<\gamma} \in \mathbb{P}_\gamma$ and $p' = (p'_i)_{i<\gamma} \in \mathbb{P}_\gamma$, then

$$p \leqslant_{\mathbb{P}_\gamma} p' \iff [(p_i)_{i<\beta} \leqslant_{\mathbb{P}_\beta} (p'_i)_{i<\beta}] \wedge [(p'_i)_{i<\beta} \Vdash (p_\beta \leqslant_{\rho_\beta} p'_\beta)]$$

If γ is a limit, $\gamma \leqslant \kappa^+$, let

$$\mathbb{P}_\gamma = \{(p_i)_{i<\gamma} \mid \forall\beta(\beta < \gamma \to (p_i)_{i<\beta} \in \mathbb{P}_\beta) \wedge (|\operatorname{sprt}(p_i)_{i<\gamma}| < \kappa)\},$$

where sprt means support, see Section II.1.2 on page 6. For every α, let $\{\sigma_{\beta\alpha\gamma} \mid \beta < \kappa^+\}$ list all $\mathbb{P}_\beta$-names for codes for a reduction. It is easily seen that $\mathbb{P}_\gamma$ is $< \kappa$-closed and has the κ^+-c.c. for all $\gamma \leqslant \kappa^+$

We claim that $\mathbb{P}_{\kappa^+}$ forces that for all α, $K(\{\alpha\}) \nleqslant_B K(\kappa \setminus \{\alpha\})$ which suffices by the discussion in the beginning of the proof, see $(**)$ for the notation.

Let G be $\mathbb{P}_{\kappa^+}$-generic and let $G_\gamma =$ "$G \cap \mathbb{P}_\gamma$" for every $\gamma < \kappa$. Then G_γ is $\mathbb{P}_\gamma$-generic.

Suppose that in $V[G]$, $f\colon 2^\kappa \to 2^\kappa$ is a reduction $K(\{\alpha\}) \leqslant_B K(\kappa \setminus \{\alpha\})$ and (Q, R) is the corresponding code for a reduction. By [**19**] Theorem VIII.5.14, there is a $\delta < \kappa^+$ such that $(Q, R) \in V[G_\delta]$. Let δ_0 be the smallest such δ.

Now there exists $\sigma_{\gamma\alpha\delta_0}$, a $\mathbb{P}_{\delta_0}$-name for (Q, R). By the definition of h, there exists a $\delta > \delta_0$ with $h(\delta) = (\gamma, \alpha, \delta_0)$. Thus

$$\mathbb{P}_{\delta+1} \Vdash \text{“}\sigma_{\gamma\alpha\delta_0} \text{ is not a code for a reduction”},$$

i.e. $V[G_{\delta+1}] \models (Q, R)$ is not a code for a reduction. Now one of the items (1)–(9) fails for (Q, R) in $V[G_{\delta+1}]$. We want to show that then one of them fails in $V[G]$. The conditions (1)–(8) are absolute, so if one of them fails in $V[G_{\delta+1}]$, then we are done. Suppose (1)–(8) hold but (9) fails. Then there is an $\eta \in R^*$ such that $Q^*(\eta \cap S_{\{\alpha\}}) \cap S_{\kappa\setminus\alpha}$ is stationary but $\eta \cap S_{\{\alpha\}}$ is not or vice versa. In $V[G_{\delta+1}]$ define

$$\mathbb{P}^{\delta+1} = \{(p_i)_{i<\kappa^+} \in \mathbb{P}_{\kappa^+} \mid (p_i)_{i<\delta+1} \in G_{\delta+1}\}.$$

Then $\mathbb{P}^{\delta+1}$ is $< \kappa$-closed. Thus it does not kill stationarity of any set. So if $G^{\delta+1}$ is $\mathbb{P}_{\delta+1}$-generic over $V[G_{\delta+1}]$, then in $V[G_{\delta+1}][G^{\delta+1}]$, (Q, R) is not a code for a reduction. Now it remains to show that $V[G] = V[G_{\delta+1}][G^{\delta+1}]$ for some $G^{\delta+1}$. In fact putting $G^{\delta+1} = G$ we get $\mathbb{P}^{\delta+1}$-generic over $V[G_{\delta+1}]$ and of course $V[G_{\delta+1}][G] = V[G]$ (since $G_{\delta+1} \subset G$). $\square_{\text{Theorem 47}}$

IV.4.1. Reducibility Between Different Cofinalities

In this section we will prove the following two theorems:

50. THEOREM. *Suppose that κ is a weakly compact cardinal and that $V = L$. Then*

- *(A) $E_{S^\kappa_\lambda} \leqslant_c E_{\mathrm{reg}(\kappa)}$ for any regular $\lambda < \kappa$, where $\mathrm{reg}(\kappa) = \{\lambda < \kappa \mid \lambda \text{ is regular}\}$,*
- *(B) In a forcing extension $E_{S^{\omega_2}_\omega} \leqslant_c E_{S^{\omega_2}_{\omega_1}}$. Similarly for λ, λ^+ and λ^{++} instead of ω, ω_1 and ω_2 for any regular $\lambda < \kappa$.*

REMARK. Notation is as in Definition 69.

51. THEOREM ($V = L$). *For a cardinal κ which is a successor of a regular cardinal, there is a cardinal preserving forcing extension in which for all regular $\lambda < \kappa$, the relations $E_{S^\kappa_\lambda}$ are $\leqslant_B$-incomparable with each other.*

Let us begin by proving the latter.

Proof of Theorem 51. Let us show that there is a forcing extension of L in which $E_{S^{\omega_2}_{\omega_1}}$ and $E_{S^{\omega_2}_\omega}$ are incomparable. The general case is similar.

We shall use Lemma 48 with $\mu_1 = \omega$ and $\mu_2 = \omega_1$ and vice versa, and then a similar iteration as in the end of the proof of Theorem 47. Since κ is a successor of a regular cardinal, the lemma is applicable. Assume $V = L$. Then there is a stationary set $S \subset S^{\omega_2}_\omega$ such that for all $\alpha \in S^{\omega_2}_{\omega_1}$, $\alpha \cap S$ is non-stationary in α (follows from $\square_{\omega_1}$). Also for all $\alpha \in S^{\omega_2}_\omega$, $\alpha \cap S^{\omega_2}_{\omega_1}$ is non-stationary.

By Lemma 48, for each code for a reduction from E_S to $E_{S^{\omega_2}_{\omega_1}}$ there is a $< \omega_2$-closed ω_3-c.c. forcing which kills it. Similarly for each code for a reduction from $E_{S^{\omega_2}_{\omega_1}}$ to $E_{S^{\omega_2}_\omega}$. Making an ω_3-long iteration, similarly as in the end of the proof of Theorem 47, we can kill all codes for reductions from E_S to $E_{S^{\omega_2}_{\omega_1}}$ and from $E_{S^{\omega_2}_{\omega_1}}$ to $E_{S^{\omega_2}_\omega}$. Thus, in the extension there are no reductions from $E_{S^{\omega_2}_{\omega_1}}$ to $E_{S^{\omega_2}_\omega}$ and no reductions from $E_{S^{\omega_2}_\omega}$ to $E_{S^{\omega_2}_{\omega_1}}$. (Suppose there is one of a latter kind, $f \colon 2^{\omega_2} \to 2^{\omega_2}$. Then $g(\eta) = f(\eta \cap S)$ is a reduction from E_S to $E_{S^{\omega_2}_{\omega_1}}$.) $\square_{\text{Theorem 51}}$

52. DEFINITION. Let X, Y be subsets of κ and suppose Y consists of ordinals of uncountable cofinality. We say that X *$\diamond$-reflects to Y* if there exists a sequence $\langle D_\alpha \rangle_{\alpha \in Y}$ such that

1. $D_\alpha \subset \alpha$ is stationary in α,
2. if $Z \subset X$ is stationary, then $\{\alpha \in Y \mid D_\alpha = Z \cap \alpha\}$ is stationary.

53. THEOREM. *If* X $\diamond$*-reflects to* Y*, then* $E_X \leqslant_c E_Y$*.*

Proof. Let $\langle D_\alpha \rangle_{\alpha \in Y}$ be the sequence of Definition 52. For a set $A \subset \kappa$ define

$$f(A) = \{\alpha \in Y | A \cap X \cap D_\alpha \text{ is stationary in } \alpha\}. \qquad (i)$$

We claim that f is a continuous reduction. Clearly f is continuous. Assume that $(A \triangle B) \cap X$ is non-stationary. Then there is a cub set $C \subset \kappa \setminus [(A \triangle B) \cap X]$. Now $A \cap X \cap C = B \cap X \cap C$ (ii). The set $C' = \{\alpha < \kappa \mid C \cap \alpha \text{ is unbounded in } \alpha\}$ is also cub and if $\alpha \in Y \cap C'$, we have that $D_\alpha \cap C$ is stationary in α. Therefore for $\alpha \in Y \cap C'$ (iii) we have the following equivalences:

$$\begin{aligned}
\alpha \in f(A) &\iff A \cap X \cap D_\alpha \text{ is stationary} \\
&\overset{(iii)}{\iff} A \cap X \cap C \cap D_\alpha \text{ is stationary} \\
&\overset{(ii)}{\iff} B \cap X \cap C \cap D_\alpha \text{ is stationary} \\
&\overset{(iii)}{\iff} B \cap X \cap D_\alpha \text{ is stationary} \\
&\overset{(i)}{\iff} \alpha \in f(B)
\end{aligned}$$

Thus $(f(A) \triangle f(B)) \cap Y \subset \kappa \setminus C'$ and is non-stationary.

Suppose $A \triangle B$ is stationary. Then either $A \setminus B$ or $B \setminus A$ is stationary. Without loss of generality suppose the former. Then

$$S = \{\alpha \in Y \mid (A \setminus B) \cap X \cap \alpha = D_\alpha\}$$

is stationary by the definition of the sequence $\langle D_\alpha \rangle_{\alpha \in Y}$. Thus for $\alpha \in S$ we have that $A \cap X \cap D_\alpha = A \cap X \cap (A \setminus B) \cap X \cap \alpha = (A \setminus B) \cap X \cap \alpha$ is stationary in α and $B \cap X \cap D_\alpha = B \cap X \cap (A \setminus B) \cap X \cap \alpha = \varnothing$ is not stationary in α. Therefore $(f(A) \triangle f(B)) \cap Y$ is stationary (as it contains S). $\square$

FACT (Π^1_1-reflection). *Assume that* κ *is weakly compact. If* R *is any binary predicate on* V_κ *and* $\forall A \varphi$ *is some* Π^1_1*-sentence where* φ *is a first-order sentence in the language of set theory together with predicates* $\{R, A\}$ *such that* $(V_\kappa, R) \models \forall A \varphi$*, then there exists stationary many* $\alpha < \kappa$ *such that* $(V_\alpha, R \cap V_\alpha) \models \forall A \varphi$*.*

We say that X *strongly reflects to* Y if for all stationary $Z \subset X$ there exist stationary many $\alpha \in Y$ with $X \cap \alpha$ stationary in α.

54. THEOREM. *Suppose* $V = L$*,* κ *is weakly compact and that* $X \subset \kappa$ *and* $Y \subset \operatorname{reg} \kappa$*. If* X *strongly reflects to* Y*, then* X $\diamond$*-reflects to* Y*.*

Proof. Define D_α by induction on $\alpha \in Y$. For the purpose of the proof also define C_α for each α as follows. Suppose (D_β, C_β) is defined for all $\beta < \alpha$. Let (D, C) be the L-least[1] pair such that

(1) C is cub subset of α.
(2) D is a stationary subset of $X \cap \alpha$
(3) for all $\beta \in Y \cap C$, $D \cap \beta \neq D_\beta$

If there is no such pair then set $D = C = \varnothing$. Then let $D_\alpha = D$ and $C_\alpha = C$. We claim that the sequence $\langle D_\alpha \rangle_{\alpha \in Y}$ is as needed. To show this, let us make a counter assumption: there is a stationary subset Z of X and a cub subset C of κ such that

$$C \cap Y \subset \{\alpha \in Y \mid D_\alpha \neq Z \cap \alpha\}. \tag{$\star$}$$

Let (Z, C) be the L-least such pair. Let $\lambda > \kappa$ be regular and let M be an elementary submodel of L_λ such that

(1) $|M| < \kappa$,
(2) $\alpha = M \cap \kappa \in Y \cap C$,
(3) $Z \cap \alpha$ is stationary in α,
(4) $\{Z, C, X, Y, \kappa\} \subset M$

(2) and (3) are possible by the definition of strong reflection.Let $\bar{M}$ be the Mostowski collapse of M and let $G \colon M \to \bar{M}$ be the Mostowski isomorphism. Then $\bar{M} = L_\gamma$ for some $\gamma > \alpha$. Since $\kappa \cap M = \alpha$, we have

$$G(Z) = Z \cap \alpha,\ G(C) = C \cap \alpha,\ G(X) = X \cap \alpha,\ G(Y) = Y \cap \alpha \text{ and } G(\kappa) = \alpha,\ (\star\star).$$

Note that by the definability of the canonical ordering of L, the sequence $\langle D_\beta \rangle_{\beta < \kappa}$ is definable. Let $\varphi(x, y, \alpha)$ be the formula which says

"(x, y) is the L-least pair such that x is contained in $X \cap \alpha$, x is stationary in α, y is cub in α and $x \cap \beta \neq D_\beta$ for all $\beta \in y \cap Y \cap \alpha$."

By the assumption,

$$L \models \varphi(Z, C, \kappa), \text{ so } M \models \varphi(Z, C, \kappa) \text{ and } L_\gamma \models \varphi(G(Z), G(C), G(\kappa)).$$

Let us show that this implies $L \models \varphi(G(Z), G(C), G(\kappa))$, i.e. $L \models \varphi(Z \cap \alpha, C \cap \alpha, \alpha)$. This will be a contradiction because then $D_\alpha = Z \cap \alpha$ which contradicts the assumptions (2) and ($\star$) above.

By the relative absoluteness of being the L-least, the relativised formula with parameters $\varphi^{L_\gamma}(G(Z), G(C), G(\kappa))$ says

---

[1] The least in the canonical definable ordering on L, see [**19**].

“$(G(Z), G(C))$ is the L-least pair such that $G(Z)$ is contained in $G(X)$, $G(Z)$ is $(\text{stationary})^{L_\gamma}$ in $G(\kappa)$, $G(C)$ is cub in $G(\kappa)$ and $G(Z) \cap \beta \neq D_\beta^{L_\gamma}$ for all $\beta \in G(C) \cap G(Y) \cap G(\kappa)$.”

Written out this is equivalent to

“$(Z \cap \alpha, C \cap \alpha)$ is the L-least pair such that $Z \cap \alpha$ is contained in $X \cap \alpha$, $Z \cap \alpha$ is $(\text{stationary})^{L_\gamma}$ in α, $C \cap \alpha$ is cub in α and $Z \cap \beta \neq D_\beta^{L_\gamma}$ for all $\beta \in C \cap Y \cap \alpha$.”

Note that this is true in L. Since $Z \cap \alpha$ is stationary in α also in L by (3), it remains to show by induction on $\beta \in \alpha \cap Y$ that $Z \cap \alpha$ $D_\beta^{L_\gamma} = D_\beta^L$ and $C_\beta^{L_\gamma} = C_\beta^L$ and we are done. Suppose we have proved this for $\delta \in \beta \cap Y$ and $\beta \in \alpha \cap Y$. Then $(D_\beta^{L_\gamma}, C_\beta^{L_\gamma})$ is

(a) $(\text{the least } L\text{-pair})^{L_\gamma}$ such that
(b) $(C_\beta \text{ is a cub subset of } \beta)^{L_\gamma}$,
(c) $(D_\beta \text{ is a stationary subset of } \beta)^{L_\gamma}$
(d) and for all $\delta \in Y \cap \beta$, $(D_\beta \cap \delta \neq D_\delta)^{L_\gamma}$.
(e) Or there is no such pair and $D_\beta = \varnothing$.

The L-order is absolute as explained above, so (a) is equivalent to $(\text{the least } L\text{-pair})^L$. Being a cub subset of α is also absolute for L_γ so (b) is equivalent to $(C_\beta \text{ is a cub subset of } \alpha)^L$. All subsets of β in L are elements of $L_{|\beta|^+}$ (see [**19**]), and since α is regular and $\beta < \alpha \leqslant \gamma$, we have $\mathcal{P}(\beta) \subset L_\gamma$. Thus

$$(D_\beta \text{ is stationary subset of } \beta)^{L_\gamma} \iff (D_\beta \text{ is stationary subset of } \beta)^L.$$

Finally the statement of (d), $(D_\beta \cap \delta \neq D_\delta)^{L_\gamma}$ is equivalent to $D_\beta \cap \delta \neq D_\delta^{L_\gamma}$ as it is defining D_β, but by the induction hypothesis $D_\delta^{L_\gamma} = D_\delta^L$, so we are done. For (e), the fact that

$$\mathcal{P}(\beta) \subset L_{|\beta|^+} \subset L_\alpha \subset L_\gamma$$

as above implies that if there is no such pair in L_γ, then there is no such pair in L. □

Proof of Theorem 50. In the case (A) we will show that S^κ_λ strongly reflects to $\text{reg}(\kappa)$ in L which suffices by Theorems 53 and 54. For (B) we will assume that κ is a weakly compact cardinal in L and then collapse it to ω_2 to get a $\diamond$-sequence which witnesses that $S^{\omega_2}_\omega$ $\diamond$-reflects to $S^{\omega_2}_{\omega_1}$ which is sufficient by Theorem 53. In the following we assume: $V = L$ and κ is weakly compact.

(A): Let us use Π^1_1-reflection. Let $X \subset S^\kappa_\lambda$. We want to show that the set

$$\{\lambda \in \operatorname{reg}(\kappa) \mid X \cap \lambda \text{ is stationary in } \lambda\}$$

is stationary. Let $C \subset \kappa$ be cub. The sentence

$$\text{“}(X \text{ is stationary in } \kappa) \wedge (C \text{ is cub in } \kappa) \wedge (\kappa \text{ is regular})\text{”}$$

is a Π^1_1-property of (V_κ, X, C). By Π^1_1-reflection we get $\delta < \kappa$ such that $(V_\delta, X \cap \delta, C \cap \delta)$ satisfies it. But then δ is regular, $X \cap \delta$ is stationary and δ belongs to C.

