Isoperimetric Inequality

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Bees, then, know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each. But we, claiming a greater share of wisdom than the bees, will investigate a somewhat wider problem, namely that, of all equilateral and equiangular plane figures having the same perimeter, that which has the greater number of angles is always greater, and the greatest of them all is the circle having its perimeter equal to them.

Pappus of Alexandria (ca 300 A.D.)

Abstract. We shall claim an even greater share of wisdom and investigate the isoperimetric inequality, which states that among all bounded subsets of $\mathbb{R}^n$ with fixed Lebesgue outer measure, the sphere minimizes the $(n-1)$-dimensional Hausdorff measure of the boundary. Under some assumtpions such as $n = 2$ and the boundary being piecewisely C^1 the proof can be reduced to a very intuitive argument. In this presentation we however focus on a more general measure theoretic proof.

In the presentation we use following notation:

- For $c \in \mathbb{R}_+$ and $A \subset \mathbb{R}^n$, $cA = \{cx \in \mathbb{R}^n \mid x \in A\}$
- For $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, $x + A = \{x + y \in \mathbb{R}^n \mid y \in A\}$
- For $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$, $A + B = \{x + y \mid x \in A \text{ and } y \in B\}$
- n-dimensional Hausdorff measure of the set A is denoted $\mathcal{H}^n(A)$. It is assumed to be normalized such that if $A \subset \mathbb{R}^n$, it equals to the n-dimensional Lebesgue-measure of A. That is why we might speek of the Lebesgue measure of some set while writing $\mathcal{H}^n(A)$.

1. Statement of the Problem

How could we possibly connect the n-dimensional Hausdorff measure of a set to the $(n-1)$-dimensional Hausdorff measure of its boundary? Consider a square in the plane, say $I^2 = [0,1] \times [0,1]$. We have $\mathcal{H}^2(I^2) = 1$ and $\mathcal{H}^1(\partial I^2) = 4$. Then for $c > 0$ we have $\mathcal{H}^2(cI^2) = c^2$ and $\mathcal{H}^1(\partial cI^2) = 4c$. Thus we can not require linear dependance $\mathcal{H}^2(A) \leq k\mathcal{H}^1(A)$, for $k \in \mathbb{R}$, but we can require linear dependance of some exponents of these quantities. Before stating the inequality, let us define the Minkowski content of a boundary by the Minkowski-Steiner formula.

1. DEFINITION. For a set $A \subset \mathbb{R}^n$, the $(n-1)$-*dimensional Minkowski content* of ∂A is

$$M^{n-1}(\partial A) = \liminf_{r \to 0^+} \frac{\mathcal{H}^n(A + B(0,r)) - \mathcal{H}^n(A)}{r}.$$

2. THEOREM. *If $A \subset \mathbb{R}^n$ is such that ∂A is closed $(n-1)$-rectifiable, then $\mathcal{H}^{n-1}(\partial A) = M^{n-1}(\partial A)$.*

PROOF. Can be found in [**Federer**], theorem 3.2.39. □

3. THEOREM (Isoperimetric inequality). *Let $A \in \mathbb{R}^n$ be such that $\mathcal{H}^n(\bar{A}) < \infty$ and denote $\Omega_n = \mathcal{H}^n(B(0,1))$. Then*

$$n\Omega_n^{1/n}\mathcal{H}^n(A)^{(n-1)/n} \leq M^{n-1}(\partial A), \tag{1}$$

which can be written

$$\mathcal{H}^n(A)^{(n-1)/n} \leq c_n M^{n-1}(\partial A), \tag{2}$$

where c_n is the obvious constant. The equality

$$\mathcal{H}^n(A)^{(n-1)/n} = c_n M^{n-1}(\partial A),$$

holds if and only if $A = B(0,1)$.

REMARK. Let us denote $\omega_{n-1} = \mathcal{H}^{n-1}(\partial B(0,1))$. I remind the reader that

$$\Omega_n = \int_0^1 \mathcal{H}^{n-1}(\partial B(0,r))dr = \int_0^1 r^{n-1}\omega_{n-1}dr = \frac{\omega_{n-1}}{n}$$

and hence

$$\omega_{n-1} = n\Omega_n.$$

Thus (1) can be written

$$\omega_{n-1}\mathcal{H}^n(A)^{(n-1)/n} \leq \Omega_n^{(n-1)/n} M^{n-1}(\partial A).$$

It is clear that if $A = B(0,1)$, then the equality holds, because then the inequality above takes form:

$$\omega_{n-1}\Omega_n^{(n-1)/n} \leq \Omega_n^{(n-1)/n}\omega_{n-1}.$$

We shall first prove the following theorem.

