Weak Ehrenfeucht-Fraïssé Games

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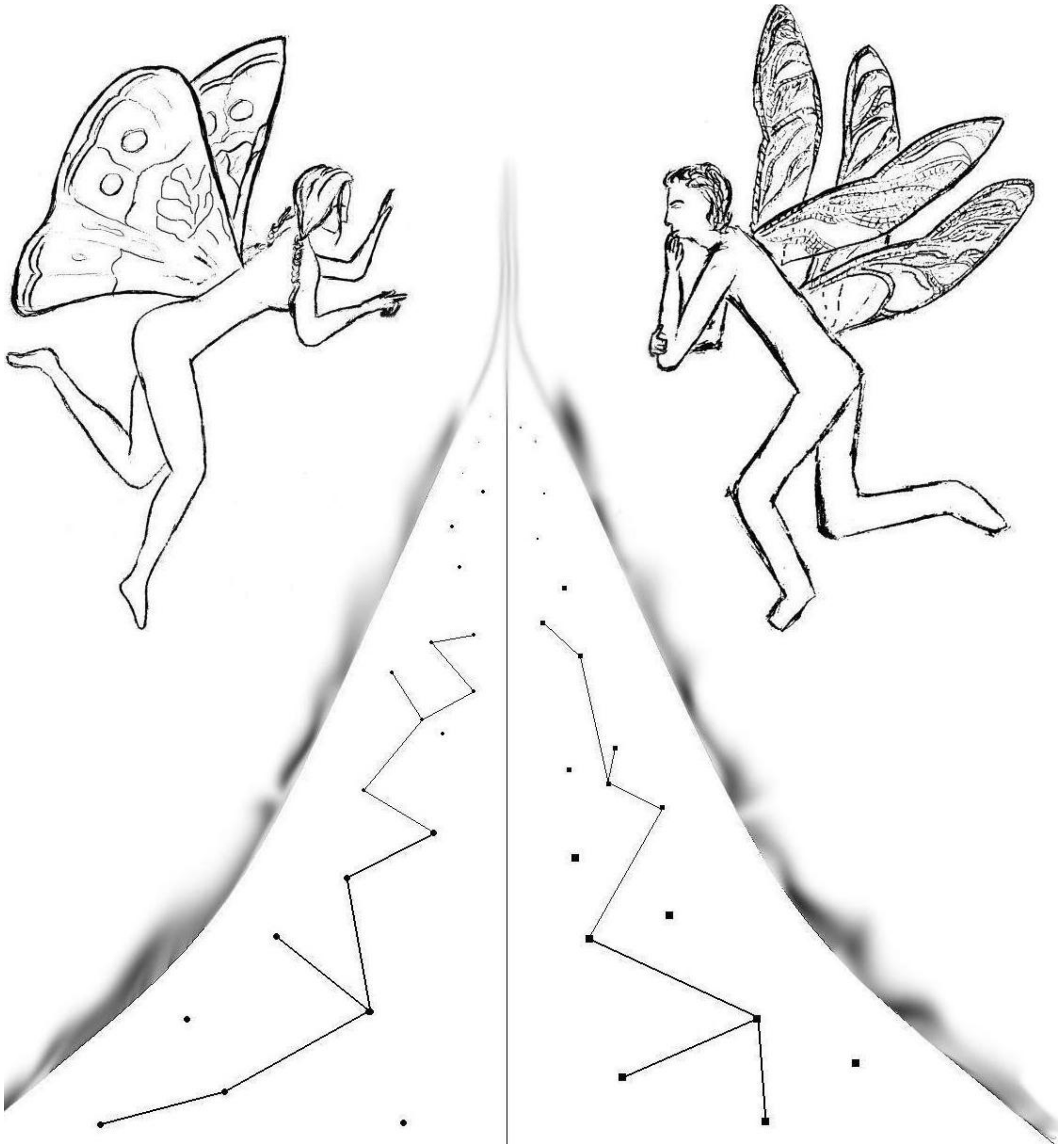

I want to thank Jouko Väänänen, whose idea it was to investigate these concepts and specially my supervisor Tapani Hyttinen for continuous rewarding discussion on the subject and useful critic. For almost all proofs of this paper I am indebted to his attention and help.

Contents

Introduction

History and games in logic

In mathematics abstract games are widely used to model biology, economics, computer science and logic. The first known discussion of game theory occurred in a letter written by James Waldegrave in 1713. In this letter Waldegrave provides a minimax strategy solution to a two-player version of the card game le Her. It was not until the publication of Antoine Augustin Cournot's research paper [**Cournot**] in 1838 that a general game theoretic analysis was pursued. In this work Cournot considers a duopoly[1] and presents a solution that is a restricted version of the Nash equilibrium.

However game theory did not really exist as a unique field until John von Neumann published a series of papers in 1928. Although Borel did some earlier work on games, von Neumann can rightfully be credited as the inventor of game theory. Von Neumann's work in game theory culminated in [**vonNeumann**]. This work contains methods for finding optimal solutions to two-person zero-sum games. During this period work on game theory was primarily focused on cooperative games, which analyzes optimal strategies for groups of individuals, presuming that they can enforce agreements between them about proper strategies.

Games appeared in logic in 1930's, when Leon Henkin introduced the notion of game semantics, later developed by Paul Lorenzen in the 1950's. The idea is to climb up the semantic tree of a logical sentence. A semantic tree branches at quantifiers and at the signs $\wedge$ and $\vee$. Here branching at a quantifier means checking all possible values of the quantified variable. For example let $\psi = (\forall x_0 R(x_0)) \wedge (\exists x_1 R(x_1))$ and let the structure $\mathcal{A}$ be such that $A = \{a, b\}$ and $R^{\mathcal{A}} = \{a\}$. The semantic tree will look like this:

$$\begin{array}{ccc}
R(a) \quad R(b) & & R(a) \quad R(b) \\
\backslash \quad / & & \backslash \quad / \\
\forall x_0 R(x_0) & & \exists x_1 R(x_1) \\
& \backslash \quad / & \\
& (\forall x_0 R(x_0)) \wedge (\exists x_1 R(x_1)) &
\end{array}$$

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[1] A market form where only two producers exist in one market.

The game starts at the root. If the quantifier $\forall$ or the sign $\wedge$ is in question, then player **I** chooses which branch to continue along, otherwise **2** chooses. If a negation occurs, then it is dropped and the players change roles. They end up with an element in the structure for each quantifier encountered on their way and an atomic formula into which the elements are substituted. If the atomic formula with this substitution is true, then **2** wins. The sentence is defined to be true if and only if **2** has a winning strategy and false if and only if **I** has a winning strategy.

In 1954 Roland Fraïssé introduced in his thesis [**Fraïssé**] a back-and-forth method, which was soon reformulated by Andrzej Ehrenfeucht in terms of a game, today known as the Ehrenfeucht-Fraïssé game. This back-and-forth method was a characterization of elementary equivalence. The point is to look at isomorphisms one at a time and how might they be extended to larger finite isomorphisms. If the isomorphism player wins the Ehrenfeucht-Fraïssé game of any finite length, i.e. the finite isomorphisms can be extended "far enough", then the structures are elementarily equivalent and the converse holds if the vocabulary is finite and relational. In [**Karp**] Carol Karp proved that **2** wins the game of length ω (which means that there exists a back-and-forth system) if and only if the structures are $L_{\infty\omega}$-equivalent. Later on, in the 1980's, this result was generalized to languages with infinitely deep quantifying by Karttunen, see [**Karttunen**], and Hyttinen, see [**Hyttinen**]. Those languages were introduced by Vaught, Karttunen, Hintikka and Rantala. In them the truth of a sentence is defined through the above-described semantic game or some version of it. In fact when the amount of quantifiers can be infinite, game semantics turn out to be in many cases the best way to handle the truth definition.

It turns out that long games such as games of length ω_1 are not generally determined. That is, it is possible that neither of the players has a winning strategy. In particular non-determined sentences (sentences whose truth value can not be determined) of infinitary languages exist as pointed out by Hyttinen in [**Hyttinen**]. It was shown in [**MekSheVä**] that the Ehrenfeucht-Fraïssé game of length ω_1 is non-determined on certain structures unless we require their size to be $\aleph_2$. In the case when structures are of size $\aleph_2$ it is independent of ZFC whether EF_{ω_1} is determined.

Although the notion of a game in mathematics generally serves merely as a guide to intuition, these games have become central in logic and have been studied independently. This has led to games which do not arise from languages or truth definitions but from other, purely model theoretic features. In this thesis we study such games, which we call *weak Ehrenfeucht-Fraïssé games*. The name refers to the fact that it is like the EF-game, but easier for the isomorphism player. There were two attempts to define this game, although one of them is leading throughout the paper. While in the ordinary EF-game the isomorphism player has to extend the isomorphism at each move, in the weak game she has only to ensure that the chosen structures at the end of the game are isomorphic. Evidently whether one of the players has a winning strategy in this game reveals something about the similarity of the structures, and this thesis is an attempt to understand what kind of similarity this is. At the first place it is shown that at certain game lengths the weak game is very similar (and sometimes equivalent) to the ordinary EF-game.

In [**Kueker**] Kueker showed that structures $\mathcal{A}$ and $\mathcal{B}$ are $L_{\infty\omega}$-equivalent if and only if the set

$$S = \{X \subset \mathcal{A} \cap \mathcal{B} \mid |X| \le \omega,\, X \cap \mathcal{A} \cong X \cap \mathcal{B}\}$$

contains a closed unbounded subset of $\mathcal{P}_{\omega_1}(A \cup B)$, the collection of all countable subsets of $A \cup B$ ordered by inclusion. This corresponds to the weak Ehrenfeucht-Fraïssé game of length ω studied in this paper, i.e. isomorphism player has a winning strategy in that game if and only if S contains a closed unbounded set.

We take a look also at games of ordinal length, but a major part of the paper is devoted to studying the games of length ω_1.

The paper is divided to parts according to different game lengths going eventually to longer and longer games. In chapter II of definitions some general observations are given, then the game of length ω is revealed after which games of ordinal lengths are considered. Finally various consistency results are proved considering the games of larger cardinal lengths as that of ω_1. In particular we proved that it is consistent that the weak EF-game is always determined on structures of size $\aleph_2$ and that it is also consistent that there are structures of this cardinality on which this game is not determined.

Prerequisites

Prerequisites for chapters I–IV are that the reader is familiar with the basics of ordinal and cardinal arithmetic and with elementary model theory. The appendix A is for deeper information on ordinals and cardinals.

With these prerequisites the reader is able to understand also the shapes of ideas of chapters V and VI, but complete understanding requires some familiarity with mathematical logic (such as Gödel's incompleteness and completeness theorems, which are stated without proofs), axiomatic set theory (ZFC, GCH, independence etc..) and group theory.

Sufficient group theory is though given in appendix B. Any familiarity with ZFC and independence will be helpful but avoidable until the end of chapter V, where the reader needs to have gone through the basics of forcing. Even there the reader can skip the forcing part, believe in the consistency of the obtained statements and read further. Of course, in this case the understanding will be incomplete.

Notations

I use standard set theoretical notations in this paper. Thus $A \subset B$ means that A is a subset of B or is equal to B. For a proper subset I use $A \subsetneq B$. Union, intersection and set theoretical difference are denoted respectively by $A \cup B$, $A \cap B$ and $A \setminus B$. For larger unions and intersections $\bigcup_{i\in I} A_i$ etc.

If $f\colon A \to B$ is a map, then $f(x)$ is the value of the function f at $x \in A$ and fX is the image of a subset $X \subset A$. If confusion is possible (as in the case $X = A \cup B$), I write $f[X]$. Similarly $f^{-1}X$ and $f^{-1}[X]$ for pre-image. On the other hand if it is clear from the context that f has an inverse, I use f^{-1} to denote that inverse. A map $f\colon X \to Y$ is said to be *injective* or *an injection* or just *one-to-one* if $x \neq y \Rightarrow f(x) \neq f(y)$. *Surjective* or *onto* means $f[X] = Y$. A surjective injection is a *bijection*.

Usually the greek letters κ, λ and μ will stand for cardinals and α, β and γ for ordinals, but this is not strict. The cofinality of an ordinal α (the least ordinal β for which there exists an increasing unbounded function $f\colon \beta \to \alpha$) is denoted by $\operatorname{cf}(\alpha)$. As in [**Jech**] I will write S^m_n to mean $\{\alpha < \omega_m \mid \operatorname{cf}(\alpha) = \omega_n\}$ (where $\omega_0 = \omega$).

I follow [**Hodges**] in the sense that 'I' means I and 'we' means we.

I Preliminaries

First I introduce the necessary notions from model theory. As mentioned in the introduction, I assume the basic notions of set theory such as ordinal and cardinal numbers as well as their elementary arithmetic[1], although in appendix A some more demanding cardinal arithmetic is reviewed on account of chapter V. In that chapter I also assume a basic familiarity with the set of axioms ZFC in which we work.

The languages $L_{\kappa\omega}$ and $L_{\infty\omega}$ will now be defined in usual few steps (essentially I follow the treatment in [**Hodges**]).

I.1. Languages

I.1. Definition (Vocabulary). Let κ, λ and μ be cardinals (possibly finite). A *vocabulary* is a set $V = \{R_i \mid i < \kappa\} \cup \{f_i \mid i < \lambda\} \cup \{c_i \mid i < \mu\}$ which consists of κ relation symbols R_i, λ function symbols f_i and μ constant symbols c_i.

If one of the cardinals is 0, then there is no respective symbols in the vocabulary. Each function symbol and relation symbol is of some positive finite *valence* (some authors prefer the word *arity*). In the next three definitions a vocabulary V and a countable set $X = \{x_i \mid i < \omega\}$ of variables are given. By R, F and C we denote respectively the sets of relations, functions and constants of V.

I.2. Definition (Terms). The set of *terms* over V is the set T of finite strings defined as follows.

(*i*) If $x \in X$ then $x \in T$.
(*ii*) If $c \in C$ then $c \in T$.
(*iii*) If $f \in F$ has valence n and $t_1, \dots, t_n \in T$ then $f(t_1, \dots, t_n) \in T$.
(*iv*) Nothing else is in T.

The last requirement (*iv*) is necessary if we want to keep control on what we define. It is not listed in the following definitions although it is implicitly assumed.

I.3. Definition (Atomic formulas). The set A of *atomic formulas* is another set of finite strings and is defined by

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[1] The axiom of choice is accepted throughout the paper but CH is not involved unless specified.

(*i*) If $s, t \in T$ then $s = t \in A$ (i.e. $s = t$ is an atomic formula).
(*ii*) If $P \in R$ has valence n and $t_1, \ldots, t_n \in T$, then $P(t_1, \ldots, t_n) \in A$.

And finally:

I.4. Definition (Language). Let κ be a regular cardinal. The *language* $L_{\kappa\omega}$ over the vocabulary V, denoted by L if no confusion is expected, is the collection of (not necessary finite) strings obtained from atomic formulas using connectives $\wedge$, $\vee$, negation $\neg$ and quantifiers $\forall$ and $\exists$. (The connective $=$ is already embedded in the definition of atomic formulas).

(*i*) Every atomic formula is in $L_{\kappa\omega}$, $A \subset L_{\kappa\omega}$.
(*ii*) If $\varphi \in L_{\kappa\omega}$ then $\neg\varphi \in L_{\kappa\omega}$.
(*iii*) Let $\lambda < \kappa$ be a cardinal. If $\varphi_i \in L_{\kappa\omega}$ for all $i < \lambda$, then $\bigwedge_{i<\lambda} \varphi_i \in L$ and $\bigvee_{i<\lambda} \varphi_i \in L$.
(*iv*) If $\varphi \in L$ and $x \in X$, then $\forall x\varphi \in L$ and $\exists x\varphi \in L$

The language $L_{\omega\omega}$ is the so called *first-order* language and consists only of finite strings. We say that φ belongs to $L_{\infty\omega}$ if $\varphi \in L_{\kappa\omega}$ for some κ. The second ω refers to the accepted amount of quantifying. We will talk only about languages with finite quantifying and so no need in defining $L_{\kappa\lambda}$, $\lambda > \omega$.

A variable x that occurs in a formula such that there is no $\forall x$ or $\exists x$ before that variable is called a free variable. Otherwise it is bounded. A formula with no free variables is sometimes called a sentence. From now on we use L for a vocabulary as well as for a language obtained from it if there is no danger of confusion. Now we are ready to define the basic concept that we shall use.

I.2. Models

First of all I mention that technically *Model* and *Structure* are synonyms. There are two words just for convenience. The word model is more often used as "Let $\mathcal{M}$ be a model of φ", which then means the same as "Let $\mathcal{M}$ be a structure such that $\mathcal{M} \models \varphi$". For definitions of these concepts, read further.

I.5. Definition (Structure). Let L be a vocabulary. Then an *L-structure* $\mathcal{A}$ consists of the following things

(*i*) A set A called the domain of $\mathcal{A}$.
(*ii*) A named element (a constant) $c^{\mathcal{A}} \in A$ for each constant symbol $c \in L$.
(*iii*) A function $f^{\mathcal{A}}\colon A^n \to A$ for each function symbol $f \in L$, which has valence n.

(*iv*) A relation $R^{\mathcal{A}} \subset A^n$ for each relation symbol $R \in L$, which has valence n.

Note that the above definition involves only a vocabulary, but not a language. The definition is very abstract and covers all significant mathematical structures such as groups, vector spaces, topologies (as lattices) and in a sense even the whole mathematics itself (as models of ZFC). I will use letters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ for structures and A, B, C to denote respectively their domains. By a structure we will mean an L-structure, where L is either an arbitrary vocabulary or is clear from the context.

I.6. DEFINITION. Let $\mathcal{A}$ be an L-structure. A *substructure* $\mathcal{B}$ of $\mathcal{A}$ is an L-structure such that

(*i*) $B \subset A$.
(*ii*) If $c \in L$ is a constant symbol, then $c^{\mathcal{B}} = c^{\mathcal{A}}$ and $c^{\mathcal{B}} \in B$.
(*iii*) If $f \in L$ is a function symbol of valence n, then B is closed under $f^{\mathcal{A}}$, i.e. $f^{\mathcal{A}}[B^n] \subset B$, and $f^{\mathcal{B}} = f^{\mathcal{A}} \restriction B^n$.
(*iv*) If $R \in L$ is a relation symbol of valence n, then $R^{\mathcal{B}} = R^{\mathcal{A}} \cap B^n$.

Assume $\mathcal{A}$ is a structure and $B \subset A$. Then $\langle B \rangle$ is the smallest substructure of $\mathcal{A}$, such that its domain includes B. Sometimes I write $\langle a_\alpha \rangle_{\alpha<\gamma}$ instead of $\langle \{a_\alpha \mid \alpha < \gamma\} \rangle$.

I.7. DEFINITION. An *isomorphism* between two L-structures $\mathcal{A}$ and $\mathcal{B}$ is a function $F \colon A \to B$ such that

(*i*) F is bijective.
(*ii*) For all constant symbols c in L, $F(c^{\mathcal{A}}) = c^{\mathcal{B}}$.
(*iii*) For all function symbols f of valence n in L and $x = (x_1, \dots, x_n) \in A^n$, $F(f^{\mathcal{A}}(x)) = f^{\mathcal{B}}(F(x_1), \dots, F(x_n))$.
(*iv*) For all relation symbols R in L, $x \in R^{\mathcal{A}} \iff F(x) \in R^{\mathcal{B}}$. Here $x \in A^n$, where n is the valence of R and $F(x)$ is really $(F(x_1), \dots, F(x_n)) \in B^n$.

Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures and $A' \subset A$, $B' \subset B$ and $f \colon A' \to B'$. If there exists an isomorphism $g \colon \langle \mathcal{A}' \rangle \to \langle \mathcal{B}' \rangle$ such that $g \restriction \mathcal{A} = f$, then f is called a *partial isomorphism* $\mathcal{A} \to \mathcal{B}$.

If there exists an isomorphism $\mathcal{A} \to \mathcal{B}$ then $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic* and this fact is denoted by $\mathcal{A} \cong \mathcal{B}$. The relation $\cong$ is an equivalence relation in the class of L-structures (it is elementary to check that a composition of isomorphisms is an isomorphism) and usually one identifies structures which belong to the same equivalence class.

We now lack only the definition of truth for full equipment.

I.8. Definition (Tarski). Let $\mathcal{A}$ be an L-structure. We define that a formula φ (no free variables) of the language $L = L_{\kappa\omega}$ is true in $\mathcal{A}$ or that $\mathcal{A}$ satisfies φ or just $\mathcal{A} \models \varphi$ in a following manner:

(*i*) If φ is the formula $s = t$, where s, t are terms, then $A \models \varphi \iff s^{\mathcal{A}} = t^{\mathcal{A}}$.

(*ii*) If φ is the formula $R(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms, then $\mathcal{A} \models \varphi \iff (t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) \in R$.

(*iii*) If φ is of the form $\bigwedge_{i<\lambda} \psi_i$, then $\mathcal{A} \models \varphi \iff$ for all $i < \lambda$, $\mathcal{A} \models \psi_i$.

(*iv*) If φ is of the form $\bigvee_{i<\lambda} \psi_i$, then $\mathcal{A} \models \varphi \iff$ for some $i < \lambda$, $\mathcal{A} \models \psi_i$.

(*v*) If φ is of the form $\neg\psi$,then $\mathcal{A} \models \varphi$ if and only if it is not the case that $\mathcal{A} \models \psi$.

(*vi*) Assume φ is of the form $\forall x \psi$. Then if x does not occur freely in ψ, then $\mathcal{A} \models \varphi \iff \mathcal{A} \models \psi$. On the other hand, if x occurs freely in $\varphi = \varphi(x)$, then let c be a constant symbol which is not in L and let $L' = L \cup \{c\}$. We define $\mathcal{A} \models \varphi \iff \mathcal{A} \models \psi(c)$ no matter what $c^{\mathcal{A}}$ is. Here $\psi(c)$ is the formula that is obtained from ψ by changing all free occurrences of x by c. It is easy to see by induction on the length of the formula that $\psi(c) \in L'$.

(*vii*) The truth of $\varphi = \exists x \psi$ is defined in a similar way. The difference is that "no matter what $c^{\mathcal{A}}$ is" is changed to "for some interpretation $c^{\mathcal{A}}$ of c".

Let $\lambda \leq \kappa$. If $\varphi(x)$ has free variables $x = (x_i)_{i<\lambda}$ and $a = (a_i)_{i<\lambda} \in A^\lambda$, then we define the truth of $\varphi(a)$ as follows. Let $c = (c_i)_{i<\lambda}$ be constants that are not in the vocabulary of L and let $L' = L \cup \{c_i \mid i < \lambda\}$ and let $\mathcal{A}'$ be the L'-structure, which agrees with $\mathcal{A}$ with respect to L and which is such that $c_i^{\mathcal{A}'} = a_i$ for each $i < \lambda$. Then $\mathcal{A} \models \varphi(a) \iff \mathcal{A}' \models \varphi(c)$. Note that the left part has an L-formula with elements of the model substituted for variables but the right hand side has just an L'-formula without free variables.