(B): Let κ be weakly compact and let us Levy-collapse κ to ω_2 with the following forcing:

$$\mathbb{P} = \{f\colon \operatorname{reg}\kappa \to \kappa^{<\omega_1} \mid \operatorname{ran}(f(\mu)) \subset \mu,\ |\{\mu \mid f(\mu) \neq \varnothing\}| \leqslant \omega\}.$$

Order $\mathbb{P}$ by $f < g$ if and only if $f(\mu) \subset g(\mu)$ for all $\mu \in \operatorname{reg}(\kappa)$. For all μ put $\mathbb{P}_\mu = \{f \in \mathbb{P} \mid \operatorname{sprt} f \subset \mu\}$ and $\mathbb{P}^\mu = \{f \in \mathbb{P} \mid \operatorname{sprt} f \subset \kappa \setminus \mu\}$, where sprt means support, see Section II.1.2 on page 6.

Claim 1. For all regular μ, $\omega < \mu \leqslant \kappa$, $\mathbb{P}_\mu$ satisfies the following:

(a) If $\mu > \omega_1$, then $\mathbb{P}_\mu$ has the μ-c.c.,
(b) $\mathbb{P}_\mu$ and $\mathbb{P}^\mu$ are $< \omega_1$-closed,
(c) $\mathbb{P} = \mathbb{P}_\kappa \Vdash \omega_2 = \check{\kappa}$,
(d) If $\mu < \kappa$, then $\mathbb{P} \Vdash \operatorname{cf}(\check{\mu}) = \omega_1$,
(e) if $p \in \mathbb{P}$, σ a name and $p \Vdash$ “σ is cub in ω_2”, then there is cub $E \subset \kappa$ such that $p \Vdash \check{E} \subset \sigma$.

Proof. Standard (see for instance [**15**]). □

We want to show that in the generic extension $S^{\omega_2}_\omega$ $\diamond$-reflects to $S^{\omega_2}_{\omega_1}$. It is sufficient to show that $S^{\omega_2}_\omega$ $\diamond$-reflects to some stationary $Y \subset S^{\omega_2}_{\omega_1}$ by letting $D_\alpha = \alpha$ for $\alpha \notin Y$. In our case $Y = \{\mu \in V[G] \mid (\mu \in \operatorname{reg}(\kappa))^V\}$. By (d) of Claim 1, $Y \subset S^{\omega_2}_{\omega_1}$, $(\operatorname{reg}(\kappa))^V$ is stationary in V (for instance by Π^1_1-reflection) and by (e) it remains stationary in $V[G]$.

It is easy to see that $\mathbb{P} \cong \mathbb{P}_\mu \times \mathbb{P}^\mu$. Let G be a $\mathbb{P}$-generic over (the ground model) V. Define

$$G_\mu = G \cap \mathbb{P}_\mu.$$

and

$$G^\mu = G \cap \mathbb{P}^\mu.$$

Then G_μ is $\mathbb{P}_\mu$-generic over V. Also G^μ is $\mathbb{P}^\mu$-generic over $V[G_\mu]$ and $V[G] = V[G_\mu][G^\mu]$.

Let

$$E = \{p \in \mathbb{P} \mid (p > q) \wedge (p_\mu \Vdash p^\mu \in \dot{D})\}$$

Then E is dense above q: If $p > q$ is arbitrary element of $\mathbb{P}$, then $q \Vdash \exists p' > \check{p}^\mu (p' \in \dot{D})$ by (#). Thus there exists $q' > q$ with $q' > p_\mu$, $q' \in \mathbb{P}_\mu$ and $p' > p, p' \in \mathbb{P}^\mu$ such that $q' \Vdash p' \in \dot{D}$ and so $(q' \restriction \mu) \cup (p' \restriction (\kappa \setminus \mu))$ is above p and in E. So there is $p \in G \cap E$. But then $p_\mu \in G_\mu$ and $p^\mu \in G^\mu$ and $p_\mu \Vdash p^\mu \in \dot{D}$, so $G^\mu \cap D \neq \varnothing$. Since D was arbitrary, this shows that G^μ is $\mathbb{P}^\mu$-generic over $V[G_\mu]$. Clearly $V[G]$ contains both G_μ and G^μ. On the other hand, $G = G_\mu \cup G^\mu$, so $G \in V[G_\mu][G^\mu]$. By minimality of forcing extensions, we get $V[G] = V[G_\mu][G^\mu]$.

For each $\mu \in \operatorname{reg}(\kappa) \setminus \{\omega, \omega_1\}$ let

$$k_\mu \colon \mu^+ \to \{\sigma \mid \sigma \text{ is a nice } \mathbb{P}_\mu \text{ name for a subset of } \mu\}$$

be a bijection. A nice $\mathbb{P}_\mu$ name for a subset of $\check{\mu}$ is of the form

$$\bigcup \{\{\check{\alpha}\} \times A_\alpha \mid \alpha \in B\},$$

where $B \subset \check{\mu}$ and for each $\alpha \in B$, A_α is an antichain in $\mathbb{P}_\mu$. By (a) there are no antichains of length μ in $\mathbb{P}_\mu$ and $|\mathbb{P}_\mu| = \mu$, so there are at most $\mu^{<\mu} = \mu$ antichains and there are μ^+ subsets $B \subset \mu$, so there indeed exists such a bijection k_μ (these cardinality facts hold because $V = L$ and μ is regular). Note that if σ is a nice $\mathbb{P}_\mu$-name for a subset of $\check{\mu}$, then $\sigma \subset V_\mu$.

Let us define

$$D_\mu = \begin{cases} \Big[k_\mu\Big([(\cup G)(\mu^+)](0)\Big)\Big]_G & \text{if it is stationary} \\ \mu & \text{otherwise.} \end{cases}$$

Now D_μ is defined for all $\mu \in Y$, recall $Y = \{\mu \in V[G] \mid (\mu \in \operatorname{reg}\kappa)^V\}$. We claim that $\langle D_\mu \rangle_{\mu \in Y}$ is the needed $\diamond$-sequence. Suppose it is not. Then there is a stationary set $S \subset S^{\omega_2}_\omega$ and a cub $C \subset \omega_2$ such that for all $\alpha \in C \cap Y$, $D_\alpha \neq S \cap \alpha$. By (e) there is a cub set $C_0 \subset C$ such that $C_0 \in V$. Let $\dot{S}$ be a nice name for S and p' such that p' forces that $\dot{S}$ is stationary. Let us show that

$$H = \{q \geqslant p' \mid q \Vdash D_\mu = \dot{S} \cap \check{\mu} \text{ for some } \mu \in C_0\}$$

is dense above p' which is obviously a contradiction. For that purpose let $p > p'$ be arbitrary and let us show that there is $q > p$ in H. Let us now use Π^1_1-reflection. First let us redefine $\mathbb{P}$. Let $\mathbb{P}^* = \{q \mid \exists r \in \mathbb{P}(r \restriction \operatorname{sprt} r = q)\}$. Clearly $\mathbb{P}^* \cong \mathbb{P}$ but the advantage is that $\mathbb{P}^* \subset V_\kappa$ and $\mathbb{P}^*_\mu = \mathbb{P}^* \cap V_\mu$ where

$\mathbb{P}^*_\mu$ is defined as $\mathbb{P}_\mu$. One easily verifies that all the above things (concerning $\mathbb{P}_\mu$, $\mathbb{P}^\mu$ etc.) translate between $\mathbb{P}$ and $\mathbb{P}^*$. From now on denote $\mathbb{P}^*$ by $\mathbb{P}$. Let

$$R = (\mathbb{P} \times \{0\}) \cup (\dot{S} \times \{1\}) \cup (C_0 \times \{2\}) \cup (\{p\} \times \{3\})$$

Then $(V_\kappa, R) \models \forall A\varphi$, where φ says: "(if A is closed unbounded and $r > p$ arbitrary, then there exist $q > r$ and α such that $\alpha \in A$ and $q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{S}$)." So basically $\forall A\varphi$ says "$p \Vdash$ ($\dot{S}$ is stationary)". It follows from (e) that it is enough to quantify over cub sets in V. Let us explain why such a formula can be written for (V_κ, R). The sets (classes from the viewpoint of V_κ) $\mathbb{P}$, $\dot{S}$ and C_0 are coded into R, so we can use them as parameters. That $r > p$ and $q > r$ and A is closed and unbounded is expressible in first-order as well as $\alpha \in A$. How do we express $q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{S}$? The definition of $\check{\alpha}$ is recursive in α:

$$\check{\alpha} = \{(\check{\beta}, 1_{\mathbb{P}}) \mid \beta < \alpha\}$$

and is absolute for V_κ. Then $q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{S}$ is equivalent to saying that for each $q' > q$ there exists $q'' > q'$ with $(\check{\alpha}, q'') \in \dot{S}$ and this is expressible in first-order (as we have taken R as a parameter).

By Π^1_1-reflection there is $\mu \in C_0$ such that $p \in \mathbb{P}_\mu$ and $(V_\mu, R) \models \forall A\varphi$. Note that we may require that μ is regular, i.e. $(\check{\mu}_G \in Y)^{V[G]}$ and such that $\alpha \in S \cap \mu$ implies $(\check{\alpha}, \check{p}) \in \dot{S}$ for some $p \in \mathbb{P}_\mu$. Let $\dot{S}_\mu = \dot{S} \cap V_\mu$.

Thus $p \Vdash_{\mathbb{P}_\mu}$ "$\dot{S}_\mu$ is stationary". Define q as follows: $\operatorname{dom} q = \operatorname{dom} p \cup \{\mu^+\}$, $q \restriction \mu = p \restriction \mu$ and $q(\mu^+) = f$, $\operatorname{dom} f = \{0\}$ and $f(0) = k_\mu^{-1}(\dot{S}_\mu)$. Then $q \Vdash_{\mathbb{P}} \dot{S}_\mu = D_\mu$ provided that $q \Vdash_{\mathbb{P}}$ "$\dot{S}_\mu$ is stationary". The latter holds since $\mathbb{P}^\mu$ is $< \omega_1$-closed., and does not kill stationarity of $(\dot{S}_\mu)_{G_\mu}$ so $(\dot{S}_\mu)_{G_\mu}$ is stationary in $V[G]$ and by the assumption on μ, $(\dot{S}_\mu)_{G_\mu} = (\dot{S}_\mu)_G$. Finally, it remains to show that in $V[G]$, $(\dot{S}_\mu)_G = S \cap \mu$. But this again follows from the definition of μ.

Instead of collapsing κ to ω_2, we could do the same for λ^{++} for any regular $\lambda < \kappa$ and obtain a model in which $E_{S^{\lambda^{++}}_{\lambda}} \leqslant_c E_{S^{\lambda^{++}}_{\lambda^+}}$. □

Open Problem. Can there be two equivalence relations, E_1 and E_2 on 2^κ, $\kappa > \omega$ such that E_1 and E_2 are Borel and incomparable, i.e. $E_1 \not\leqslant_B E_2$ and $E_2 \not\leqslant_B E_1$?

V Complexity of Isomorphism Relations

Let T be a countable complete theory. Let us turn to the question discussed in Section I: "How is the set theoretic complexity of $\cong_T$ related to the stability theoretic properties of T?". The following theorems give some answers. As pointed out in Section I, the assumption that κ is uncountable is crucial in the following theorems. For instance the theory of dense linear orderings without end points is unstable, but $\cong_T$ is an open set in case $\kappa = \omega$, while we show below that for unstable theories T the set $\cong_T$ cannot be even Δ_1^1 when $\kappa > \omega$. Another example introduced by Martin Koerwien in his Ph.D. thesis and in [**18**] shows that there are classifiable shallow theories whose isomorphism is not Borel when $\kappa = \omega$, although we prove below that the isomorphism of such theories is always Borel, when $\kappa > \omega$. This justifies in particular the motivation for studying the space κ^κ for model theoretic purposes: the set theoretic complexity of $\cong_T$ positively correlates with the model theoretic complexity of T.

The following model theoretical notions will be used: stable, superstable, DOP, OTOP, shallow, $\lambda(T)$ and $\kappa(T)$. Classifiable means superstable with no DOP nor OTOP.

The main theme in this section is exposed in the following two theorems:

55. THEOREM ($\kappa^{<\kappa} = \kappa$). *Assume that κ is not weakly inaccessible. A theory T is classifiable and shallow if and only if its isomorphism relation on structures of size κ is Borel.*

56. THEOREM ($\kappa^{<\kappa} = \kappa$). *Assume that for all $\lambda < \kappa$, $\lambda^\omega < \kappa$ and $\kappa > \omega_1$. Then Then in the forcing extension after adding κ^+ Cohen subsets of κ we have: for any theory T, T is classifiable if and only if $\cong_T$ is Δ_1^1.*

The two theorems above are proved in many subtheorems below. Our results are stronger than those given by 55 and 56 (for instance the cardinality assumption $\kappa > \omega_1$ is needed only in the case where T is superstable with DOP and the stable unsuperstable ase is the only one for which Theorem

56 cannot be proved in ZFC). Theorem 55 follows from Theorems 60, 61. Theorem 56 follows from Theorems 62, 63 and 64.

V.1. Preliminary Results

The following Theorems 57 and 59 will serve as bridges between the set theoretic complexity and the model theoretic complexity of an isomorphism relation.

57. Theorem ($\kappa^{<\kappa} = \kappa$). *For a theory T, the set $\cong_T$ is Borel if and only if the following holds: there exists a $\kappa^+\omega$-tree t such that for all models $\mathcal{A}$ and $\mathcal{B}$ of T, $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$.*

Proof. First suppose that there exists a $\kappa^+\omega$-tree t such that for all models $\mathcal{A}$ and $\mathcal{B}$ of T, $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$. Let us show that there exists a $\kappa^+\omega$-tree u which constitutes a Borel code for $\cong_T$ (see Remark 15 on page 16).

Let u be the tree of sequences of the form

$$\langle (p_0, A_0), f_0, (p_1, A_1), f_1, \dots, (p_n, A_n), f_n \rangle$$

such that for all $i \leqslant n$

(1) (p_i, A_i) is a move of player $\mathbf{I}$ in EF_t^κ, i.e. $p_i \in t$ and $A_i \subset \kappa$ with $|A_i| < \kappa$,

(2) f_i is a move of player $\mathbf{II}$ in EF_t^κ, i.e. it is a partial function $\kappa \to \kappa$ with $|\operatorname{dom} f|, |\operatorname{ran} f| < \kappa$

(3) $\langle (p_0, A_0), f_0, (p_1, A_1), f_1, \dots, (p_n, A_n), f_n \rangle$ is a valid position of the game, i.e. $(p_i)_{i \leqslant n}$ is an initial segment of a branch in t and $A_i \subset A_j$ and $f_i \subset f_j$ whenever $i < j \leqslant n$.

Order u by end extension. The tree u is a $\kappa^+\omega$-tree (because t is and by (3)).

Let us now define the function

$$h \colon \{\text{branches of } u\} \to \{\text{basic open sets of } \kappa^\kappa\}.$$

Let $b \subset u$ be a branch,

$$b = \{\varnothing, \langle (p_0, A_0) \rangle, \langle (p_0, A_0), f_0 \rangle, \dots, \langle (p_0, A_0), f_0, \dots, (p_k, A_k), f_k \rangle\}.$$

It corresponds to a unique EF-game between some two structures with domains κ. In this game the players have chosen some set $A = \bigcup_{i<k} A_i \subset \kappa$ and some partial function $f = \bigcup_{i<k} f_i \colon \kappa \to \kappa$. Let $h(b)$ be the set of all

pairs $(\eta,\xi)\in(\kappa^\kappa)^2$ such that $f\colon \mathcal{A}_\eta\restriction A\cong\mathcal{A}_\xi\restriction A$ is a partial isomorphism. This is clearly an open set:

$$(\eta,\xi)\in h(b)\Rightarrow N_{\eta\restriction((\sup A)+1)}\times N_{\xi\restriction((\sup A)+1)}\subset h(b).$$

Finally we claim that $\mathcal{A}_\eta\cong\mathcal{A}_\xi \iff \mathbf{II}\uparrow G(u,h,(\eta,\xi))$. Here G is the game as in Definition 14 of Borel* sets, page 15 but played on the product $\kappa^\kappa\times\kappa^\kappa$. Assume $\mathcal{A}_\eta\cong\mathcal{A}_\xi$. Then $\mathbf{II}\uparrow \mathrm{EF}^\kappa_t(\mathcal{A}_\eta,\mathcal{A}_\xi)$. Let υ denote the winning strategy. In the game $G(u,h,(\eta,\xi))$, let us define a winning strategy for player $\mathbf{II}$ as follows. By definition, at a particular move, say n, $\mathbf{I}$ chooses a sequence

$$\langle(p_0,A_0),f_0,\dots(p_n,A_n)\rangle.$$

Next $\mathbf{II}$ extends it according to υ to

$$\langle(p_0,A_0),f_0,\dots(p_n,A_n),f_n\rangle,$$

where $f_n=\upsilon((p_0,A_0),\dots,(p_n,A_n))$. Since υ was a winning strategy, it is clear that $f=\bigcup_{i<k}f_i$ is going to be a isomorphism between $\mathcal{A}_\eta\restriction A$ and $\mathcal{A}_\xi\restriction A$, so $(\eta,\xi)\in h(b)$.

Assume that $\mathcal{A}_\eta\not\cong\mathcal{A}_\xi$. Then by the assumption there is no winning strategy of $\mathbf{II}$, so player $\mathbf{I}$ can play in such a way that $f=\bigcup_k f_i$ is not an isomorphism between $\mathcal{A}_\eta\restriction\cup A_i$ and $\mathcal{A}_\xi\restriction\cup A_i$, so (η,ξ) is not in $h(b)$. This completes the proof of the direction "$\Leftarrow$"

Let us prove "$\Rightarrow$". Suppose $\cong_T$ is Borel and let us show that there is a tree as in the statement of the theorem. We want to use Theorem 22 and formalize the statement "$\cong_T$ is definable in $L_{\kappa^+\kappa}$" by considering the space consisting of pairs of models.

