Compact not sequentially compact and vice versa

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The space I^I, $I = [0,1]$ is such. Proof: Let $f_n \in I^I$, $n \in \mathbb{N}$, be a sequence such that $f_n(x) = n$:th binary digit of x. Let f_{n_j}, $(n_0 < n_1 < ... \in \mathbb{N})$, be any subsequence. Then we can find a point $y \in I$ whose n_j:th binary digit is 0 if j is even and 1 if j is odd. Then $f_{n_j}(x)$ converges not. Hence f_{n_j} does not either. I^I is compact by Tychonoff. (Without uncountable axiom of choice, can we find such a space?)

On the other hand ω_1 with the order topology is sequentially compact but not compact. Moreover ω_2 is sequentially compact but not even N_1. $\omega_2 + 1$ is compact but not N_1 and contains a point (the highest point) which has no neighbourhood basis of cardinality ω_1. If κ is a regular cardinal, then the ordinal $\kappa + 1$ with order topology is compact but there is a point which has no neighbourhood basis of cardinality $< \kappa$.