I.9. Definition. Two L-structures $\mathcal{A}$ and $\mathcal{B}$ are κ-elementarily equivalent, $\mathcal{A} \equiv_{\kappa\omega} \mathcal{B}$, if for all $\varphi \in L_{\kappa\omega}$ with no free variables,

$$\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi$$

holds. If $\mathcal{A} \equiv_{\omega\omega} \mathcal{B}$, then we say just that they are elementarily equivalent and sometimes forget the index $_{\omega\omega}$ from the notation: $\mathcal{A} \equiv \mathcal{B}$. If on the other hand $\mathcal{A} \equiv_{\kappa\omega} \mathcal{B}$ for all cardinals κ, then we write $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$.

As a special case, if $\mathcal{A} \subset \mathcal{B}$, $\mathcal{A} \equiv \mathcal{B}$ and for every formula $\varphi(x_1, \ldots, x_n)$,

$$\mathcal{A} \models \varphi(a_1, \ldots, a_n) \iff \mathcal{B} \models \varphi(a_1, \ldots, a_n),$$

where $a_1, \ldots, a_n$ are new constant symbols and $a_i^{\mathcal{A}} = a_i^{\mathcal{B}}$ for all i. This is equivalent to saying that the inclusion map $\mathcal{A} \hookrightarrow \mathcal{B}$ is an *elementary embedding.* Then we write $\mathcal{A} \prec \mathcal{B}$ and say that $\mathcal{A}$ is an *elementary substructure* of $\mathcal{B}$

The relations $\equiv_{\kappa\omega}$ are clearly equivalence relations in the class of L-structures.

I.10. LEMMA. *If structures $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, then they are κ-elementarily equivalent for any κ, that is $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$.* □

We shall not use the following except at one tiny point, but it is still worth mentioning:

I.11. THEOREM (Löwenheim-Skolem). *Let $\mathcal{A}$ be an L-structure, where L is a countable language, and $X \subset \mathcal{A}$ a countable subset. Then there is a countable elementary submodel $\mathcal{A}' \prec \mathcal{A}$ such that $X \subset \mathcal{A}'$.* □

II Weak Ehrenfeucht-Fraïssé Games

II.1. Definitions

II.1. Definition. A *game* $G_\gamma(S)$ consists of a set S, game length γ (an ordinal) and a winning set $W \subset (S \times S)^\gamma$. It is played between two players, $\mathfrak{I}$ (he) and $\mathfrak{2}$ (she). At the move $\beta < \gamma$ player $\mathfrak{I}$ chooses $a_\beta \in S$ and then $\mathfrak{2}$ chooses $b_\beta \in S$. Player $\mathfrak{2}$ wins if and only if $(a_i, b_i)_{i<\gamma} \in W$.

This definition covers all games defined in this paper, though these games will not be defined by formally describing W. The above definition is rather for convenience when defining general concepts such as winning strategy. For example in the following S would be the union of the domains of $\mathcal{A}$ and $\mathcal{B}$.

II.2. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be structures and γ an ordinal. The *Ehrenfeucht-Fraïssé game* of length γ, $\mathrm{EF}_\gamma(\mathcal{A},\mathcal{B})$, is played as follows. On the move α, $\alpha < \gamma$, player $\mathfrak{I}$ chooses an element $a_\alpha \in A$ (or $b_\alpha \in B$). Then $\mathfrak{2}$ answers by choosing an element $b_\alpha \in B$ (or $a_\alpha \in A$). $\mathfrak{2}$ wins if the function f, which takes a_α to b_α for each $\alpha < \gamma$ is a partial isomorphism $\mathcal{A} \to \mathcal{B}$. Otherwise $\mathfrak{I}$ wins.

The players $\mathfrak{I}$ and $\mathfrak{2}$ are referred respectively as male and female.

II.3. Definition. Let $\mathcal{A}$, $\mathcal{B}$ and γ be as in II.2. The *weak Ehrenfeucht-Fraïssé game* of length γ, $\mathrm{EF}^*_\gamma(\mathcal{A},\mathcal{B})$, is played as follows.

Player $\mathfrak{I}$: On the β:th move ($\beta = \alpha + 1 < \gamma$) player $\mathfrak{I}$ chooses an element
$a_\beta \in A$, if he chose an element from $\mathcal{B}$ on the previous move,
$b_\beta \in B$, if he chose an element from $\mathcal{A}$ on the previous move.
If α is a limit ordinal or 0, then $\mathfrak{I}$ decides which structure he picks the element from.

Player $\mathfrak{2}$: On the β:th move she chooses
$b_\alpha \in B$, if $\mathfrak{I}$ chose from $\mathcal{A}$,
$a_\alpha \in A$, if $\mathfrak{I}$ chose from $\mathcal{B}$.

Player $\mathfrak{2}$ wins if the substructures $\langle a_\alpha \rangle_{\alpha<\gamma}$ and $\langle b_\alpha \rangle_{\alpha<\gamma}$ are isomorphic. Otherwise $\mathfrak{I}$ wins.

As we will see in section II.2, when κ is a cardinal, EF^*_κ is equivalent to an easier defined game $\overset{*}{\mathrm{EF}}{}^{1,1}_\kappa(\mathcal{A},\mathcal{B})$, where on the move β

Player I: chooses an element $a_\beta \in A \cup B$
Player II: chooses an element $b_\beta \in A \cup B$.

In the original game that we studied there was no restriction for player I from which structure to choose but a restriction for player II. That game was somewhat in the middle of the games of definitions II.2 and II.3. It turned out that the main results we proved for that game hold for the above defined EF^* as well.

II.4. Definition. The game, which is exactly as in II.3, but where I can freely choose the structure from which to play, will be denoted $\mathrm{EF}^\circ_\gamma(\mathcal{A},\mathcal{B})$.

By *the* weak Ehrenfeucht-Fraïssé game we will refer to the game EF^* defined in II.3 and by the weak EF-*games* we will refer to all, EF^*, EF° and $\overset{*}{\mathrm{EF}}$, which will be defined in the next section.

II.5. Definition. A *strategy* for player I in some game $G_\gamma(S)$ is a function $\tau\colon S^{<\gamma} \to S$. A strategy τ for player I is *winning* if player I wins the game $G_\gamma(S)$ by playing the element $\tau((b_\alpha)_{\alpha<\beta})$ on the $(\beta+1)$:th move, where b_α are the elements that player II has chosen, for each $\beta < \gamma$.

Note that in the case of Ehrenfeucht-Fraïssé games on structures $\mathcal{A}$ and $\mathcal{B}$ a strategy is a function $\tau\colon (A\cup B)^{<\gamma} \to (A\cup B)$. The concepts of a strategy and a winning strategy are defined analogously for player II. A game is said to be determined if one of the players has a winning strategy. Otherwise not determined or non-determined.

II.6. Definition. Assume that τ is a strategy of player I and σ is a strategy of player II. Consider the game, where I uses τ and II uses σ. If II wins, we say that σ *beats* τ and vice versa.

II.7. Lemma. *A game G is non-determined if and only if for every strategy τ of I there exists a strategy of II, which beats τ and for every strategy σ of II there exists a strategy of I which beats σ.*

If II has a winning strategy in the (weak) EF-game (of length γ) on the structures $\mathcal{A}$ and $\mathcal{B}$ then we say that $\mathcal{A}$ and $\mathcal{B}$ are *(weakly)* EF_γ*-equivalent.* Some notations to be used:

$X \uparrow G$	Player X has a winning strategy in the game G.
$\mathcal{A} \cong \mathcal{B}$	$\mathcal{A}$ and $\mathcal{B}$ are isomorphic.
$\mathcal{A} \sim_\gamma \mathcal{B}$	means the same as $\mathbf{2} \uparrow \mathrm{EF}_\gamma(\mathcal{A}, \mathcal{B})$.
$\mathcal{A} \sim^\circ_\gamma \mathcal{B}$	means the same as $\mathbf{2} \uparrow \mathrm{EF}^\circ_\gamma(\mathcal{A}, \mathcal{B})$.
$\mathcal{A} \sim^*_\gamma \mathcal{B}$	means the same as $\mathbf{2} \uparrow \mathrm{EF}^*_\gamma(\mathcal{A}, \mathcal{B})$.

All of the relations, $\sim_\gamma$, $\sim^\circ_\gamma$ and $\sim^*_\gamma$ are equivalence relations in the class of L-structures. As an example of how to prove it, let us show

II.8. THEOREM. *The relation $\sim^*_\gamma$ is an equivalence relation in the class of L-structures.*

PROOF. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be structures of the same vocabulary.
Reflexivity: $\mathcal{A} \sim^*_\gamma \mathcal{A}$.
Proof: The winning strategy for $\mathbf{2}$ is to play the same elements as $\mathbf{I}$ did. Thus at the move β, if $\mathbf{I}$ chooses $a \in \mathcal{A}$, $\mathbf{2}$ must choose also $a \in \mathcal{A}$. Clearly $\mathbf{2}$ wins. $\square_{\text{Refl.}}$

Symmetry: $\mathcal{A} \sim^*_\gamma \mathcal{B} \iff \mathcal{B} \sim^*_\gamma \mathcal{A}$
Proof: The definition of EF^* did not in any manner distinguish between $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$. $\square_{\text{Symm.}}$

Transitivity $\mathcal{A} \sim^*_\gamma \mathcal{B} \wedge \mathcal{B} \sim^*_\gamma \mathcal{C} \Rightarrow \mathcal{A} \sim^*_\gamma \mathcal{C}$
Proof: Let σ_1 be a winning strategy of $\mathbf{2}$ in $\mathrm{EF}^*_\gamma(\mathcal{A}, \mathcal{B})$ and σ_2 respectively in $\mathrm{EF}^*_\gamma(\mathcal{B}, \mathcal{C})$. In the game $\mathrm{EF}^*_\gamma(\mathcal{A}, \mathcal{C})$ she imagines that she is actually playing all three games at a time. At the move β some sequences $(x_i)_{i<\beta}$, $(b_i)_{i<\beta}$, $(b'_i)_{i<\beta}$ and $(y_i)_{i<\beta}$ are picked (all empty, when $\beta = 0$). Suppose $\mathbf{I}$ chooses $x_\beta \in A$. Then $\mathbf{2}$ first defines $b_\beta = x_\beta$ and picks $b'_\beta = \sigma_1((b_i)_{i\leq\beta}) \in B$ and then answers in the actual EF^*_γ by $y_\beta = \sigma_2((b'_i)_{i\leq\beta}) \in C$. On the other hand, if $\mathbf{I}$ picks $y_0 \in C$, then $\mathbf{2}$ first defines $b'_\beta = y_\beta$ and picks $b_\beta = \sigma_2((b'_i)_{i\leq\beta}) \in B$ and then answers in the actual EF^*_γ by $x_\beta = \sigma_1((b_i)_{i\leq\beta}) \in A$. Thus $\mathbf{2}$ thinks of b_i as the moves of player $\mathbf{I}$ in $\mathrm{EF}^*_\gamma(\mathcal{A}, \mathcal{B})$ and at the same time as moves of $\mathbf{2}$ in $\mathrm{EF}^*_\gamma(\mathcal{B}, \mathcal{C})$. Similarly for b'_i's.

In the end, because σ_1 and σ_2 are winning, we have

$$\{x_i \mid i < \gamma\} \overset{\sigma_1}{\cong} \{b_i, b'_i \mid i < \gamma\} \cap \mathcal{B} \overset{\sigma_2}{\cong} \{y_i \mid i < \gamma\},$$

thus we have constructed a strategy for $\mathbf{2}$ in $\mathrm{EF}^*_\gamma(\mathcal{A}, \mathcal{C})$. $\square_{\text{Trans.}}$
We conclude that $\sim^*_\gamma$ is an equivalence relation. $\square$

It is clear that

$$\mathbf{2} \uparrow \mathrm{EF}_\gamma(\mathcal{A},\mathcal{B}) \Rightarrow \mathbf{2} \uparrow \mathrm{EF}^\circ_\gamma(\mathcal{A},\mathcal{B}) \Rightarrow \mathbf{2} \uparrow \mathrm{EF}^*_\gamma(\mathcal{A},\mathcal{B})$$

and

$$\mathbf{I} \uparrow \mathrm{EF}_\gamma(\mathcal{A},\mathcal{B}) \Leftarrow \mathbf{I} \uparrow \mathrm{EF}^\circ_\gamma(\mathcal{A},\mathcal{B}) \Leftarrow \mathbf{I} \uparrow \mathrm{EF}^*_\gamma(\mathcal{A},\mathcal{B}).$$

The converses are those that are interesting. Because of these implications I shall prefer to prove that $\mathbf{2} \uparrow \mathrm{EF}^\circ(\mathcal{A},\mathcal{B})$ though the intended result might be that $\mathbf{2} \uparrow \mathrm{EF}^*(\mathcal{A},\mathcal{B})$. Similarly it is enough to show that $\mathbf{I} \uparrow \mathrm{EF}^*(\mathcal{A},\mathcal{B})$ in order to prove that $\mathbf{I} \uparrow \mathrm{EF}^\circ(\mathcal{A},\mathcal{B})$. We say that EF^* is easier than EF° for player $\mathbf{2}$ and that is also the reason to say that EF^* is weaker than EF°. Similarly for EF° and EF etc.

An easy example shows that $\mathrm{EF}_k(\mathcal{A},\mathcal{B})$ and $\mathrm{EF}^*_k(\mathcal{A},\mathcal{B})$ are non-equivalent games for finite $k > 1$. Let $A = \mathbb{N}$ and $B = \mathbb{Z}$ equipped with the usual ordering on both. Then $\mathbf{I}$ wins $\mathrm{EF}_k(\mathcal{A},\mathcal{B})$ by playing first $0 \in \mathbb{N}$ and then $n-1 \in \mathbb{Z}$, where n is the first move by $\mathbf{2}$, so $\mathbf{I} \uparrow \mathrm{EF}_k(\mathcal{A},\mathcal{B})$. On the other hand all finite linear orderings are isomorphic if and only if their cardinality is the same. Thus $\mathbf{2} \uparrow \mathrm{EF}^\circ_k(\mathcal{A},\mathcal{B})$ and so also $\mathbf{2} \uparrow \mathrm{EF}^*_k(\mathcal{A},\mathcal{B})$. In fact $\mathbf{2} \uparrow \mathrm{EF}^*_k(\mathcal{A},\mathcal{B})$ for all $k < \omega$ and linear orders $\mathcal{A}$ and $\mathcal{B}$. Another example are the abelian groups $\mathcal{A} = \langle \mathbb{Q}, + \rangle$ and $\mathcal{B} = \langle \mathbb{Z}, + \rangle$: player $\mathbf{I}$ easily wins $\mathrm{EF}_3(\mathcal{A},\mathcal{B})$ by playing subsequent elements from $\mathcal{B}$ and then in the middle of the answers of $\mathbf{2}$. On the other hand all finitely generated subgroups of $\mathcal{A}$ or $\mathcal{B}$ are isomorphic to $\langle \mathbb{Z}, + \rangle$, so $\mathbf{2} \uparrow \mathrm{EF}^*_n(\mathcal{A},\mathcal{B})$ for all $n < \omega$.

We turn now to investigate infinite game lengths.

II.2. Equivalent and similar definitions

Let κ be a cardinal. Consider the game $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$. Let $S = \{X \subset A \cup B \mid |X| \leq \kappa,\ X \cap \mathcal{A} \cong X \cap \mathcal{B}\}$. Player $\mathbf{2}$ has a winning strategy in $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$ if and only if S contains a κ-cub set and player $\mathbf{I}$ has a winning strategy if and only if the complement of S, e.g. $\mathcal{P}_{\kappa^+}(A \cup B) \setminus S$ contains a κ-cub set. The used concepts will be defined first.

In appendix A a κ-cub set is defined as a subset of an ordinal number. The next definition is a generalization.

II.9. Definition. Let $(X, <)$ be a partial order. That means we have transitivity, antisymmetry and irreflexivity:

- $\forall x, y, z \in X (x < y \wedge y < z \rightarrow x < z)$
- $\forall x, y \in X (x < y \rightarrow y \not< x)$
- $\forall x \in X (x \not< x)$.

We say that a subset $C \subset X$ is a $<$-λ-cub or just λ-cub if the conditions below are satisfied:

- Closeness: if $(c_i)_{i<\lambda}$ is a $<$-increasing chain of elements of C and there exists an element $c \in X$ such that for all $i \in \lambda$ $c_i < c$ and for all $c' \in X$ either $c' > c$ or there exists $i < \lambda$ such that $c_i > c'$, then $c \in C$. The element c is called the *supremum* of the chain $(c_i)_{i<\lambda}$.
- Unboundness: For each $c \in X$ there exists $c' \in C$ such that $c < c'$.

It is a trivial task to see that the set $\mathcal{P}_{\kappa^+}(X) = \{Y \subset X \mid |Y| < \kappa^+\}$ equipped with the proper subset relation $Y < Y' \iff Y \subsetneq Y'$ is a partially ordered set according to the above definition, so it is understood what is meant by a λ-cub subset of $\mathcal{P}_{\kappa^+}(X)$. A set C is cub if it is λ-cub for all $\lambda < |C|$. Equivalently one can say that a set C is cub if and only if it is unbounded and closed in the order topology.

Let $\mathcal{A}$ and $\mathcal{B}$ be two structures and let $S = \{X \subset A \cup B \mid |X| \le \kappa,\ X \cap \mathcal{A} \cong X \cap \mathcal{B}\} \subset \mathcal{P}_{\kappa^+}(A \cup B)$.

II.10. THEOREM. *If S (resp. $\mathcal{P}_{\kappa^+}(A \cup B) \setminus S$) contains a κ-cub set, then $\mathbf{2}$ (resp. $\mathbf{I}$) has a winning strategy in $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$. If $\kappa^{<\kappa} = \kappa$, then the converse is also true: if $\mathbf{2}$ (resp. $\mathbf{I}$) wins the game $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$, then S (resp. $\mathcal{P}_{\kappa^+}(A \cup B) \setminus S$) contains a κ-cub set.*

PROOF. Assume S contains a κ-cub set C.

The strategy for player $\mathbf{2}$ in $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$ is to make sure that if in the end of the game the resulting substructures of $\mathcal{A}$ and $\mathcal{B}$ are respectively X and Y, then $X \cup Y \in S$.

Assume that on the move α the sequences sequences $(a_i)_{i\le\alpha}$ (moves by $\mathbf{I}$) and $(b_i)_{i<\alpha}$ (moves by $\mathbf{2}$) are played from $A \cup B$ and an increasing chain of elements of S is defined $(X_i)_{i<\alpha}$ (empty in the beginning). Player $\mathbf{2}$ defines X_α to be an element of S, which contains the set $\bigcup_{i<\alpha} X_i \cup \{a_i \mid i \le \alpha\} \cup \{b_i \mid i < \alpha\}$, which exists by unboundness of S. Then $\mathbf{2}$ plays some element b_α. The way she plays is revealed in what follows.

Because the sets X_i are of cardinality $\le \kappa$, we can write

$$X_\beta = \{x_{(\alpha,\beta)} \mid \alpha < \kappa\}.$$

Set $f\colon \kappa \times \kappa \to \kappa \setminus \{0\}$ to be a bijection such that $f(x,y) > y$ for all $x, y \in \kappa$. Player $\mathbf{2}$ plays $a_\alpha = x_{f^{-1}(z)}$, where $z < \kappa$ is the least such ordinal that $x_{f^{-1}(z)}$ is not played yet and is in the structure from which $\mathbf{2}$ is ought to play. Definitely we have $z \le \alpha$, hence $x_{f^{-1}(z)} \in X_\beta$ with $\beta < \alpha$. In the end the set $X = \bigcup_{\alpha<\kappa} X_\alpha$ has been enumerated and that set belongs to S, because S is closed. So $X \cap \mathcal{A} \cong X \cap \mathcal{B}$ and $\mathfrak{2}$ wins.

The proof for $\mathfrak{I}$, when the complement of S contains a κ-sub set is exactly the same. One just changes roles of the players.

Let us now show that under $\kappa^{<\kappa} = \kappa$ if $\mathfrak{2} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$, then S contains a κ-cub set. In fact the set $W = \{X \subset A \cup B \mid X \text{ is closed under } \sigma \text{ and } |X| \leq \kappa\}$ is such a set. $W \subset S$, because player $\mathfrak{I}$ can enumerate any element X of W by his moves and because that element is closed under σ, player $\mathfrak{2}$ answers only in X and because σ is winning, we have $X \cap \mathcal{A} \cong X \cap \mathcal{B}$. It is unbounded, because if we take $Y \subset A \cup B$, we can take closure of that set under σ, which by A.5 is of size less or equal to κ. If we take a κ-sequence $(Y_i)_{i<\kappa}$ of sets that are closed under σ, then obviously their union is also closed, because the domain of σ contains only sequences of size strictly less than κ, so any such sequence is inside some Y_i (κ is regular by the assumption $\kappa^{<\kappa} = \kappa$), which itself is closed under σ.

On the other hand if τ is a winning strategy for $\mathfrak{I}$, then take $W = \{X \subset A \cup B \mid X \text{ is closed under } \tau \text{ and } |X| \leq \kappa\}$ and continue similarly as above. □

II.11. Corollary. *If $\mathfrak{I}$ (resp. $\mathfrak{2}$) does not have a winning strategy in* $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$, *then S (resp. $\mathcal{P}_{\kappa^+} \setminus S$) is κ-stationary (intersects all κ-cub sets).* □

This gives an alternative definition for the game at cardinal length.

II.12. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be some structures of the same vocabulary and $\lambda, \mu \leq \kappa$ non-zero cardinals. Let us define the game $\overset{*}{\mathrm{EF}}{}^{\lambda,\mu}_{\kappa}(\mathcal{A}, \mathcal{B})$, which is played between $\mathfrak{I}$ and $\mathfrak{2}$ as follows. At the move α,

Player $\mathfrak{I}$: chooses $X_\alpha \subset A \cup B$ such that $|X_\alpha| \leq \lambda$ and then
Player $\mathfrak{2}$: chooses $Y_\alpha \subset A \cup B$ such that $|X_\alpha| < \mu$

In the end $\mathfrak{2}$ wins if the substructures generated by $A \cap \bigcup_{\alpha<\kappa} X_\alpha \cup Y_\alpha$ and $B \cap \bigcup_{\alpha<\kappa} X_\alpha \cup Y_\alpha$ are isomorphic. Otherwise $\mathfrak{I}$ wins.

In II.3, EF^*_α was defined for ordinals α. We shall see now that when $\alpha = \kappa$ is an infinite cardinal, the defined games coincide.

II.13. Theorem. *Let λ, μ and κ be non-zero cardinals such that $\lambda, \mu \leq \kappa$ and κ infinite. Player $\mathfrak{I}/\mathfrak{2}$ wins the game $\overset{*}{\mathrm{EF}}{}^{\lambda,\mu}_{\kappa}(\mathcal{A}, \mathcal{B})$ if and only if he/she wins the game* $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$.

Proof. Fix a bijective map $f \colon \kappa \times \kappa \to \kappa \setminus \{0\}$ such that for each α we have $f(\alpha, \beta) > \alpha$.