Denote the vocabulary of $\mathcal{A}$ and $\mathcal{B}$ as usual by L. Let P be a unary relation symbol not in L. We will now discuss two distinct vocabularies, L and $L\cup\{P\}$ at the same time, so we have to introduce two distinct codings. Fix an $\eta\in 2^\kappa$. Let $\mathcal{A}_\eta$ denote the L-structure as defined in Definition 11 of our usual coding. Let $\rho\colon\kappa\cup\kappa^{<\omega}\to\kappa$ be a bijection and define $\mathcal{A}^\eta$ to be the model with $\operatorname{dom}\mathcal{A}^\eta=\kappa$ and if $a\in\operatorname{dom}\mathcal{A}^\eta$, then $\mathcal{A}^\eta\models P(a)\iff\eta(\rho(a))=1$ such that if $(a_1,\dots,a_n)\in(\operatorname{dom}\mathcal{A}^\eta)^n$, then $\mathcal{A}^\eta\models P_n(a_1,\dots,a_n)\iff\eta(\rho(a_1,\dots,a_n))=1$. Note that we are making a distinction here between κ and $\kappa^{\{0\}}$.

Claim 1. The set $W=\{\eta\in 2^\kappa\mid \kappa=|P^{\mathcal{A}^\eta}|=|\kappa\setminus P^{\mathcal{A}^\eta}|\}$ is Borel.

Proof of Claim 1. Let us show that the complement is Borel. By symmetry it is sufficient to show that

$$B = \{\eta \mid \kappa > |P^{\mathcal{A}^\eta}|\}$$

is Borel. Let $I \subset \kappa$ be a subset of size $< \kappa$. For $\beta \notin I$ define $U(I, \beta)$ to be the set

$$U(I,\beta) = \{\eta \mid \eta(\rho(\beta)) = 0\}.$$

Clearly $U(I, \beta)$ is open for all I, β. Now

$$B = \bigcup_{I \in [\kappa]^{<\kappa}} \bigcap_{\beta \notin I} U(I,\beta).$$

By the assumption $\kappa^{<\kappa} = \kappa$, this is Borel (in fact a union of closed sets). $\square_{\text{Claim 1}}$

Define a mapping $h\colon W \to (2^\kappa)^2$ as follows. Suppose $\xi \in W$. Let

$$r_1\colon \kappa \to P^{\mathcal{A}^\xi}$$

and

$$r_2\colon \kappa \to \kappa \setminus P^{\mathcal{A}^\xi}$$

be the order preserving bijections (note $P^{\mathcal{A}^\eta} \subset \kappa = \operatorname{dom} \mathcal{A}^\eta$).

Let η_1 be such that r_1 is an isomorphism

$$\mathcal{A}_{\eta_1} \to (\mathcal{A}^\xi \cap P^{\mathcal{A}^\xi}) \restriction L$$

and η_2 such that r_2 is an isomorphism

$$\mathcal{A}_{\eta_2} \to (\mathcal{A}^\xi \setminus P^{\mathcal{A}^\xi}) \restriction L.$$

Clearly η_1 and η_2 are unique, so we can define $h(\xi) = (\eta_1, \eta_2)$.

Claim 2. h is continuous.

Proof of Claim 2. Let $U = N_p \times N_q$ be a basic open set of $(2^\kappa)^2$, $p, q \in 2^{<\kappa}$ and let $\xi \in h^{-1}[U]$. Let $P^{\mathcal{A}^\xi} = \{\beta_i \mid i < \kappa\}$ be an enumeration such that $\beta_i < \beta_j \iff i < j$ and similarly $\kappa \setminus P^{\mathcal{A}^\xi} = \{\gamma_i \mid i < \kappa\}$. Let $\alpha = \max\{\beta_{\operatorname{dom} p}, \gamma_{\operatorname{dom} q}\} + 1$. Then $N_{\xi \restriction \alpha} \subset h^{-1}[U]$. Thus arbitrary ξ in $h^{-1}[U]$ have an open neighbourhood in $h^{-1}[U]$, so it is open. $\square_{\text{Claim 2}}$

Recall our assumption that $E = \{(\eta, \xi) \in 2^\kappa \mid \mathcal{A}_\eta \cong \mathcal{A}_\xi\}$ is Borel. Since h is continuous and in particular Borel, this implies that

$$E' = \{\eta \mid \mathcal{A}_{h_1(\eta)} \cong \mathcal{A}_{h_2(\eta)}\} = h^{-1}E$$

is Borel in W. Because W is itself Borel, E' is Borel in 2^κ. Additionally, E' is closed under permutations: if $\mathcal{A}^\eta$ is isomorphic to $\mathcal{A}^\xi$, then $\mathcal{A}^\eta \cap P^{\mathcal{A}^\eta}$ is isomorphic to $\mathcal{A}^\xi \cap P^{\mathcal{A}^\xi}$ and $\mathcal{A}^\eta \setminus P^{\mathcal{A}^\eta}$ is isomorphic to $\mathcal{A}^\xi \setminus P^{\mathcal{A}^\xi}$, so if $\mathcal{A}^\eta \in E'$, then also $\mathcal{A}^\xi \in E'$ (and note that since $\eta \in W$, also $\xi \in W$). By Theorem 22, there is a sentence θ of $L_{\kappa^+\kappa}$ over $L \cup \{P\}$ that defines E'. Thus, there is t such that

$$\text{if } \eta \in E' \text{ and } \xi \notin E' \text{, then } \mathbf{II} \not\uparrow \mathrm{EF}^\kappa_t(\mathcal{A}^\eta, \mathcal{A}^\xi). \qquad \odot$$

We claim that t is as needed, i.e. for all models $\mathcal{A}, \mathcal{B}$ of T

$$\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B}).$$

Suppose not. Then there are models $\mathcal{A} \ncong \mathcal{B}$ such that $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B})$. Let η and ξ be such that $\mathcal{A}_{h_1(\eta)} = \mathcal{A}_{h_2(\eta)} = \mathcal{A}_{h_1(\xi)} = \mathcal{A}$ and $\mathcal{A}_{h_2(\xi)} = \mathcal{B}$. Clearly $\eta \in E'$, but $\xi \notin E'$, so by $\odot$ there is no winning strategy of $\mathbf{II}$ in $\mathrm{EF}^\kappa_t(\mathcal{A}^\eta, \mathcal{A}^\xi)$ which is clearly a contradiction, because $\mathbf{II}$ can apply her winning strategies in $\mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B})$ and $\mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{A})$ to win in $\mathrm{EF}^\kappa_t(\mathcal{A}^\eta, \mathcal{A}^\xi)$. $\square_{\text{Theorem 57}}$

We will use the following lemma from [**22**]:

58. Lemma. *If $t \subset (\kappa^{<\kappa})^2$ is a tree and $\xi \in \kappa^\kappa$, denote*

$$t(\xi) = \{p \in \kappa^{<\kappa} \mid (p, \xi \restriction \operatorname{dom} p) \in t\}$$

Similarly if $t \in (\kappa^{<\kappa})^3$, then

$$t(\eta, \xi) = \{p \in \kappa^{<\kappa} \mid (p, \eta \restriction \operatorname{dom} p, \xi \restriction \operatorname{dom} p) \in t\}.$$

Assume that Z is Σ^1_1. Then Z is Δ^1_1 if and only if for every tree $t \subset (\kappa^{<\kappa})^2$ such that

$$t(\xi) \text{ has a } \kappa\text{-branch} \iff \xi \in Z$$

there exists a $\kappa^+\kappa$-tree t' such that $\xi \in Z \iff t(\xi) \not\leq t'$.

59. Theorem. *Let T be a theory and assume that for every $\kappa^+\kappa$-tree t there exist $(\eta, \xi) \in (2^\kappa)^2$ such that $\mathcal{A}_\eta, \mathcal{A}_\xi \models T$, $\mathcal{A}_\eta \ncong \mathcal{A}_\xi$ but $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}_\eta, \mathcal{A}_\xi)$. Then $\cong_T$ is not Δ^1_1.*

Proof. Let us abbreviate some statements:

$A(t)$: $t \subset (\kappa^{<\kappa})^3$ is a tree and for all $(\eta, \xi) \in (\kappa^\kappa)^2$,

$$(\eta, \xi) \in \cong_T \iff t(\eta, \xi) \text{ contains a } \kappa\text{-branch}\ .$$

$B(t,t')$: $t \subset (\kappa^{<\kappa})^3$ is a $\kappa^+\kappa$-tree and for all $(\eta,\xi) \in \kappa^\kappa$,

$$(\eta,\xi) \in {\cong_T} \iff t(\eta,\xi) \not\leqslant t'.$$

Now Lemma 58 implies that if $\cong_T$ is Δ^1_1, then $\forall t[A(t) \to \exists t' B(t,t')]$. We will show that $\exists t[A(t) \wedge \forall t' \neg B(t,t')]$, which by Lemma 58 suffices to prove the theorem. Let us define t. In the following, ν_α, η_α and ξ_α stand respectively for $\nu \restriction \alpha$, $\eta \restriction \alpha$ and $\xi \restriction \alpha$.

$t = \{(\nu_\alpha, \eta_\alpha, \xi_\alpha) \mid \alpha < \kappa$ and ν codes an isomorphism between $\mathcal{A}_\eta$ and $\mathcal{A}_\xi\}$.

Using Theorem 12 it is easy to see that t satisfies $A(t)$. Assume now that t' is an arbitrary $\kappa^+\kappa$-tree. We will show that $B(t,t')$ does not hold. For that purpose let $u = \omega \times t'$ be the tree defined by the set $\{(n,s) \mid n \in \omega, s \in t'\}$ and the ordering

$$(n_0,s_0) <_u (n_1,s_1) \iff \big(s_0 <_{t'} s_1 \vee (s_0 = s_1 \wedge n_0 <_\omega n_1)\big). \qquad (1)$$

This tree u is still a $\kappa^+\kappa$-tree, so by the assumption of the theorem there is a pair (ξ_1,ξ_2) such that the corresponding models are non-isomorphic, i.e. $(\xi_1,\xi_2) \notin E$, but are EF^κ_u-equivalent

It is now sufficient to show that $t(\xi_1,\xi_2) \not\leqslant t'$.

Claim 1. There is no order preserving function

$$\sigma t' \to t',$$

where $\sigma t'$ is defined in Definition 28.

Proof of Claim 1. Assume $g\colon \sigma t' \to t'$, is order preserving. Define $x_0 = g(\varnothing)$ and

$$x_\alpha = g(\{y \in t' \mid \exists \beta < \alpha (y \leqslant x_\beta)\}) \text{ for } 0 < \alpha < \kappa$$

Then $(x_\alpha)_{\alpha<\kappa}$ contradicts the assumption that t' is a $\kappa^+\kappa$-tree. $\square_{\text{Claim 1}}$

Claim 2. There is an order preserving function

$$\sigma t' \to t(\xi_1,\xi_2).$$

Proof of Claim 2. The idea is that players **I** and **II** play an EF-game for each branch of the tree t' and **II** uses her winning strategy in $\mathrm{EF}^\kappa_u(\mathcal{A}_{\xi_1},\mathcal{A}_{\xi_2})$to embed that branch into the tree of partial isomorphisms. A problem is that the winning strategy gives arbitrary partial isomorphisms while we are interested in those which are coded by functions defined on page 14. Now the tree u of (1) above becomes useful.

Let σ be a winning strategy of player **II** in $\mathrm{EF}^\kappa_u(\mathcal{A}_{\xi_1}, \mathcal{A}_{\xi_2})$. Let us define $g\colon \sigma t' \to t(\xi_1, \xi_2)$ recursively. Recall the function π from Definition 11 and define

$$C = \{\alpha \mid \pi[\alpha^{<\omega}] = \alpha\}.$$

Clearly C is cub. If $s \subset t'$ is an element of $\sigma t'$, then we assume that g is defined for all $s' <_{\sigma t'} s$ and that EF^κ_u is played up to $(0, \sup s) \in u$. If s does not contain its supremum, then put $g(s) = \bigcup_{s' < s} g(s')$. Otherwise let them continue playing the game for ω more moves; at the n^{th} of these moves player **I** picks $(n, \sup s)$ from u and a $\beta < \kappa$ where β is an element of C above

$$\max\{\operatorname{ran} f_{n-1}, \operatorname{dom} f_{n-1}\}$$

where f_{n-1} is the previous move by **II**. (If $n = 0$, it does not matter what **I** does.) In that way the function $f = \bigcup_{n<\omega} f_n$ is a partial isomorphism such that $\operatorname{dom} f = \operatorname{ran} f = \alpha$ for some ordinal α. It is straightforward to check that such an f is coded by some $\nu_\alpha\colon \alpha \to \kappa$. It is an isomorphism between $\mathcal{A}_{\xi_1} \cap \alpha$ and $\mathcal{A}_{\xi_2} \cap \alpha$ and since α is in C, there are ξ_1' and ξ_2' such that $\xi_1 \restriction \alpha \subset \xi_1'$, $\xi_2 \restriction \alpha \subset \xi_2'$ and there is an isomorphism $\mathcal{A}_{\xi_1'} \cong \mathcal{A}_{\xi_2'}$ coded by some ν such that $\nu_\alpha = \nu \restriction \alpha$. Thus $\nu_\alpha \in t(\xi_1, \xi_2)$ is suitable for setting $g(s) = \nu_\alpha$. $\square_{\text{Claim 2}}$

$\square_{\text{Theorem 59}}$

V.2. Classifiable

Throughout this section κ is a regular cardinal satisfying $\kappa^{<\kappa} = \kappa > \omega$.

60. Theorem. *If the theory T is classifiable and shallow, then $\cong_T$ is Borel.*

Proof. If T is classifiable and shallow, then from [**25**] Theorem XIII.1.5 it follows that the models of T are characterized by the game EF^κ_t up to isomorphism, where t is some $\kappa^+\omega$-tree (in fact a tree of descending sequences of an ordinal $\alpha < \kappa^+$). Hence by Theorem 57 the isomorphism relation of T is Borel. $\square$

61. Theorem. *If the theory T is classifiable but not shallow, then $\cong_T$ is not Borel. If κ is not weakly inaccessible and T is not classifiable, then $\cong_T$ is not Borel.*

Proof. If T is classifiable but not shallow, then by [**25**] XIII.1.8, the $L_{\infty\kappa}$-Scott height of models of T of size κ can be any ordinal $< \kappa^+$ (see Definition 7 on page 12). Because any $\kappa^+\omega$-tree can be embedded into $t_\alpha = \{\text{decreasing sequences of } \alpha\}$ for some α (see Fact II.2.1 on page 8), this implies that for any $\kappa^+\omega$-tree t there exists a pair of models $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \not\cong \mathcal{B}$ but $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B})$. Theorem 57 now implies that the isomorphism relation is not Borel.

If T is not classifiable κ is not weakly inaccessible, then by [**26**] Theorem 0.2 (Main Conclusion), there are non-isomorphic models of T of size κ which are $L_{\infty\kappa}$-equivalent, so the same argument as above, using Theorem 57, gives that $\cong_T$ is not Borel. □

62. Theorem. *If the theory T is classifiable, then $\cong_T$ is Δ^1_1.*

Proof. Shelah's theorem [**25**] XIII.1.4 implies that if a theory T is classifiable, then any two models that are $L_{\infty\kappa}$-equivalent are isomorphic. But $L_{\infty\kappa}$ equivalence is equivalent to $\mathrm{EF}^\kappa_\omega$-equivalence (see Theorem 10 on page 13). So in order to prove the theorem it is sufficient to show that if for any two models $\mathcal{A}$, $\mathcal{B}$ of the theory T it holds that $\mathbf{II} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B}) \iff \mathcal{A} \cong \mathcal{B}$, then the isomorphism relation is Δ^1_1. The game $\mathrm{EF}^\kappa_\omega$ is a closed game of length ω and so determined. Hence we have $\mathbf{I} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B}) \iff \mathcal{A} \not\cong \mathcal{B}$. By Theorem 6 the set

$$\{(\nu, \eta, \xi) \in (\kappa^\kappa)^3 \mid \nu \text{ codes a winning strategy for } \mathbf{I} \uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}_\eta, \mathcal{A}_\xi))\}$$

is closed and thus $\{(\eta, \xi) \mid \mathcal{A}_\eta \not\cong \mathcal{A}_\xi\}$ is Σ^1_1, which further implies that $\cong_T$ is Δ^1_1 by Corollary 13. □

V.3. Unclassifiable

V.3.1. The Unstable, DOP And OTOP Cases

As before, κ is a regular cardinal satisfying $\kappa^{<\kappa} = \kappa > \omega$.

63. Theorem. *(1) If T is unstable then $\cong_T$ is not Δ^1_1.*

(2) If T is stable with OTOP, then $\cong_T$ is not Δ^1_1.

(3) If T is superstable with DOP and $\kappa > \omega_1$, then $\cong_T$ is not Δ^1_1.

(4) If for all $\lambda < \kappa$, $\lambda^\omega < \kappa$ and T is stable and unsuperstable, then $\cong_T$ is not Δ^1_1 in the forcing extension after adding κ^+ Cohen subsets of κ.

(5) If T is stable with DOP and $\lambda = \mathrm{cf}(\lambda) = \lambda(T) + \lambda^{<\kappa(T)} \geqslant \omega_1, \kappa > \lambda^+$ and for all $\xi < \kappa$, $\xi^\lambda < \kappa$, then $\cong_T$ is not Δ^1_1.

Proof. For (4), by Theorem 80 the equivalence relation $E_{S^\kappa_\omega}$ can be embedded to $\cong_T$. So (4) follows from Corollary 43.