4. THEOREM (Brunn-Minkowski inequality). *If A and B are any subsets of $\mathbb{R}^n$, then*

$$\mathcal{H}^n(A)^{1/n} + \mathcal{H}^n(B)^{1/n} \leq \mathcal{H}^n(A+B)^{1/n}.$$

PROOF. The proof proceeds in five steps. First we prove the inequality for intervals, then for their finite disjoint unions, then for compact sets, then for measurable sets and finally for arbitrary sets.

(i) By an interval in $\mathbb{R}^n$ we mean $I = I_1 \times \cdots \times I_n$, where I_k is an open bounded interval of $\mathbb{R}$: $I_k = (a_k, b_k)$. Assume that A and B are intervals $A = I_1 \times \cdots \times I_n$, $B = J_1 \times \cdots \times J_n$. Obviously then

$$A + B = \prod_{i=1}^{n} (I_i + J_i).$$

Using the facts that

(F1) $\prod_{i=1}^{n} x_i^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i$

(F2) $\mathcal{H}^n \left(\prod_{i=1}^{n} I_i \right) = \prod_{i=1}^{n} \mathcal{H}^1(I_i)$ and denoting

(F3) $a_i = \dfrac{\mathcal{H}^n(I_i)}{\mathcal{H}^n(I_i + J_i)}$ and $b_i = \dfrac{\mathcal{H}^n(J_i)}{\mathcal{H}^n(I_i + J_i)}$,

(F4) For every i we have $a_i + b_i = 1$ (because $\mathcal{H}^1(x, y) + \mathcal{H}^1(x', y') = \underbrace{|x - y| + |x' - y'|}_{\text{both positive}} = |(x + x') - (y + y')|$).

we get

$$\begin{aligned}
\frac{\mathcal{H}^n(A)^{1/n} + \mathcal{H}^n(B)^{1/n}}{\mathcal{H}^n(A + B)^{1/n}} &\overset{\text{F2,F3}}{=} \prod_{i=1}^{n} a_i^{1/n} + \prod_{i=1}^{n} b_i^{1/n} \\
&\overset{\text{F1}}{\leq} \frac{1}{n} \sum_{i=1}^{n} a_i^{1/n} + \frac{1}{n} \sum_{i=1}^{n} b_i^{1/n} \\
&= \frac{1}{n} \sum_{i=1}^{n} (a_i^{1/n} + b_i^{1/n}) \overset{\text{F4}}{=} \frac{1}{n} \sum_{i=1}^{n} 1 = 1.
\end{aligned}$$

Thus $\mathcal{H}^n(A)^{1/n} + \mathcal{H}^n(B)^{1/n} \leq \mathcal{H}^n(A + B)^{1/n}$ and we are done.

(ii) Assume that $P = \bigcup_{i=1}^{p} I_i$ and $Q = \bigcup_{i=1}^{q} J_i$ are disjoint. Note that in this case I_i is a subset of $\mathbb{R}^n$ unlike before (a subset of $\mathbb{R}$). The proof is by induction on $p + q$. If $p + q = 2$, the statement is either clear ($p = 2$ or $q = 2$) or follows from the previous step ($p = q = 1$). Assume $p + q > 2$, whence we can assume that $p \geq 2$. Because the intervals are disjoint, we can choose $i \in \{1, 2, \ldots, n\}$ and $a \in \mathbb{R}$ such that both

$$P_{-} = P \cap \{x \mid x_i < a\} \text{ and } P_{+} = P \cap \{x \mid x_i > a\}$$

contain one of the sets I_k, $1 \leq k \leq p$. Then by Bolzano's theorem we can find $b \in \mathbb{R}$ such that if $Q_{-} = Q \cap \{x \mid x_i < b\}$ and $Q_{+} = Q \cap \{x \mid x_i > b\}$, then

$$\frac{\mathcal{H}^n(P_{-})}{\mathcal{H}^n(P)} = \frac{\mathcal{H}^n(Q_{-})}{\mathcal{H}^n(Q)} \text{ and } \frac{\mathcal{H}^n(P_{+})}{\mathcal{H}^n(P)} = \frac{\mathcal{H}^n(Q_{+})}{\mathcal{H}^n(Q)}. \qquad (*)$$