Assume first that **2** has a winning strategy in the game $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}$. Then the strategy of **2** in $\mathrm{EF}_{\kappa}^{*}(\mathcal{A},\mathcal{B})$ is as follows. She imagines that she is playing $\overset{*}{\mathrm{EF}}$ against **I**. At each move she chooses $X_\alpha \subset A \cup B$ according to her strategy in the game $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}$ and anytime he chooses an element $x_\alpha \in A \cup B$, she considers it as the set $\{x_\alpha\}$ being played by **I** in her imaginary game. Also she enumerates all these sets $X_\alpha = \{x_{\alpha,\beta} \mid \beta < \kappa\}$ (enumeration need not be one-to-one) and at the γ:th move she plays $x_{f^{-1}(\gamma)}$ in the actual game. Thus she eventually enumerates the same set as she would in $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}$.

On the other hand if **2** wins $\mathrm{EF}_{\kappa}^{*}(\mathcal{A},\mathcal{B})$ the strategy for her in $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}$ is somewhat a converse reasoning to the previous: she imagines that they are playing EF_{κ}^{*}. Everytime he chooses a set $X_\alpha \in A \cup B$, she enumerates it $X_\alpha = \{x_{\alpha,\beta} \mid \beta < \kappa\}$ and imagines that he played $x_{f^{-1}(\alpha)}$ in the game EF_{κ}^{*} and in the actual game she plays $\{x_\gamma\}$, where x_γ is according to the winning strategy in EF_{κ}^{*}. Eventually the same sets are enumerated as they were playing the imaginary game of **2**. So the resulting substructures are isomorphic as she used a winning strategy.

The proofs for **I** are exactly the same. □

Remark. This shows that actually all games $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\lambda,\mu}(\mathcal{A},\mathcal{B})$ are equivalent to the game $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\kappa,\kappa}(\mathcal{A},\mathcal{B})$.

It is also not difficult to see that in $\overset{*}{\mathrm{EF}}{}_{\kappa}^{\kappa,\kappa}(\mathcal{A},\mathcal{B})$ we could require player **2** to choose at each move such an $X \subset A \cup B$ that $X \cap \mathcal{A} \cong X \cap \mathcal{B}$ and it would not change the game (i.e. **2** wins exactly on the same structures as before as well as **I**).

The question ”Can $\mathrm{EF}_{\kappa}^{*}(\mathcal{A},\mathcal{B})$ be non-determined?” is given a positive answer in chapter V.

III Similarity of EF_κ and EF^*_κ

First I state a simple result of model theory spiced with weak equivalence.

III.1. Theorem. *Let κ be a cardinal and let the structures $\mathcal{A}$ and $\mathcal{B}$ be such that $|A| = |B| = \kappa$. Then the following are equivalent.*

(*i*) $\mathcal{A} \cong \mathcal{B}$

(*ii*) $\mathfrak{2} \uparrow \mathrm{EF}_\kappa(\mathcal{A}, \mathcal{B})$

(*iii*) $\mathfrak{2} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$

(*iv*) $\mathfrak{1}$ *has no winning strategy in* $\mathrm{EF}_\kappa(\mathcal{A}, \mathcal{B})$

(*v*) $\mathfrak{1}$ *has no winning strategy in* $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$.

Proof. We shall prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: The strategy for $\mathfrak{2}$ is based on going along the isomorphism between the structures. Let $f\colon A \to B$ be an isomorphism. If $\mathfrak{1}$ plays $a \in A$, $\mathfrak{2}$ answers $f(a) \in B$. If $\mathfrak{1}$ plays $b \in B$, then $\mathfrak{2}$ answers $f^{-1}(b) \in A$.

$(ii) \Rightarrow (iii)$: Player $\mathfrak{2}$ just uses the same strategy in EF^*_κ as she would use in EF_κ.

$(iii) \Rightarrow (iv)$: Because $\mathfrak{2}$ has a winning strategy, she wins the game in which $\mathfrak{1}$ fills both structures with his moves. Thus the structures are isomorphic and so by $(i) \Rightarrow (ii)$ $\mathfrak{2} \uparrow \mathrm{EF}_\kappa(\mathcal{A}, \mathcal{B})$, which implies $\neg \mathfrak{1} \uparrow \mathrm{EF}_\kappa(\mathcal{A}, \mathcal{B})$.

$(iv) \Rightarrow (v)$: Assume $\mathfrak{1}$ has a winning strategy in $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$. So he can play such that the resulting substructures are non-isomorphic. Thus the same strategy would work for $\mathrm{EF}_\kappa(\mathcal{A}, \mathcal{B})$.

$(v) \Rightarrow (i)$: Assume on contrary that $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic. Then $\mathfrak{1}$ has a winning strategy: he can fill the structures with his moves.

$\square$

This theorem implies that both games, EF_κ and EF^*_κ, are determined when played on structures of size κ and we have $\mathfrak{2} \uparrow \mathrm{EF}_\kappa(\mathcal{A}, \mathcal{B}) \iff \mathfrak{2} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$. In such case I say that the games $\mathrm{EF}_\kappa(\mathcal{A}, \mathcal{B})$ and $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$ are equivalent.

Since the weak game is easier for the second player, we know a priori the implications, shown on the Figure 1.

One more implication can be proved:

III.2. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be any structures and κ a cardinal such that $\kappa^{<\kappa} = \kappa$. Then* $\mathfrak{1} \uparrow \mathrm{EF}_\kappa(\mathcal{A}, \mathcal{B}) \Rightarrow \mathfrak{1} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$.

$$\begin{tikzcd}
 & \mathfrak{2} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B}) \arrow[dd] & \\
\neg \mathfrak{I} \uparrow \mathrm{EF}_\kappa(\mathcal{A},\mathcal{B}) \arrow[dr] & & \mathfrak{2} \uparrow \mathrm{EF}_\kappa(\mathcal{A},\mathcal{B}) \arrow[ul] \arrow[ll] \\
 & \neg \mathfrak{I} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B}) &
\end{tikzcd}$$

FIGURE 1. Implications that follow directly from the definitions of the games.

For later needs we shall prove a slightly more general result:

III.3. THEOREM. *Let $\mathcal{A}$ and $\mathcal{B}$ be any structures, κ a cardinal and λ an ordinal such that $\kappa^{<\lambda} = \kappa$ (see lemma A.5). Then $\mathfrak{I} \uparrow \mathrm{EF}_\lambda(\mathcal{A},\mathcal{B}) \Rightarrow \mathfrak{I} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$.*

PROOF. Assume $\tau\colon (A \cup B)^{<\lambda} \to (A \cup B)$ is the winning strategy of player $\mathfrak{I}$. We now claim that the set

$$W = \{X \in \mathcal{P}_{\kappa^+}(A \cup B) \mid X \text{ is closed under } \tau \text{ and } \tau(\varnothing) \in X\} \subset \mathcal{P}_{\kappa^+}(A \cup B)$$

is κ-cub. To see this, use lemma A.5:

(*i*) If $X \in \mathfrak{S}$, then by A.5 there exist $X' \subset A \cup B$, such that $|X'| = \kappa$, X' is closed under τ and which satisfy $X \cup \{\tau(\varnothing)\} \subset X'$. So $X < X' \in W$.

(*ii*) Assume $(X_\alpha)_{\alpha<\kappa}$ is increasing and each X_α is closed under τ. To see that $\bigcup_{\alpha<\kappa} X_\alpha$ is also closed under τ, let $k \in \left(\bigcup_{\alpha<\kappa} X_\alpha\right)^{<\kappa}$. Then $k \in (X_\alpha)^\gamma$ for some $\alpha < \kappa$ and $\gamma < \kappa$, but $X_\alpha \cup Y_\alpha$ is closed under τ.

Now it remains to show that if $X \cup Y \in W$ ($X \subset A$, $Y \subset B$) then X and Y cannot be isomorphic. By definition of W the set $X \cup Y$ is closed under τ, the winning strategy of $\mathfrak{I}$ in $\mathrm{EF}_\lambda(\mathcal{A},\mathcal{B})$. If were $f\colon X \cong Y$, then $\mathfrak{2}$ could win the game $\mathrm{EF}_\lambda(\mathcal{A},\mathcal{B})$ when $\mathfrak{I}$ uses τ: she plays according to the isomorphism f. Note that the first move of $\mathfrak{I}$ $\tau(\varnothing)$ is in $X \cup Y$ again by definition of W and since W is closed under this strategy, also all subsequent moves are there. A contradiction. So W is a κ-cub set outside the set S of theorem II.10.

Now by theorem II.10 $\mathfrak{I}$ has a winning strategy in the game $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$ and so also in the game $\mathrm{EF}^\circ_\kappa(\mathcal{A},\mathcal{B})$. □

III.4. COROLLARY. *If κ is such that $\kappa^{<\kappa} = \kappa$ and $\mathrm{EF}_\kappa(\mathcal{A},\mathcal{B})$ is determined, then $\mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$ as well as $\mathrm{EF}^\circ_\kappa(\mathcal{A},\mathcal{B})$ are determined and*

$$\mathfrak{2} \uparrow \mathrm{EF}_\kappa(\mathcal{A},\mathcal{B}) \Leftrightarrow \mathfrak{2} \uparrow \mathrm{EF}^\circ_\kappa(\mathcal{A},\mathcal{B}) \Leftrightarrow \mathfrak{2} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B}).$$

Proof. When EF-game is determined, the diagram on Figure 1 endowed with theorem III.2 looks like the diagram on Figure 2.

$$\begin{tikzcd}
 & \mathfrak{2} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B}) \arrow[dd] & \\
\neg\mathfrak{I} \uparrow \mathrm{EF}_\kappa(\mathcal{A},\mathcal{B}) \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow, "III.2"] & & \mathfrak{2} \uparrow \mathrm{EF}_\kappa(\mathcal{A},\mathcal{B}) \arrow[ul] \\
 & \neg\mathfrak{I} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B}) &
\end{tikzcd}$$

Figure 2. Implication of theorem III.2 added

and can be straightforwardly completed to the form

$$\begin{tikzcd}
 & \mathfrak{2} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B}) \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\neg\mathfrak{I} \uparrow \mathrm{EF}_\kappa(\mathcal{A},\mathcal{B}) \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[ur, leftrightarrow] & & \mathfrak{2} \uparrow \mathrm{EF}_\kappa(\mathcal{A},\mathcal{B}) \arrow[dl, leftrightarrow] \\
 & \neg\mathfrak{I} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B}) &
\end{tikzcd}$$

□

IV Countable Games

IV.1. Applications to EF^*_ω

Let $C = \{X \subset A \cup B \mid X \cap \mathcal{A} \cong X \cap \mathcal{B} \text{ and } |X| \leq \omega\}$ for some structures $\mathcal{A}$ and $\mathcal{B}$. Recall that $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ means that for all $\varphi \in L_{\infty\omega}$, $\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi$. It was proved in [**Kueker**] (Theorem 3.5) that

(a) $\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \iff C$ contains a cub subset of $\mathcal{P}_{\omega_1}(A \cup B)$
(b) $\mathcal{A} \not\equiv_{\infty\omega} \mathcal{B} \iff \mathcal{P}_{\omega_1}(A \cup B) \setminus C$ contains a cub subset of $\mathcal{P}_{\omega_1}(A \cup B)$

This can be reformulated by theorem II.10 as follows:

(a) $\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \iff \mathbf{II} \uparrow \mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$
(b) $\mathcal{A} \not\equiv_{\infty\omega} \mathcal{B} \iff \mathbf{I} \uparrow \mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$

We will prove the former formulation of Kueker's theorem for the sake of completeness. Already Carol Karp proved [**Karp**] that this formulation holds if the weak game is replaced by the ordinary EF_ω.

IV.1. Definition. Recall the general definition II.1 of a game $G_\alpha(A)$. The game is said to be *closed* if the winning set W or its complement is closed in the product topology of $(A \times A)^\alpha$.

The game $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ can be seen as $G_\omega(A \cup B)$ with an appropriate W. By the Gale-Stewart theorem, [**GaleStwrt**], which says that every closed game of length ω is determined we have the following.

IV.2. Theorem. *EF_ω is determined.*

Proof. By Gale-Stewart theorem it is enough to show that the winning set for player $\mathbf{I}$ is open in the product topology. Let $s \in (A \cup B)^\omega$ be a sequence of moves such that $\mathbf{I}$ won the game. Assume he won on the n:th move. Then

$$\{s' \in (A \cup B)^\omega \mid s' \restriction (n+1) = s \restriction (n+1)\}$$

is open and contains only winning sequences for $\mathbf{I}$. So every point in the winning set for $\mathbf{I}$ has an open neighbourhood in that set. $\square$

IV.3. Corollary. *The games* $\mathrm{EF}^{\circ}_{\omega}(\mathcal{A},\mathcal{B})$ *and* $\mathrm{EF}^{*}_{\omega}(\mathcal{A},\mathcal{B})$ *are determined for every* $\mathcal{A}$ *and* $\mathcal{B}$ *and*

$$\mathcal{A} \sim_{\omega} \mathcal{B} \iff \mathcal{A} \sim^{\circ}_{\omega} \mathcal{B} \iff \mathcal{A} \sim^{*}_{\omega} \mathcal{B}.$$

Proof. Because $\omega^{<\omega} = \omega$, we can apply III.4. □

IV.4. Corollary. *For any structures* $\mathcal{A}$ *and* $\mathcal{B}$ *we have* $\mathcal{A} \sim^{*}_{\omega} \mathcal{B} \Rightarrow \mathcal{A} \equiv \mathcal{B}$

Proof. By corollary IV.3 $\mathcal{A} \sim^{*}_{\omega} \mathcal{B}$ implies $\mathcal{A} \sim_{\omega} \mathcal{B}$, which in turn implies $\mathcal{A} \equiv \mathcal{B}$. A proof for the last fact can be found in [**Hodges**], but is also a consequence of the next corollary. □

IV.5. Theorem ([**Karp**]). *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be any structures. Then* $\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \iff \mathcal{A} \sim_{\omega} \mathcal{B}$.

Proof. The proof of $\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \Leftarrow \mathcal{A} \sim_{\omega} \mathcal{B}$ is by induction on the length of the formula $\varphi \in L_{\infty\omega}$ and can be found in [**Karp**] and in [**Hodges**] (page 80, theorem 3.2.4). It now follows, again by IV.3 that $\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \Leftarrow \mathcal{A} \sim^{*}_{\omega} \mathcal{B}$.

Let us prove the converse: $\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \Rightarrow \mathcal{A} \sim^{*}_{\omega} \mathcal{B}$. We will construct a winning strategy for **2** in the game $\mathrm{EF}_{\omega}(\mathcal{A},\mathcal{B})$ and apply once more IV.3.

Assume sequences $a_0,\dots,a_{n-1} \in A$, $b_0,\dots,b_{n-1} \in B$ are played such that

$$(\mathcal{A}, a_0,\dots,a_{n-1}) \equiv_{\infty\omega} (\mathcal{B}, b_0,\dots,b_{n-1})$$

(in the beginning $n = 0$ and the sequences are empty). Let **I** play $a_n \in A$ (the situation being symmetric, the same works for $b_n \in B$). We want to show that **2** can find $b_n \in B$ such that $\mathcal{A} \models \varphi(a_0,\dots,a_n) \iff \mathcal{B} \models \varphi(b_0,\dots,b_n)$ for all $\varphi \in L_{\infty\omega}$. Assume on contrary that for each $b_n \in B$ there is $\varphi_{b_n} \in L_{\infty\omega}$ such that $\mathcal{A} \models \varphi_{b_n}(a_0,\dots,a_n)$ but $\mathcal{B} \not\models \varphi_{b_n}(b_0,\dots,b_n)$. In the following the notation $\varphi(x_0,\dots,x_n)$ means that $x_0,\dots,x_n$ are the only free variables of φ; in particular that there are $n+1$ free variables. We let

$$\Phi_n = \{\varphi_{b_n}(x_0,\dots,x_n) \in L_{\infty\omega} \mid \mathcal{A} \models \varphi_{b_n}(a_0,\dots,a_n),\ \mathcal{B} \not\models \varphi_{b_n}(b_0,\dots,b_n),\ b_n \in B\}.$$

Because now

$$\mathcal{A} \models \bigwedge \Phi_n(a_0,\dots,a_{n-1},a_n)$$

as well as

$$\mathcal{A} \models \exists x \bigwedge \Phi_n(a_0,\dots,a_{n-1},x),$$

we conclude by the induction hypothesis that

$$\mathcal{B} \models \exists x \bigwedge \Phi_n(b_0,\dots,b_{n-1},x).$$

But this is a contradiction by the definition of Φ_n, which implies

$$\mathcal{B} \not\models \exists x \bigwedge \Phi_n(b_0, \dots, b_{n-1}, x),$$

so $\mathbf{II}$ finds $b_n \in B$ such that $(\mathcal{A}, a_0, \dots, a_n) \equiv_{\infty\omega} (\mathcal{B}, b_0, \dots, b_n)$. ☐

IV.6. Corollary ([**Kueker**]). *Let $\mathcal{A}$ and $\mathcal{B}$ be any structures. Then $\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \iff \mathcal{A} \sim^*_\omega \mathcal{B}$.*

Proof. By IV.5 $\mathcal{A} \equiv_{\infty\omega} \mathcal{B} \iff \mathcal{A} \sim_\omega \mathcal{B}$ and by IV.3 we obtain the statement. ☐

IV.7. Corollary. *The following are equivalent*

- *$\mathcal{A}$ and $\mathcal{B}$ are back-and-forth-equivalent*
- $\mathbf{II} \uparrow \mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$
- $\neg\mathbf{I} \uparrow \mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$.

☐

As an example we use the weak EF-game to prove a well known theorem IV.8:

IV.8. Theorem. *If $\mathcal{A}$ and $\mathcal{B}$ are models of a countable ω-categorical theory T, then $\mathcal{A}$ and $\mathcal{B}$ are back-and-forth-equivalent, that is $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$.*

Remark. An ω-categorical theory is a theory whose all countable models are isomorphic.

Proof. First of all $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent. If on contrary $\mathcal{A} \models T \cup \varphi$ and $\mathcal{B} \models T \cup \neg\varphi$, then by the Löwenheim-Skolem theorem I.11 there are countable submodels $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{B}' \subset \mathcal{B}$ such that $\mathcal{A}' \models T \cup \varphi$ and $\mathcal{B}' \models T \cup \neg\varphi$. But now $\mathcal{A}'$ and $\mathcal{B}'$ are countable models of T and non-isomorphic, which is a contradiction with ω-categoricity.

By the Tarski-Vaught-criterion (see [**Hodges**] or [**SlamanWoodin**] for details) if $(\mathcal{A}_i)_{i<\omega}$ is an increasing sequence of elementary substructures of $\mathcal{A}$, then $\bigcup_{i<\omega} \mathcal{A}_i$ is an elementary substructure of $\mathcal{A}$. Also by the Löwenheim-Skolem theorem each countable $X \subset \mathcal{A}$ extends to an elementary countable substructure. Hence the set

$$C = \{X \subset A \cup B \mid X \cap \mathcal{A} \prec \mathcal{A},\ X \cap \mathcal{B} \prec \mathcal{B} \text{ and } |X| = \aleph_0\}$$

is a cub-set. Moreover by ω-categoricity $\forall X \in C(X \cap \mathcal{A} \cong X \cap \mathcal{B})$, so by II.13 we have $\mathbf{II} \uparrow \mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B})$, now apply IV.7. ☐

IV.2. Counterexamples for game length α, $\omega < \alpha < \omega_1$

As mentioned in chapter II, the result of theorem IV.3 does not work for finite ordinals and it does not generally extend for example to ordinals $\omega < \alpha < \omega_1$ either.

Let us give some examples with different nature to show that for games of countable length $> \omega$, the theorems III.2, IV.3 fail.

A simple example, which uses the fact that players are allowed to play same points over and over again shows that there are structures such that $\mathfrak{2}$ wins $\mathrm{EF}^{\circ}_{\alpha}$ for every countable ordinal α but looses $\mathrm{EF}_{\omega+1}$:

IV.9. Example. Let $A = B = \omega_1$, R a unary relation such that $R^{\mathcal{A}} = \omega$, $R^{\mathcal{B}} = \omega_1 \setminus \omega$. Now clearly $\mathcal{A} \sim_\omega \mathcal{B}$. Also if $\mathfrak{I}$ fills the set $\omega \subset A$ during the first ω moves, the second player loses the ordinary EF game on the next move. But $\mathfrak{2}$ survives in the weak game. She survives as long as the length of the game is countable, because only thing she has to do is choose points with the same property as $\mathfrak{I}$ (property means R or $\neg R$).

Although the previous example reveals some specific nature of the weak game, I do not consider it very informative, because the order of the moves is not involved and $\mathfrak{2}$ does not have to take to account the previous moves of $\mathfrak{I}$, only the last one.

The next example, though more complicated, shows how the weak game can be won due to its order-of-the-moves-blindness and $f(a_i) = b_i$ keeps to be one-to-one (but not isomorphism, that is $\mathfrak{2}$ loses EF).

IV.10. Example. Consider the structures constructed in [**NadelStavi**]: For $B \subset \omega_1$ let

$$\Phi(B) = \bigcup_{\alpha<\omega_1} \{\alpha\} \times \tau_\alpha,$$

where $\tau_\alpha = 1 + \mathbb{Q}$ if $\alpha \in B$ and $\tau_\alpha = \mathbb{Q}$ if $\alpha \notin B$. The order on Φ is lexicographical, that is $(\alpha, q) < (\beta, p)$ if $\alpha < \beta$ or $\alpha = \beta$ and $q < p$. We set now $\mathcal{A} = \Phi(\varnothing)$ and $\mathcal{B} = \Phi(\omega_1 \setminus \omega)$. The game $\mathrm{EF}_{\omega+2}(\mathcal{A}, \mathcal{B})$ is a win for $\mathfrak{I}$, which implies the same for $\mathrm{EF}_{\omega+n}(\mathcal{A}, \mathcal{B})$, where $n \geq 2$. Proof: The strategy for player $\mathfrak{I}$ is as follows. Assume the last pair of moves consists of $(\alpha, p) \in A$ and $(\beta, q) \in B$. Then if $\alpha \leq \beta$, then player $\mathfrak{I}$ plays $(\beta + 1, p) \in A$ and otherwise $(\alpha + 1, q) \in B$. Thus after ω moves the chosen sets in $\mathcal{A}$ and $\mathcal{B}$ are respectively of the form $\{(\alpha_1, p_1), (\alpha_2, p_2), \dots\} \subset A$ and $\{(\beta_1, q_1), (\beta_2, q_2), \dots\} \subset B$, where $\alpha_1 < \alpha_2 < \cdots$ and $\beta_1 < \beta_2 < \cdots$. Let $\alpha = \sup_n \alpha_n$ and $\beta = \sup_n \beta_n$. On the $\omega + 1$:st move let $\mathfrak{I}$ choose the element $(\beta, a) \in B$. Here a denotes the least element of the set $1 + \mathbb{Q}$. Player $\mathfrak{2}$ is now forced to answer an element (γ, r), such that $(\gamma, r) > (\alpha_n, p_n)$ in $\mathcal{A}$ for all $n < \omega$, in particular $\gamma \geq \alpha$. Here $r \in \mathbb{Q}$, so $\mathfrak{I}$ can play $(\gamma, r - 1) \in A$, which also satisfies

$(\gamma, r-1) > (\alpha_n, p_n)$ for all n. But there is no corresponding element in B, because if $(\gamma', r') < (\beta, a)$, then $\gamma' < \beta_n$ for some n.