For (3) we need a result by Hyttinen and Tuuri, Theorem 6.2. from [**14**]:

FACT (Superstable with DOP). *Let T be a superstable theory with DOP and $\kappa^{<\kappa} = \kappa > \omega_1$. Then there exists a model $\mathcal{A}$ of T of cardinality κ with the following property: for every $\kappa^+\kappa$-tree t there is a model $\mathcal{B}$ of T of cardinality κ such that $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A},\mathcal{B})$ and $\mathcal{A} \ncong \mathcal{B}$.*

For (5) we will need a result by Hyttinen and Shelah from [**13**]:

FACT (Stable with DOP). *Let T be a stable theory with DOP and $\lambda = \mathrm{cf}(\lambda) = \lambda(T) + \lambda^{<\kappa(T)} \geqslant \omega_1$, $\kappa^{<\kappa} = \kappa > \lambda^+$ and for all $\xi < \kappa$, $\xi^\lambda < \kappa$. Then there is a model $\mathcal{A}$ of T of power κ such that the following is true: for all $\kappa^+\kappa$-trees t there is a model $\mathcal{B}$ of power κ such that $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A},\mathcal{B})$ and $\mathcal{A} \ncong \mathcal{B}$.*

For (1) a result by Hyttinen and Tuuri Theorem 4.9 from [**14**]:

FACT (Unstable). *Let T be an unstable theory. Then there exists a model $\mathcal{A}$ of T of cardinality κ with the following property: for every $\kappa^+\kappa$-tree t there is a model $\mathcal{B}$ of T of cardinality κ such that $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A},\mathcal{B})$ and $\mathcal{A} \ncong \mathcal{B}$.*

And for (2) another result by Hyttinen and Tuuri [**14**]:

FACT (Stable with OTOP). *Suppose T is a stable theory with OTOP. Then there exists a model $\mathcal{A}$ of T of cardinality κ with the following property: for every $\kappa^+\kappa$-tree t there is a model $\mathcal{B}$ of T of cardinality κ such that $\mathbf{II} \uparrow \mathrm{EF}^\kappa_t(\mathcal{A},\mathcal{B})$ and $\mathcal{A} \ncong \mathcal{B}$.*

Now (1), (2) and (5) follow immediately from Theorem 59. □

V.3.2. Stable Unsuperstable

We assume $\kappa^{<\kappa} = \kappa > \omega$ in all theorems below.

64. THEOREM. *Assume that for all $\lambda < \kappa$, $\lambda^\omega < \kappa$.*

(1) If T is stable unsuperstable, then $\cong_T$ is not Borel.
(2) It is consistent that if κ is as above and T is stable unsuperstable then $\cong_T$ is not Δ^1_1.

Proof. By Theorem 80 on page 100 the relation $E_{S^\kappa_\omega}$ can be reduced to $\cong_T$. The theorem follows now from Corollary 43 on page 54. □

On the other hand, stable unsuperstable theories sometimes behave nicely to some extent:

65. Lemma. *Assume that T is a theory and t a $\kappa^+\kappa$-tree such that if $\mathcal{A}$ and $\mathcal{B}$ are models of T, then $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$. Then $\cong$ of T is Borel*.*

Proof. Similar to the proof of Theorem 57. □

66. Theorem. *Assume $\kappa \in I[\kappa]$ and $\kappa = \lambda^+$ ("$\kappa \in I[\kappa]$" is known as the Approachability Property and follows from $\lambda^{<\lambda} = \lambda$). Then there exists an unsuperstable theory T whose isomorphism relation is Borel*.*

Proof. In [**11**] and [**12**] Hyttinen and Shelah show the following (Theorem 1.1 of [**12**], but the proof is essentially in [**11**]):

> Suppose $T = ((\omega^\omega, E_i)_{i<\omega})$, where $\eta E_i \xi$ if and only if for all $j \leqslant i$, $\eta(j) = \xi(j)$. If $\kappa \in I[\kappa]$, $\kappa = \lambda^+$ and $\mathcal{A}$ and $\mathcal{B}$ are models of T of cardinality κ, then $\mathcal{A} \cong \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}_{\lambda\cdot\omega+2}^\kappa(\mathcal{A}, \mathcal{B})$, where $+$ and $\cdot$ denote the ordinal sum and product, i.e. $\lambda \cdot \omega + 2$ is just an ordinal.

So taking the tree t to be $\lambda \cdot \omega + 2$ the claim follows from Lemma 65. □

Open Problem. We proved that the isomorphism relation of a theory T is Borel if and only if T is classifiable and shallow. Is there a connection between the depth of a shallow theory and the Borel degree of its isomorphism relation? Is one monotone in the other?

VI Reductions

Recall that in Chapter IV we obtained a provable characterization of theories which are both classifiable and shallow in terms of the definability of their isomorphism relations. Without the shallowness condition we obtained only a consistency result. In this chapter we improve this to a provable characterization by analyzing isomorphism relations in terms of Borel reducibility.

67. Definition. Let $E_0 \subset X_0^2$ and $E_1 \subset X_1^2$ be equivalence relations respectively on $X_0 \subset 2^\kappa$ and $X_1 \subset 2^\kappa$. We say that E_0 is continuously (or Borel) reducible to E_1, denoted $E_0 \leqslant_c E_1$ (respectively $E_0 \leqslant_B E_1$), if there is a continuous (respectively Borel) function $f\colon X_0 \to X_1$ such that for all $\eta, \xi \in X_0$

$$(\eta,\xi) \in E_0 \iff (f(\eta), f(\xi)) \in E_1.$$

We say that relations E_0 and E_1 are (Borel) *bireducible* to each other if $E_0 \leqslant_B E_1$ and $E_1 \leqslant_B E_0$.

68. Theorem. *Suppose $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$. Let T be a theory. Then T is classifiable if and only if for all regular $\mu < \kappa$, $E_{S^\kappa_\mu} \not\leqslant_B \cong_T$.*

69. Definition. Let $X \subset \kappa$ be a stationary subset. The equivalence relation E_X is defined by

$$\forall \eta, \xi \in 2^\kappa (\eta E_X \xi \iff (\eta^{-1}\{1\} \triangle \xi^{-1}\{1\}) \cap X \text{ is non-stationary}).$$

These equivalence relations are Σ_1^1 ($A E_X B$ if and only if *there exists* a cub subset of $\kappa \setminus (X \cap (A \triangle B))$).

Simple conclusions can readily be made from the following observation that roughly speaking, set theoretic complexity of a relation does not decrease under reductions:

Fact. *If E_1 is a Borel (or Δ_1^1) equivalence relation and E_0 is an equivalence relation with $E_0 \leqslant_B E_1$, then E_0 is Borel (respectively Δ_1^1 if E_1 is Δ_1^1).* $\square$

VI.1. Classifiable Theories

The following follows from [**25**] Theorem XIII.1.4.

70. Theorem ([**25**]). *If a theory T is classifiable and $\mathcal{A}$ and $\mathcal{B}$ are non-isomorphic models of T, then $\mathbf{I}\uparrow \mathrm{EF}^{\kappa}_{\omega}(\mathcal{A},\mathcal{B})$.* $\square$

71. Theorem ($\kappa^{<\kappa}=\kappa$). *If a theory T is classifiable, then for all $\lambda<\kappa$*

$$E_{S^{\kappa}_{\lambda}} \not\leqslant_B \cong_T .$$

Proof. Let $\mathrm{NS}\in\{E_{S^{\kappa}_{\lambda}}\mid \lambda\in\mathrm{reg}(\kappa)\}$.

Suppose $r\colon 2^{\kappa}\to 2^{\kappa}$ is a Borel function such that

$$\forall\eta,\xi\in 2^{\kappa}(\mathcal{A}_{r(\eta)}\models T\wedge\mathcal{A}_{r(\xi)}\models T\wedge(\eta\,\mathrm{NS}\,\xi\iff\mathcal{A}_{r(\eta)}\cong\mathcal{A}_{r(\xi)})). \qquad (\nabla)$$

By Theorem 44, let D be an intersection of κ-many dense open sets such that $R=r\restriction D$ is continuous. D can be coded into a function $v\colon\kappa\times\kappa\to\kappa^{<\kappa}$ such that $D=\bigcap_{i<\kappa}\bigcup_{j<\kappa}N_{v(i,j)}$. Since R is continuous, it can also be coded into a single function $u\colon\kappa^{<\kappa}\times\kappa^{<\kappa}\to\{0,1\}$ such that

$$R(\eta)=\xi\iff(\forall\alpha<\kappa)(\exists\beta<\kappa)[u(\eta\restriction\beta,\xi\restriction\alpha)=1].$$

(For example define $u(p,q)=1$ if $D\cap N_p\subset R^{-1}[N_q]\}$.) Let

$$\varphi(\eta,\xi,u,v)=(\forall\alpha<\kappa)(\exists\beta<\kappa)[u(\eta\restriction\beta,\xi\restriction\alpha)=1]\wedge(\forall i<\kappa)(\exists j<\kappa)[\eta\in N_{v(i,j)}].$$

It is a formula of set theory with parameters u and v. It is easily seen that φ is absolute for transitive elementary submodels M of $H(\kappa^+)$ containing κ, u and v with $(\kappa^{<\kappa})^M=\kappa^{<\kappa}$. Let $\mathbb{P}=2^{<\kappa}$ be the Cohen forcing. Suppose $M\preccurlyeq H(\kappa^+)$ is a model as above, i.e. transitive, $\kappa,u,v\in M$ and $(\kappa^{<\kappa})^M=\kappa^{<\kappa}$. Note that then $\mathbb{P}\cup\{\mathbb{P}\}\subset M$. Then, if G is $\mathbb{P}$-generic over M, then $\cup G\in D$ and there is ξ such that $\varphi(\cup G,\xi,u,v)$. By the definition of φ and u, an initial segment of ξ can be read from an initial segment of $\cup G$. That is why there is a nice $\mathbb{P}$-name τ for a function (see [**19**]) such that

$$\varphi(\cup G,\tau_G,u,v)$$

whenever G is $\mathbb{P}$-generic over M.

Now since the game $\mathrm{EF}^{\kappa}_{\omega}$ is determined on all structures, (at least) one of the following holds:

(1) there is p such that $p\Vdash\mathbf{II}\uparrow\mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}_{\tau},\mathcal{A}_{r(\bar{0})})$
(2) there is p such that $p\Vdash\mathbf{I}\uparrow\mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}_{\tau},\mathcal{A}_{r(\bar{0})})$

where $\bar{0}$ is the constant function with value 0. Let us show that both of them lead to a contradiction.

Assume (1). Fix a nice $\mathbb{P}$-name σ such that

$$p \Vdash \text{“}\sigma \text{ is a winning strategy of } \mathbf{II} \text{ in } \mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}_{\tau}, \mathcal{A}_{r(\bar{0})})\text{”}$$

A strategy is a subset of $([\kappa]^{<\kappa})^{<\omega} \times \kappa^{<\kappa}$ (see Definition 5), and the forcing does not add elements to that set, so the nice name can be chosen such that all names in $\operatorname{dom}\sigma$ are standard names for elements that are in $([\kappa]^{<\kappa})^{<\omega} \times \kappa^{<\kappa} \in H(\kappa^{+})$.

Let M be an elementary submodel of $H(\kappa^{+})$ of size κ such that

$$\{u, v, \sigma, r(\bar{0}), \tau, \mathbb{P}\} \cup (\kappa + 1) \cup M^{<\kappa} \subset M.$$

Listing all dense subsets of $\mathbb{P}$ in M, it is easy to find a $\mathbb{P}$-generic G over M which contains p and such that $(\cup G)^{-1}\{1\}$ contains a cub. Now in V, $\cup G \not{\mathrm{NS}}\, \bar{0}$. Since $\varphi(\cup G, \tau_G, u, v)$ holds, we have by (∇):

$$\mathcal{A}_{\tau_G} \not\cong \mathcal{A}_{r(\bar{0})}. \tag{i}$$

Let us show that σ_G is a winning strategy of player $\mathbf{II}$ in $\mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}_{\tau_G}, \mathcal{A}_{r(\bar{0})})$ (in V) which by Theorem 70 above is a contradiction with (i).

Let μ be any strategy of player $\mathbf{I}$ in $\mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}_{\tau_G}, \mathcal{A}_{r(\bar{0})})$ and let us show that σ_G beats it. Consider the play $\sigma_G * \mu$ and assume for a contradiction that it is a win for $\mathbf{I}$. This play is well defined, since the moves made by μ are in the domain of σ_G by the note after the definition of σ, and because $([\kappa]^{<\kappa})^{<\omega} \times \kappa^{<\kappa} \subset M$.

The play consists of ω moves and is a countable sequence in the set $([\kappa]^{<\kappa}) \times \kappa^{<\kappa}$ (see Definition of EF-games 5). Since $\mathbb{P}$ is $< \kappa$ closed, there is $q_0 \in \mathbb{P}$ which decides $\sigma_G * \mu$ (i.e. $\sigma_{G_0} * \mu = \sigma_{G_1} * \mu$ whenever $q_0 \in G_0 \cap G_1$). Assume that G' is a $\mathbb{P}$-generic over V with $q_0 \in G'$. Then

$$(\sigma_{G'} * \mu)^{V[G']} = (\sigma_G * \mu)^{V[G']} = (\sigma_G * \mu)^{V}$$

(again, because $\mathbb{P}$ does not add elements of $\kappa^{<\kappa}$) and so

$$(\sigma_{G'} * \mu \text{ is a win for } \mathbf{I})^{V[G']}$$

But $q_0 \Vdash$ “$\sigma * \mu$ is a win for $\mathbf{II}$”, because q_0 extends p and by the choice of σ.

The case (2) is similar, just instead of choosing $\cup G$ such that $(\cup G)^{-1}\{1\}$ contains a cub, choose G such that $(\cup G)^{-1}\{0\}$ contains a cub. Then we should have $\mathcal{A}_{\tau_G} \cong \mathcal{A}_{r(\bar{0})}$ which contradicts (2) by the same absoluteness argument as above. □

VI.2. Unstable And Superstable Theories

In this section we use Shelah's ideas on how to prove non-structure theorems using Ehrenfeucht-Mostowski models, see [**26**]. We use the definition of Ehrenfeucht-Mostowski models from [**14**], Definition 4.2.

72. Definition. In the following discussion of linear orderings we use the following concepts.

- *Coinitiality* or *reverse cofinality* of a linear order η, denoted $\operatorname{cf}^*(\eta)$ is the smallest ordinal α such that there is a map $f\colon \alpha \to \eta$ which is strictly decreasing and $\operatorname{ran} f$ has no (strict) lower bound in η.
- If $\eta = \langle \eta, < \rangle$ is a linear ordering, by η^* we denote its mirror image: $\eta^* = \langle \eta, <^* \rangle$ where $x <^* y \iff y < x$.
- Suppose λ is a cardinal. We say that an ordering η is λ-dense if for all subsets A and B of η with the properties $\forall a \in A \forall b \in B (a < b)$ and $|A| < \lambda$ and $|B| < \lambda$ there is $x \in \eta$ such that $a < x < b$ for all $a \in A$, $b \in B$. Dense means ω-dense.

73. Theorem. *Suppose that $\kappa = \lambda^+ = 2^\lambda$ such that $\lambda^{<\lambda} = \lambda$. If T is unstable or superstable with OTOP, then $E_{S^\kappa_\lambda} \leqslant_c \cong_T$. If additionally $\lambda \geqslant 2^\omega$, then $E_{S^\kappa_\lambda} \leqslant_c \cong_T$ holds also for superstable T with DOP.*

Proof. We will carry out the proof for the case where T is unstable and shall make remarks on how certain steps of the proof should be modified in order this to work for superstable theories with DOP or OTOP. First for each $S \subset S^\kappa_\lambda$, let us construct the linear orders $\Phi(S)$ which will serve a fundamental role in the construction. The following claim is Lemma 7.17 in [**8**]:

Claim 1. For each cardinal μ of uncountable cofinality there exists a linear ordering $\eta = \eta_\mu$ which satisfies:

(1) $\eta \cong \eta + \eta$,
(2) for all $\alpha \leqslant \mu$, $\eta \cong \eta \cdot \alpha + \eta$,
(3) $\eta \cong \eta \cdot \mu + \eta \cdot \omega_1^*$,
(4) η is dense,
(5) $|\eta| = \mu$,
(6) $\operatorname{cf}^*(\eta) = \omega$.

Proof of Claim 1. Exactly as in [**8**]. $\square_{\text{Claim 1}}$

For a set $S \subset S^\kappa_\lambda$, define the linear order $\Phi(S)$ as follows:

$$\Phi(S) = \sum_{i<\kappa} \tau(i,S),$$

where $\tau(i,S) = \eta_\lambda$ if $i \notin S$ and $\tau(i,S) = \eta_\lambda \cdot \omega_1^*$, if $i \in S$. Note that $\Phi(S)$ is dense. For $\alpha < \beta < \kappa$ define

$$\Phi(S,\alpha,\beta) = \sum_{\alpha \leqslant i < \beta} \tau(i,S).$$

(These definitions are also as in [**8**] although the idea dates back to J. Conway's Ph.D. thesis from the 1960's; they are first referred to in [**23**]). From now on denote $\eta = \eta_\lambda$.

Claim 2. If $\alpha \notin S$, then for all $\beta \geqslant \alpha$ we have $\Phi(S,\alpha,\beta+1) \cong \eta$ and if $\alpha \in S$, then for all $\beta \geqslant \alpha$ we have $\Phi(S,\alpha,\beta+1) \cong \eta \cdot \omega_1^*$.

Proof of Claim 2. Let us begin by showing the first part, i.e. assume that $\alpha \notin S$. This is also like in [**8**]. We prove the statement by induction on $\operatorname{OTP}(\beta \setminus \alpha)$. If $\beta = \alpha$, then $\Phi(S,\alpha,\alpha+1) = \eta$ by the definition of Φ. If $\beta = \gamma+1$ is a successor, then $\beta \notin S$, because S contains only limit ordinals, so $\tau(\beta,S) = \eta$ and

$$\Phi(S,\alpha,\beta+1) = \Phi(S,\alpha,\gamma+1+1) = \Phi(S,\alpha,\gamma+1) + \eta$$

which by the induction hypothesis and by (1) is isomorphic to η. If $\beta \notin S$ is a limit ordinal, then choose a continuous cofinal sequence $s\colon \operatorname{cf}(\beta) \to \beta$ such that $s(\gamma) \notin S$ for all $\gamma < \operatorname{cf}(\beta)$. This is possible since S contains only ordinals of cofinality λ. By the induction hypothesis $\Phi(S,\alpha,s(0)+1) \cong \eta$,

$$\Phi(S,s(\gamma)+1,s(\gamma+1)+1) \cong \eta$$

for all successor ordinals $\gamma < \operatorname{cf}(\beta)$,

$$\Phi(S,s(\gamma),s(\gamma+1)+1) \cong \eta$$

for all limit ordinals $\gamma < \operatorname{cf}(\beta)$ and so now

$$\Phi(S,\alpha,\beta+1) \cong \eta \cdot \operatorname{cf}(\beta) + \eta$$

which is isomorphic to η by (2). If $\beta \in S$, then $\operatorname{cf}(\beta) = \lambda$ and we can again choose a cofinal sequence $s\colon \lambda \to \beta$ such that $s(\alpha)$ is not in S for all $\alpha < \lambda$. By the induction hypothesis. as above,

$$\Phi(S,\alpha,\beta+1) \cong \eta \cdot \lambda + \tau(\beta,S)$$

and since $\beta \in S$ we have $\tau(\beta, S) = \eta \cdot \omega_1^*$, so we have

$$\Phi(S, \alpha, \beta + 1) \cong \eta \cdot \lambda + \eta \cdot \omega_1^*$$

which by (3) is isomorphic to η.