Thus

$$P_- = \bigcup_{i=1}^{p} I_i \cap P_- = \bigcup_{i=1}^{p'} I_i',$$

where $p' < p$, similarily for P_+, and

$$Q_- = \bigcup_{i=1}^{q} I_i \cap Q_- = \bigcup_{i=1}^{q'} I_i',$$

where $q' \leq q$. Moreover $P_- + Q_-$ and $P_+ + Q_+$ are disjoint and of course measurable, hence by induction hypothesis

$$\begin{aligned}
\mathcal{H}^n(P+Q) &\geq \mathcal{H}^n(P_- + Q_-) + \mathcal{H}^n(P_+ + Q_+) \\
&\geq \left(\mathcal{H}^n(P_-)^{1/n} + \mathcal{H}^n(Q_-)^{1/n}\right)^n + \left(\mathcal{H}^n(P_+)^{1/n} + \mathcal{H}^n(Q_+)^{1/n}\right)^n \\
(*) \quad &= \mathcal{H}^n(P_-)\left(1 + \left(\frac{\mathcal{H}^n(Q)}{\mathcal{H}^n(P)}\right)^{1/n}\right)^n + \mathcal{H}^n(P_+)^{1/n}\left(1 + \left(\frac{\mathcal{H}^n(Q)}{\mathcal{H}^n(P)}\right)^{1/n}\right)^n \\
&= \left(\mathcal{H}^n(P_-) + \mathcal{H}^n(P_+)^{1/n}\right)\left(1 + \left(\frac{\mathcal{H}^n(Q)}{\mathcal{H}^n(P)}\right)^{1/n}\right)^n \\
&= \mathcal{H}^n(P)\left(1 + \left(\frac{\mathcal{H}^n(Q)}{\mathcal{H}^n(P)}\right)^{1/n}\right)^n \\
&= \left(\mathcal{H}^n(P)^{1/n} + \mathcal{H}^n(Q)^{1/n}\right)^n.
\end{aligned}$$

(iii) Let $A, B \subset \mathbb{R}^n$ be compact and let $\varepsilon > 0$. Let $\mathcal{D}_k$ be the following collection of all such intervals, whose side length is 2^{-k} and the projection to any coordinate is an interval $(m, m+1) \subset \mathbb{R}$ with m an integer.

$$\mathcal{D}_k = \{I_1 \times \cdots \times I_n \mid I_k = \left(m_k 2^{-k}, (m_k+1)2^{-k}\right), m_k \in \mathbb{Z}\}.$$

Now define

$$A_k = \bigcup\{P \in \mathcal{D}_k \mid \bar{P} \cap A \neq \varnothing\}$$

and

$$B_k = \bigcup\{P \in \mathcal{D}_k \mid \bar{P} \cap B \neq \varnothing\}.$$

Clearly the sets $A \setminus A_k$ and $B \setminus B_k$ are of measure zero and the sets A_k, B_k are disjoint finite unions of open n-intervals. Let $\varepsilon > 0$. Since

$A + B = \bigcap_{k\in\mathbb{N}}(\bar{A}_k + \bar{B}_k)$, there exists k such that

$$\begin{aligned}
\mathcal{H}^n(A+B)^{1/n} + \varepsilon &\geq \mathcal{H}^n(\bar{A}_k + \bar{B}_k)^{1/n} \\
&\geq \mathcal{H}^n(A_k + B_k)^{1/n} \\
\text{step } \textbf{(ii)} &\geq \mathcal{H}^n(A_k)^{1/n} + \mathcal{H}^n(B_k)^{1/n} \\
&\geq \mathcal{H}^n(A)^{1/n} + \mathcal{H}^n(B)^{1/n}.
\end{aligned}$$

Because this can be done for every ε, we have $\mathcal{H}^n(A+B)^{1/n} \geq \mathcal{H}^n(A)^{1/n} + \mathcal{H}^n(B)^{1/n}$.

(iv) Assume that A and B are Lebesgue-measurable. Let $\varepsilon > 0$ and K_A, K_B compact subsets of A and B such that $\mathcal{H}^n(A \setminus K_A) + \mathcal{H}^n(B \setminus K_B) < \varepsilon$. Now

$$\begin{aligned}
\mathcal{H}^n(A+B)^{1/n} &\geq \mathcal{H}^n(K_A + K_B)^{1/n} \\
\text{step } \textbf{(iii)} &\geq \mathcal{H}^n(K_A)^{1/n} + \mathcal{H}^n(K_B)^{1/n} \\
&\geq \big(\mathcal{H}^n(A) - \varepsilon\big)^{1/n} + \big(\mathcal{H}^n(B) - \varepsilon\big)^{1/n}.
\end{aligned}$$

As ε is again arbitrary, we get the intended statement.