On the other hand consider $\mathrm{EF}^*_{\omega+n}(\mathcal{A}, \mathcal{B})$. For $\alpha < \omega_1$ define the substructures

$$\mathcal{A}_\alpha = \{(\gamma, x) \in \mathcal{A} \mid (\gamma, x) < (\alpha, 0)\} \subset \mathcal{A}$$

and

$$\mathcal{B}_\alpha = \{(\gamma, x) \in \mathcal{B} \mid (\gamma, x) < (\alpha, 0)\} \subset \mathcal{B}.$$

The corresponding domains will be denoted by A_α and B_α. For any $\alpha, \beta < \omega_1$ the structures $\mathcal{A}_\alpha$, $\mathcal{B}_\beta$ are both countable dense linear orderings without end points and so isomorphic. Player **2** will construct a sequence of partial isomorphisms $f_0 < f_1 < \cdots$ during the game, where $f < g$ means $g \restriction \mathrm{dom}(f) = f$.

Claim: If $\alpha, \beta, \alpha', \beta' < \omega_1$, $\alpha < \alpha'$, $\beta < \beta'$ and $f \colon A_\alpha \to B_\beta$ an isomorphism, then there exists an isomorphism $g \colon A_{\alpha'} \to B_{\beta'}$ which extends f, i.e. $f < g$.

Proof of the claim: The sets $A_{\alpha'} \setminus A_\alpha$ and $B_{\beta'} \setminus B_\beta$ are both countable dense linear orderings with a "first point" but without the "last point" with respect to the ordering respectively in $\mathcal{A}$ and $\mathcal{B}$. Thus we obtain a partial isomorphism

$$h \colon A_{\alpha'} \setminus A_\alpha \to B_{\beta'} \setminus B_\beta.$$

Set

$$g(\gamma, r) = \begin{cases} f(\gamma, r), & \text{if } (\gamma, r) \in A_\alpha \\ h(\gamma, r), & \text{if } (\gamma, r) \in A_{\alpha'} \setminus A_\alpha \end{cases}$$

$\square_{\text{Claim}}$

We now provide a strategy for player **2**. After the very first move of player **I** (say he played (α, p) no matter from which structure), set $f_0 \colon A_{\alpha+1} \to B_{\alpha+1}$ to be an isomorphism and the first move of **2** will be $f_0(\alpha, p)$ or $f_0^{-1}(\alpha, p)$ depending in which structure **I** played. Assume n moves are played and partial isomorphisms $f_0 < \cdots < f_n$ are constructed and assume that **I** plays an element (β, q), no matter in which structure. If (β, q) is in the range or the domain of f_n, then set $f_{n+1} = f_n$ and player **2** chooses her element according to this isomorphism ($f_{n+1}(\beta, q)$ or $f_{n+1}^{-1}(\beta, q)$). If $(\beta, q) \in A \setminus \mathrm{dom}\, f_n$ or $(\beta, q) \in B \setminus \mathrm{Rng}\, f_n$, then set γ to be

$$\max\big\{\beta+1, \sup\{\alpha \mid \exists x \in \mathbb{Q}\, (\alpha, x) \in \mathrm{dom}\, f_n\}+1, \sup\{\alpha \mid \exists x \in 1+\mathbb{Q}\, (\alpha, x) \in \mathrm{Rng}\, f_n\}+1\big\}.$$

Now by the claim above there exists a partial isomorphism $f_{n+1} \colon A_\gamma \to B_\gamma$, which extends f_n. Now **2** can play again according to the isomorphism f_{n+1}. After ω moves are done, set $f_\omega = \cup_{n<\omega} f_n$. Finitely moves are left. On each of these moves **2** proceeds similarly: if **I** plays in $A \setminus \mathrm{dom}\, f_\omega$, then she answers in $B \setminus \mathrm{Rng}\, f_\omega$ and vise versa. If he plays in $\mathrm{dom}\, f_\omega$ or $\mathrm{Rng}\, f_\omega$ then she follows the partial isomorphism f_ω.

In similar way we can construct structures $\mathcal{A}$ and $\mathcal{B}$ for which the game $\mathrm{EF}_{\kappa+n}(\mathcal{A},\mathcal{B})$ is non-determined but in the game $\mathrm{EF}^*_{\kappa+n}(\mathcal{A},\mathcal{B})$ player $\mathfrak{2}$has a winning strategy. Assume $\kappa > \omega$ and $\kappa^{<\kappa} = \kappa$ and let $A \subset \{x < \kappa^+ \mid \mathrm{cf}(x) = \kappa\}$ be κ-stationary such that $\{x < \kappa^+ \mid \mathrm{cf}(x) = \kappa\} \setminus A$ is also κ-stationary. Now put

$$\Phi_\kappa(A) = \bigcup_{\alpha<\kappa^+} \{\alpha\} \times \tau_\alpha,$$

where τ_α is the saturated dense linear ordering η of cardinality κ without end points if $\alpha \in A$ and $1+\eta$ if $\alpha \notin A$. In the article [**Hyttinen**] it is shown that $\mathrm{EF}_{\kappa+n}(\Phi_\kappa(A), \Phi_\kappa(\varnothing))$ is not determined. But the proof of the previous example shows that $\mathfrak{2} \uparrow \mathrm{EF}^*_{\kappa+n}(\Phi_\kappa(A), \Phi_\kappa(\varnothing))$. (Everywhere is assumed $1 < n < \omega$.) For more discussion and the proof why

$$\mathrm{EF}_{\kappa+n}(\Phi_\kappa(A), \Phi_\kappa(\varnothing))$$

is not determined, see [**Hyttinen**].

One might think that if $\mathfrak{2}$ wins EF^*_ω, then she wins also EF^*_α for any $\alpha < \omega_1$, because she is anyway seeking only for isomorphic countable structures and $|\alpha| = |\omega|$ whenever $\omega < \alpha < \omega_1$. This is the case in the example IV.9 and the same intuition is offered in IV.10. However this doesn't hold as the example below shows. This example is also the first one to manifest that it is not true in the weak game that if $\mathfrak{2}$ wins a long game, she also wins a shorter one, though in EF-game it is clearly the case and is one of the most important properties of the ordinary EF-game.

IV.11. Example. Let $\mathcal{A} = \langle \mathbb{R}, < \rangle$ be the real numbers with the usual ordering and $\mathcal{B}$ with domain $B = \mathbb{R} \times \omega_1$ and lexicographical ordering $((x,\alpha) < (y,\beta) \iff \alpha < \beta \vee (\alpha = \beta \wedge x < y))$. These are dense linear orderings and are EF_ω-equivalent as a simple back-and-forth argument shows. However $\mathfrak{I} \uparrow \mathrm{EF}^*_{\omega+1}(\mathcal{A},\mathcal{B})$: he can play such that an unbounded set of $\mathcal{A}$ is chosen. But since any countable subset of $\mathcal{B}$ is bounded, $\mathfrak{I}$ can play an upper bound on the last move $\omega+1$. But when the length of the game is increased again to $\omega+\omega$, $\mathfrak{2}$ wins again using the same method. Thus actually $\mathfrak{I} \uparrow \mathrm{EF}^*_\alpha(\mathcal{A},\mathcal{B})$ for successors $\omega < \alpha < \omega$ and $\mathfrak{2} \uparrow \mathrm{EF}^*_\alpha(\mathcal{A},\mathcal{B})$ for limits α.

Inspired by this example we prove a theorem:

IV.12. Theorem. *Assume $\mathfrak{2} \uparrow \mathrm{EF}^*_\kappa(\mathcal{A},\mathcal{B})$, where κ is a cardinal such that $\kappa^{<\kappa} = \kappa$ and let $\alpha < \kappa^+$ be an ordinal with $\mathrm{cf}(\alpha) = \kappa$. Then $\mathfrak{2} \uparrow \mathrm{EF}^*_\alpha(\mathcal{A},\mathcal{B})$.*

Proof. By II.10 there is a κ-cub set $S \subset \mathcal{P}_{\kappa^+}(A \cup B)$ of such X that $\mathcal{A} \cap X$ and $\mathcal{B} \cap X$ are isomorphic. Let $\eta \colon \kappa \to \alpha$ be an increasing cofinal map. By II.13 player $\mathfrak{2}$ has a winning strategy in the game $\overset{*}{\mathrm{EF}}{}^{\kappa,1}_{\kappa}(\mathcal{A},\mathcal{B})$. Assume that the move number is α. If $\alpha \notin \eta[\kappa]$, then $\mathfrak{2}$ chooses anything. If $\alpha \in \eta[\kappa]$, she thinks of the game $\overset{*}{\mathrm{EF}}{}^{\kappa,1}_{\kappa}$ and assumes that the previous move of $\mathfrak{I}$ in $\overset{*}{\mathrm{EF}}{}^{\kappa,1}_{\kappa}$ is the set of all previously played points in the actual EF^{*}_{α} and then she chooses her point according to the winning strategy in $\overset{*}{\mathrm{EF}}{}^{\kappa,1}_{\kappa}$. □

If we let $\mathfrak{I}$ to choose freely the structure from wich he plays, the following example will work.

IV.13. Example. Let $\mathcal{A} = (\mathbb{R}, <)$ and $\mathcal{B} = (\mathbb{R} \setminus \{0\}, <)$. Then $\mathfrak{2} \uparrow \mathrm{EF}^{\circ}_{\omega}(\mathcal{A}, \mathcal{B})$ but $\mathfrak{I} \uparrow \mathrm{EF}^{\circ}_{\omega+1}(\mathcal{A}, \mathcal{B})$. These are EF_{ω}-equivalent being again dense linear orderings. Still $\mathfrak{I}$ wins $\mathrm{EF}^{\circ}_{\omega+1}(\mathcal{A}, B)$: he plays the Cauchy sequence

$$\left\{ \frac{(-1)^n}{n+1} \mid n < \omega \right\}$$

in $\mathcal{B}$. Player $\mathfrak{2}$ has to response with a similar one in $\mathcal{A}$. Now by the completeness of the ordering of the real numbers $\mathfrak{I}$ finds a point in $\mathcal{A}$, which corresponds to the missing 0 in $\mathcal{B}$. Apparently $\mathfrak{2}$ looses.

The questions whether games EF^{*}_{κ}, $\mathrm{EF}^{\circ}_{\kappa}$ and EF_{κ} are different becomes relevant, when $\kappa > \omega$ since if $\kappa = \omega$, the games are equivalent. The next section is devoted to establishing the differences between these games, when $\kappa = \omega_1$.

V Weak Games of Length ω_1

In this section I use the methods of [**MekSheVä**] to show that assuming the consistency of a measurable cardinal it is consistent that $\mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$ is determined for all structures $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\leq \aleph_2$ and that it is also consistent that Continuum Hypothesis holds and there are structures $\mathcal{F}$ and $\mathcal{G}$ of cardinality $\aleph_2$ such that $\mathrm{EF}_{\omega_1}(\mathcal{F},\mathcal{G})$ is not determined but $\mathbf{2}\uparrow \mathrm{EF}^*_{\omega_1}(\mathcal{F},\mathcal{G})$ (and $\mathbf{2}\uparrow \mathrm{EF}^\circ_{\omega_1}(\mathcal{F},\mathcal{G})$ with an extra assumption). Then, in section V.3 models $\mathcal{A}$ and $\mathcal{B}$ are constructed such that in $\mathrm{EF}^\circ_{\omega_1}(\mathcal{A},\mathcal{B})$ player $\mathbf{2}$ does not have a winning strategy, but in $\mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$ player $\mathbf{2}$ does have a winning strategy. Finally another structures $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_2$ are made up using a forcing argument, such that $\mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$ is not determined. Thus the assumptions of theorem III.2 are not enough to prove the equivalence of the games from point of view of player $\mathbf{2}$ though enough to prove it from the point of view of player $\mathbf{I}$.

Continuum Hypothesis and the statement $\aleph_2^{<\omega_2} = \aleph_2$ are assumed throughout this chapter except section V.2.

Some background on consistency results is provided first.

V.1. On consistency results

Kurt Gödel gave the fundamental result of the whole modern theory of consistency and independence proofs: the second incompleteness theorem. The result was first stated in Gödel's paper [**Gödel**]. Applied to ZFC it states:

V.1. THEOREM. *The statement Con(ZFC)="ZFC is a consistent set of formulas" can be proved in ZFC alone if and only if ZFC is inconsistent.* □

Thus if we assume Con(ZFC) and are able to prove ZFC+A⇒Con(ZFC), where A is some axiom, then A is not provable from ZFC (in particular it might contradict ZFC). Otherwise we had ZFC⇒ZFC+A⇒Con(ZFC) which by V.1 implies ¬Con(ZFC). What kind of an axiom A should be such that ZFC+A⇒Con(ZFC)? Answers to this question would yield an intuition to *what are we actually assuming when assuming Con(ZFC).*

Another result of Gödel called the completeness theorem, to contrast the previous one, will be helpful:

V.2. Theorem. *A set of sentences is consistent if and only if it has a model.* ☐

In terms of our paper T is a consistent set of sentences if and only if there is a structure $\mathcal{M}$ such that for every $\varphi \in T$ $\mathcal{M} \models \varphi$. By V.1, if we assume Con(ZFC), then it follows that the existence of $\mathcal{M}$, which would satisfy all of ZFC, is not provable from ZFC. Can we make A to be the axiom "there exists $\mathcal{M}$ such that $\mathcal{M} \models$ZFC"? Well, this is almost the case. What is needed is a large enough set (to be the domain of $\mathcal{M}$). Thus an assumption that there exists a large enough cardinal number would be enough.

V.3. Definition. An uncountable cardinal κ is *weakly inaccessible* if it is a regular limit cardinal. It is *strongly inaccessible* if it is regular and for all $\lambda < \kappa$, $2^{\lambda} < \kappa$.

Clearly strongly inaccessibles are weakly inaccessibles and under GCH[1] the notions coincide. From now on inaccessible means strongly inaccessible.

For an intuition what is going on, let us consider ZFC–Inf (ZFC with the axiom of infinity deleted). Note that if the assumption that κ is uncountable is dropped, then ω is inaccessible. Assuming only Con(ZFC–Inf) we cannot deduce the existence of a model which satisfies ZFC–Inf, but intuitively if we take the collection of all hereditary finite sets (the sets obtained from $\varnothing$ by iterating union and power set operations a finite number of times), it would satisfy our axioms. But assuming Inf, namely the existence of ω, the "inaccessible" from the point of view of finite sets (taking power sets of finite sets we can never "access" ω), we can make the existence of the universe of hereditary finite sets precise and it is not difficult to show that it satisfies ZFC–Inf. Thus ω is large enough cardinal to prove the consistency of ZFC–Inf.

It turns out that an inaccessible cardinal is large enough to imply consistency of ZFC: assuming the existence of such a cardinal κ we can construct the set of all $< \kappa$ hereditary sets and show that it satisfies ZFC.

V.4. Theorem. *Assuming ZFC and that there exists an inaccessible cardinal, one can prove Con(ZFC).* ☐

In general we can call a cardinal number κ whose existence implies consistency of ZFC *a large cardinal*. A measurable cardinal is such. We will not go here into the definition of a measurable cardinal. A comprehensive review of the subject can be found in [**Jech**]. Yet I shall state some facts about measurable cardinals and the first one is: from ZFC it follows that a measurable cardinal is inaccessible. Hence (assuming consistency of ZFC) from ZFC one cannot prove that Con(ZFC) implies

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[1] The Generalized Continuum Hypothesis: $2^{\kappa} = \kappa^{+}$ for all infinite cardinals κ.

Con(ZFC+there exists a measurable cardinal). To see this, denote I="there exists an inaccessible" and make a counter assumption that we can prove

$$\text{Con(ZFC)} \Rightarrow \text{Con(ZFC+I)}. \qquad \boxtimes$$

Now because ZFC+I⇒Con(ZFC), we get by Gödel's theorem V.1 applied to ZFC+I (substitute ZFC+I instead of ZFC in the formulation of V.1) and ⊠ that ZFC+I is inconsistent. But then using ⊠ again, we get that ZFC is inconsistent. It follows that it is consistent that no inaccessibles exist.

In the section V.2 I will state another fact concerning measurable cardinals, namely that the statement ♡ (defined in V.2) implies the consistency of the existence of a measurable cardinal and thus cannot be proved in ZFC.

V.2. All games can be determined on structures of size $\aleph_2$

Here we show that it is consistent with ZFC that all the games EF_{ω_1}, $\text{EF}^{\circ}_{\omega_1}$ and $\text{EF}^{*}_{\omega_1}$ are determined and equivalent when played on structures whose cardinality is $\leq \aleph_2$.

The assumption that a measurable cardinal exists can be weakened to the assumption that there is a weakly compact cardinal. This is shown in [**HytSheVä**].

Nevertheless we follow [**MekSheVä**] and make the following assumption.

> ♡ Let $\mathcal{S}$ be the set of all ω_1-stationary subsets of ω_2. Then there exists a σ-closed dense $K \subset \mathcal{S}$. That is, for every $s \in \mathcal{S}$ there exists $k \in K$ such that $k \subset s$ and if $k_i \in K$ for all $i < \omega$, then $\cap_i k_i \in K$.

As remarked in [**MekSheVä**] this implies that the ω_1-nonstationary ideal on ω_2 is precipitous, so the consistency of this statement implies the consistency of a measurable cardinal. For the result that this assumption is consistent relative to the consistency of a measurable cardinal, the reader is referred to [**JechMagMit**].

Let $\mathcal{A}$ and $\mathcal{B}$ be of cardinality $\aleph_2$. We can assume that the domain of both structures is ω_2. Let us write $\mathcal{A}_\alpha = \alpha \cap \mathcal{A}$ and $\mathcal{B}_\alpha = \alpha \cap \mathcal{B}$.

V.5. LEMMA. *If* $\mathfrak{l}$ *does not have a winning strategy in* $\text{EF}^{*}_{\omega_1}(\mathcal{A},\mathcal{B})$ *and* $|\mathcal{A}| = |\mathcal{B}| = \omega_2$*, then* $\{\alpha < \omega_2 \mid \mathcal{A}_\alpha \cong \mathcal{B}_\alpha\}$ *is* ω_1*-stationary.*

PROOF. This is essentially as theorem II.11. Assume there is an ω_1-cub set C outside $\{\alpha < \kappa \mid \mathcal{A}_\alpha \cong \mathcal{B}_\alpha\}$. Then by II.10, player $\mathfrak{l}$ can make sure that in the end of the game substructures $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$ are chosen, where $\alpha \in C$. □

V.6. THEOREM. *Assuming* ♡*, the game* $\text{EF}^{*}_{\omega_1}(\mathcal{A},\mathcal{B})$ *is determined for any structures* $\mathcal{A}$ *and* $\mathcal{B}$ *of cardinality* $\aleph_2$*.*

Proof. Let us assume that $\mathbf{I}$ does not have a winning strategy. By V.5 the set $S = \{\alpha < \omega_2 \mid \mathcal{A}_\alpha \cong \mathcal{B}_\alpha\}$ is ω_1-stationary. For each $\alpha \in S$ let $h_\alpha \colon \mathcal{A}_\alpha \cong \mathcal{B}_\alpha$ be the isomorphism. We will now describe a winning strategy for player $\mathbf{2}$ in $\mathrm{EF}_{\omega_1}(\mathcal{A}, \mathcal{B})$ (the ordinary EF-game). At the first move assume $\mathbf{I}$ played $a_0 \in A$. Then by pressing down lemma (see appendix A, lemma A.4) the set $S_0 = \{i < \omega_2 \mid h_i(a_0) = b_0\}$ is ω_1-stationary for some $b_0 \in B$. Let $\mathbf{2}$ choose that one b_0. Assume a sequence $(a_i, b_i)_{i<\alpha}$ has been played, $\alpha < \omega_1$. Player $\mathbf{2}$ has constructed a decreasing sequence of stationary sets $(S_j)_{j<\alpha}$ which all belong to K, which is defined in $\heartsuit$. Thus $S' = \bigcap_{j<\alpha} S_j \in K$ and is ω_1-stationary. Assume player $\mathbf{I}$ plays now $a_\alpha \in A$. Then the set $S'' = \{i \in S' \mid h_i(a_\alpha) = b_\alpha\}$ is ω_1-stationary for some $b_\alpha \in B$ (again by pressing down lemma). The next move by $\mathbf{2}$ is b_α and S_α is an ω_1-stationary subset of S'', which is also an element of K (which is dense). □

Note that we have described a strategy for player $\mathbf{2}$ in the ordinary game, so we can conclude that under $\heartsuit$ the games EF_{ω_1} and $\mathrm{EF}^*_{\omega_1}$ are equivalent for structures of cardinality $\leq \aleph_2$.

V.7. Corollary. *If $\heartsuit$ and $\mathcal{A}$ and $\mathcal{B}$ are of cardinality $\aleph_2$, then*

$$\mathbf{2} \uparrow \mathrm{EF}_{\omega_1}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathbf{2} \uparrow \mathrm{EF}^\circ_{\omega_1}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathbf{2} \uparrow \mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$$

and all games are determined. □

I remind the reader that the assumption about a measurable cardinal can be weakened to the one about a weakly compact but the proof is more complicated and is offered in [**HytSheVä**].

V.3. $\mathcal{A} \sim^*_{\omega_1} \mathcal{B} \not\Rightarrow \mathcal{A} \sim_{\omega_1} \mathcal{B}$ on structures of size $\aleph_2$

The weak EF-game is easier for player $\mathbf{2}$ and we have proved the equivalence of EF^*_κ and EF_κ from the point of view of player $\mathbf{I}$, under $\kappa^{<\kappa} = \kappa$. In this section I will present two groups $\mathcal{F}$ and $\mathcal{G}$, constructed assuming $\Box_{\omega_1}$ which are such that

- $\mathrm{EF}_{\omega_1}(\mathcal{F}, \mathcal{G})$ is not determined.
- If the groups $\mathcal{F}$ and $\mathcal{G}$ are presented as L-structures with $L = \{+, -, 0\}$, where $+$ and $-$ are function symbols (as in appendix B), then $\mathbf{2} \uparrow \mathrm{EF}^\circ_{\omega_1}(\mathcal{F}, \mathcal{G})$. (see section V.4)
- $\mathbf{2} \uparrow \mathrm{EF}^*_{\omega_1}(\mathcal{F}, \mathcal{G})$ even if the vocabulary is relational (ternary relation instead of the binary function $+$ and no distinct symbol for $-$).

Before we construct the groups $\mathcal{F}$ and $\mathcal{G}$, we shall make the *square principle* $\Box_{\omega_1}$ familiar.

V.3.1. The square principle $\square_{\omega_1}$

V.8. Definition. The statement $\square_{\omega_1}$ says that there exists a sequence $\langle C_\alpha \mid \alpha < \omega_2, \cup\alpha = \alpha\rangle$ of sets such that

(*i*) C_α is a closed and unbounded subset of α.
(*ii*) If $\mathrm{cf}(\alpha) = \omega$, then $|C_\alpha| = \omega$.
(*iii*) If γ is a limit point of C_α, then $C_\gamma = C_\alpha \cap \gamma$.

For the proof of the next theorem the reader is referred to [**Jech**] or to the primary source of this result by Jensen [**Jensen**].

V.9. Theorem. *If $V = L$ then $\square_{\omega_1}$ holds.* □

This square principle, $\square_{\omega_1}$, implies the existence of a non-reflecting stationary set E on ω_2, which we will use to construct our groups. Recall that $S^2_0 = \{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$.