Suppose $\alpha \in S$. Then $\alpha + 1 \notin S$, so by the previous part we have

$$\Phi(S, \alpha, \beta + 1) \cong \tau(\alpha, S) + \Phi(S, \alpha + 1, \beta + 1) = \eta \cdot \omega_1^* + \eta = \eta \cdot \omega_1^*.$$

$\square_{\text{Claim 2}}$

This gives us a way to show that the isomorphism type of $\Phi(S)$ depends only on the $E_{S^\kappa_\lambda}$-equivalence class of S:

Claim 3. If $S, S' \subset S^\kappa_\lambda$ and $S \triangle S'$ is non-stationary, then $\Phi(S) \cong \Phi(S')$.

Proof of Claim 3. Let C be a cub set outside $S \triangle S'$. Enumerate it $C = \{\alpha_i \mid i < \kappa\}$ where $(\alpha_i)_{i<\kappa}$ is an increasing and continuous sequence. Now $\Phi(S) = \bigcup_{i<\kappa} \Phi(S, \alpha_i, \alpha_{i+1})$ and $\Phi(S') = \bigcup_{i<\kappa} \Phi(S', \alpha_i, \alpha_{i+1})$. Note that by the definitions these are disjoint unions, so it is enough to show that for all $i < \kappa$ the orders $\Phi(S, \alpha_i, \alpha_{i+1})$ and $\Phi(S', \alpha_i, \alpha_{i+1})$ are isomorphic. But for all $i < \kappa$ $\alpha_i \in S \iff \alpha_i \in S'$, so by Claim 2 either

$$\Phi(S, \alpha_i, \alpha_{i+1}) \cong \eta \cong \Phi(S', \alpha_i, \alpha_{i+1})$$

(if $\alpha_i \notin S$) or

$$\Phi(S, \alpha_i, \alpha_{i+1}) \cong \eta \cdot \omega_1^* \cong \Phi(S', \alpha_i, \alpha_{i+1})$$

(if $\alpha_i \in S$). $\square_{\text{Claim 3}}$

74. Definition. K^λ_{tr} is the set of L-models $\mathcal{A}$ where $L = \{<, \lessdot, (P_\alpha)_{\alpha \leqslant \lambda}, h\}$, with the properties

- $\operatorname{dom} \mathcal{A} \subset I^{\leqslant \lambda}$ for some linear order I.
- $\forall x, y \in A(x < y \iff x \subset y)$.
- $\forall x \in A(P_\alpha(x) \iff \text{length}(x) = \alpha)$.
- $\forall x, y \in A[x \lessdot y \iff \exists z \in A((x, y \in \text{Succ}(z)) \wedge (I \models x < y))]$
- $h(x, y)$ is the maximal common initial segment of x and y.

For each S, define the tree $T(S) \in K^\lambda_{tr}$ by

$$T(S) = \Phi(S)^{<\lambda} \cup \{\eta \colon \lambda \to \Phi(S) \mid \eta \text{ increasing and } \operatorname{cf}^*(\Phi(S) \setminus \{x \mid (\exists y \in \operatorname{ran} \eta)(x < y)\}) = \omega_1\}.$$

The relations $<$, $\lessdot$, P_n and h are interpreted in the natural way.

Clearly an isomorphism between $\Phi(S)$ and $\Phi(S')$ induces an isomorphism between $T(S)$ and $T(S')$, thus $T(S) \cong T(S')$ if $S \triangle S'$ is non-stationary.

Claim 4. Suppose T is unstable in the vocabulary v. Let T_1 be T with Skolem functions in the Skolemized vocabulary $v_1 \supset v$. Then there is a function $\mathcal{P}(S^\kappa_\lambda) \to \{\mathcal{A}^1 \mid \mathcal{A}^1 \models T_1, |\mathcal{A}^1| = \kappa\}$, $S \mapsto \mathcal{A}^1(S)$ which has following properties:

(a) There is a mapping $T(S) \to (\operatorname{dom} \mathcal{A}^1(S))^n$ for some $n < \omega$, $\eta \mapsto a_\eta$, such that $\mathcal{A}^1(S)$ is the Skolem hull of $\{a_\eta \mid \eta \in T(S)\}$, i.e. $\{a_\eta \mid \eta \in T(S)\}$ is the skeleton of $\mathcal{A}^1(S)$. Denote the skeleton of $\mathcal{A}$ by $\mathrm{Sk}(\mathcal{A})$.

(b) $\mathcal{A}(S) = \mathcal{A}^1(S) \restriction v$ is a model of T.

(c) $\mathrm{Sk}(\mathcal{A}^1(S))$ is indiscernible in $\mathcal{A}^1(S)$, i.e. if $\bar\eta, \bar\xi \in T(S)$ and $\mathrm{tp}_{\text{q.f.}}(\bar\eta/\varnothing) = \mathrm{tp}_{\text{q.f.}}(\bar\xi/\varnothing)$, then $\mathrm{tp}(a_{\bar\eta}/\varnothing) = \mathrm{tp}(a_{\bar\xi}/\varnothing)$ where $a_{\bar\eta} = (a_{\eta_1}, \dots, a_{\eta_{\text{length}\,\bar\eta}})$. This assignment of types in $\mathcal{A}^1(S)$ to q.f.-types in $T(S)$ is independent of S.

(d) There is a formula $\varphi \in L_{\omega\omega}(v)$ such that for all $\eta, \nu \in T(S)$ and $\alpha < \lambda$, if $T(S) \models P_\lambda(\eta) \wedge P_\alpha(\nu)$, then $T(S) \models \eta > \nu$ if and only if $\mathcal{A}(S) \models \varphi(a_\eta, a_\nu)$.

Proof of Claim 4. The following is known:

(F1) Suppose that T is a complete unstable theory. Then for each linear order η, T has an Ehrenfeucht-Mostowski model $\mathcal{A}$ of vocabulary v_1, where $|v_1| = |T| + \omega$ and order is definable by a first-order formula, such that the template (assignment of types) is independent of η.[1]

It is not hard to see that for every tree $t \in K^\omega_{tr}$ we can define a linear order $L(t)$ satisfying the following conditions:

(1) $\mathrm{dom}(L(t)) = (\operatorname{dom} t \times \{0\}) \cup (\operatorname{dom} t \times \{1\})$,

(2) for all $a \in t$, $(a, 0) <_{L(t)} (a, 1)$,

(3) if $a, b \in t$, then $a <_t b \iff [(a,0) <_{L(t)} (b,0)] \wedge [(b,1) <_{L(t)} (a,1)]$,

(4) if $a, b \in t$, then

$$(a \not\leqslant b) \wedge (b \not\leqslant a) \iff [(b,1) <_{L(t)} (a,0)] \vee [(a,1) <_{L(t)} (b,0)].$$

Now for every $S \subset \kappa$, by (F1), there is an Ehrenfeucht-Mostowski model $\mathcal{A}^1(S)$ for the linear order $L(T(S))$ where order is definable by the formula ψ

---

[1] This is from [**27**]; there is a sketch of the proof also in [**14**], Theorem 4.7.

which is in $L_{\infty\omega}$. Suppose $\bar\eta = (\eta_0, \dots, \eta_n)$ and $\bar\xi = (\xi_0, \dots, \xi_n)$ are sequences in $T(S)$ that have the same quantifier free type. Then the sequences

$$\langle(\eta_0, 0), (\eta_0, 1), (\eta_1, 0), (\eta_1, 1), \dots, (\eta_n, 0), (\eta_n, 1)\rangle$$

and

$$\langle(\xi_0, 0), (\xi_0, 1), (\xi_1, 0), (\xi_1, 1), \dots, (\xi_n, 0), (\xi_n, 1)\rangle$$

have the same quantifier free type in $L(T(S))$ (refer to this property as (#)). Now let the canonical skeleton of $\mathcal{A}^1(S)$ given by (F1) be $\{a_x \mid x \in L(T(S))\}$. Define the $T(S)$-skeleton of $\mathcal{A}^1(S)$ to be the set

$$\{a_{(\eta,0)}{}^\frown a_{(\eta,1)} \mid \eta \in T(S)\}.$$

Let us denote $b_\eta = a_{(\eta,0)}{}^\frown a_{(\eta,1)}$. This guarantees that (a), (b) and (c) are satisfied.

For (d) suppose that the order $L(T(S))$ is definable in $\mathcal{A}(S)$ by the formula $\psi(\bar u, \bar c)$, i.e. $\mathcal{A}(S) \models \psi(a_x, a_y) \iff x < y$ for $x, y \in L(T(S))$. Let $\varphi(x_0, x_1, y_0, y_1)$ be the formula

$$\psi(x_0, y_0) \wedge \psi(y_1, x_1).$$

Suppose $\eta, \nu \in T(S)$ are such that $T(S) \models P_\lambda(\eta) \wedge P_\alpha(\nu)$. Then

$$\varphi((a_\nu, 0), (a_\nu, 1), (a_\eta, 0), (a_\eta, 1))$$

holds in $\mathcal{A}(S)$ if and only if $\nu <_{T(S)} \eta$. $\square_{\text{Claim 4}}$

Claim 5. Suppose $S \mapsto \mathcal{A}(S)$ is a function as described in Claim 4 with the identical notation. Suppose further that $S, S' \subset S^\kappa_\lambda$. Then $S \bigtriangleup S'$ is non-stationary if and only if $\mathcal{A}(S) \cong \mathcal{A}(S')$.

Proof of Claim 5. Suppose $S \bigtriangleup S'$ is non-stationary. Then by Claim 3 $T(S) \cong T(S')$ which implies $L(T(S)) \cong L(T(S'))$ (defined in the proof of Claim 4) which in turn implies $\mathcal{A}(S) \cong \mathcal{A}(S')$.

Let us now show that if $S \bigtriangleup S'$ is stationary, then $\mathcal{A}(S) \not\cong \mathcal{A}(S')$. Let us make a counter assumption, namely that there is an isomorphism

$$f \colon \mathcal{A}(S) \cong \mathcal{A}(S')$$

and that $S \bigtriangleup S'$ is stationary, and let us deduce a contradiction. Without loss of generality we may assume that $S \setminus S'$ is stationary. Denote

$$X_0 = S \setminus S'$$

For all $\alpha < \kappa$ define $T^\alpha(S)$ and $T^\alpha(S')$ by

$$T^\alpha(S) = \{\eta \in T(S) \mid \operatorname{ran}\eta \subset \Phi(S,0,\beta+1) \text{ for some } \beta < \alpha\}$$

and

$$T^\alpha(S') = \{\eta \in T(S) \mid \operatorname{ran}\eta \subset \Phi(S',0,\beta+1) \text{ for some } \beta < \alpha\}.$$

Then we have:

(i) if $\alpha < \beta$, then $T^\alpha(S) \subset T^\beta(S)$
(ii) if γ is a limit ordinal, then $T^\gamma(S) = \bigcup_{\alpha<\gamma} T^\alpha(S)$

The same of course holds for S'. Note that if $\alpha \in S \setminus S'$, then there is $\eta \in T^\alpha(S)$ cofinal in $\Phi(S,0,\alpha)$ but there is no such $\eta \in T^\alpha(S')$ by definition of Φ: a cofinal function η is added only if $\mathrm{cf}^*(\Phi(S',\alpha,\kappa)) = \omega_1$ which it is not if $\alpha \notin S'$ This is the key to achieving the contradiction.

But the clauses (i),(ii) are not sufficient to carry out the following argument, because we would like to have $|T^\alpha(S)| < \kappa$. That is why we want to define a different kind of filtration for $T(S)$, $T(S')$.

For all $\alpha \in X_0$ fix a function

$$\eta^\alpha_\lambda \in T(S) \tag{\#\#}$$

such that $\operatorname{dom}\eta^\alpha_\lambda = \lambda$, for all $\beta < \lambda$, $\eta^\alpha_\lambda \restriction \beta \in T^\alpha(S)$ and $\eta^\alpha_\lambda \notin T^\alpha(S)$.

For arbitrary $A \subset T(S) \cup T(S')$ let $\mathrm{cl}_{\mathrm{Sk}}(A)$ be the set $X \subset \mathcal{A}(S) \cup \mathcal{A}(S')$ such that $X \cap \mathcal{A}(S)$ is the Skolem closure of $\{a_\eta \mid \eta \in A \cap T(S)\}$ and $X \cap \mathcal{A}(S')$ the Skolem closure of $\{a_\eta \mid \eta \in A \cap T(S')\}$. The following is easily verified:

There exists a λ-cub set C and a set $K^\alpha \subset T^\alpha(S) \cup T^\alpha(S')$ for each $\alpha \in C$ such that

(i') If $\alpha < \beta$, then $K^\alpha \subset K^\beta$
(ii') If γ is a limit ordinal in C, then $K^\gamma = \bigcup_{\alpha \in C\cap\gamma} K^\alpha$
(iii) for all $\beta < \alpha$, $\eta^\beta_\lambda \in K^\alpha$. (see (##) above)
(iv) $|K^\alpha| = \lambda$.
(v) $\mathrm{cl}_{\mathrm{Sk}}(K^\alpha)$ is closed under $f \cup f^{-1}$.
(vi) $\{\eta \in T^\alpha(S) \cup T^\alpha(S') \mid \operatorname{dom}\eta < \lambda\} \subset K^\alpha$.
(vii) K^α is downward closed.

Denote $K^\kappa = \bigcup_{\alpha<\kappa} K^\alpha$. Clearly K^κ is closed under $f \cup f^{-1}$ and so f is an isomorphism between $\mathcal{A}(S) \cap \mathrm{cl}_{\mathrm{Sk}}(K^\kappa)$ and $\mathcal{A}(S') \cap \mathrm{cl}_{\mathrm{Sk}}(K^\kappa)$. We will derive a contradiction from this, i.e. we will actually show that $\mathcal{A}(S) \cap \mathrm{cl}_{\mathrm{Sk}}(K^\kappa)$ and $\mathcal{A}(S') \cap \mathrm{cl}_{\mathrm{Sk}}(K^\kappa)$ cannot be isomorphic by f. Clauses (iii), (v), (vi) and (vii) guarantee that all elements we are going to deal with will be in K^κ.

Let

$$X_1 = X_0 \cap C.$$

For $\alpha \in X_1$ let us use the following abbreviations:

- By $\mathcal{A}_\alpha(S)$ denote the Skolem closure of $\{a_\eta \mid \eta \in K^\alpha \cap T(S)\}$.
- By $\mathcal{A}_\alpha(S')$ denote the Skolem closure of $\{a_\eta \mid \eta \in K^\alpha \cap T(S')\}$.
- $K^\alpha(S) = K^\alpha \cap T(S)$.
- $K^\alpha(S') = K^\alpha \cap T(S')$.

In the following we will often deal with finite sequences. When defining such a sequence we will use a bar, but afterwards we will not use the bar in the notation (e.g. let $a = \bar{a}$ be a finite sequence...).

Suppose $\alpha \in X_1$. Choose

$$\xi^\alpha_\lambda = \bar{\xi}^\alpha_\lambda \in T(S') \tag{\#\#\#}$$

to be such that for some (finite sequence of) terms $\pi = \bar{\pi}$ we have

$$\begin{aligned} & f(a_{\eta^\alpha_\lambda}) = \pi(a_{\xi^\alpha_\lambda}) \\ = \quad & \langle \pi_1(a_{\xi^\alpha_\lambda(1)}, \dots, a_{\xi^\alpha_\lambda(\mathrm{length}(\bar{\xi}^\alpha_\lambda))}), \dots \pi_{\mathrm{length}\,\bar{\pi}}(a_{\xi^\alpha_\lambda(1)}, \dots, a_{\xi^\alpha_\lambda(\mathrm{length}(\xi^\alpha_\lambda))})\rangle. \end{aligned}$$

Note that ξ^α_λ is in K^κ by the definition of K^α's.

$$\text{Let us denote by } \eta^\alpha_\beta\text{, the element } \eta^\alpha_\lambda \restriction \beta. \tag{\#\#\#\#}$$

Let

$$\xi^\alpha_* = \{\nu \in T(S') \mid \exists \xi \in \xi^\alpha_\lambda (\nu < \xi)\}.$$

Also note that $\xi^\alpha_* \subset K^\beta$ for some β.

Next define the function $g\colon X_1 \to \kappa$ as follows. Suppose $\alpha \in X_1$. Let $g(\alpha)$ be the smallest ordinal β such that $\xi^\alpha_* \cap K^\alpha(S') \subset K^\beta(S')$. We claim that $g(\alpha) < \alpha$. Clearly $g(\alpha) \leqslant \alpha$, so suppose that $g(\alpha) = \alpha$. Since ξ^α_λ is finite, there must be a $\xi^\alpha_\lambda(i) \in \xi^\alpha_\lambda$ such that for all $\beta < \alpha$ there exists γ such that $\xi^\alpha_\lambda(i) \restriction \gamma \in K^\alpha(S') \setminus K^\beta(S')$, i.e. $\xi^\alpha_\lambda(i)$ is cofinal in $\Phi(S', 0, \alpha)$ which it cannot be, because $\alpha \notin S'$.

Now by Fodor's lemma there exists a stationary set

$$X_2 \subset X_1$$

and γ_0 such that $g[X_2] = \{\gamma_0\}$.

Since there is only $< \kappa$ many finite sequences in $\mathcal{A}_{\gamma_0}(S')$, there is a stationary set

$$X_3 \subset X_2$$

and a finite sequence $\xi = \bar{\xi} \in K^{\gamma_0}(S')$ such that for all $\alpha \in X_3$ we have $\xi_*^\alpha \cap K^{\gamma_0}(S') = \xi_*$ where ξ_* is the set

$$\xi_* = \{\nu \in T(S') \mid \nu \leqslant \zeta \text{ for some } \zeta \in \bar{\xi}\} \subset K^{\gamma_0}(S').$$

Let us fix a (finite sequence of) term(s) $\pi = \bar{\pi}$ such that the set

$$X_4 = \{\alpha \in X_3 \quad | \quad f(a_{\eta_\lambda^\alpha}) = \pi(a_{\xi_\lambda^\alpha})\}$$

is stationary (see (##)). Here $f(\bar{a})$ means $\langle f(a_1), \dots, f(a_{\text{length}\,\bar{a}})\rangle$ and $\bar{\pi}(\bar{b})$ means

$$\langle \pi_1(b_1, \dots, b_{\text{length}\,\bar{a}}), \dots, \pi_{\text{length}\,\pi}(b_1, \dots, b_{\text{length}\,\bar{a}})\rangle.$$

We can find such π because there are only countably many such finite sequences of terms.