(v) Let $E \subset \mathbb{R}^n$ be arbitrary. The *Lebesgue hull* of E is the set

$$E^* \left\{ x \in \mathbb{R}^n \mid \lim_{r\to 0^+} \frac{\mathcal{H}^n(E \cap B(x,r))}{\mathcal{H}^n(B(x,r))} = 1 \right\},$$

which is a Borel set and $\mathcal{H}^n(E^*) = \mathcal{H}^n(E)$. It is not difficult to see that $A^* + B^* \subset (A+B)^*$, namely if $a \in A$ and $b \in B$, then

$$\frac{\mathcal{H}^n((A+B) \cap B(a+b,r))}{\mathcal{H}^n(B(a+b,r))} \geq \frac{\mathcal{H}^n((A+b) \cap B(a+b,r))}{\mathcal{H}^n(B(a+b,r))} = \frac{\mathcal{H}^n(A \cap B(a,r))}{\mathcal{H}^n(B(a,r))} = 1.$$

Hence

$$\begin{aligned}
\mathcal{H}^n(A+B)^{1/n} &= \mathcal{H}^n((A+B)*)^{1/n} \\
&\geq \mathcal{H}^n(A^* + B^*)^{1/n} \\
\text{step } \textbf{(iv)} &\geq \mathcal{H}^n(A^*)^{1/n} + \mathcal{H}^n(B^*)^{1/n} = \mathcal{H}^n(A)^{1/n} + \mathcal{H}^n(B)^{1/n}
\end{aligned}$$

and we are done. □

Proof of the isoperimetric inequality (theorem 3). If we assume that $M^{n-1}(\partial A)$, then it follows $\mathcal{H}^n(\partial A) = 0$ and so it in suffices to show the inequality

$$n\Omega_n^{1/n} \mathcal{H}^n(\bar{A})^{(n-1)/n} \leq M^{n-1}(\partial A).$$

This can be shown as follows. From now on A is some fixed subset of $\mathbb{R}^n$, which is closed and has finite $\mathcal{H}^n$-measure. Applying Brunn-Minkowski inequality we obtain the following:

$$\begin{aligned}
\mathcal{H}^n(A+B(0,r)) &\geq [\mathcal{H}^n(\bar{A})^{1/n}+\mathcal{H}^n(B(0,r))^{1/n}]^n \\
&= [\mathcal{H}^n(\bar{A})^{1/n}+r\Omega_n^{1/n}]^n \\
&\geq \mathcal{H}^n(\bar{A})+n\mathcal{H}^n(\bar{A})^{(n-1)/n}\Omega_n^{1/n}r \\
\Longrightarrow \mathcal{H}^n(A+B(0,r))-\mathcal{H}^n(A) &\geq n\mathcal{H}^n(A)^{(n-1)/n}\Omega_n^{1/n}r
\end{aligned}$$

Let $\varepsilon > 0$. There exists r such that

$$\begin{aligned}
M^{n-1}(\partial A)+\varepsilon &\geq \frac{1}{r}\Big[\mathcal{H}^n(A+B(0,r))-\mathcal{H}^n(A)\Big] \\
&\geq \frac{1}{r}\Big[n\mathcal{H}^n(A)^{(n-1)/n}\Omega_n^{1/n}r\Big] \\
&= n\mathcal{H}^n(A)^{(n-1)/n}\Omega_n^{1/n} \\
\stackrel{\varepsilon\to 0}{\longrightarrow} M^{n-1}(\partial A) &\geq \mathcal{H}^n(\bar{A})^{(n-1)/n}n\Omega_n^{1/n}
\end{aligned}$$

as intended. □

2. Relation to the Sobolev Inequality

Sobolev inequality states (for p=1) that there is a constant C_n such that

$$\|u\|_{L^{\frac{n}{n-1}}} \leq C_n\|\nabla u\|_{L^1}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Substituting an approximation for $u = \chi_A$, it is not difficult to deduce the isoperimetric inequality for A. On thenother hand Sobolev inequality can be deduced from isoperimetric inequality such that the constant of isoperimetric $c_n = C_n$ the constant of Sobolev.

Bibliography

[Federer] Herbert Federer, Geometric Measure Theory, Springer-Verlag, 1969.