V.10. Lemma. *Assume $\square_{\omega_1}$. Then there exists an ω-stationary set $E \subset S^2_0$ such that for every ordinal $\gamma < \omega_2$ of cofinality ω_1, the set $E \cap \gamma$ is non-λ-stationary on γ for any λ.*

Proof. Let $\langle C_\alpha \mid \alpha < \omega_2, \cup\alpha = \alpha\rangle$ be as in the definition of $\square_{\omega_1}$ and let $\mathrm{ORD}(C_\alpha)$ denote the ordinal, which is order isomorphic to C_α, i.e. the order type. Then the mapping $\alpha \mapsto \mathrm{ORD}(C_\alpha)$ is regressive for all $\alpha \in S^2_0 \setminus \omega_1$, since by the definition of $\square_{\omega_1}$, $|C_\alpha| = \omega$, when $\alpha \in S^2_0$ and we have $\mathrm{ORD}(C_\alpha) < \omega_1$. Now by lemma A.4 there is $\beta \in \omega_1$ such that the set

$$E_\beta = \{\alpha \in S^2_0 \mid \mathrm{ORD}(C_\alpha) = \beta\} \qquad ※$$

is ω-stationary. Denote that E_β by just E. We have to show that E does not reflect, namely that if γ is an ordinal of cofinality ω_1, then $E \cap \gamma$ is non-stationary on γ. If $\mathrm{cf}(\gamma) > \omega$, then C_γ is cub on γ and the set of limit ordinals $\alpha \in C_\gamma$ that are above the β:th element of C_γ form a cub set and for each of them we have $\mathrm{ORD}(C_\alpha) > \beta$, because $C_\alpha = C_\gamma \cap \alpha$. □

V.3.2. Construction of the groups $\mathcal{F}$ and $\mathcal{G}$

Now we are ready to construct the groups we talked about in the beginning of this section. We shall use some well-known facts about free groups, direct products etc. These are reviewed in appendix B.

The groups we shall define has been constructed in [**MekSheVä**] in order to show that it is consistent that EF_{ω_1} is non-determined. The principle $\square_{\omega_1}$ (specifically the set E of lemma V.10) is used in the construction. As both, $\square_{\omega_1}$ and CH hold if $V = L$, the use of CH makes no contradiction. One of the groups is rather simple: $\mathcal{F}$ will be the free abelian group generated by ω_2:

$$\mathcal{F} = \bigoplus_{i<\omega_2} \mathbb{Z}.$$

Another group will be a so called *almost free* abelian group. The idea is that an almost free group G is the union $G = \cup_{i<\alpha} G_i$ of its subgroups G_i such that

- Each G_i is free.
- $G_i \subset G_j$ whenever $i < j$
- G is not free.

V.11. Definition. A subgroup S of an abelian group G (write it additively) is *pure* if for all $x \in S$ $(\exists y \in G(ny = x)) \to (\exists y \in S(ny = x))$. That is, if $x \in S$ is divisible in G, it has to be divisible in S.

Let $\mathbb{Z}^{\omega_2}$ stand for the direct product $\Pi_{\alpha<\omega_2}\mathbb{Z}$ of ω_2 copies of integers. By x_γ we shall denote the element of $\mathbb{Z}^{\omega_2}$ which is zero on coordinates $\neq \gamma$ and 1 on the coordinate γ.

For each $\delta \in E$ (defined in ⨳) let us fix an increasing cofinal function $\eta_\delta \colon \omega \to \delta$ such that $\eta_\delta[\omega] \cap E = \varnothing$ (for instance take successor ordinals only). Define

$$z_\delta = \sum_{n=0}^{\infty} 2^n x_{\eta_\delta(n)} \in \mathbb{Z}^{\omega_2}.$$

For each $\alpha \leq \omega_2$ let $\mathcal{G}_\alpha$ be the smallest pure subgroup of $\mathbb{Z}^{\omega_2}$, which contains the set $\{x_\gamma \mid \gamma < \alpha\} \cup \{z_\delta \mid \delta \in E \cap \alpha\}$. We set $\mathcal{G} = \mathcal{G}_{\omega_2}$. Let also $\mathcal{F}_\alpha$ be the free group generated by $\{x_\gamma \mid \gamma < \alpha\}$ and set $\mathcal{F} = \mathcal{F}_{\omega_2}$. We shall denote by $\langle y_\alpha \mid \alpha < \beta\rangle$ the group generated by the set $\{y_\alpha \mid \alpha < \beta\}$.

V.12. Lemma. *For each $\alpha < \omega_2$ the group $\mathcal{G}_\alpha$ is free and if $\beta \in \alpha \setminus E$, then any free basis of $\mathcal{G}_\beta$ can be extended to a free basis of $\mathcal{G}_\alpha$.*

Proof: Induction on α. When $\alpha = 0$, we have $\mathcal{G}_\alpha = \{0\}$, which is free. Let the induction hypothesis be the following

$\mathscr{I.H.}$ For all $\beta < \alpha$, $\mathcal{G}_\beta$ is free and if $\beta < \varepsilon < \alpha$ and $\beta \in \alpha \setminus E$, then any free basis of $\mathcal{G}_\beta$ extends to a free basis of $\mathcal{G}_\varepsilon$. That is, $\mathcal{G}_\varepsilon = \mathcal{G}_\beta \oplus H$ for some H and moreover if $\varepsilon \in E$, then $H = K \oplus K'$, where K' is generated by $\{x_{\eta_\varepsilon(n)} \mid n_0 < n < \omega\}$ for some n_0.

Then the claim is the same thing for $\alpha + 1$. We have distinguished three cases:

C1: $\alpha = \gamma + 1$ is a successor and $\gamma \notin E$. By $\mathscr{I}.\mathscr{H}.$ $\mathcal{G}_\gamma$ is free and by definition does not contain any element of the group $\langle x_\gamma \rangle$. On the other hand E does not contain successor ordinals, so $\mathcal{G}_\alpha = \mathcal{G}_\gamma \oplus \langle x_\gamma \rangle$. Also if $\beta < \gamma$ and $\beta \notin E$, then by $\mathscr{I}.\mathscr{H}.$ $\mathcal{G}_{\gamma+1} = \mathcal{G}_\gamma \oplus \langle x_\gamma \rangle = \mathcal{G}_\beta \oplus K \oplus \langle x_\gamma \rangle$, where K is some free group (recall theorem B.6).

C2: $\alpha = \delta + 1$ and $\delta \in E$. Let $\beta \in \delta \setminus E$, whence by $\mathscr{I}.\mathscr{H}.$ we have $\mathcal{G}_\beta \oplus K \oplus K' = \mathcal{G}_\delta$, where K' is generated by $\{x_{\eta_\delta}(n) \mid n_0 < n < \omega\}$ for some n_0. Then $\mathcal{G}_{\delta+1} = \mathcal{G}_\beta \oplus K \oplus K''$, where

$$K'' = \langle \sum_{n=m}^{\infty} 2^{n-m} x_{\eta_\delta(n)} \mid n > n_0 \rangle.$$

It is easy to see that $K' \subset K''$ and that $K'' \cap K = \{0\}$. Also

$$\sum_{n=m}^{\infty} 2^{n-m} x_{\eta_\delta(n)} - \sum_{n=m'}^{\infty} 2^{n-m'} x_{\eta_\delta(n)} \neq 0,$$

for all $m' \neq m$, so the above makes sense.

C3: α is a limit ordinal. Let $\beta < \alpha$ be such that $\beta \in \alpha \setminus E \neq \varnothing$. Then one can take a continuous cofinal sequence $(\alpha_i)_{i<\mathrm{cf}(\alpha)}$ in α with the property $\alpha_0 = \beta$ and $\alpha_i \in \alpha \setminus E$. It is possible to take a continuous sequence, because there is a cub set $\subset \alpha \setminus E$. If $\alpha \in E$, then choose

$$\begin{cases} \alpha_n = \eta_\alpha(n), & \text{when } n > n_0 \text{ and } n_0 \text{ is such that } \eta_\alpha(n_0) > \beta \\ \alpha_n = \beta, & \text{when } n \leq n_0. \end{cases}$$

Now by $\mathscr{I}.\mathscr{H}.$ and the fact that if a base of a free abelian group R is extended to a base of a free abelian group R', then there exists a free abelian group Γ such that $R' = R \oplus \Gamma$ (see theorem B.6) we have

- $\mathcal{G}_{\alpha_{i+1}} = \mathcal{G}_{\alpha_i} \oplus H_i$ for some H_i
- $\mathcal{G}_{\alpha_\gamma} = \bigcup_{i<\gamma}$ for a limit γ.

We obtain

$$\mathcal{G}_\alpha = \mathcal{G}_\beta \oplus \bigoplus_{i<\mathrm{cf}(\alpha)} H_i. \tag{$\circledast$}$$

It remains to consider the last statement of the claim: For all $\beta < \varepsilon < \alpha$ and $\varepsilon \in E$ $\mathcal{G}_\varepsilon = \mathcal{G}_\beta \oplus H$, where $H = K \oplus K'$ and K' is generated by $\{x_{\eta_\varepsilon(n)} \mid n_0 < n < \omega\}$ for some n_0.

By theorem B.6 it is enough to show that $\{x_{\eta_\varepsilon(n)} \mid n_0 < n < \omega\} \subset H$ for some n_0. Let n_0 be the smallest such that $\eta_\varepsilon(n_0) > \beta$. By $\circledast$ we have $G_\varepsilon = G_\beta \oplus \bigoplus_{i<\omega} H_i$, where H_i is of the form $\mathcal{G}_{\eta_\varepsilon(n+1)}$ $(n > n_0)$ and contains

$\mathcal{G}_{\eta_\varepsilon(n)+1}$ (since $\eta_\varepsilon(n)+1 \leq \eta_\varepsilon(n+1)$). Then by the definition of η_ε, $\mathcal{G}_{\eta_\varepsilon(n)+1}$ is defined according to C1 and thus contains $x_{\eta_\varepsilon(n)}$. □

V.13. THEOREM. *If* $\square_{\omega_1}$ *and CH, in particular if* $V = L$, *then there exist structures* $\mathcal{F}$ *and* $\mathcal{G}$ *such that* $\mathrm{EF}_{\omega_1}(\mathcal{F}, \mathcal{G})$ *is not determined.*

PROOF. The structures are of course the groups (constructed using $\square_{\omega_1}$) $\mathcal{F}$ and $\mathcal{G}$.
Player I does not win:
Assume the opposite: τ is a winning strategy for I. Now by CH the set

$$\{\alpha \in S^2_1 \mid \mathcal{F}_\alpha \cup \mathcal{G}_\alpha \text{ is closed under } \tau\}$$

is an ω_1-cub set. Let now α be such that $\mathrm{cf}(\alpha) = \omega_1$ and $\mathcal{F}_\alpha \cup \mathcal{G}_\alpha$ is closed under τ. But both, $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ are free and thus isomorphic. So 2 can play according to the isomorphism and τ cannot be a winning strategy.
Player 2 does not win:
Suppose that σ is a winning strategy for 2. We will construct a strategy for I which beats σ. Let us denote σ_ω the restriction of σ to the first ω moves, i.e. $\sigma_\omega = \sigma \restriction (F \cup G)^{<\omega}$. The set

$$\{\alpha < \omega_2 \mid \mathcal{F}_\alpha \cup \mathcal{G}_\alpha \text{ is closed under } \sigma_\omega\}$$

is ω-cub. Since E is ω-stationary, we can find δ such that $\delta \in E$ and $\mathcal{F}_\delta \cup \mathcal{G}_\delta$ is closed under first ω moves of 2. If $n < \omega$, player I will play the element $x_{\eta_\delta(n)} \in \mathcal{B}_\delta$ on the $2n$:th move and the odd moves he plays in $\mathcal{F}_\delta$. He plays his elements from $\mathcal{F}_\delta$ in such a way that after ω moves some direct summand K of $\mathcal{F}_\delta$ is picked. Assume now that $J \subset \mathcal{G}_\delta$ is the countable set that the players enumerated during the first ω moves. Let A be the smallest pure subgroup of $\mathcal{G}$ containing $J \cup \{z_\delta\}$ and let I enumerate that group during the next ω moves. Because σ is winning, there should be an isomorphism $f\colon A \to H$, where H is the set that the player chose from $\mathcal{F}$ during these first $\omega + \omega$ moves. Moreover, because the ordinary game is in question it should hold that $fJ = K$. But this is a contradiction, because H/K is free, but A/J is not free: the element $z_\delta + J$ is infinitely divisible by 2. □

V.14. THEOREM. *Player 2 wins* $\mathrm{EF}^*_{\omega_1}(\mathcal{F}, \mathcal{G})$.

PROOF. Recall theorem II.13. In the game $\overset{*}{\mathrm{EF}}{}^{1,\omega_1}_{\omega_1}$ player 2 can choose at each move $\mathcal{G}_\beta$, where β is such that all elements played before this move are in $\mathcal{G}_\beta$. Eventually substructures $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ are enumerated in the end of the game. By lemma V.12 they are isomorphic. □

V.4. $\mathcal{A} \sim^*_{\omega_1} \mathcal{B} \not\Rightarrow \mathcal{A} \sim^\circ_{\omega_1} \mathcal{B} \not\Rightarrow \mathcal{A} \sim_{\omega_1} \mathcal{B}$, $|\mathcal{A}| = |\mathcal{B}| = \aleph_2$

Here we shall show that all these games can be different on structures of size $\aleph_2$. This section is separated from the previous one because here we use some extra assumptions we did not need to prove directly $\mathcal{A} \sim^*_{\omega_1} \mathcal{B} \not\Rightarrow \mathcal{A} \sim_{\omega_1} \mathcal{B}$.

To prove that $\mathrm{EF}^\circ_{\omega_1}$ is different from EF_{ω_1}, we use vocabulary with function symbols whereas to prove that $\mathrm{EF}^*_{\omega_1}$ is different from $\mathrm{EF}^\circ_{\omega_1}$, we allow vocabulary to contain uncountably many relation symbols. The author is nevertheless fairly sure that these assumptions are not necessary.

V.4.1. $\mathcal{A} \sim^\circ_{\omega_1} \not\Rightarrow \mathcal{A} \sim_{\omega_1} \mathcal{B}$

Let L be a language which contains function symbols $f_1, \ldots, f_n$, such that f_i is a k_i-place function and $\mathcal{A}$ and $\mathcal{B}$ L-structures. Let L' be another language, which is obtained from L by changing each function symbol f_i to a $k_i + 1$-place relation symbol R_i. Let now $\mathcal{A}'$ and $\mathcal{B}'$ be L'-structures such that

- $\mathcal{A}' \restriction L \cap L' = \mathcal{A} \restriction L \cap L'$ and $\mathcal{B}' \restriction L \cap L' = \mathcal{B} \restriction L \cap L'$
- For all $i \in \{1, \ldots, n\}$, $R_i^{\mathcal{A}'}(x_1, \ldots, x_{k_i}, x_{k_i+1}) \iff f_i^{\mathcal{A}}(x_1, \ldots, x_{k_i}) = x_{k_i+1}$ and similarly for $\mathcal{B}'$ and $\mathcal{B}$.

V.15. LEMMA. *Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures and $\mathcal{A}'$ and $\mathcal{B}'$ L'-structures as above. Then if σ is a winning strategy for player **2** in $\mathrm{EF}_\gamma(\mathcal{A}, \mathcal{B})$, then σ is a winning strategy for **2** in $\mathrm{EF}_\gamma(\mathcal{A}', \mathcal{B}')$.*

PROOF. Recall the definitions of EF-game II.2, of generated substructures I.6 and of a partial isomorphism I.7. If f is a partial isomorphism $\mathcal{A} \to \mathcal{B}$, then it is clearly a partial isomorphism $\mathcal{A}' \to \mathcal{B}'$ (less is required). ☐

Thus the proof that **I** does not have a winning strategy in $\mathrm{EF}_{\omega_1}(\mathcal{F}, \mathcal{G})$ in section V.3 is independent of how the groups are presented: with function symbols $+$ and $-$ or with a ternary relation, which replaces them. Also at cardinal game lengths we have:

V.16. LEMMA. *Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures and $\mathcal{A}'$ and $\mathcal{B}'$ L'-structures as above and κ an infinite cardinal. Then if a player has a winning strategy in $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$, then the same player has a winning strategy in $\mathrm{EF}^*_\kappa(\mathcal{A}', \mathcal{B}')$.*

PROOF. The generated substructures at the move $\alpha < \kappa$ will have cardinality $|\alpha + \omega| < \kappa$ (countable language and function symbols of finite valence), so it is easy to see that if e.g. player **I** wins the game with function symbols, he will win the game $\overset{*}{\mathrm{EF}}{}^{\kappa,1}_\kappa$ (see II.13) with relation symbols: he will just choose the substructures that the previous moves would generate. Similarly for **2**. ☐

It remains open to the author how EF° behaves in this context at cardinal game length. The generated substructures may carry much less information than the particular sets, which generate them. In finite case it makes a difference. For example any finitely generated subgroups of additive groups of rationals $\mathbb{Q}$ and integers $\mathbb{Z}$ are isomorphic and player **2** wins a finite game if viewed with function symbols, but if viewed with relation symbols, the situation is not that clear (**I** at least wins EF on these structures if game is long enough).

V.17. Theorem. *Let $\mathcal{F}'$ and $\mathcal{G}'$ be the groups constructed in V.3 presented with function symbols $+$, $-$. Then $\mathrm{EF}_{\omega_1}(\mathcal{F}', \mathcal{G}')$ is non-determined.*

Proof. By lemma V.15 and theorem V.13, player **2** does not have a winning strategy in $\mathrm{EF}_{\omega_1}(\mathcal{F}', \mathcal{G}')$. On the other hand player **I** does not have a winning strategy in this game by exactly the same argument as in the proof of V.13. If τ is any strategy of **I**, then there is α such that $\mathcal{F}'_\alpha \cup \mathcal{G}'_\alpha$ is closed under τ (and it is always closed under the function symbols $+$ and $-$, because $\mathcal{F}'_\alpha$ and $\mathcal{G}'_\alpha$ are subgroups) and **2** can just follow the isomorphism. □

V.18. Theorem. *Let $\mathcal{F}'$ and $\mathcal{G}'$ be the groups constructed in V.3 presented with function symbols $+$, $-$. Then player **2** wins $\mathrm{EF}^\circ_{\omega_1}(\mathcal{F}', \mathcal{G}')$.*

Proof. Note that now any substructure is a subgroup. Assume the players chose $X \subset \mathcal{F}'$ and $Y \subset \mathcal{G}'$ and the subgroups $\langle X \rangle$ and $\langle Y \rangle$ are isomorphic. Assume that **I** picks next $x \in \mathcal{F}'$. If

$$\dim\langle X \cup \{x\}\rangle > \dim\langle X \rangle,$$

then obviously

$$\dim\langle X \cup \{x\}\rangle = \dim\langle X \rangle + 1$$

(see definition B.8 and corollary B.9 in appendix B) wherefore let **2** pick an element $y \in \mathcal{G}'$ such that

$$\dim\langle Y \cup \{y\}\rangle = \dim\langle X \cup \{x\}\rangle$$

(it is possible since X and Y are still countable subsets of $\mathcal{F}'$ and $\mathcal{G}'$, which are of cardinality $\aleph_2$). On the other hand, if x is such that $\dim\langle X \cup \{x\}\rangle = \dim\langle X \rangle$, then we have three cases:

C1: $\dim\langle X \rangle < \omega$. **2** has to pick an element, which is already in Y.
C2: $\dim\langle X \rangle = \omega$ and $x \in X$. **2** has to pick an element, which is already in Y.
C3: $\dim\langle X \rangle = \omega$ and $x \notin X$. **2** has to pick an element, which is in $\mathcal{G}' \setminus Y$.

If **I** picks an element from $\mathcal{G}'$ instead of $\mathcal{F}'$, the reasoning for player **2** would be exactly the same with the structures switched.

This strategy guarantees that at each move the groups generated by the played sequences remain isomorphic and simultaneously it guarantees that if **I** picks at the end of the game an uncountable amount (i.e. $\aleph_1$) of points from one of the structures, then also uncountable amount is picked from the other one, while the chosen groups are also isomorphic, because their sets of generators must be of cardinality $\aleph_1$ (see appendix B). $\square$

Thus $\mathcal{F}' \sim^{\circ}_{\omega_1} \mathcal{G}'$, however by theorem V.17, we have $\mathcal{F}' \not\sim_{\omega_1} \mathcal{G}'$ even with function symbols. Thus the intended result is proved.

V.4.2. $\mathcal{A} \sim^{*}_{\omega_1} \mathcal{B} \not\Rightarrow \mathcal{A} \sim^{\circ}_{\omega_1} \mathcal{B}$

Let us consider two structures, $\mathcal{A}$ and $\mathcal{B}$ such that $\mathrm{EF}_{\omega_1}(\mathcal{A},\mathcal{B})$ is non-determined, but $\mathbf{2} \uparrow \mathrm{EF}^{*}_{\omega_1}$. Using these structures, we shall construct new structures $M(\mathcal{A})$ and $M(\mathcal{B})$ such that $\mathrm{EF}^{\circ}_{\omega_1}(M(\mathcal{A}),M(\mathcal{B}))$ is non-determined but $\mathbf{2} \uparrow \mathrm{EF}^{*}_{\omega_1}$. Such structures $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_2$ were constructed in V.3.2, thus we can assume that $\mathcal{A} = \mathcal{F}$ and $\mathcal{B} = \mathcal{G}$ (as in V.3). Under the assumption $\aleph_2^{<\omega_1} = \aleph_2$, we will have $|M(\mathcal{A})| = |M(\mathcal{B})| = \aleph_2$.

V.19. Definition. Let $\mathcal{A}$ be an L-structure. Let

$$L^{+} = L \cup \{<\} \cup \{P_\alpha \mid \alpha < \omega_1,\ P_\alpha \text{ is a unary relation symbol}\},$$

where the new symbols are not in L. We define $M(\mathcal{A})$ to be the L^{+}-structure with the domain

$$\mathrm{dom}(M(\mathcal{A})) = \{f \colon \alpha + 1 \to A \mid \alpha < \omega_1\}$$

and if $f_i \in \mathrm{dom}(M(\mathcal{A}))$, $i < n$ and R is an n-place relation symbol of the vocabulary, we define

$$R^{M(\mathcal{A})}(f_0, \dots, f_{n-1}) \iff R^{\mathcal{A}}(f_0(\alpha_0), \dots, f_{n-1}(\alpha_{n-1})),$$

where α_i is the maximum of the domain of f_i. The partial order $f < g$ is defined for $f, g \in M(\mathcal{A},\mathcal{B})$ if $f \subset g$, that is $g \restriction \mathrm{dom}(f) = f$. The relations are interpreted as $P_\alpha^{M(\mathcal{A})} = \{f \mid f \colon \alpha + 1 \to \mathcal{A}\}$.

Note that if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, then $M(\mathcal{A})$ and $M(\mathcal{B})$ are isomorphic. Also if $(f_i)_{i<\alpha}$ is an increasing chain, then the reduction of the substructure $\{f_i \mid i < \alpha\} \subset M(\mathcal{A})$ to L is isomorphic to the substructure $\{f_i\big(\max(\mathrm{dom}(f_i))\big) \mid i < \alpha\} \subset \mathcal{A}$. But if we have a chain $\{f_i \mid i < \alpha\}$ in $M(\mathcal{A})$ and another chain $\{g_i \mid i < \alpha\}$ in $M(\mathcal{B})$, then if there is an isomorphism $\{f_i \mid i < \alpha\} \to \{g_i \mid i < \alpha\}$, then it has to be order preserving.