We claim that in $T(S')$ there are at most λ many quantifier free types over ξ_*. All types from now on are quantifier free. Let us show that there are at most λ many 1-types; the general case is left to the reader. To see this, note that a type p over ξ_* is described by the triple

$$(\nu_p, \beta_p, m_p) \tag{$\star$}$$

defined as follows: if η satisfies p, then ν_p is the maximal element of ξ_* that is an initial segment of η, β_p is the level of η and m_p tells how many elements of $\xi_* \cap P_{\text{dom}\,\nu_p+1}$ are there $<$-below $\eta(\text{dom}\,\nu_p)$ (recall the vocabulary from Definition 74).

Since $\nu_p \in \xi_*$ and ξ_* is of size λ, $\beta_p \in (\lambda+1) \cup \{\infty\}$ and $m_p < \omega$, there can be at most λ such triples.

Recall the notations (##), (###) and (####) above.

We can pick ordinals $\alpha < \alpha'$, $\alpha, \alpha' \in X_4$, a term τ and an ordinal $\beta < \lambda$ such that

$$\eta_\beta^{\alpha'} \neq \eta_\beta^\alpha,$$

$$f(\eta_\beta^\alpha) = \tau(a_{\xi_\beta^\alpha}) \text{ and } f(\eta_\beta^{\alpha'}) = \tau(a_{\xi_\beta^{\alpha'}}) \text{ for some } \xi_\beta^\alpha, \xi_\beta^{\alpha'}$$

$$\text{tp}(\xi_\lambda^\alpha/\xi_*) = \text{tp}(\xi_\lambda^{\alpha'}/\xi_*)$$

and

$$\text{tp}(\xi_\beta^\alpha/\xi_*) = \text{tp}(\xi_\beta^{\alpha'}/\xi_*).$$

We claim that then in fact

$$\text{tp}(\xi_\beta^\alpha/(\xi_* \cup \{\xi_l^{\alpha'}\})) = \text{tp}(\xi_\beta^{\alpha'}/(\xi_* \cup \{\xi^{\alpha'}\})).$$

Let us show this. Denote

$$p = \text{tp}(\xi_\beta^\alpha/(\xi_* \cup \{\xi_\lambda^{\alpha'}\}))$$

and

$$p' = \mathrm{tp}(\xi^{\alpha'}_\beta / (\xi_* \cup \{\xi^{\alpha'}_\lambda\})).$$

By the same reasoning as above at $(\star)$ it is sufficient to show that these types p and p' have the same triple of the form $(\star)$. Since α and α' are in X_3 and X_2, we have $\xi^{\alpha'}_* \cap K^{\alpha'}(S') = \xi_* \subset K^{\gamma_0}(S')$. On the other hand $f \restriction \mathcal{A}_{\alpha'}(S)$ is an isomorphism between $\mathcal{A}_{\alpha'}(S)$ and $\mathcal{A}_{\alpha'}(S')$, because α and α' are in X_1, and so $\xi^{\alpha'}_\beta \in K^{\alpha'}(S')$. Thus $\nu_p = \nu_{p'} \in \xi_*$ and $m_p = m_{p'}$ follows in the same way. Clearly $\beta_p = \beta_{p'}$.

Now we have: ξ^α_λ and π are such that $f(\eta^\alpha_\lambda) = \pi(\xi^\alpha_\lambda)$ and ξ^α_β and τ are such that $f(\eta^\alpha_\beta) = \tau(\xi^\alpha_\beta)$. Similarly for α'. The formula φ is defined in Claim 4.

We know that

$$\mathcal{A}(S) \models \varphi(a_{\eta^{\alpha'}_\lambda}, a_{\eta^{\alpha'}_\beta})$$

and because f is isomorphism, this implies

$$\mathcal{A}(S') \models \varphi(f(a_{\eta^{\alpha'}_\lambda}), f(a_{\eta^{\alpha'}_\beta}))$$

which is equivalent to

$$\mathcal{A}(S') \models \varphi(\pi(a_{\xi^{\alpha'}_\lambda}), \tau(a_{\xi^{\alpha'}_\beta}))$$

(because α, α' are in X_4). Since $T(S')$ is indiscernible in $\mathcal{A}(S')$ and $\xi^{\alpha'}_\beta$ and ξ^α_β have the same type over over $(\xi_* \cup \{\xi^{\alpha'}_\lambda\})$, we have

$$\mathcal{A}(S') \models \varphi(\pi(a_{\xi^{\alpha'}_\lambda}), \tau(a_{\xi^{\alpha'}_\beta})) \iff \varphi(\pi(a_{\xi^{\alpha'}_\lambda}), \tau(a_{\xi^{\alpha}_\beta})) \tag{$*$}$$

and so we get

$$\mathcal{A}(S') \models \varphi(\pi(a_{\xi^{\alpha'}_\lambda}), \tau(a_{\xi^{\alpha}_\beta}))$$

which is equivalent to

$$\mathcal{A}(S') \models \varphi(f(a_{\eta^{\alpha'}_\lambda}), f(a_{\eta^{\alpha}_\beta}))$$

and this in turn is equivalent to

$$\mathcal{A}(S) \models \varphi(a_{\eta^{\alpha'}_\lambda}, a_{\eta^{\alpha}_\beta})$$

The latter cannot be true, because the definition of β, α and α' implies that $\eta^{\alpha'}_\beta \neq \eta^\alpha_\beta$. $\square_{\text{Claim 5}}$

Thus, the above Claims 1 – 5 justify the embedding of $E_{S^\kappa_\lambda}$ into the isomorphism relation on the set of structures that are models for T for unstable T. This embedding combined with a suitable coding of models gives a continuous map.

DOP and OTOP cases. The above proof was based on the fact (F1) that for unstable theories there are Ehrenfeucht-Mostowski models for any linear order such that the order is definable by a first-order formula φ and is indiscernible relative to $L_{\omega\omega}$, (see (c) on page 85); it is used in $(*)$ above. For the OTOP case, we use instead the fact (F2):

(F2) Suppose that T is a theory with OTOP in a countable vocabulary v. Then for each dense linear order η we can find a model $\mathcal{A}$ of a countable vocabulary $v_1 \supset v$ such that $\mathcal{A}$ is an Ehrenfeucht-Mostowski model of T for η where order is definable by an $L_{\omega_1\omega}$-formula.[2]

Since the order $\Phi(S)$ is dense, it is easy to argue that if $T(S)$ is indiscernible relative to $L_{\omega\omega}$, then it is indiscernible relative to $L_{\infty\omega}$ (define this as in (c) on page 85 changing tp to $\mathrm{tp}_{L_{\infty\omega}}$). Other parts of the proof remain unchanged, because although the formula φ is not first-order anymore, it is still in $L_{\infty\omega}$.

In the DOP case we have the following fact:

(F3) Let T be a countable superstable theory with DOP of vocabulary v. Then there exists a vocabulary $v_1 \supset v$, $|v_1| = \omega_1$, such that for every linear order η there exists a v_1-model $\mathcal{A}$ which is an Ehrenfeucht-Mostowski model of T for η where order is definable by an $L_{\omega_1\omega_1}$-formula.[3]

Now the problem is that φ is in $L_{\infty\omega_1}$. By (c) of Claim 4, $T(S)$ is indiscernible in $\mathcal{A}(S)$ relative to $L_{\omega\omega}$ and by the above relative to $L_{\infty\omega}$. If we could require $\Phi(S)$ to be ω_1-dense, we would similarly get indiscernible relative to $L_{\infty\omega_1}$. Let us show how to modify the proof in order to do that. Recall that in the DOP case,we assume $\lambda \geqslant 2^{\omega}$.

In Claim 1 (page 82), we have to replace clauses (3), (4) and (6) by (3'), (4') and (6'):

(3') $\eta \cong \eta \cdot \mu + \eta \cdot \omega^*$,
(4') η is ω_1-dense,
(6') $\mathrm{cf}^*(\eta) = \omega_1$.

The proof that such an η exists is exactly as the proof of Lemma 7.17 [**8**] except that instead of putting $\mu = (\omega_1)^V$ put $\mu = \omega$, build θ-many functions with domains being countable initial segments of ω_1 instead of finite initial

---

[2] Contained in the proof of Theorem 2.5. of [**24**]; see also [**14**], Theorem 6.6.

[3] This is essentially from [**28**] Fact 2.5B; a proof can be found also in [**14**] Theorem 6.1.

segments of ω and instead of $\mathbb{Q}$ (the countable dense linear order) use an ω_1-saturated dense linear order – this order has size 2^ω and that is why the assumption $\lambda \geqslant 2^\omega$ is needed.

In the definition of $\Phi(S)$ (right after Claim 1), replace ω_1^* by ω^* and η by the new η satisfying (3'), (4') and (6') above. Note that $\Phi(S)$ becomes now ω_1-dense. In Claim 2 one has to replace ω_1^* by ω^*. The proof remains similar. In the proof of Claim 3 (page 84) one has to adjust the use of Claim 2. Then, in the definition of $T(S)$ replace ω_1 by ω.

Claim 4 for superstable T with DOP now follows with (c) and (d) modified: instead of indiscernible relative to $L_{\omega\omega}$, demand $L_{\infty\omega_1}$ and instead of $\varphi \in L_{\omega\omega}$ we have now $\varphi \in L_{\infty\omega_1}$. The proof is unchanged except that the language is replaced by $L_{\infty\omega_1}$ everywhere and fact (F1) replaced by (F3) above.

Everything else in the proof, in particular the proof of Claim 5, remains unchanged modulo some obvious things that are evident from the above explanation. $\square_{\text{Theorem 73}}$

VI.3. Stable Unsuperstable Theories

In this section we provide a tree construction (Lemma 79) which is similar to Shelah's construction in [**26**] which he used to obtain (via Ehrenfeucht-Mostowski models) many pairwise non-isomorphic models. Then using a prime-model construction (proof of Theorem 80) we will obtain the needed result.

75. Definition. Let I be a tree of size κ. Suppose $(I_\alpha)_{\alpha<\kappa}$ is a collection of subsets of I such that

- For each $\alpha < \kappa$, I_α is a downward closed subset of I
- $\bigcup_{\alpha<\kappa} I_\alpha = I$
- If $\alpha < \beta < \kappa$, then $I_\alpha \subset I_\beta$
- If γ is a limit ordinal, then $I_\gamma = \bigcup_{\alpha<\gamma} I_\alpha$
- For each $\alpha < \kappa$ the cardinality of I_α is less than κ.

Such a sequence $(I_\alpha)_{\alpha<\kappa}$ is called *κ-filtration* or just *filtration* of I.

76. Definition. Recall K^λ_{tr} from Definition 74 on page 84. Let $K^\lambda_{tr*} = \{A \restriction L^* \mid A \in K^\lambda_{tr}\}$, where L^* is the vocabulary $\{<\}$.

77. Definition. Suppose $t \in K^{\omega}_{tr*}$ is a tree of size κ (i.e. $t \subset \kappa^{\leqslant\omega}$) and let $\mathcal{I} = (I_\alpha)_{\alpha<\kappa}$ be a filtration of t. Define

$$S_{\mathcal{I}}(t) = \Big\{\alpha < \kappa \mid (\exists \eta \in t)\big[(\operatorname{dom}\eta = \omega) \wedge \forall n < \omega(\eta\restriction n \in I_\alpha) \wedge (\eta \notin I_\alpha)\big]\Big\}$$

By $S \sim_{\mathrm{NS}} S'$ we mean that $S \mathbin{\triangle} S'$ is not ω-stationary

78. Lemma. *Suppose trees t_0 and t_1 are isomorphic, and $\mathcal{I} = (I_\alpha)_{\alpha<\kappa}$ and $\mathcal{J} = (J_\alpha)_{\alpha<\kappa}$ are κ-filtrations of t_0 and t_1 respectively. Then $S_{\mathcal{I}}(t_0) \sim_{\mathrm{NS}} S_{\mathcal{J}}(t_1)$.*

Proof. Let $f\colon t_0 \to t_1$ be an isomorphism. Then $f\mathcal{I} = (f[I_\alpha])_{\alpha<\kappa}$ is a filtration of t_1 and

$$\alpha \in S_{\mathcal{I}}(t_0) \iff \alpha \in S_{f\mathcal{I}}(t_1). \tag{$\star$}$$

Define the set $C = \{\alpha \mid f[I_\alpha] = J_\alpha\}$. Let us show that it is cub. Let $\alpha \in \kappa$. Define $\alpha_0 = \alpha$ and by induction pick $(\alpha_n)_{n<\omega}$ such that $f[I_{\alpha_n}] \subset J_{\alpha_{n+1}}$ for odd n and $J_{\alpha_n} \subset f[I_{\alpha_{n+1}}]$ for even n. This is possible by the definition of a κ-filtration. Then $\alpha_\omega = \bigcup_{n<\omega} \alpha_n \in C$. Clearly C is closed and $C \subset \kappa \setminus S_{f\mathcal{I}}(t_1) \mathbin{\triangle} S_{\mathcal{J}}(t_1)$, so now by ($\star$)

$$S_{\mathcal{I}}(t_0) = S_{f\mathcal{I}}(t_1) \sim_{\mathrm{NS}} S_{\mathcal{J}}(t_1). \qquad \square$$

79. Lemma. *Suppose for $\lambda < \kappa$, $\lambda^\omega < \kappa$ and $\kappa^{<\kappa} = \kappa$. There exists a function $J\colon \mathcal{P}(\kappa) \to K^{\omega}_{tr*}$ such that*

- *$\forall S \subset \kappa(|J(S)| = \kappa)$.*
- *If $S \subset \kappa$ and $\mathcal{I}$ is a κ filtration of $J(S)$, then $S_{\mathcal{I}}(J(S)) \sim_{\mathrm{NS}} S$.*
- *If $S_0 \sim_{\mathrm{NS}} S_1$, then $J(S_0) \cong J(S_1)$.*

Proof. Let $S \subset S^\kappa_\omega$ and let us define a preliminary tree $I(S)$ as follows. For each $\alpha \in S$ let C_α be the set of all strictly increasing cofinal functions $\eta\colon \omega \to \alpha$. Let $I(S) = \underline{[\kappa]}^{<\omega} \cup \bigcup_{\alpha\in S} C_\alpha$ where $\underline{[\kappa]}^{<\omega}$ is the set of strictly increasing functions from finite ordinals to κ.

For ordinals $\alpha < \beta \leqslant \kappa$ and $i < \omega$ we adopt the notation:

- $[\alpha,\beta] = \{\gamma \mid \alpha \leqslant \gamma \leqslant \beta\}$
- $[\alpha,\beta) = \{\gamma \mid \alpha \leqslant \gamma < \beta\}$
- $\tilde{f}(\alpha,\beta,i) = \bigcup_{i\leqslant j\leqslant\omega}\{\eta\colon [i,j) \to [\alpha,\beta) \mid \eta \text{ strictly increasing}\}$

For each $\alpha, \beta < \kappa$ let us define the sets $P^{\alpha,\beta}_\gamma$, for $\gamma < \kappa$ as follows. If $\alpha = \beta = \gamma = 0$, then $P^{0,0}_0 = I(S)$. Otherwise let $\{P^{\alpha,\beta}_\gamma \mid \gamma < \kappa\}$ enumerate all downward closed subsets of $\tilde{f}(\alpha,\beta,i)$ for all i, i.e.

$$\{P^{\alpha,\beta}_\gamma \mid \gamma<\kappa\}=\bigcup_{i<\omega}\mathcal{P}(\tilde{f}(\alpha,\beta,i))\cap\{A \mid A \text{ is closed under inital segments}\}.$$

Define

$$\tilde{n}(P^{\alpha,\beta}_\gamma)$$

to be the natural number i such that $P^{\alpha,\beta}_\gamma\subset\tilde{f}(\alpha,\beta,i)$. The enumeration is possible, because by our assumption $\kappa^{<\kappa}=\kappa$ we have

$$\begin{array}{rcl}\Big|\bigcup_{i<\omega}\mathcal{P}(\tilde{f}(\alpha,\beta,i))\Big| & \leqslant & \omega\times|\mathcal{P}(\tilde{f}(0,\beta,0))| \\ & \leqslant & \omega\times|\mathcal{P}(\beta^\omega)| \\ & = & \omega\times 2^{\beta^\omega} \\ & \leqslant & \omega\times\kappa \\ & = & \kappa\end{array}$$

Let $S\subset\kappa$ be a set and define $J(S)$ to be the set of all $\eta\colon s\to\omega\times\kappa^4$ such that $s\leqslant\omega$ and the following conditions are met for all $i,j<s$:

(1) η is strictly increasing with respect to the lexicographical order on $\omega\times\kappa^4$.
(2) $\eta_1(i)\leqslant\eta_1(i+1)\leqslant\eta_1(i)+1$
(3) $\eta_1(i)=0\to\eta_2(i)=\eta_3(i)=\eta_4(i)=0$
(4) $\eta_1(i)<\eta_1(i+1)\to\eta_2(i+1)\geqslant\eta_3(i)+\eta_4(i)$
(5) $\eta_1(i)=\eta_1(i+1)\to(\forall k\in\{2,3,4\})(\eta_k(i)=\eta_k(i+1))$
(6) if for some $k<\omega$, $[i,j)=\eta_1^{-1}\{k\}$, then
$\eta_5\restriction[i,j)\in P^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}$
(7) if $s=\omega$, then either
$(\exists m<\omega)(\forall k<\omega)(k>m\to\eta_1(k)=\eta_1(k+1))$
or
$\sup\operatorname{ran}\eta_5\in S$.
(8) Order $J(S)$ by inclusion.

