Isomorphic (finite) Graphs and Eigenvalues

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Abstract

This paper is considered on proving that the eigenvalues of representative matrices of two graphs A and B are same if A and B are isomorphic. Thus it is an invariant of graphs.

1 Some basic definitions and lemmas

Definition Define a graph to be a finite set V together with a binary relation R on this set, which is symmetric and irreflexive, that is

$$(\nexists x \in V)(xRx) \wedge (\forall (x,y) \in V^2)(xRy \Leftrightarrow yRx)$$

Definition A *permutation matrix* is a square matrix such that each row and each column containes exactly one 1 and all other zeros. For example the unit matrix is a (trivial) permutation matrix.

It is easy to see that a permutation matrix permutates the rows of a matrix if multiplied on the left and does the same to the columns if multiplied on the right.

Lemma 1 *Let $s : I_n \to I_n$ be a permutation of a finite set*

$$I_n = \{k \in \mathbb{N} | 1 \leq k \leq n\}$$

and let a^i be a column vector, whose i:th component is 1 and other are zeros, for each $1 \leq i \leq n$. Then the matrix $P_s = (a_{s(1)}, ..., a_{s(n)})$, whose rows are the vectors a_i in the order defined by s, is a permutation matrix. Moreover if B is a matrix $(b^1, ..., b^n)$ (b^i being column vectors) then $BP_s = (b^{s(1)}, ..., b^{s(n)})$ and if $C = (c_1, ..., c_n)$ (c_i being row vectors), then $P_sC = (c_{s(0)}, ..., c_{s(n-1)})$

Proof

2 Proving the theorem

Let us now define the representative matrix of a graph.

Definition Let (V, R) be a finite graph with say n vertices. Then it is clear that we have a bijection α from the set I_n to the set V of vertices. Now let us from the matrix $m(A)$ such that the element of i:th row and j:th column is

$$m(A)_{ij} = \begin{cases} 1, & \text{if } \alpha(i)R\alpha(j) \\ 0, & \text{if not } \alpha(i)R\alpha(j) \end{cases}$$

That is, $m(A)_{ij}$ is 1 if and only if vertexes numbered i and j are connected, 0 otherwise. For example every such matrix has zeros on its diagonal: $\forall i\, m(A)_{ii} = 0$

Remark Given a matrix A, $m(A)$ is not uniquely determined. It ofcourse depends on the choice of the enumerating – the function α. To specify that we can note $m_\alpha(A)$ to underline that this matrix is obtained using the function α.

Theorem 2 *If symmetric $n \times n$-matrices A and B has the same eigenvalues then there exists an invertible orthogonal $n \times n$-matrix M such that $A = MBM^{-1}$*

Proof Let D be the diagonal matrix with the eigenvalues of A (thus also of B) as the non-zero elements. Now there exists orthonormal matrices O and N such that

$$\begin{aligned} & A &= ODO^T \\ \text{and} \quad & B &= NDN^T. \end{aligned}$$

There from follows that $A = ON^TDNO^T$ and we can set $M = ON^T$. Now clearly M is orthogonal and $M^T = NO^T$ and $A = MBM^T$.

Theorem 3 *Let A be a square matrix and N an invertible orthogonal matrix of the same size. Then the eigenvalues of A are precisely those of NAN^{-1}*

Thus the only way two matrices of a same graph can differ is that the enumerating is different. Let us think, what happens to the matrix $m(A)$ when the vertices are permutated.

Theorem 4 *If A and B are isomorphic graphs then the eigenvalues of $m(A)$ and $m(B)$ coinside.*

Proof As we have seen, if A and B are isomorphic, there exists a permutation matrix P such that $m(A) = Pm(B)P^{-1}$ ($P = P^{-1}$ for any permutation matrix P). Because P is orthonormal, it does not affect the eigenvalues.

Remark There are non-isomorphic graphs with the same sets of eigenvalues. But there is not many such graphs, so one might look consider the probability that graphs are isomorphic very low if the eigenvalues are different.