Assume $\mathcal{A}$ and $\mathcal{B}$ are the groups $\mathcal{F}$ and $\mathcal{G}$ of section V.3.2. We need only the fact that $\mathrm{EF}_{\omega_1}(\mathcal{A},\mathcal{B})$ is non-determined and $\mathbf{2} \uparrow \mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$. We claim now that $\mathrm{EF}^\circ_{\omega_1}(M(\mathcal{A}), M(\mathcal{B}))$ is non-determined.

V.20. THEOREM. *Player* $\mathbf{2}$ *does not have a winning strategy in* $\mathrm{EF}^\circ_{\omega_1}(M(\mathcal{A}), M(\mathcal{B}))$.

PROOF. Assume that σ is a winning strategy of $\mathbf{2}$. Player $\mathbf{I}$ will play such that the played elements form a $<$-chain. This will force σ to do the same: if on some move $\mathbf{2}$ plays such that the chosen elements of say $M(\mathcal{A})$ fail to form a chain, the chosen elements of $\mathcal{B}$ still form a chain and $\mathbf{I}$ will play all subsequent moves from $M(\mathcal{B})$ continuing that chain. Apparently in the end the structures will not be isomorphic with respect to $<$. Also if player $\mathbf{I}$ plays an element f on the move α, then $\mathrm{dom}(f) = \alpha + 1$. This forces $\mathbf{2}$ to do the same because of P_α's.

Now player $\mathbf{I}$, as playing $\mathrm{EF}^\circ_{\omega_1}(M(\mathcal{A}), M(\mathcal{B}))$, imagines that they are playing the game $\mathrm{EF}_{\omega_1}(\mathcal{A},\mathcal{B})$: whenever $\mathbf{2}$ picks $f \in M(\mathcal{A})$ or $M(\mathcal{B})$, he imagines that she played $f(\max \mathrm{dom}(f))$ from $\mathcal{A}$ or $\mathcal{B}$. Let τ be a strategy of $\mathbf{I}$ which wins that game $\mathrm{EF}_{\omega_1}(\mathcal{A},\mathcal{B})$ (strategy of $\mathbf{2}$ is fixed by σ). He will pick elements according to this strategy except that interpreting them as functions in the structures $M(\mathcal{A})$ and $M(\mathcal{B})$ in the way described above.

Because τ wins in $\mathrm{EF}_{\omega_1}(\mathcal{A},\mathcal{B})$, the chosen structures are not isomporphic by the isomorphism which respects the order of moves. But the order of moves is the same as that induced by the ordering in $M(\mathcal{A})$ and $M(\mathcal{B})$. □

However it is necessary for $\mathbf{I}$ to be able to choose from which structure to play:

V.21. THEOREM. *Player* $\mathbf{2}$ *has a winning strategy in* $\mathrm{EF}^*_{\omega_1}(M(\mathcal{A}), M(\mathcal{B}))$.

PROOF. Again, the only thing we use about $\mathcal{A}$ and $\mathcal{B}$ is that $\mathrm{EF}_{\omega_1}(\mathcal{A},\mathcal{B})$ is non-determined but $\mathbf{2} \uparrow \mathrm{EF}^*_{\omega_1}(\mathcal{A},\mathcal{B})$.

If $X \subset \mathcal{A} \cup \mathcal{B}$, let

$$N(X) = \{f \in M(\mathcal{A}) \cup M(\mathcal{B}) \mid \mathrm{Rng}\, f \subset X\}$$

and if $Y \subset M(\mathcal{A}) \cup M(\mathcal{B})$, then

$$N^{-1}(Y) = \{x \in \mathcal{A} \cup \mathcal{B} \mid x \in \mathrm{Rng}\, f \text{ for some } f \in Y\}.$$

Realize that for all $X, X' \subset \mathcal{A} \cup \mathcal{B}$, $Y, Y' \subset M(\mathcal{A}) \cup M(\mathcal{B})$ we have

- $|X| \le \omega_1 \iff N(X) \le \omega_1$
- $N(N^{-1}(Y)) \supset Y$
- $N^{-1}(N(X)) = X$
- $N(X \cap \mathcal{A}) = N(X) \cap M(A)$ and $N(X \cap \mathcal{B}) = N(X) \cap M(B)$
- $X \cong X' \iff N(X) \cong N(X')$.

By II.10 it is enough to show that there is an ω_1-cub set

$$C \subset S = \{X \subset M(\mathcal{A}) \cup M(\mathcal{B}) \mid X \cap M(\mathcal{A}) \cong X \cap M(\mathcal{B}),\ |X| \leq \omega_1\}.$$

We know that $S' = \{X \subset \mathcal{A} \cup \mathcal{B} \mid X \cap \mathcal{A} \cong X \cap \mathcal{B},\ |X| \leq \omega_1\}$ contains a cub set. Let it be denoted by C'. We claim that the set

$$C = \{Y \subset M(\mathcal{A}) \cup M(\mathcal{B}) \mid Y = N(X),\ X \in C\} \subset S$$

is cub.

Let $Y \in C$. Then there is $X \in C'$ such that $X \supset N^{-1}(Y)$. Then $N(X) \supset N(N^{-1}(Y)) \supset Y$. And on the other hand, because $X \cap \mathcal{A} \cong X \cap \mathcal{B}$, we get

$$N(X) \cap M(\mathcal{A}) = N(X \cap \mathcal{A}) \cong N(X \cap \mathcal{B}) = N(X) \cap M(\mathcal{B}).$$

Thus C is unbounded.

Assume that $(Y_i)_{i<\omega_1} = (N(X_i))_{i<\omega_1}$ is an increasing chain in C. Then X_i is in fact an increasing chain in C'. Thus we know $\bigcup_{i<\omega_1} X_i \in C'$. But then $N\left(\bigcup_{i<\omega_1} X_i\right) \in C$ and it easy to see that

$$N\left(\bigcup_{i<\omega_1} X_i\right) = \bigcup_{i<\omega_1} N(X_i).$$

It is easy to see because the functions have always a countable domain, so if $f \in N\left(\bigcup_{i<\omega_1} X_i\right)$, then surely $f \in N\left(\bigcup_{i<\alpha} X_i\right)$ for some $\alpha < \omega_1$ and since the chain is increasing this implies $f \in X_\alpha$. □

V.5. All games can be non-determined on structures of size $\aleph_2$

The structures that are constructed in this and the next chapters were introduced to me by my supervisor Tapani Hyttinen.

Recall that, by II.11, in order to construct $\mathcal{A}$ and $\mathcal{B}$ such that $\mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$ is non-determined, we have to find models $\mathcal{A}$ and $\mathcal{B}$ such that the set $\{X \subset A \cup B \mid X \cap \mathcal{A} \cong X \cap \mathcal{B}\}$ is ω_1-bistationary i.e. a stationary set whose complement is also stationary. For that purpose we will force a set $S \subset S^2_0$, which is such that the set

$$E = \{\alpha \in S^2_1 \mid \alpha \cap S \text{ contains an } \omega\text{-cub in } \alpha\}$$

is bistationary.

V.5.1. Forcing a bistationary set

The aim is to produce a stationary $S \subset S^2_0$, which reflects to a bistationary set: i.e. the set

$$E = \{\alpha \in S^2_1 \mid \alpha \cap S \text{ contains an } \omega\text{-cub set}\}$$

is ω_1-stationary and $S^2_1 \setminus E$ is as well.

We will have a ground model M and a generic extension $M[G]$. If K is an object in $M[G]$, then $\dot K$ denotes a name for that object. If H is an object in M, then $\hat H$ will denote the standard name for H. We will use $p \le q$ to mean q is stronger than p and $\varnothing$ is the least element of the partial order and at the same time the weakest condition.

Let

$$\mathbb{P} = \{p \colon \alpha \to \{0,1\} \mid \alpha < \omega_2\}$$

be the forcing notion.

V.22. Lemma. *Let $\mathbb{P}$ be as above and G a $\mathbb{P}$-generic. Assume that in the ground model $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Then in the generic extension the same holds: $(2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_2)^{M[G]}$. Moreover $\mathbb{P}$ preserves cardinals.*

Proof. We will do the case $2^{\aleph_0} = \aleph_1$, the other case being similar.

We will show that for all $p \in \mathbb{P}$ and a name $\dot f$ such that $p \Vdash \dot f \in \widehat{2^\omega}$ there exists $q \ge p$ and a function $g \in 2^\omega$ such that $q \Vdash \dot f = \hat g$.

This shows that for each $f \in M[G]$ the set of conditions which force $\dot f = \hat g$ for some g in 2^ω is dense and so G must intersect it.

Let p be such that $p \Vdash \dot f \in \widehat{2^\omega}$. Let us construct q and g by induction as follows. Define $q_0 = p$ and assume that $q_0 \le q_1 \le \cdots \le q_n$ and $g(0), g(1), \ldots, g(n-1)$ are defined (the sequence $g(0), g(1), \ldots, g(n-1)$ being empty in the beginning). Let us define $g(n)$ and q_{n+1}. If

$$q_n \Vdash \dot f(0) = \hat 0 \vee q_n \Vdash \dot f(0) = \hat 1,$$

then let $q_{n+1} = q_n$ and $g(n) = 0$ if $q_{n+1} \Vdash \dot f(n) = \hat 0$ and $g(n) = 1$ if $q_{n+1} \Vdash \dot f(n) = \hat 1$. On the other hand, if

$$q_n \nVdash \dot f(n) = \hat 0 \wedge q_n \nVdash \dot f(n) = \hat 1,$$

then find $q_{n+1} > p$ such that

$$q_{n+1} \Vdash \dot f(n) = \hat 0 \vee q_{n+1} \Vdash \dot f(n) = \hat 1$$

and set $g(n) = 0$ if $q_{n+1} \Vdash \dot f(n) = \hat 0$ and $g(n) = 1$ if $q_{n+1} \Vdash \dot f(n) = \hat 1$.

Now it is clear that $q = \bigcup_{n<\omega} q_n$ is a condition such that

$$q \Vdash \dot f = \hat g.$$

Because there is only $\aleph_1$ functions $g \in 2^\omega$ in the ground model, there can then be at most the same amount in $M[G]$.

For the moreover-part, note that under GCH $\mathbb{P}$ has ω_3-chain condition and thus preserves cardinals $\geq \omega_3$. On the other hand $\mathbb{P}$ is ω_2-closed and thus preserves cardinals $\leq \omega_2$. For details, see Chapter VII, §6 of [**Kunen**]. $\square$

V.23. Lemma. *Let* $\mathbb{P}$ *be as above and* G *a* $\mathbb{P}$*-generic. Then* $S = S^2_0 \cap (\cup G)^{-1}\{1\}$ *reflects to a bistationary set.*

Proof. We have to show that $E = \{\alpha \in S^2_1 \mid \alpha \cap S \text{ contains an } \omega\text{-cub set}\}$ is stationary. By a symmetric argument the same will follow for the complement: $E = S^2_1 \setminus E$ is stationary.

Assume that $\dot{C}$ is a name of an ω_1-cub set C, i.e. $\varnothing \Vdash \dot{C}$ is cub. We want to show that the set

$$D = \{p \in \mathbb{P} \mid p \text{ forces } C \cap E \neq \varnothing\}$$

is dense. For that it is enough to show

> ✠ for all $q \in \mathbb{P}$ there exists $p > q$ and an ordinal $c \in S^2_1$ such that $p \Vdash \hat{c} \in \dot{C}$ and the set $\{\alpha \in S^2_0 \cap c \mid p(\alpha) = 1\}$ contains a cub set in c.

Note that the set $D_\alpha = \{p \in \mathbb{P} \mid \exists \beta > \alpha(p(\beta) = 1)\}$ is dense, so we can always extend a condition p to such which has value 1 at a point $> \operatorname{dom} p$.

To prove ✠, let $q \in \mathbb{P}$ be arbitrary. We have $\varnothing \Vdash \exists x \in \dot{C} \cap \{\beta \mid \beta > \widehat{\operatorname{dom}}\, q\}$. Now using note above we can find a sequence $(p_i)_{i<\omega_1}$ as follows.

(*i*) p_0 is such that $p_0 > q$, $p \Vdash \hat{c}_0 \in \dot{C}$ and $p(\alpha_0) = 1$ for some $\alpha_0 > c_0 > \operatorname{dom} q$.

(*ii*) p_{i+1} is such that $p_{i+1} > p_i$, $p_{i+1} \Vdash \hat{c}_{i+1} \in \dot{C}$ for some $c_{i+1} > \operatorname{dom}(p_i)$ and $p_{i+1}(\alpha_{i+1}) = 1$ for some $\alpha_{i+1} > c_{i+1}$, $\operatorname{cf}(\alpha_{i+1}) = \omega$.

(*iii*) Assume that γ is a limit ordinal. Then p_γ is such that $p_\gamma > \cup_{i<\gamma} p_i$ and $p_\gamma(\alpha_\gamma) = 1$, where $\alpha_\gamma = \cup_{i<\gamma} \operatorname{dom}(p_i) = \cup_{i<\gamma} \alpha_i$.

Now let $p = p_{\omega_1}$ and $c = \operatorname{dom}(p_{\omega_1})$. Then

- For all $i < \omega_1$, $p \Vdash \hat{c}_{i+1} \in \dot{C}$, because $p > p_{i+1}$.
- $p \Vdash$ "$\dot{C}$ is an ω_1-cub".
- From the above two it follows that $p \Vdash \hat{c} \in \dot{C}$.
- The set $\{\alpha \in S^2_0 \cap c \mid p(\alpha) = 1\}$ contains a cub set. Indeed: let $F = \{\alpha_i \mid i < \omega_1\}$, and $(\beta_n)_{n<\omega}$ be an increasing sequence in F. Assume $\beta_n = \alpha_{i_n}$ and $\gamma = \cup_{n<\omega} i_n$. Then α_γ is as defined in (*iii*) is in F and is the supremum of $(\beta_n)_{n<\omega}$.

Thus ✠ holds and we are done. $\square$

V.5.2. On determinacy of cub-games

The next definition is a modification of that given in [**HytSheTu**].

V.24. Definition. Let $\lambda \leq \alpha < \kappa$ be such that λ and κ are regular cardinals and α an ordinal. Then let $A \subset \kappa$. The cub-game $G^{\alpha}_{\lambda}(A)$ is the following game by players **I** and **2**. At the move $\gamma < \alpha$ first player **I** picks $x_\gamma \in \kappa$, such that x_γ is greater than any element played so far in the game and then player **2** chooses $y_\gamma \in A$ such that $y_\gamma > x_\gamma$. Finally sequences $(x_\gamma)_{\gamma<\alpha}$ and $(y_\gamma)_{\gamma<\alpha}$ are formed. Player **2** wins if (1) she could play according the rules and (2) $\mathrm{cl}_\lambda\{y_\gamma \mid \gamma < \alpha\} \subset A$, Where $\mathrm{cl}_\lambda B$ is the smallest λ-closed set, which contains B.

The concepts of strategy, winning strategy and beating strategy are defined according to section II.1.

Remark. Clearly if **I** wins the game $G^{\alpha}_{\lambda}(S)$, then he wins also the game $G^{\beta}_{\lambda}(S)$ for every $\beta > \alpha$.

Let $S \subset S^2_0$ be as forced in the above paragraph and $E = \{\alpha \in S^2_1 \mid S \cap \alpha \text{ contains cub}\}$. We are interested in the game $G^{\alpha}_{\omega}(S)$, where $\omega_1 \leq \alpha < \omega_2$ and $\mathrm{cf}(\alpha) = \omega_1$.

V.25. Lemma ([**HytSheTu**]). *Assume $A \subset \omega_2$. If* **I** *does not have a winning strategy for $G^{\omega_1}_{\omega}(A)$, then he has no winning strategy for $G^{\alpha}_{\omega}(A)$ for any $\omega_1 < \alpha < \omega_2$ either.*

Proof. I will use notation $[\kappa]^\alpha = \{\text{increasing sequences in } \kappa \text{ of length } \alpha\}$ and $[\kappa]^{<\lambda} = \cup_{\alpha<\lambda}[\kappa]^\alpha$, which in fact can be identified with $\mathcal{P}_\lambda(\kappa)$

Assume on contrary that **I** has a winning strategy for some $\alpha > \omega_1$. Then by the remark above, there is the smallest $\bar{\alpha}$ for which **I** has a winning strategy. We will show that $\mathrm{cf}(\bar{\alpha}) = \bar{\alpha}$, which implies $\bar{\alpha} = \omega_1$ or $\bar{\alpha} = \omega$, which will produce a contradiction.

Clearly $\bar{\alpha}$ is a limit ordinal. Assume $\mathrm{cf}(\bar{\alpha}) < \bar{\alpha}$ and let $\tau\colon [\omega_2]^{<\bar{\alpha}} \to \omega_2$ be a winning strategy for **I**. Let $\eta\colon \mathrm{cf}(\bar{\alpha}) \to \bar{\alpha}$ be a continuous increasing cofinal sequence. By the counter assumptions we have $\mathrm{cf}(\bar{\alpha}) < \bar{\alpha}$ and if $\gamma < \mathrm{cf}(\bar{\alpha})$ and β_γ is such that $\eta(\gamma+1) = \eta(\gamma) + \beta_\gamma$, then $\beta_\gamma < \bar{\alpha}$. We will use the fact that **I** does not have winning strategies for $G^{\beta_\gamma}_{\omega}(A)$ for $\gamma < \mathrm{cf}(\bar{\alpha})$ and produce a winning strategy for $G^{\mathrm{cf}(\bar{\alpha})}_{\omega}(A)$.

Let $x \in [\omega_2]^{<\mathrm{cf}(\bar{\alpha})}$, say $x\colon \delta \to \omega_2$. We will define a strategy $\sigma^x\colon [\omega_2]^{<\eta(\delta)} \to \omega_2$, which beats $\tau \restriction [\omega_2]^{<\eta(\delta)}$ in the game $G^{\eta(\delta)}_{\omega}(A)$.

$$\begin{aligned} \sigma^x_0 &= \varnothing \\ \sigma^x_\gamma &= \cup_{i<\gamma}\sigma_i \text{ if } \gamma \text{ is a limit ordinal} \end{aligned}$$

The case σ^x_{i+1} goes as follows. Let β_i be such that $\eta(i)+\beta_i=\eta(i+1)$. Then we can define

$$\tau^i_x\colon [\omega_2]^{<\beta_i}\to\omega_2$$

to be such that if $(y_j)_{j<\eta(i)}$ are the choices of **2** in the game $G^{\eta(i)}_\omega(A)$, where **I** plays according to $\tau\restriction[\omega_2]^{<\eta(i)}$ and **2** according to σ^x_i, then

$$\tau^i_x((y_j)_{j<\eta(i)+k})=\tau((y_j)_{j<\eta(i)}{}^\frown x(i){}^\frown(y_j)_{\eta(i)\le j<\eta(i)+k})$$

(implicitly assuming $x(i)<\gamma_{\eta(i)}<\cdots$) and

$$\tau^i_x((y_j)_{j<k})=\tau((y_j)_{j<k})$$

if $k<\eta(i)$. Then $\hat\tau^i_x\colon[\omega_2]^{<\beta_i}\to\omega_2$ defined by

$$\hat\tau^i_x((y_i)_{i<k})=\tau^i_x((y_j)_{j<\eta(i)}{}^\frown(y_j)_{\eta(i)\le j<\eta(i)+k})$$

is a strategy for **I** in the game $G^{\beta_i}_\omega(A\cap\{\alpha>x(i)\})$ and there exists a strategy

$$\sigma^x_*\colon[\omega_2]^{<\beta_i}\to\omega_2$$

for **2**, which beats it. Define now

$$\sigma^x_{i+1}((x_i)_{i<k})=\sigma^x_i((x_i)_{i<k})$$

if $k<\eta(i)$ and

$$\sigma^x_{i+1}((x_j)_{j<\eta(i)+k})=\sigma^x_*((x_j)_{\eta(i)\le j<\eta(i)+k}).$$

Finally set $\sigma_x=\sigma^x_\delta$. We see that if $x\subset x'$, then $\sigma^x\subset\sigma^{x'}$. Let us now define

$$\tau^*\colon\omega_2^{<\mathrm{cf}(\bar\alpha)}\to\omega_2$$

such that $\tau^*(x)$ is the least element which is greater than any element played in the game $G^{\eta(\mathrm{dom}(x))}_\omega(A)$, when **I** uses τ restricted to $\eta(\mathrm{dom}(x))$ and **2** uses σ^x.

Now because τ is winning, we see that τ^* is in fact a winning strategy of **I** in $G^{\mathrm{cf}(\bar\alpha)}_\omega(A)$. □

V.26. Corollary. *Let α be such that $\omega_1\le\alpha<\omega_2$. Player* **I** *does not have a winning strategy in $G^\alpha_\omega(S)$. (S is the set forced above.)*

Proof. By previous lemma it is enough to show that **I** does not win $G^{\omega_1}_\omega(\mathrm{S})$.

Let $\tau\colon[\omega_2]^{<\omega_1}\to\omega_2$ be any strategy for **I** in $G^{\omega_1}_\omega$. Then the set

$$C=\{\beta<\omega_2\mid\beta\text{ is closed under }\tau\}$$

is ω_1-cub, so we find $\beta_0\in C\cap E$, whence $S\cap\beta_0$ contains an ω-cub in β_0. Let C_{β_0} be that ω-cub set. Now **2** can pick elements from C_{β_0} and **I** will be forced to answer by elements $<\beta_0$ by the definition of β_0. Clearly **2** wins (note $\mathrm{cf}(\beta_0)=\omega_1$). □

V.27. LEMMA. *The set*

$$T = \{\alpha \in E \mid \text{ there is an } \omega\text{-cub } C_\alpha \subset S \cap \alpha \text{ s.t. } \mathrm{ORD}(C_\alpha) = \alpha\}$$

is ω_1-stationary.