Note that it follows from the definition of $P^{\alpha,\beta}_\gamma$ and the conditions (6) and (4) that for all $i<j<\operatorname{dom}\eta$, $\eta\in J(S)$:

(9) $i<j\to\eta_5(i)<\eta_5(j)$.

For each $\alpha < \kappa$ let

$$J^\alpha(S) = \{\eta \in J(S) \mid \operatorname{ran}\eta \subset \omega \times (\beta+1)^4 \text{ for some } \beta < \alpha\}.$$

Then $(J^\alpha(S))_{\alpha<\kappa}$ is a κ-filtration of $J(S)$ (see Claim 2 below). For the first item of the lemma, clearly $|J(S)| = \kappa$.

Let us observe that if $\eta \in J(S)$ and $\operatorname{ran}\eta_1 = \omega$, then

$$\sup\operatorname{ran}\eta_4 \leqslant \sup\operatorname{ran}\eta_2 = \sup\operatorname{ran}\eta_3 = \sup\operatorname{ran}\eta_5 \qquad (\#)$$

and if in addition to that, $\eta \restriction k \in J^\alpha(S)$ for all k and $\eta \notin J^\alpha(S)$ or if $\operatorname{ran}\eta_1 = \{0\}$, then

$$\sup\operatorname{ran}\eta_5 = \alpha. \qquad (\circledast)$$

To see $(\#)$ suppose $\operatorname{ran}\eta_1 = \omega$. By (9), $(\eta_5(i))_{i<\omega}$ is an increasing sequence. By (6) $\sup\operatorname{ran}\eta_3 \geqslant \sup\operatorname{ran}\eta_5 \geqslant \sup\operatorname{ran}\eta_2$. By (4), $\sup\operatorname{ran}\eta_2 \geqslant \sup\operatorname{ran}\eta_3$ and again by (4) $\sup\operatorname{ran}\eta_2 \geqslant \sup\operatorname{ran}\eta_4$. Inequality $\sup\operatorname{ran}\eta_5 \leqslant \alpha$ is an immediate consequence of the definition of $J^\alpha(S)$, so $(\circledast)$ follows now from the assumption that $\eta \notin J^\alpha(S)$.

Claim 1. Suppose $\xi \in J^\alpha(S)$ and $\eta \in J(S)$. Then if $\operatorname{dom}\xi < \omega$, $\xi \subsetneq \eta$ and $(\forall k \in \operatorname{dom}\eta \setminus \operatorname{dom}\xi)\big(\eta_1(k) = \xi_1(\max\operatorname{dom}\xi) \wedge \eta_1(k) > 0\big)$, then $\eta \in J^\alpha(S)$.

Proof of Claim 1. Suppose $\xi, \eta \in J^\alpha(S)$ are as in the assumption. Let us define $\beta_2 = \xi_2(\max\operatorname{dom}\xi)$, $\beta_3 = \xi_2(\max\operatorname{dom}\xi)$, and $\beta_4 = \xi_4(\max\operatorname{dom}\xi)$. Because $\xi \in J^\alpha(S)$, there is β such that $\beta_2, \beta_3, \beta_4 < \beta+1$ and $\beta < \alpha$. Now by (5) $\eta_2(k) = \beta_2$, $\eta_3(k) = \beta_3$ and $\eta_4(k) = \beta_4$, for all $k \in \operatorname{dom}\eta \setminus \operatorname{dom}\xi$. Then by (6) for all $k \in \operatorname{dom}\eta \setminus \operatorname{dom}\xi$ we have that $\beta_2 < \eta_5(k) < \beta_3 < \beta+1$. Since $\xi \in J^\alpha(S)$, also $\beta_4 < \beta+1$, so $\eta \in J^\alpha(S)$. $\square_{\text{Claim 1}}$

Claim 2. $|J(S)| = \kappa$, $(J^\alpha(S))_{\alpha<\kappa}$ is a κ-filtration of $J(S)$ and if $S \subset \kappa$ and $\mathcal{I}$ is a κ-filtration of $J(S)$, then $S_{\mathcal{I}}(J(S)) \sim_{\mathrm{NS}} S$.

Proof of Claim 2. For all $\alpha \leqslant \kappa$, $J^\alpha(S) \subset (\omega \times \alpha^4)^{\leqslant\omega}$, so by the cardinality assumption of the lemma, the cardinality of $J^\alpha(S)$ is $< \kappa$ if $\alpha < \kappa$ ($J^\kappa(S) = J(S)$). Clearly $\alpha < \beta$ implies $J^\alpha(S) \subset J^\beta(S)$. Continuity is verified by

$$\begin{aligned}\bigcup_{\alpha<\gamma} J^\alpha(S) &= \{\eta \in J(S) \mid \exists\alpha < \gamma, \exists\beta < \alpha(\operatorname{ran}\eta \subset \omega \times (\beta+1)^4)\} \\ &= \{\eta \in J(S) \mid \exists\beta < \cup\gamma(\operatorname{ran}\eta \subset \omega \times (\beta+1)^4)\}\end{aligned}$$

which equals $J^\gamma(S)$ if γ is a limit ordinal. By Lemma 78 it is enough to show $S_{\mathcal{I}}(J(S)) \sim_{\mathrm{NS}} S$ for $\mathcal{I} = (J^\alpha(S))_{\alpha<\kappa}$, and we will show that if $\mathcal{I} = (J^\alpha(S))_{\alpha<\kappa}$, then in fact $S_{\mathcal{I}}(J(S)) = S$.

Suppose $\alpha \in S_{\mathcal{I}}(J(S))$. Then there is $\eta \in J(S)$, $\operatorname{dom}\eta = \omega$, such that $\eta\restriction k \in J^\alpha(S)$ for all $k < \omega$ but $\eta \notin J^\alpha(S)$. Thus there is no $\beta < \alpha$ such that $\operatorname{ran}\eta \subset \omega \times (\beta+1)^4$ but on the other hand for all $k < \omega$ there is β such that $\operatorname{ran}\eta\restriction k \subset \omega \times (\beta+1)^4$. By (5) and (6) this implies that either $\operatorname{ran}\eta_1 = \omega$ or $\operatorname{ran}\eta_1 = \{0\}$. By $(\circledast)$ on page 95 it now follows that $\sup\operatorname{ran}\eta_5 = \alpha$ and by (7), $\alpha \in S$.

Suppose then that $\alpha \in S$. Let us show that $\alpha \in S_{\mathcal{I}}(J(S))$. Fix a function $\eta_\alpha \colon \omega \to \kappa$ with $\sup\operatorname{ran}\eta_\alpha = \alpha$. Then $\eta_\alpha \in I(S)$ and the function η such that $\eta(n) = (0,0,0,0,\eta_\alpha(n))$ is as required. (Recall that $P_0^{0,0} = I(S)$ in the definition of $J(S)$). $\square_{\text{Claim 2}}$

Claim 3. Suppose $S \sim_{\mathrm{NS}} S'$. Then $J(S) \cong J(S')$.

Proof of Claim 3. Let $C \subset \kappa \setminus (S \triangle S')$ be the cub set which exists by the assumption. By induction on $i < \kappa$ we will define α_i and F_{α_i} such that

(a) If $i < j < \kappa$, then $\alpha_i < \alpha_j$ and $F_{\alpha_i} \subset F_{\alpha_j}$.

(b) If i is a successor, then α_i is a successor and if i is limit, then $\alpha_i \in C$.

(c) If γ is a limit ordinal, then $\alpha_\gamma = \sup_{i<\gamma}\alpha_i$,

(d) F_{α_i} is a partial isomorphism $J(S) \to J(S')$

(e) Suppose that $i = \gamma + n$, where γ is a limit ordinal or 0 and $n < \omega$ is even. Then $\operatorname{dom} F_{\alpha_i} = J^{\alpha_i}(S)$ (e1). If also $n > 0$ and $(\eta_k)_{k<\omega}$ is an increasing sequence in $J^{\alpha_i}(S)$ such that $\eta = \bigcup_{k<\omega}\eta_k \notin J(S)$, then $\bigcup_{k<\omega} F_{\alpha_i}(\eta_k) \notin J(S')$ (e2).

(f) If $i = \gamma + n$, where γ is a limit ordinal or 0 and $n < \omega$ is odd, then $\operatorname{ran} F_{\alpha_i} = J^{\alpha_i}(S')$ (f1). Further, if $(\eta_k)_{k<\omega}$ is an increasing sequence in $J^{\alpha_i}(S')$ such that $\eta = \bigcup_{k<\omega}\eta_k \notin J(S')$, then $\bigcup_{k<\omega} F_{\alpha_i}^{-1}(\eta_k) \notin J(S)$ (f3).

(g) If $\operatorname{dom}\xi < \omega$, $\xi \in \operatorname{dom} F_{\alpha_i}$, $\eta\restriction\operatorname{dom}\xi = \xi$ and $(\forall k \geqslant \operatorname{dom}\xi)\big(\eta_1(k) = \xi_1(\max\operatorname{dom}\xi) \wedge \eta_1(k) > 0\big)$, then $\eta \in \operatorname{dom} F_{\alpha_i}$. Similarly for $\operatorname{ran} F_{\alpha_i}$

(h) If $\xi \in \operatorname{dom} F_{\alpha_i}$ and $k < \operatorname{dom}\xi$, then $\xi\restriction k \in \operatorname{dom} F_{\alpha_i}$.

(i) For all $\eta \in \operatorname{dom} F_{\alpha_i}$, $\operatorname{dom}\eta = \operatorname{dom}(F_{\alpha_i}(\eta))$

The first step. The first step and the successor steps are similar, but the first step is easier. Thus we give it separately in order to simplify the readability. Let us start with $i = 0$. Let $\alpha_0 = \beta + 1$, for arbitrary $\beta \in C$.

Let us denote by

$$\tilde{o}(\alpha)$$

the ordinal that is order isomorphic to $(\omega \times \alpha^4, <_{\text{lex}})$. Let γ be such that there is an isomorphism $h\colon P_\gamma^{0,\tilde{o}(\alpha_0)} \cong J^{\alpha_0}(S)$ and such that $\tilde{n}(P_\gamma^{0,\alpha_0}) = 0$. Such exists by (1). Suppose that $\eta \in J^{\alpha_0}(S)$. Note that because P_γ^{0,α_0} and $J^{\alpha_0}(S)$ are closed under initial segments and by the definitions of $\tilde{n}$ and $P_\gamma^{\alpha,\beta}$, we have $\operatorname{dom} h^{-1}(\eta) = \operatorname{dom} \eta$, Define $\xi = F_{\alpha_0}(\eta)$ such that $\operatorname{dom} \xi = \operatorname{dom} \eta$ and for all $k < \operatorname{dom} \xi$

- $\xi_1(k) = 1$
- $\xi_2(k) = 0$
- $\xi_3(k) = \tilde{o}(\alpha_0)$
- $\xi_4(k) = \gamma$
- $\xi_5(k) = h^{-1}(\eta)(k)$

Let us check that $\xi \in J(S')$. Conditions (1)-(5) and (7) are satisfied because ξ_k is constant for all $k \in \{1,2,3,4\}$, $\xi_1(i) \neq 0$ for all i and ξ_5 is increasing. For (6), if $\xi_1^{-1}\{k\}$ is empty, the condition is verified since each $P_\gamma^{\alpha,\beta}$ is closed under initial segments and contains the empty function. If it is non-empty, then $k = 1$ and in that case $\xi_1^{-1}\{k\} = [0,\omega)$ and by the argument above ($\operatorname{dom} h^{-1}(\eta) = \operatorname{dom} \eta = \operatorname{dom} \xi$) we have $\xi_5 = h^{-1}(\eta) \in P_\gamma^{0,\tilde{o}(\alpha_0)} = P_{\xi_4(0)}^{\xi_2(0),\xi_3(0)}$, so the condition is satisfied.

Let us check whether all the conditions (a)-(i) are met. In (a), (b), (c), (e2) and (f) there is nothing to check. (d) holds, because h is an isomorphism. (e1) and (i) are immediate from the definition. Both $J^{\alpha_0}(S)$ and $P_\gamma^{0,\tilde{o}(\alpha_0)}$ are closed under initial segments, so (h) follows, because $\operatorname{dom} F_{\alpha_0} = J^{\alpha_0}(S)$ and $\operatorname{ran} F_{\alpha_0} = \{1\} \times \{0\} \times \{\tilde{o}(\alpha_0)\} \times \{\gamma\} \times P_\gamma^{0,\alpha_0}$. Claim 1 implies (g) for $\operatorname{dom} F_{\alpha_0}$. Suppose $\xi \in \operatorname{ran} F_{\alpha_0}$ and $\eta \in J(S')$ are as in the assumption of (g). Then $\eta_1(i) = \xi_1(i) = 1$ for all $i < \operatorname{dom} \eta$. By (5) it follows that $\eta_2(i) = \xi_2(i) = 0$, $\eta_3(i) = \xi_3(i) = \tilde{o}(\alpha_0)$ and $\eta_4(i) = \xi_4(i) = \gamma$ for all $i < \operatorname{dom} \eta$, so by (6) $\eta_5 \in P_\gamma^{0,\tilde{o}(\alpha_0)}$ and since h is an isomorphism, $\eta \in \operatorname{ran} F_{\alpha_0}$.

Odd successor step. We want to handle odd case but not the even case first, because the most important case is the successor of a limit ordinal, see $(\iota\iota\iota)$ below. Except that, the even case is similar to the odd case.

Suppose that $j < \kappa$ is a successor ordinal. Then there exist β_j and n_j such that $j = \beta_j + n_j$ and β is a limit ordinal or 0. Suppose that n_j is odd and that α_l and F_{α_l} are defined for all $l < j$ such that the conditions (a)–(i) and (1)–(9) hold for $l < j$.

Let $\alpha_j = \beta+1$ where β is such that $\beta \in C$, $\operatorname{ran} F_{\alpha_{j-1}} \subset J^{\beta}(S')$, $\beta > \alpha_{j-1}$. For convenience define $\xi(-1) = (0,0,0,0,0)$ for all $\xi \in J(S) \cup J(S')$. Suppose $\eta \in \operatorname{ran} F_{\alpha_{j-1}}$ has finite domain $\operatorname{dom}\eta = m < \omega$ and denote $\xi = F^{-1}_{\alpha_{j-1}}(\eta)$. Fix γ_η to be such that $\tilde{n}(P^{\alpha,\beta}_{\gamma_\eta}) = m$ and such that there is an isomorphism $h_\eta \colon P^{\alpha,\beta}_{\gamma_\eta} \to W$, where

$$W = \{\zeta \mid \operatorname{dom}\zeta = [m,s), m < s \leqslant \omega, \eta^\frown\langle m, \zeta(m)\rangle \notin \operatorname{ran} F_{\alpha_{j-1}}, \eta^\frown\zeta \in J^{\alpha_j}(S')\},$$

$\alpha = \xi_3(m-1) + \xi_4(m-1)$ and $\beta = \alpha + \tilde{o}(\alpha_j)$ (defined in the beginning of the First step).

We will define F_{α_j} so that its range is $J^{\alpha_j}(S')$ and instead of F_{α_j} we will define its inverse. So let $\eta \in J^{\alpha_j}(S')$. We have three cases:

(ι) $\eta \in \operatorname{ran} F_{\alpha_{j-1}}$,

($\iota\iota$) $\exists m < \operatorname{dom}\eta(\eta \restriction m \in \operatorname{ran} F_{\alpha_{j-1}} \wedge \eta \restriction (m+1) \notin F_{\alpha_{j-1}})$,

($\iota\iota\iota$) $\forall m < \operatorname{dom}\eta(\eta \restriction (m+1) \in \operatorname{ran} F_{\alpha_{j-1}} \wedge \eta \notin \operatorname{ran} F_{\alpha_{j-1}})$.

Let us define $\xi = F^{-1}_{\alpha_j}(\eta)$ such that $\operatorname{dom}\xi = \operatorname{dom}\eta$. If ($\iota$) holds, define $\xi(n) = F^{-1}_{\alpha_{j-1}}(\eta)(n)$ for all $n < \operatorname{dom}\eta$. Clearly $\xi \in J(S)$ by the induction hypothesis. Suppose that ($\iota\iota$) holds and let m witness this. For all $n < \operatorname{dom}\xi$ let

- If $n < m$, then $\xi(n) = F^{-1}_{\alpha_{j-1}}(\eta \restriction m)(n)$.
- Suppose $n \geqslant m$. Let
  - $\xi_1(n) = \xi_1(m-1) + 1$
  - $\xi_2(n) = \xi_3(m-1) + \xi_4(m-1)$
  - $\xi_3(n) = \xi_2(m) + \tilde{o}(\alpha_j)$
  - $\xi_4(n) = \gamma_{\eta\restriction m}$
  - $\xi_5(n) = h^{-1}_{\eta\restriction m}(\eta)(n)$.

Next we should check that $\xi \in J(S)$; let us check items (1) and (6), the rest are left to the reader.

(1) By the induction hypothesis $\xi \restriction m$ is increasing. Next, $\xi_1(m) = \xi_1(m-1)+1$, so $\xi(m-1) <_{\text{lex}} \xi(m)$. If $m \leqslant n_1 < n_2$, then $\xi_k(n_1) = \xi_k(n_2)$ for all $k \in \{1,2,3,4\}$ and ξ_5 is increasing.

(6) Suppose that $[i,j) = \xi_1^{-1}\{k\}$. Since $\xi_1 \restriction [m,\omega)$ is constant, either $j < m$, when we are done by the induction hypothesis, or $i = m$ and $j = \omega$. In that case one verifies that $\eta \restriction [m,\omega) \in W = \operatorname{ran} h_{\eta\restriction m}$ and then, imitating the corresponding argument in the first step, that

$$\xi_5 \restriction [m,\omega) = h^{-1}_{\eta\restriction m}(\eta \restriction [m,\omega))$$

and hence in $\operatorname{dom} h_{\eta\restriction m} = P^{\xi_2(m),\xi_3(m)}_{\xi_4(m)}$.

Suppose finally that $(\iota\iota\iota)$ holds. Then $\operatorname{dom}\eta$ must be ω since otherwise the condition $(\iota\iota\iota)$ is simply contradictory (because $\eta\restriction(\operatorname{dom}\eta-1+1)=\eta$ (except for the case $\operatorname{dom}\eta=0$, but then condition (ι) holds and we are done)). By (g), we have $\operatorname{ran}\eta_1=\omega$, because otherwise we had $\eta\in\operatorname{ran}F_{\alpha_{j-1}}$. Let $F^{-1}_{\alpha_j}(\eta)=\xi=\bigcup_{n<\omega}F^{-1}_{\alpha_{j-1}}(\eta\restriction n)$.