PROOF. Let us have an ω_1-cub set $C \subset \omega_2$. We will show that $C \cap T \neq \varnothing$. Let $\tau((y)_{i<k}) = \min C \setminus \{y_i \mid i < k\}$ be a strategy of player $\mathfrak{I}$ (no matter in how long game). Similarly as in the previous proof, for each countable sequence $c = (\alpha_i)_{i<\beta}$ in C, $\beta < \omega_1$, let σ_c be a strategy of player $\mathfrak{2}$ which beats τ in the game $G_\omega^{\sup c}(S)$ (game of length $\sup c < \omega_2$) in such a way that if $c \subset c'$, then $\sigma_c \subset \sigma_{c'}$ and the choice of σ_c at move i is greater than α_i. Denote by C_c the ω-closure of the set of the choices made by $\mathfrak{2}$ in the game $G_\omega^\beta(S)$ where $\mathfrak{I}$ used τ and $\mathfrak{2}$ used σ_c. Because σ_c beats τ, we have that $C_c \subset S$. Note also that

$$\mathrm{ORD}(C_c) = \sup c. \qquad \bigstar$$

Now let τ^* be a strategy of player $\mathfrak{I}$ in $G_\omega^{\omega_1}(S)$ which is such that $\tau^*(c) = \sup C_c$. Now let σ^* be a strategy of $\mathfrak{2}$ in $G_\omega^{\omega_1}(S)$, which beats τ^*. At move γ player $\mathfrak{I}$ will choose an element $x_\gamma \in C$ and $\mathfrak{2}$ answers by $\alpha_\gamma = \sigma^*((x_i)_{i \le \gamma})$. Define $c_j = (\alpha_i)_{i<j}$. Thus $c_0 \subset c_1 \subset c_2 \subset \cdots$. Then $C = \cup_{j<\omega_1} C_{c_j}$ is the desired cub-set. Why:

- It is a cub set.
  Let $(a_i)_{i<\omega}$ be an increasing ω-sequence in C. We can assume that it is either contained in one of C_{c_j} or in $\{\alpha \mid \alpha = \sup_{j<\omega_1} C_{c_j} \text{ for some } c\}$. But both are chosen using a beating strategy of $\mathfrak{2}$.
- It satisfies $\mathrm{ORD}\, C = \sup C$.
  From the construction of C and note $\bigstar$ we have that

  $$\sup \mathrm{ORD}(C_{c_j}) \le \sup c_{j+1} \overset{\bigstar}{=} \mathrm{ORD}(C_{c_{j+1}}) \le \sup \mathrm{ORD}(C_{c_{j+1}}).$$

  But this implies that at limits γ we have $\sup \mathrm{ORD}(C_{c_\gamma}) = \mathrm{ORD}(C_{c_\gamma})$.
- Finally $\sup C \in E$.
  This is clear, because C is a cub set of $\sup C$ and this is now by definition in E. □

V.5.3. Construction of the structures

Let $S \subset S_0^2$ be as forced in the paragraph V.5.1. Notation: if $f\colon X \to Y \times Z$ is function we can write $f = (f_1, f_2)$, where $f_1\colon X \to Y$ and $f_2\colon X \to Z$ are the components in the natural way.

In the cartesian product $\omega_2 \times \omega$ we use the reversed alphabetical order, that is $(\alpha, n) < (\beta, m) \iff [(m < n) \vee (m = n \wedge \alpha < \beta)]$. The projection map $\mathrm{pr}_1 \colon \omega_2 \times \omega \to \omega_2$ is defined by $\mathrm{pr}_1(\alpha, n) = \alpha$.

$$\begin{aligned} A \; = \{f \colon \alpha + 1 \to \; & \omega_2 \times \omega \mid \alpha < \omega_2, \\ & f \text{ is strictly increasing}, \\ (+) \qquad & \text{for each } n < \omega \text{ the set } \mathrm{pr}_1[\mathrm{Rng}(f_1) \cap (\omega_2 \times \{n\})] \\ & \quad \text{is closed in } \omega_2 \text{ and is contained in } S\} \end{aligned}$$

Now let us define

$$\begin{aligned} B \; = \{f \colon \alpha + 1 \to \; & \omega_2 \times \omega \mid \alpha < \omega_2, \\ & f \text{ is strictly increasing}, \\ (*) \qquad & \text{for each } n < \omega \text{ the set } \mathrm{pr}_1[\mathrm{Rng}(f_1) \cap (\omega_2 \times \{n\})] \\ & \quad \text{is closed as a subset of } \omega_2 \text{ and } \mathbf{if} \;\; n > 0, \; \mathbf{then} \text{ is contained in } S\} \end{aligned}$$

The structures $\mathcal{A} = \langle A, \leq \rangle$ and $\mathcal{B} = \langle B, \leq \rangle$ are formed from the above defined sets with an ordering added. This ordering is the subset ordering of functions: $f \leq g \iff f \subset g$. Clearly $\mathcal{A}$ is a substructure of $\mathcal{B}$ as the condition $(*)$ is weaker than $(+)$. Some of the elements that are in $\mathcal{B}$ but not in $\mathcal{A}$ are of the form $f \colon \alpha \to \alpha \times \{0\}$, $\forall \gamma < \alpha (f(\gamma) = \gamma)$. These will be denoted by id_α.

V.28. THEOREM. *Player* **2** *cannot have a winning strategy in* $\mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$.

PROOF. Let σ be any strategy of **2**. Recall that $\mathcal{P}_{\omega_2}(A \cup B) = \{F \subset A \cup B \mid |F| < \omega_2\}$. Define a function $G \colon \mathcal{P}_{\omega_2}(A \cup B) \to \omega_2$ such that $G(F) = \sup\{\mathrm{Rng}(f_1) \mid f \in F\}$ Let

$$\begin{aligned} C_1 &= \{F \in \mathcal{P}_{\omega_2}(A \cup B) \mid F \text{ is closed under } \sigma\}, \\ C_2 &= \{F \in \mathcal{P}_{\omega_2}(A \cup B) \mid \forall \alpha < G(F)(\mathrm{id}_\alpha \in F)\} \\ C_3 &= \{F \in \mathcal{P}_{\omega_2}(A \cup B) \mid \forall \alpha < G(F)(\alpha + \alpha < G(F))\} \end{aligned}$$

By CH these sets are ω_1-cub in $\mathcal{P}_{\omega_2}(A \cup B)$, so also $C = C_1 \cap C_2 \cap C_3$ is cub. Hence the set $G[C]$ contains an ω_1-cub in ω_2. Let $\beta \in S^2_1 \cap G[C] \setminus E$. Now if σ were winning, then for $F \in C$ with $G(F) = \beta$ we should have $F \cap \mathcal{A} \cong F \cap \mathcal{B}$, but in $\mathcal{B} \cap F$ there is increasing β-sequence (the sequence $(\mathrm{id}_\alpha)_{\alpha < \beta}$), but in $\mathcal{A} \cap F$ there is no such, because $S \cap \beta$ contains no closed unbounded set and by definition of $\mathcal{A}$ and by the property introduced by C_3 there cannot be such a sequence. □

V.29. THEOREM. *Player* **I** *cannot have a winning strategy in* $\mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$.

PROOF. First we prove two claims. Let

$$\mathcal{A}_\alpha = \{f \in \mathcal{A} \mid \mathrm{Rng}(f_1) \subsetneq \alpha\}$$

and similarly

$$\mathcal{B}_\alpha = \{f \in \mathcal{B} \mid \mathrm{Rng}(f_1) \subsetneq \alpha\}.$$

A map $g\colon \alpha \to \alpha$ is ω-continuous if for every increasing sequence $(x_k)_{k<\omega}$ in α $g(\cup_{k<\omega} x_k) = \cup_{k<\omega} g(x_k)$. Define $\mathfrak{C}$ to be the set of such functions h:

$$\mathfrak{C} = \{h\colon \alpha \to S \cap \alpha \mid \alpha \in S, h \text{ is } \omega\text{-continuous increasing and unbounded}\}$$

and

$$\mathfrak{C}_\alpha = \{h \in \mathfrak{C} \mid \mathrm{dom}(h) < \alpha\}.$$

Claim 1: For each $h \in \mathfrak{C}$, there exists an isomorphism $F_h\colon \mathcal{A}_\alpha \cong \mathcal{B}_\alpha$ in such a way that if $h \subset h'$, then $F_h \subset F_{h'}$.

Proof of Claim 1. Let $h\colon \alpha \to S \cap \alpha$ be as in the assumption. Then in particular h is an order isomorphism $\alpha \to h[\alpha]$ and the former is a closed unbounded subset of α. Hence we can write h^{-1} for the inverse $h[\alpha] \to \alpha$. For defining the isomorphism $F_h\colon \mathcal{A}_\alpha \to \mathcal{B}_\alpha$, let $f \in \mathcal{A}_\alpha$ be arbitrary, say $f\colon \delta \to S \times \omega$, $\delta < \alpha$. Put

$$\beta_f = \min\{\beta \mid f(\beta) \notin h[\alpha] \times \{0\}\} \cup \{\delta\}.$$

Now for all $\gamma < \beta_f$ let $F_h(f)(\gamma) = (h^{-1}(f_1(\gamma)), 0)$ and for all $\gamma \geq \beta_f$ define

$$F_h(f)(\gamma) = \begin{cases} (f_1(\gamma), f_2(\gamma)+1), & \text{if } f_1(\beta_f) \notin h[\alpha], \\ (f_1(\gamma), f_2(\gamma)) = f(\gamma), & \text{if } f_1(\beta_f) \in h[\alpha]. \end{cases}$$

Clearly $F_h(f) \in \mathcal{B}_\alpha$ and in fact $F_h(f)\colon \delta \to \alpha \times \omega$ (same domain as that of f). We will show that F_h is an isomorphism.

(1) F_h is one-to-one. Assume $f \neq g$ are in $\mathcal{A}$. If $\mathrm{dom}(f) \neq \mathrm{dom}(g)$, then clearly $F_h(f) \neq F_h(g)$, so we can assume $\mathrm{dom}(f) = \mathrm{dom}(g)$. If $\beta_f < \beta_g$, then $F_h(f)_2(\beta_f) > 0$ but $F_h(g)_2(\beta_f) = 0$, thus $F_h(f) \neq F_h(g)$, so we can even assume $\beta_f = \beta_g$. Now by $f \neq g$ there exists γ such that $f(\gamma) \neq g(\gamma)$.

Case 1.: $\gamma < \beta_f = \beta_g$. Now $F_h(f)_1(\gamma) = h^{-1}(f(\gamma))$ and $F_h(g)_1(\gamma) = h^{-1}(g(\gamma))$. Because h^{-1} is bijection and $f(\gamma) \neq g(\gamma)$, we conclude that $F_h(f)_1(\gamma) \neq F_h(g)_1(\gamma)$, hence $F_h(f) \neq F_h(g)$.

Case 2.: $\gamma \geq \beta_f = \beta_g$. If $f_1(\beta_f) \in h[\alpha]$ but $g_1(\beta_g) \notin h[\alpha]$ or vice versa, then we are done, since we assume $\beta_f = \beta_g$. Hence we have either $F_h(f)(\gamma) = f(\gamma)$ and $F_h(g)(\gamma) = g(\gamma)$ or $F_h(f)(\gamma) = (f_1(\gamma), f_2(\gamma)+1)$ and $F_h(g)(\gamma) = (g_1(\gamma), g_2(\gamma)+1)$; in both cases the assumption $f(\gamma) \neq g(\gamma)$ gives $F_h(f) \neq F_h(g)$.

(2) F_h is onto. Let $g \in \mathcal{B}_\alpha$ be arbitrary, $g\colon \delta \to \omega_2 \times \omega$. Let $\beta_0 = \min\{\beta \mid g_2(\beta) \neq 0\} \cup \{\delta\}$ and let $f\colon \delta \to S \times \omega$ be such that

$$f(\gamma) = \begin{cases} h(g(\gamma)), & \text{if } \gamma < \beta_0, \\ g(\gamma), & \text{if } \gamma \geq \beta_0 \text{ and } g_1(\beta_0) \in h[\alpha], \\ (g_1(\gamma), g_2(\gamma) - 1), & \text{if } \gamma \geq \beta_0 \text{ and } g_1(\beta_0) \notin h[\alpha], \end{cases}$$

It is not difficult to check that $f \in \mathcal{A}_\alpha$ and $F_h(f) = g$.

(3) F_h preserves ordering. Assume $f < g$. If $\beta_g \geq \operatorname{dom}(f)$, then for all $\gamma < \operatorname{dom}(f)$ we have $F_h(f)(\gamma) = h^{-1}(f(\gamma)) = h^{-1}(g(\gamma)) = F_h(g)(\gamma)$, thus $F_h(f) < F_h(g)$. So assume then $\beta_g < \operatorname{dom}(f)$, in which case $\beta_f = \beta_g$ and $f_1(\beta_f) \in h[\alpha] \iff g_1(\beta_g) \in h[\alpha]$. Hence clearly $F_h(f)(\gamma) = F_h(g)(\gamma)$ whenever $\beta_f \leq \gamma < \operatorname{dom}(f)$. The case $\gamma < \beta_f$ as above.

By (1), (2) and (3) F_h is an isomorphism. $\square_{\text{Claim 1}}$

Claim 2: Let $h \in \mathfrak{C}$ and $\gamma \geq \operatorname{dom}(h)$. Then there exists $h' \in \mathfrak{C}$, which extends h and $\gamma \leq \operatorname{dom}(h')$.

Proof of Claim 2. Let β be such that

- $\beta \in E$,
- There is a cub-set $W \subset S \cap \beta$ of order type β,
- $h \in \mathfrak{C}_\beta$.

Assume $\eta\colon \beta \to W$ is an order isomorphism. Let $\alpha_0 = \min(W \setminus \gamma)$ and

$$\alpha_{n+1} = \eta(\alpha_n) \text{ and } \alpha_\omega = \cup_{n<\omega}\alpha_n.$$

Then $\eta \restriction (\alpha, \alpha_\omega)$ is a function from (α, α_ω) to $W \cap (\alpha, \alpha_\omega)$. Thus we can define

$$h' = h \cup \eta \restriction (\alpha, \alpha_\omega).$$

Then $h'\colon \alpha_\omega \to S \cap \alpha_\omega$ and $h' \in \mathfrak{C}_\beta$. $\square_{\text{Claim 2}}$

Let now τ be any strategy of player $\mathfrak{I}$ in $\mathrm{EF}^*_{\omega_1}(\mathcal{A}, \mathcal{B})$. Let $G\colon \mathcal{P}_{\omega_2}(A \cup B) \to \omega_2$ be as in the proof of the theorem V.28.

If $h \in \mathfrak{C}$ then by Claim 1, there is a unique isomorphism $F_h\colon \mathcal{A}_{\operatorname{dom}(h)} \to \mathcal{B}_{\operatorname{dom}(h)}$. Notation: if $f\colon X^{<\kappa} \to X$ is a function ($\kappa \geq 1$) and $P \subset X$, let $f_{\mathrm{cl}}[P]$ denote the closure of P under f:

$$f_{\mathrm{cl}}[P] = \text{the smallest subset of } X \text{, which contains } P \text{ and is closed under } f.$$

If $h \in \mathfrak{C}$, define $H(h, \gamma)$ to be the h' given by Claim 2 and $h_0 \in \mathfrak{C}$ arbitrary. Let $(\gamma_i)_{i<k}$ be an increasing sequence. If $C = \mathrm{cl}_\omega\{\gamma_i \mid i < k\} \subset S$, we can define a new increasing sequence $(\gamma_i)_{i<k}$, which enumerates C and a sequence of functions

$$h_0, h_1 = H(h_0, \gamma_0'), \dots, h_\omega = \cup_i h_i', \dots .$$

Denote the function $\cup_{i<k} h_i = h[(\gamma_i)_{i<k}]$. Thus we have a mapping $(\gamma_i)_{i<k} \mapsto h[(\gamma_i)_{i<k}] \in \mathfrak{C}$ defined, when $\mathrm{cl}_\omega\{\gamma_i \mid i < k\} \subset S$. Define $\tau^* \colon \omega_2^{<\omega_1} \to \omega_2$ and X_i, $i < \omega_2$, to be as follows. Because I want the following to fit the page, I shall write $\mathrm{s}(\alpha)$ instead of ”α is a successor ordinal”.

$$\begin{aligned} X_0 &= \tau_{\mathrm{cl}}[(F_{h_0})_{\mathrm{cl}}(\tau(\varnothing))] \\ \tau^*(\varnothing) &= G(X_0) \end{aligned}$$

$$X_\alpha = \begin{cases} \text{anything}, & \text{if } \mathrm{cl}_\omega\{y_i \mid i < \alpha\} \not\subset S \text{ and } \mathrm{s}(\alpha) \\ \tau_{\mathrm{cl}}[(F_{h[y_i]_{i<\alpha}} \cup F^{-1}_{h[y_i]_{i<\alpha}})_{\mathrm{cl}}[\{\mathrm{id}_{y_i}\}]], & \text{if } \mathrm{cl}_\omega\{y_i \mid i < \alpha\} \subset S \text{ and } \mathrm{s}(\alpha) \\ \cup_{j<\alpha} X_j, & \alpha \text{ is a limit} \end{cases}$$

$$\tau^*((y_i)_{i<\alpha}) = \begin{cases} \text{anything} > \sup\{y_i \mid i < \alpha\}, & \text{if } \mathrm{cl}_\omega\{y_i \mid i < \alpha\} \not\subset S \text{ and } \mathrm{s}(\alpha) \\ G(X_\alpha), & \text{otherwise} \end{cases}$$

Let σ^* be a strategy of $\mathbf{2}$ which beats τ^* in $G_\omega^{\omega_1}$. Then the cases ”anything” are never used and $X_0 \subset X_1 \subset \cdots$ and each X_i is closed under τ and under the isomorphism $F_{\cup_{i<\omega_1} h_i}$ (which exists because σ^* is winning). Thus

$$X = \bigcup_{i<\omega_1} X_i$$

is closed under τ, but $\mathcal{A} \cap X \cong \mathcal{B} \cap X$. Our conclusion is that τ cannot be a winning strategy. $\square$

VI Do Long Games Determine Short Games?

Evidently $\mathcal{A} \sim_\beta \mathcal{B} \Rightarrow \mathcal{A} \sim_\alpha \mathcal{B}$ for $\alpha < \beta$. It is one of the most central properties of Ehrenfeucht-Fraïssé games. In the case of the weak game situation is different. As example IV.11 showed it might happen, that player **2** has a winning strategy for a limit ordinal but **I** has a winning strategy for a smaller successor ordinal. Also the theorem IV.12 provided a result for the opposite direction: if player **2** wins the game $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$ for $\kappa^{<\kappa} = \kappa$, then she wins the game for $\alpha < \kappa^+$ with $\mathrm{cf}(\alpha) = \kappa$. If $n \le \omega$, the assertion $\mathrm{EF}^*_\omega(\mathcal{A}, \mathcal{B}) \Rightarrow \mathrm{EF}^*_n(\mathcal{A}, \mathcal{B})$ follows from III.2. For regular cardinal game lengths an example as good as that of IV.11 is impossible due to

VI.1. Theorem. *Let κ be a regular cardinal and λ such an ordinal that $\kappa^{<\lambda} = \kappa$. Then if* **I** *has a winning strategy in $\mathrm{EF}^*_\lambda(\mathcal{A}, \mathcal{B})$ then he has one also in $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$.*

Proof. Assume **I** wins $\mathrm{EF}^*_\lambda(\mathcal{A}, \mathcal{B})$. Then he wins $\mathrm{EF}_\lambda(\mathcal{A}, \mathcal{B})$ and then by theorem III.3 he wins also $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$. □

If furthermore $\mathrm{EF}^*_\lambda(\mathcal{A}, \mathcal{B})$ happens to be determined, this implies $\mathcal{A} \sim^*_\kappa \mathcal{B} \Rightarrow \mathcal{A} \sim^*_\lambda \mathcal{B}$ for such cardinals κ and ordinals λ. In particular if GCH holds, then $\mathcal{A} \sim^*_\kappa \mathcal{B} \Rightarrow \mathcal{A} \sim^*_\lambda \mathcal{B}$ holds for all regular cardinals κ and ordinals $\lambda < \kappa$ provided that EF^*_λ determined. Theorem VI.1 in ensemble with III.2 also implies (GCH not needed) that for all regular cardinals κ we have $\mathcal{A} \sim^*_\kappa \mathcal{B} \Rightarrow \mathcal{A} \sim_\omega \mathcal{B}$.

Despite all this, it is possible that there are structures $\mathcal{A}$ and $\mathcal{B}$ and cardinals $\kappa < \lambda$ such that **2** has a winning strategy in $\mathrm{EF}^*_\lambda(\mathcal{A}, \mathcal{B})$ but not in $\mathrm{EF}^*_\kappa(\mathcal{A}, \mathcal{B})$. We will now provide a similar construction to that made in section V.5, which will give such an example, where however λ is singular. In fact we will have an infinite sequence of cardinals

$$\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots$$

such that $\mathcal{A} \not\sim^*_{\alpha_n} \mathcal{B}$ but $\mathcal{A} \sim^*_{\beta_n} \mathcal{B}$.

VI.1. Structures with non-reflecting winning strategies

As mentioned, the construction will be very much as that in V.5. Let $\mu = \omega^+_{\omega\cdot\omega}$. This is a cardinal number of size $\aleph_{\omega\cdot\omega+1}$, a successor of the singular cardinal $\aleph_{\omega\cdot\omega}$. In this section we assume GCH.

Define: $\alpha_n = \omega_{\omega\cdot n+1}$ and $\beta_n = \omega_{\omega\cdot(n+1)}$. Then $\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots$ and α_n's are regular and β_n's singular of cofinality ω.

Using $\mathbb{P} = \{p\colon \alpha \to \{0,1\} \mid \alpha < \mu\}$ one can force a set $S \subset \mu$ such that

$$E_n = \{\beta < \mu \mid \mathrm{cf}(\beta) = \alpha_n, \beta \cap S \text{ contains a cub set of order type } \alpha_n\}$$

is bistationary for every $n \geq 0$. The proof is exactly as in paragraph V.5.1 with indices changed a bit.

Then define as in paragraph V.5.3:

$$\begin{aligned} A \ = \{f\colon \alpha+1 \to \ & \mu \times \omega \mid \alpha < \omega_2, \\ & f \text{ is strictly increasing,} \\ & \text{for each } n < \omega \text{ the set } \mathrm{pr}_1[\mathrm{Rng}(f_1) \cap (\mu \times \{n\})]] \\ & \quad \text{is closed in } \mu \text{ and is contained in } S\} \end{aligned}$$

and

$$\begin{aligned} B \ = \{f\colon \alpha+1 \to \ & \mu \times \omega \mid \alpha < \omega_2, \\ & f \text{ is strictly increasing,} \\ & \text{for each } n < \omega \text{ the set } \mathrm{pr}_1[\mathrm{Rng}(f_1) \cap (\mu \times \{n\})]] \\ & \quad \text{is closed as a subset of } \mu \text{ and } \textbf{if } \ n > 0, \textbf{ then} \text{ is contained in } S\} \end{aligned}$$

VI.2. THEOREM. *Player* **2** *can not have a winning strategy in the game* $\mathrm{EF}^*_{\alpha_n}(\mathcal{A}, \mathcal{B})$.

PROOF. Almost precisely as the proof of theorem V.28. Use $\mathcal{P}_{\alpha_n^+}(A \cup B)$ instead of $\mathcal{P}_{\omega_2}(A \cup B)$, E_n instead of E and by GCH one can close under the strategy σ of **2**. Thus proceeding as in V.28 we obtain a set $C \subset A \cup B$ of size α_n, which is closed under the strategy σ and

$$G(C) = \sup\{\mathrm{dom}\, f \mid f \in C\} \in \{\alpha < \mu \mid \mathrm{cf}(\alpha) = \alpha_n\} \setminus E_n.$$

Similarly as in V.28, we can add a branch of length α_n to $C \cap \mathcal{B}$ but $C \cap \mathcal{A}$ does not contain such a branch. □

VI.3. THEOREM. *If* $\mathrm{cf}(\lambda) = \omega$, $\lambda < \mu$ *(for example* $\lambda = \beta_n$*), then player* **2** *has a winning strategy in the game* $\mathrm{EF}^*_\lambda(\mathcal{A}, \mathcal{B})$.

PROOF. As in section V.5 there are isomorphisms $F_\beta\colon \mathcal{A}_\beta \to \mathcal{B}_\beta$ for each β in a particular subset of $\cup_{n<\omega} E_n$. For our purpose it is enough to know that this subset is unbounded. In the game $\overset{*}{\mathrm{EF}}{}^{1,\lambda}_{\lambda}$ player **2** will take a cofinal $\eta\colon \omega \to \lambda$ and at the move $\eta(n)$ assume that X_n is the set of already picked elements. By the methods of section V.5 she can choose an isomorphism F_{β_n} such that β_n is greater than $\sup\{\operatorname{dom} f \mid f \in X\}$ keeping in mind that in the end of the game $\cup_{k<\omega} F_{\beta_k}$ should be a partial isomorphism. Then she will pick a set, which contains X and is closed under each F_{β_k}, $k \in \{0, \ldots, n\}$. $\square$

Thus the sequence

$$\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots ,$$

where $\alpha_n = \omega_{\omega \cdot n+1}$ and $\beta_n = \omega_{\omega\cdot(n+1)}$ is such that $\mathcal{A} \not\sim^*_{\alpha_n} \mathcal{B}$ but $\mathcal{A} \sim^*_{\beta_n} \mathcal{B}$.

A Appendix: On Cardinal Arithmetic

In this appendix I introduce some of the basic notions of cardinal arithmetic, which are used throughout the paper.

A.1. Definition.

(*i*) A subset C of an ordinal is *closed unbounded* or *cub* if for every increasing sequence $(a_\alpha)_{\alpha<\lambda}$ the point $\sup\{a_\alpha \mid \alpha < \lambda\}$ is in C and C is unbounded in α.

(*ii*) A subset S of an ordinal α is *stationary* if it intersects all cub sets of α.

(*iii*) $C \subset \alpha$ is λ-cub, where $\lambda < \alpha$ is a (fixed) cardinal, if C is unbounded and closed under increasing sequences of length λ.

(*iv*) $S \subset \alpha$ is λ-stationary if it intersects all λ-cub subsets of α.

A.2. Lemma. *Let κ be regular, $\mu < \kappa$, and $(C_\alpha)_{\alpha<\mu}$ a sequence of λ-cub sets. Then $C = \bigcap_{\alpha<\mu} C_\alpha$ is λ-cub.*

Proof. If $(a_\gamma)_{\gamma<\lambda}$ is increasing in C, then it is increasing in every C_α and thus $\sup_\gamma a_\gamma$ is in every C_α. Hence C is closed.

Let $\alpha < \kappa$. Let $\beta_\xi(\alpha)$ be the least element of C_ξ which is greater than α. We define $f(\alpha) = \sup_{\xi<\mu} \beta_\xi(\alpha)$. By regularity $f(\alpha) \in \kappa$. Let us now define $f^{\gamma+1}(\alpha) = f(f^\gamma(\alpha))$ and if γ is limit $f^\gamma(\alpha) = \bigcup_{\beta<\gamma} f^\beta(\alpha)$.

It is clear that for each ξ there is now a sequence $\beta^0_\xi(\alpha) < \beta^1_\xi(\alpha) < \cdots$ such that $f^\lambda(\alpha) = \sup_{\gamma<\lambda} \beta^\gamma_\xi(\alpha)$ for each $\xi < \mu$. Thus $f^\lambda(\alpha)$ is in C. $\square$

A.3. Lemma. *Intersection of a λ-cub set and a λ-stationary set is λ-stationary.*

Proof. Let C be λ-cub and S be λ-stationary. Let C' be any λ-cub. We want to show that $(C \cap S) \cap C'$ is non-empty. But $C \cap C'$ is λ-cub by A.2 and $(C \cap S) \cap C' = (C \cap C') \cap S$, so by stationarity of S this is non-empy. $\square$

The following is a modification of Fodor's lemma. Fodor's lemma is stated for stationary and cub sets like in i. and ii. of the definition A.1. In the proof I thereabouts follow [**Jech**], theorems 8.4, 8.7.

A.4. Lemma (Fodor). *Let κ be a regular cardinal and $S \subset \kappa$ a λ-stationary subset. Assume that $f\colon S \to \kappa$ is a mapping such that for all $\alpha \in S$, $f(\alpha) < \alpha$, i.e. a regressive mapping. Then there exists $\beta \in \kappa$ such that the set $f^{-1}\{\beta\}$ is λ-stationary.*

Proof. Let us assume that this is not true. Then for each $\alpha \in fS$ there is a λ-cub set C_α outside $f^{-1}\{\alpha\}$. We will now deduce that there is $\beta \in S$ such that $f(\beta) \geq \beta$. Let

$$C = \{\xi < \kappa \mid \xi \in \bigcap_{\alpha<\xi} C_\alpha\}. \tag{1}$$

It is clear that

$$\bigcap_{\alpha<\xi} C_\alpha = \bigcap_{\alpha<\xi} \bigcap_{\beta<\alpha} C_\beta$$

(intersection over the same indices), so we can write

$$C = \{\xi < \kappa \mid \xi \in \bigcap_{\alpha<\xi} C'_\alpha\},$$

where $C'_\alpha = \bigcap_{\beta<\alpha} C$ and the main advantage is that

$$C'_\alpha \supset C'_\beta \text{ whenever } \alpha < \beta. \tag{2}$$

Note that C'_α is λ-cub for each α by A.2.

Claim: C is λ-cub.

Proof of the claim: Let $(a_\alpha)_{\alpha<\lambda}$ be an increasing λ-sequence in C. We have to show that $\sup\{a_\alpha\} \in C'_\beta$ for each $\beta < \sup\{\alpha\}$. Let $\beta < \sup\{a_\alpha\}$. Then there exists some $\gamma \in \lambda$ such that if $\alpha > \gamma$, then $a_\alpha > \beta$ (because $\beta < \sup\{a_\alpha\}$). We know that $a_\alpha \in C$ and thus $a_\alpha \in C'_\beta$ for all $\alpha > \gamma$ (by $\beta < a_\alpha$ and by definition of C). Because C'_β is cub, we have $\sup\{a_\alpha\} = \sup\{a_\alpha \mid \alpha > \gamma\} \in C'_\beta$ and we are done.

To show that C is unbounded, let $\alpha < \kappa$. We will construct a sequence as follows: $\beta_0 \in C_0$ such that $\beta > \alpha$ and let $\beta_{n+1} \in C_{\beta_n}$ be such that $\beta_{n+1} > \beta_n$. We want to show that $\beta = \sup_n \beta_n \in C$. Let $\xi < \beta$. Then there is n such that $\beta_m > \xi$ for all $m \geq n$. By (2) $\beta_m \in C_{\beta_n}$ for all $m > n$ also by (2) $C_{\beta_n} \subset C_{\beta_\xi}$ and because the latter is cub, we have $\beta \in C_{\beta_\xi}$, which proves the claim. $\square_{\text{Claim}}$

As an intersection of λ-cub set and a λ-stationary set, $C \cap S$ is λ-stationary by A.3 (thus non-empty). Let $\beta \in C \cap S$. Now by (1) $\beta \in C_\alpha \subset S \setminus f^{-1}\{\alpha\}$ for all $\alpha < \beta$, which means $f(\beta) \neq \alpha$ for all $\alpha < \beta$. Hence $f(\beta) \geq \beta$. $\square$

Another lemma which we use in proving theorems and which is closely related to cardinalities is stated next. If λ is an ordinal and κ a cardinal, we define $\kappa^{<\lambda} = \bigcup_{\beta<\lambda} |\kappa^\beta|$.

A.5. Lemma. *Let κ and λ be such that $\kappa^{<\lambda} = \kappa$. Let A be any set, $f\colon A^{<\lambda} \to A$ and $B \subset A$ a subset of cardinality $\leq \kappa$. Then there exists $X \subset A$ such that $|X| = \kappa$, $f[X^{<\lambda}] \subset X$ and $B \subset X$. That is, X is closed under f and is an extension of B.*

Proof. If $|A| = \kappa$, then take $X = A$. Assume that $|A| > \kappa$. Moreover we can assume that $|B| = \kappa$: just extend it to some set of that cardinality. Let

$$X_0 = B \subset A.$$

For each ordinal α, $0 < \alpha < \lambda$ set

$$X_\alpha = \bigcup_{\beta<\alpha} \big\{ f(k) \mid k \in X_\beta^\gamma,\, \gamma < \lambda \big\}.$$

By $\kappa^{<\lambda} = \kappa$, and by the fact that $|X_0| = \kappa$, we conclude that $|X_1| = |\{f_i(k) \mid k \in X_0^\gamma, \gamma < \lambda\big\}| \leq \kappa$. By induction we see that for each α holds $|X_\alpha| \leq \kappa$.

Now set

$$X = X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha.$$

Now

$$|X| = |\bigcup_{\alpha<\lambda} X_\alpha| \leq |\bigcup_{\alpha<\lambda} \kappa^\alpha| = \kappa^{<\lambda} = \kappa.$$

Actually $|X| = \kappa$ since $|X_0| = \kappa$. Moreover X is closed under f. Indeed, if $k \in X^\gamma$ for any $\gamma < \lambda$, then obviously $k \in X_\alpha^\gamma$ for some α. Otherwise pick $x \in X_{\alpha+1} \setminus X_\alpha$ such that x is a member of k and obtain cofinal γ-sequence in κ. Then by König's lemma (see [**Kunen**], lemma 10.40) $\kappa^{<\lambda} \geq \kappa^\gamma > \kappa$. So $f(k) \in X_{\alpha+1} \subset X$. $\square$

B Appendix: On Free Abelian Groups

In section V.3 we used some properties of free abelian groups. The aim of this appendix is clarify these notions. As we are working in model theory, we try to view groups in this context. For more details a book (e.g. [**Lang**]) by Serge Lang is recommended.

B.1. Definition. Let $L = \{+, -, 0\}$ be the language consisting of one binary function symbol $+$, unary function symbol $-$ and a constant symbol 0. Then a *group* G is an L-structure, which satisfies the L-sentences described below. Instead of $+(a, b)$ and $-(a)$ we write $a + b$ and $-a$. The sentences are as follows.

- $\forall x \forall y \forall z((x + y) + z = x + (y + z))$
- $\forall y(y + 0 = y)$
- $\forall x(x + (-x) = 0)$.

If G satisfies also

- $\forall x \forall y(x + y = y + x)$,

then it is called abelian.

A homomorphism f from a group G to a group F is a function $f \colon G \to F$ such that $f(x +^G y) = f(x) +^F f(y)$ and $f(0^G) = 0^F$. We will drop the structures from the upper indexes when no confusion is expected. Note that a bijective homomorphism is an isomorphism in the sense of definition I.7. It is easy to see that a substructure of a group (definition I.6) is also a group; we say that it is a *subgroup*.

Sometimes we shall talk about groups with a relational vocabulary. It means that the function symbols $+$, $-$ above are replaced by one ternary relation symbol $+'$ such that $+'(a, b, c) \iff a + b = c$. Then the fact $a = -b$ can be viewed as $+'(a, b, 0)$. In this case obviously submodels of groups need not be groups, so the term subgroup would mean a submodel, which is itself a group.

However in this appendix we consider only groups viewed with function symbols.

B.2. Definition. Let $\{A_i \mid i \in I\}$ be a family of pairwise disjoint groups ($i \neq j \Rightarrow A_i \cap A_j = \varnothing$). Let

$$G = \{f \colon I \to \bigcup_{i \in I} A_i \mid \forall i \in I\,(f(i) \in A_i)\}.$$

We define the addition in G by $(f+g)(i) = f(i) + g(i)$. Clearly G is now a group. It is the *direct product* of the groups A_i denoted by $\Pi_{i \in I} A_i$.

B.3. Definition. Let $\{A_i \mid i \in I\}$ be as above. The *direct sum* of the groups A_i,

$$\bigoplus_{i \in I} A_i$$

is defined to be the subgroup of $\Pi_{i \in I} A_i$ which consists of mappings $f\colon I \to A_i$ that are 0 almost everywhere. Almost everywhere means that for each $f \in \bigoplus_{i \in I} A_i$ there is finite $S \subset I$, such that for all i in $I \setminus S$ we have $f(i) = 0$.

B.4. Definition. Let S be a set, G a group and $f\colon S \to G$ a mapping. Then G is the *free group generated by* S if for every group F and a mapping $g\colon S \to F$ there exists a unique homomorphism $\varphi\colon G \to F$ such that $g = \varphi \circ f$.

If in the above definition B.4 all groups are replaced with abelian groups, then G is said to be a free abelian group (generated by S). In fact *the* is a good article for the free (abelian) group generated by S, since it is quite clear from the definition that any two free (abelian) groups generated by (the same set) S are isomorphic. Note also that f in the definition B.4 has to be one-to-one because otherwise g could not be one-to-one. This implies that actually the free group generated by S is isomorphic to the free group generated by S' if and only if $|S| = |S'|$. If G is generated by a countable set S, then it is itself countably infinite and if $|S| = \kappa \geq \omega$, then $|G| = |S| = \kappa$. We say that fS generates G or is a basis of G (f is as in the definition B.4).

We use only abelian groups in this paper, so we shall prove the following result only for them.

B.5. Theorem. *For every set S the free abelian group generated by S exists.*

Proof. For each $s \in S$ let $\mathbb{Z}_s = \{s\} \times \mathbb{Z}$, where $\mathbb{Z}$ is the additive group of integers. Let

$$F = \bigoplus_{s \in S} \mathbb{Z}_s.$$

and let $f\colon S \to F$ be such that $f(s)$ is the element, which is 1 at s and 0 everywhere else. We claim that F is the desired free group. To show this, let G be any abelian group and $g\colon S \to G$ a mapping. Because f is clearly one-to-one, we can define $\varphi(f(s)) = g(s)$ for each $f(s) \in F$. By definition every element x of F has a unique representation as a finite sum

$$x = \sum_{i \in I} z_i f(i),\ I \subset S \text{ finite},$$

where $z_i \in \mathbb{Z}$ and $z_i a$ means $\overbrace{a + \cdots + a}^{z_i\ times}$ if $z_i > 0$, 0, if $z_i = 0$ and $\overbrace{(-a) + \cdots + (-a)}^{-z_i\ times}$ if $z_i < 0$. Therefore for every $x = \sum_{i \in I} z_i f(i) \in F$ we can define

$$\varphi(x) = \sum_{i \in I} z_i \varphi(f(i)).$$

From construction it is rather obvious that φ is a homomorphism and is unique: under the requirement $\varphi \circ f = g$ there is no other way to define $\varphi(f(i))$ and under the requirement that it should be a homomorphism, there is no other way to define $\varphi(x)$ for $x \in F$. ☐

For example the group $F = \bigoplus_{i<\omega_2} \mathbb{Z}$ used in V.3 is the free group generated by a set of cardinality $\aleph_2$ (in V.3 we think that this set is ω_2). More generally one sees that a free abelian group F is always isomorphic to $\bigoplus_{i \in I} \mathbb{Z}$ for some I (for instance $I = \dim F$, see end of this appendix). Thus if H and K are free, their direct sum

$$\bigoplus_{i \in I_1} \mathbb{Z} \oplus \bigoplus_{i \in I_2} \mathbb{Z} = \bigoplus_{i \in I_1 \cup I_2} \mathbb{Z}$$

is also free. For the converse we have the theorem B.6.

The next proof is an application of Zorn's lemma, so I will state it for the sake of completeness. The proof that Zorn's lemma is equivalent to the Axiom of Choice is omitted and can be found e.g. in [**Jech**]. If λ is a cardinal and $(a_i)_{i<\lambda}$ is a sequence of sets such that $i < j \Rightarrow a_i \subset a_j$, we call it a chain. A closure of a chain $(a_i)_{i<\lambda}$ is a chain $(a_i)_{i\le\lambda}$, where $a_\lambda \supset \bigcup_{i<\lambda} a_i$ (this is not unique).

ZORN'S LEMMA. *Let X be a set and $A \subset \mathcal{P}(X)$ a subset of its power set. If for every chain of elements of A a closure of this chain is contained in A, then there is a maximal element $b \in A$, i.e. $\forall a \in A (b \not\subset a)$.*

This is equivalent to the situation, where instead of A is some arbitrary partially ordered set $(P, \le)$ (see definition II.9) chains are $\le$-chains and a closed chain is a chain with a greatest element. Partial order by inclusion is clearly a partial order, on the other hand given a partial order P one can take

$$A = \{W \in \mathcal{P}(P) \mid W \text{ is a chain closed downward and has a greatest element}\}.$$

B.6. THEOREM. *Let F be a free abelian group generated by the set $X = \{x_k \mid k \in I\}$ and let $G \subset F$ be a subgroup. Then G is free and a basis of G can be indexed by a subset of I. Moreover if G is generated by a subset of X, then there is a group $H \subset F$ such that $F = G \oplus H$.*

PROOF. This proof can be found in [**Lang**], Appendix 2 §2.

If $G = \{0\}$, the statement holds, so we can assume that G is non-trivial. First we shall prove this for finite X by induction. When $|X| = 1$, G is isomorphic to $\mathbb{Z}$ (being non-trivial) and clearly free. Assume that if a group is generated by a set of size $\leq k$, then every subgroup of it is free. Let $X = \{x_1, x_2, \dots, x_k, x_{k+1}\}$, F the free group generated by X and $G \subset F$ a subgroup. Let $\mathrm{pr}\colon G \to F$ be the projection

$$\mathrm{pr}(a_1 x_1 + \cdots + a_{k+1} x_{k+1}) = a_1 x_1.$$

If $\mathrm{Rng}(\mathrm{pr}) = \{0\}$, then G is a subset of $\langle x_2, \dots, x_{k+1}\rangle$ and free by the induction hypothesis. Thus we can assume that the range is non-trivial. Let $m > 0$ be the least such that $mx_1 \in \mathrm{Rng}(\mathrm{pr})$ and choose some x such that $\mathrm{pr}x = mx_1$. It is standard to verify that $x \notin \mathrm{Ker}(\mathrm{pr})$ and if $y \in G$, then $y = nx + k$, where $k \in \mathrm{Ker}(\mathrm{pr})$ and $n \in \mathbb{Z}$. Hence

$$G = \mathrm{Ker}(\mathrm{pr}) \oplus \langle x \rangle.$$

By the induction hypothesis $\mathrm{Ker}(\mathrm{pr})$ and $\langle x \rangle$ are free: first is isomorphic to a subgroup of $\langle x_2, \dots x_{k+1}\rangle$ and the second to $\mathbb{Z}$.

Assume now that $X = \{x_i \mid i \in I\}$ is arbitrary. For each subset J of I let F_J be the free group generated by $\{x_i \mid i \in J\}$, thus $F_J \subset F$ is a free subgroup and denote $G_J = F_J \cap G$. Now set

$$S = \{(G_J, w) \mid G_J \text{ is a non-trivial free group and } w \text{ is a basis of } G_J\}.$$

Formally w is a one-to-one mapping $w\colon J' \to G_J$ such that $w[J']$ generates G_J and $J' \subset I$.

Clearly S is non-empty: Let us have an element x in G. Then

$$x = a_1 x_{i_1} + \cdots + a_n x_{i_n}$$

and thus the free group generated by $S = \{x_{i_1}, \dots, x_{i_n}\}$ contains x and the intersection $G \cap F_J$ is a non-trivial subgroup of a finitely generated free abelian group and thus free by the induction above.

If $(G_J, w), (G_K, u) \in S$, define order $(G_J, w) \leq (G_K, u)$ if and only if $J \subset K$ and the basis u is an extension of w; formally if $w\colon J' \to G_J$ and $u\colon K' \to G_K$, then $J' \subset K'$ and $u \restriction w = w$.

If $(G_{J_r}, w_r)_{r \in L}$ is a $\leq$-chain (L is some linear order) of elements of S, then obviously

$$(\bigcup_{r \in L} G_{J_r}, \bigcup_{r \in L} w_r) \in S,$$

so we can apply Zorn's lemma and conclude that there exists a maximal (G_J, w). Since $G_I = G$, it is enough to prove now that $J = I$. Assume on contrary that there is $k \in I \setminus J$. Put $K = J \cup \{k\}$. If

$$G_K = F_K \cap G = G_J,$$

then it means that $(G_J, w) \le (G_K, w)$, but they are not equal, so (G_K, w) is bigger, which contradicts maximality of (G_J, w). Otherwise there is an element

$$nx_k + y \in G_K$$

such that $n \in \mathbb{Z} \setminus \{0\}$ and $y \in G_J \subset F_J$. The set of $n \in \mathbb{Z}$ for which there exists $y \in G_J$ such that

$$nx_k + y \in G$$

forms a subgroup of $\mathbb{Z}$. Let n_0 be a generator of this group and let

$$w_k = n_0 x_k + y \in G,$$

with $y \in F_J$. Now if $z \in G_K$, then for some $m \in Z$ $z = z - mw_k + mw_k$, where $z - mw_k \in G_J$. On the other hand clearly $w_k\mathbb{Z} \cap G_J = \{0\}$, so

$$w' = w \cup \{\langle k, w_k \rangle\}$$

is a basis of G_K, and $(G_K, w') \ge (G_J, w)$ contradicting the maximality again. Thus G is free and by definitions, the basis w of G is indexed by a subset of I.

Assume that G is generated by a subset Y of X. Let H be the group generated by $X \setminus Y$. We have $G \cap H = \{0\}$, because otherwise we had $x_i - x_j = 0$ for some distinct $x_i, x_j \in X$. On the other hand evidently $G + H = F$, hence $F = G \oplus H$. □

If F is generated by S such that $|S| > \aleph_0$, then $|F| = |S|$. On the other hand if $|F| = \lambda > \omega$ it can be generated only by a set of cardinality λ. Thus in this case obviously if F is generated by S and is isomorphic to a free group G generated by S', then $|S| = |S'|$. We can prove this also for smaller cardinals.

B.7. THEOREM. *If S and S' are two bases of a fee group F, i.e. F is isomorphic to the free group generated by S and to the one generated by S', then $|S'| = |S|$.*

PROOF. Assume that F is generated by S, G by S' and there is an isomorphism $f\colon F \to \mathcal{G}$. Assume $S \subset F$ and $S' \subset G$. Clearly now $f[S]$ generates $f[F] = G$. Because $f[F] \subset G$, by the theorem B.6 we are able to index the basis of $f[F]$ by a subset of S. Thus $|f[S']| \le |S|$. On the other hand f is bijective, so $G \subset f[F]$ and by the same argument $|S| \le |f[S']|$. Hence $|f[S']| = |S|$ and again by bijectivity of f we see $|S| = |S'|$. □

B.8. DEFINITION. Let S be a basis of a free group F. Then by above theorem $\lambda = |S|$ is a cardinal number depending only on F. The cardinal number λ is the *dimension* of the free abelian group F and is denoted by $\dim F$.

B.9. Corollary. *If G and F are free groups and $G \subset F$, then* $\dim G \leq \dim F$.

Proof. Let X be a basis of G. By theorem B.6 the basis X can be indexed by a subset of a basis of F. Since the size of a basis of F is some unique cardinal number λ, we can actually index X by a subset I of λ and thus we have

$$\begin{tikzcd} X \arrow[r, "i^{-1}"] & I \arrow[r, hook, "j"] & \lambda, \end{tikzcd}$$

where i is the indexing map and j the inclusion map. It follows that $\dim G = |X| \leq \lambda = \dim F$. $\square$

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