Let us check that it is in $J(S)$. Conditions (1)–(6) are satisfied by ξ, because they are satisfied by all its initial segments. Let us check (7).

First of all ξ cannot be in $J^{\alpha_{j-1}}(S)$, since otherwise, by (d) and (i),

$$F_{\alpha_{j-1}}(\xi)=\bigcup_{n<\omega}F_{\alpha_{j-1}}(\xi\restriction n)=\bigcup_{n<\omega}\eta\restriction n=\eta$$

were again in $\operatorname{ran}F_{\alpha_{j-1}}$. If $j-1$ is a successor ordinal, then we are done: by (b) α_{j-1} is a successor and we assumed $\eta\in J(S')$, so by (e2) we have $\xi\in J(S)$. Thus we can assume that $j-1$ is a limit ordinal. Then by (b), α_{j-1} is a limit ordinal in C and by (a), (e) and (f), $\operatorname{ran}F_{\alpha_{j-1}}=J^{\alpha_{j-1}}(S')$ and $\operatorname{dom}F_{\alpha_{j-1}}=J^{\alpha_{j-1}}(S)$. This implies that $\operatorname{ran}\eta\not\subset\omega\times\beta^4$ for any $\beta<\alpha_{j-1}$ and by (⊛) on page 95 we must have $\sup\operatorname{ran}\eta_5=\alpha_{j-1}$ which gives $\alpha_{j-1}\in S'$ by (7). Since $\alpha_{j-1}\in C\subset\kappa\setminus S\triangle S'$, we have $\alpha_{j-1}\in S$. Again by (⊛) and that $\operatorname{dom}F_{\alpha_{j-1}}=J^{\alpha_{j-1}}(S)$ by (e1), we have $\sup\operatorname{ran}\xi_5=\alpha_{j-1}$, thus ξ satisfies the condition (7).

Let us check whether all the conditions (a)-(i) are met. (a), (b), (c) are common to the cases (ι), $(\iota\iota)$ and $(\iota\iota\iota)$ in the definition of $F^{-1}_{\alpha_j}$ and are easy to verify. Let us sketch a proof for (d); the rest is left to the reader.

(d) Let $\eta_1,\eta_2\in\operatorname{ran}F_{\alpha_j}$ and let us show that

$$\eta_1\subsetneq\eta_2\iff F^{-1}_{\alpha_j}(\eta_1)\subsetneq F^{-1}_{\alpha_j}(\eta_2).$$

The case where both η_1 and η_2 satisfy $(\iota\iota)$ is the interesting one (implies all the others).

So suppose $\eta_1,\eta_2\in(\iota\iota)$. Then there exist m_1 and m_2 as described in the statement of $(\iota\iota)$. Let us show that $m_1=m_2$. We have $\eta_1\restriction(m_1+1)=\eta_2\restriction(m_1+1)$ and $\eta_1\restriction(m_1+1)\notin\operatorname{ran}F_{\alpha_{j-1}}$, so $m_2\leqslant m_1$. If $m_2\leqslant m_1$, then $m_2<\operatorname{dom}\eta_1$, since $m_1<\operatorname{dom}\eta_1$. Thus if $m_2\leqslant m_1$, then $\eta_1\restriction(m_2+1)=\eta_2\restriction(m_2+1)\notin\operatorname{ran}F_{\alpha_{j-1}}$, which implies $m_2=m_1$. According to the definition of $F^{-1}_{\alpha_j}(\eta_i)(k)$ for $k<\operatorname{dom}\eta_1$, $F^{-1}_{\alpha_j}(\eta_i)(k)$ depends only on m_i and $\eta\restriction m_i$ for $i\in\{1,2\}$. Since $m_1=m_2$ and $\eta_1\restriction m_1=\eta_2\restriction m_2$, we have $F^{-1}_{\alpha_j}(\eta_1)(k)=F^{-1}_{\alpha_j}(\eta_2)(k)$ for all $k<\operatorname{dom}\eta_1$.

Let us now assume that $\eta_1 \not\subset \eta_2$. Then take the smallest $n \in \operatorname{dom}\eta_1 \cap \operatorname{dom}\eta_2$ such that $\eta_1(n) \neq \eta_2(n)$. It is now easy to show that $F_{\alpha_j}^{-1}(\eta_1)(n) \neq F_{\alpha_j}^{-1}(\eta_2)(n)$ by the construction.

Even successor step. Namely the one where $j = \beta + n$ and n is even. But this case goes exactly as the above completed step, except that we start with $\operatorname{dom} F_{\alpha_j} = J^{\alpha_j}(S)$ where α_j is big enough successor of an element of C such that $J^{\alpha_j}(S)$ contains $\operatorname{ran} F_{\alpha_{j-1}}$ and define $\xi = F_{\alpha_j}(\eta)$. Instead of (e) we use (f) as the induction hypothesis. This step is easier since one does not need to care about the successors of limit ordinals.

Limit step. Assume that j is a limit ordinal. Then let $\alpha_j = \bigcup_{i<j} \alpha_i$ and $F_{\alpha_j} = \bigcup_{i<j} F_{\alpha_i}$. Since α_i are successors of ordinals in C, $\alpha_j \in C$, so (b) is satisfied. Since each F_{α_i} is an isomorphism, also their union is, so (d) is satisfied. Because conditions (e), (f) and (i) hold for $i < j$, the conditions (e) and (i) hold for j. (f) is satisfied because the premise is not true. (a) and (c) are clearly satisfied. Also (g) and (h) are satisfied by Claim 1 since now $\operatorname{dom} F_{\alpha_j} = J^{\alpha_j}(S)$ and $\operatorname{ran} F_{\alpha_j} = J^{\alpha_j}(S')$ (this is because (a), (e) and (f) hold for $i < j$).

Finally $F = \bigcup_{i<\kappa} F_{\alpha_i}$ is an isomorphism between $J(S)$ and $J(S')$. $\square_{\text{Claim 3}}$

$\square_{\text{Lemma 79}}$

80. Theorem. *Suppose κ is such that $\kappa^{<\kappa} = \kappa$ and for all $\lambda < \kappa$, $\lambda^\omega < \kappa$ and that T is a stable unsuperstable theory. Then $E_{S^\kappa_\omega} \leqslant_c \cong_T$.*

Proof. For $\eta \in 2^\kappa$ let $J_\eta = J(\eta^{-1}\{1\})$ where the function J is as in Lemma 79 above. For notational convenience, we assume that J_η is a downward closed subtree of $\kappa^{\leqslant\omega}$. Since T is stable unsuperstable, for all η and $t \in J_\eta$, there are finite sequences $a_t = a_t^\eta$ in the monster model such that

(1) If $\operatorname{dom}(t) = \omega$ and $n < \omega$ then
$$a_t \underset{\bigcup_{m<n} a_t\restriction m}{\not\downarrow} a_{t\restriction n}.$$

(2) For all downward closed subtrees $X, Y \subset J_\eta$,
$$\bigcup_{t\in X} a_t \underset{\bigcup_{t\in X\cap Y} a_t}{\downarrow} \bigcup_{t\in Y} a_t$$

(3) For all downward closed subtrees $X \subset J_\eta$ and $Y \subset J_{\eta'}$ the following holds: If $f\colon X \to Y$ is an isomorphism, then there is an automorphism F of the monster model such that for all $t \in X$, $F(a_t^\eta) = a_{f(t)}^{\eta'}$

Then we can find an F^f_ω-construction

$$(\bigcup_{t\in J_\eta} a_t, (b_i, B_i)_{i<\kappa})$$

(here $(t(b/C), D) \in F^f_\omega$ if $D \subset C$ is finite and $b \downarrow_D C$, see [**25**]) such that

($\star$) for all $\alpha < \kappa$, c and finite $B \subset \bigcup_{t\in J_\eta} a_t \cup \bigcup_{i<\alpha} b_i$ there is $\alpha < \beta < \kappa$ such that $B_\beta = B$ and

$$\operatorname{stp}(b_\beta/B) = \operatorname{stp}(c/B).$$

Then

$$M_\eta = \bigcup_{t\in J_\eta} a_t \cup \bigcup_{i<\kappa} b_i \models T.$$

Without loss of generality we may assume that the trees J_η and the F^f_ω-constructions for M_η are chosen coherently enough such that one can find a code ξ_η for (the isomorphism type of) M_η so that $\eta \mapsto \xi_\eta$ is continuous. Thus we are left to show that $\eta E_{S^\kappa_\omega} \eta' \iff M_\eta \cong M_{\eta'}$

"$\Rightarrow$" Assume $J_\eta \cong J_{\eta'}$. By (3) it is enough to show that F^f_ω-construction of length κ satisfying ($\star$) are unique up to isomorphism over $\bigcup_{t\in J_\eta} a_t$. But ($\star$) guarantees that the proof of the uniqueness of F-primary models from [**25**] works here.

"$\Leftarrow$" Suppose $F\colon M_\eta \to M_{\eta'}$ is an isomorphism and for a contradiction suppose $(\eta, \eta') \notin E_{S^\kappa_\omega}$. Let $(J^\alpha_\eta)_{\alpha<\kappa}$ be a filtration of J_η and $(J^\alpha_{\eta'})_{\alpha<\kappa}$ be a filtration of $J_{\eta'}$ (see Definition 75 above). For $\alpha < \kappa$, let

$$M^\alpha_\eta = \bigcup_{t\in J^\alpha_\eta} a_t \cup \bigcup_{i<\alpha} b_i$$

and similarly for η':

$$M^\alpha_{\eta'} = \bigcup_{t\in J^\alpha_{\eta'}} a_t \cup \bigcup_{i<\alpha} b_i.$$

Let C be the cub set of those $\alpha < \kappa$ such that $F \restriction M^\alpha_\eta$ is onto $M^\alpha_{\eta'}$ and for all $i < \alpha$, $B_i \subset M^\alpha_\eta$ and $B'_i \subset M^\alpha_{\eta'}$, where $(\bigcup_{t\in J_{\eta'}}, (b'_i, B'_i)_{i<b})$ is in the construction of $M_{\eta'}$. Then we can find $\alpha \in \lim C$ such that in J_η there is t^* satisfying (a)–(c) below, but in $J_{\eta'}$ there is no such t^*.:

(a) $\operatorname{dom}(t^*) = \omega$,
(b) $t^* \notin J^\alpha_\eta$,
(c) for all $\beta < \alpha$ there is $n < \omega$ such that $t^* \restriction n \in J^\alpha_\eta \setminus J^\beta_\eta$,

Note that

($\star\star$) if $\alpha \in C$ and $c \in M^{\alpha}_{\eta}$, there is a finite $D \subset \bigcup_{t \in J^{\alpha}_{\eta}} a_t$ such that $(t(c, \bigcup_{t \in J_\eta} a_t), D) \in F^f_\omega$,

Let $c = F(a_{t^*})$. By the construction we cat find finite $D \subset M^{\alpha}_{\eta'}$, and $X \subset J_{\eta'}$ such that

$$\Big(t(c, M^{\alpha}_{\eta'} \cup \bigcup_{t \in J_{\eta'}} a^{\eta'}_t), D \cup \bigcup_{t \in X} a^{\eta'}_t\Big) \in F^f_\omega.$$

But then there is $\beta \in C$, $\beta < \alpha$, such that $D \subset M^{\beta}_{\eta'}$ and if $u \leqslant t$ for some $t \in X$, then $u \in J^{\beta}_{\eta'}$ (since in $J_{\eta'}$ there is no element like t^* is in J_η). But then using ($\star\star$) and (2), it is easy to see that

$$c \underset{M^{\beta}_{\eta'}}{\downarrow} M^{\alpha}_{\eta'}.$$

On the other hand, using (1), (2), ($\star\star$) and the choice of t^* one can see that $a_{t^*} \underset{M^{\beta}_{\eta}}{\not\downarrow} M^{a}_{\eta}$, a contradiction. $\square$

Bibliography

[1] S. **Adams**, A. S. **Kechris**: *Linear algebraic groups and countable Borel equivalence relations* – Journal: J. Amer. Math. Soc. 13 (2000), 909-943.

[2] D.**Blackwell**: *Borel Sets Via Games.* – Ann. Probab. Volume 9, Number 2 (1981), 321-322.

[3] A. **Halko**: *Negligible subsets of the generalized Baire space* $\omega_1^{\omega_1}$, Ann. Acad. Sci. Ser. Diss. Math. 108 (1996).

[4] A. **Halko**, S. **Shelah**: *On strong measure zero subsets of* ${}^{\kappa}2$ – Fundamenta Math 170 (2001) 219-229

[5] L. **Harrington**, A. S. **Kechris**, A. **Louveau**: *A Glimm-Eros dichotomy theorem for Borel equivalence relations* – Journal of the American Mathematical Society, vol. 3 (1990), pp. 903–928. MR **91h**:28023

[6] B. **Hart**, E. **Hrushovski**, M. C. **Laskowski**: *The Uncountable Spectra of Countable Theories* – The Annals of Mathematics, Second Series, Vol. 152, No. 1 (Jul., 2000), pp. 207-257.

[7] G. **Hjorth**: *Group actions and countable models* – A survey article presented at the ASL European meeting in Utrecht, 1999.

[8] T. **Huuskonen**, T. **Hyttinen**, M. **Rautila**: *On potential isomorphism and non-structure* – Arch. Math. Logic 43 (2004), pp. 85-120.

[9] T. **Hyttinen**: *Model theory for infinite quantifier languages* – Fund. Math. 134 (1990), 123-140.

[10] T. **Hyttinen**, M. **Rautila**: *The canary tree revisited* – J. Symbolic Logic 66 (2001), no. 4, pp. 1677–1694.

[11] T. **Hyttinen**, S. **Shelah**: *Constructing strongly equivalent nonisomorphic models for unsuperstable theories, Part A* – The Journal of Symbolic Logic, Vol. 59, No. 3 (Sep., 1994), pp. 984-996, Published by: Association for Symbolic Logic.

[12] T. **Hyttinen**, S. **Shelah**: *Constructing strongly equivalent nonisomorphic models for unsuperstable theories, Part B* – The Journal of Symbolic Logic, Vol. 60, No. 4 (Dec., 1995), pp. 1260-1272, Published by: Association for Symbolic Logic.

[13] T. **Hyttinen**, S. **Shelah**: *Constructing strongly equivalent nonisomorphic models for unsuperstable theories, Part C* – The Journal of Symbolic Logic, Vol. 64, No. 2 (Jun., 1999), pp. 634-642, Published by: Association for Symbolic Logic.

[14] T. **Hyttinen**, H. **Tuuri**: *Constructing strongly equivalent nonisomorphic models* – Annals of Pure and Applied Logic Volume 52, Issue 3, 24 June 1991, Pages 203-248.

[15] T. **Jech**: *Set Theory* – ISBN-10 3-540-44085-7 Springer-Verlag Berlin Heidelberg New York, 2003.

[16] C. **Karp**: *Finite-quantifier equivalence* – Theory of Models (Proc. 1963 Internat. Sympos. Berkeley), pages 407–412. North-Holland, Amsterdam, 1965.

[17] M. **Karttunen**: *Model theory for infinitely deep languages* – Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, vol 64, 1987.

[18] M. **Koerwien** *A complicated ω-stable depth 2 theory*, to appear in Journal of Symbolic Logic.

[19] K. **Kunen**: *Set Theory / An Introduction to Independence Proofs* –

[20] A. **Louveau**, B. **Velickovic**: *A Note on Borel Equivalence Relations* – Proceedings of the American Mathematical Society, Vol. 120, No. 1 (Jan., 1994), pp. 255-259,

[21] A. **Mekler**, S. **Shelah**: *The Canary Tree* – Canadian J Math Journal Canadien de Mathematiques 36 (1993) 209-215.

[22] A. **Mekler**, J. **Väänänen**: *Trees And* Π^1_1*-Subsets of* ${}^{\omega_1}\omega_1$ – The Journal of Symbolic Logic, Vol. 58, No. 3 (Sep., 1993), pp. 1052-1070.

[23] M. **Nadel** and J. **Stavi**: $L_{\infty\lambda}$*-equivalence, isomorphism and potential isomorphism.* – Trans. Amer. Math. Soc., vol 236, 1978, pp. 51-74.

[24] S. **Shelah**: *A combinatorial problem; stability and order for models and theories in infinitary languages* – Pacific J Math 41 (1972) 247-261.

[25] S. **Shelah**: *Classification Theory, Revised Edition* – North-Holland Publishing Company Amsterdam New-York Oxford Tokyo, 2000.

[26] S. **Shelah**: *Existence of many* $L_{\infty,\lambda}$*-equivalent, nonisomorphic models of* T *of power* λ – Annals Pure and Applied Logic 34 (1987) 291–310.

[27] S. **Shelah**: *The number of non-isomorphic models of an unstable first-order theory* – Israel J Math 9 (1971) 473-487

[28] S. **Shelah**: *The spectrum problem I:* $\aleph_\varepsilon$*-saturated models, the main gap* – Israel J. Math. 43 (1982), 324–356.

[29] S. **Shelah**, J. **Väänänen**: *Stationary Sets and Infinitary Logic* – J Symbolic Logic 65 (2000) 1311-1320.

[30] J. H. **Silver**: *Counting the number of equivalence classes of Borel and coanalytic equivalence relations* – Ann. Math. Logic 18 (1980), 1-28.

[31] M. **Suslin**: *Sur une déinition des ensembles mesurables B sans nombres transfinis.* – C. R. Acad. Sci. Paris 164 (1917), 88–91.

[32] H. **Tuuri**: *Relative separation theorems for* $L_{\kappa^+\kappa}$. – Notre Dame J. Formal Logic Volume 33, Number 3 (1992), 383-401.

[33] J. **Väänänen**: *How complicated can structures be?* – Nieuw Archief voor Wiskunde. June 2008, 117-121.

[34] J. **Väänänen**: *Games and trees in infinitary logic: A Survey.* – In M. Krynicki, M. Mostowski and L. Szczerba, editors, Quantifiers, Kluwer Academic Publishers, 105-138, 1995.

[35] R. **Vaught**: *Invariant sets in topology and logic.* Fund. Math. 82 (1974/75), 269-294. MR 51 #167.