Analytic Sets

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Abstract. The continuous images of Borel sets are not usually Borel sets. However they are measurable and that is the main purpose of this lecture.

Main reference of this presentation is [**Jech**], chapter 11.

1. Polish spaces and the Baire space

A *Polish space* is a topological space that is homeomorphic to a complete separable metric space. For example $\mathbb{R}$ and every interval (open/closed/none) of $\mathbb{R}$ is a Polish space. The space $\mathcal{N} = \mathbb{N}^{\mathbb{N}} = \{a \colon \mathbb{N} \to \mathbb{N}\}$ endowed with the product topology is also a Polish space, called *the Baire space*. It is metrizable as a countable product of metric (discrete) spaces. Polish spaces are so named because they were first extensively studied by Polish topologists and logicians, like Sierpinski, Kuratowski, Tarski, and others. In this presentation we will concentrate on uncountable Polish spaces.

1.1. Definition. Let Seq denote the set of finite sequences of natural numbers:

$$\mathrm{Seq} = \{f : \{0,\dots,m\} \to \mathbb{N} \mid m \in \mathbb{N}\} = \{a \restriction \{0,\dots,n\} \mid a \in \mathcal{N},\, n \in \mathbb{N}\}.$$

We also assume $\varnothing \in \mathrm{Seq}$. Let us order this set by $s < t \iff t \restriction \operatorname{dom} s = s$. If $s \in \mathrm{Seq}$ and $k \in \mathbb{N}$, $s = \langle n_0,\dots,n_m\rangle$, denote by $s^\frown k = \langle n_0,\dots,n_m,k\rangle$ the extension of s by k. When $x \in \mathcal{N}$, we write $x \restriction n$ to mean $x \restriction \{0,\dots,n-1\}$. Note that $x \restriction n \in \mathrm{Seq}$ for any n.

Then for each $s \in \mathrm{Seq}$ one can define a basic open set of $\mathcal{N}$:

$$O_s = \{x \in \mathcal{N} \mid x \restriction \operatorname{dom} s = s\}.$$

1.2. Theorem. *The Baire space $\mathcal{N}$ is a Polish space.*

Proof. For $a, b \in \mathcal{N}$ define $d(a,b) = 2^{-n}$, where $n = \min\{n \mid a(n) \neq b(n)\}$. This turns $\mathcal{N}$ into a metric space. The set $\{a \in \mathcal{N} \mid \exists m \exists n_0 \in \mathbb{N} \forall n > n_0 (a(n) = m)\}$ (all eventually constant functions) is countable and dense. In this metric $\mathcal{N}$ is also complete. $\square$

REMARK. Identifying $a \in \mathcal{N}$ with continuous fraction $[a(0); a(1), a(2), \dots]$ one obtains a homeomorphism from $\mathcal{N}$ to irrational numbers.

1.3. THEOREM. *Let X be any Polish space. Then there exists a continuous surjective map $f\colon \mathcal{N} \to X$.*

PROOF. In a Polish space X it is easy to construct a set of closed balls $\{C_s \mid s \in \mathrm{Seq}\}$ indexed by elements of Seq such that $C_\varnothing = X$ and

- (i) $\mathrm{diameter}(C_s) < 1/n$, where $n = \mathrm{length}(s)$.
- (ii) For all $s, t \in \mathrm{Seq}$, if $s < t$, then $\mathrm{center}(C_t) \in C_s$
- (iii) For all $s \in \mathrm{Seq}$, $C_s \subset \bigcup_{k=0}^{\infty} C_{s^\frown k}$.

Now for $a \in \mathcal{N}$ one can define $f(x)$ to be the unque element in $\bigcap_{n=0}^{\infty} C_{x\restriction n}$. It is not difficult to see that f is continuous and by the construction of C_s it is onto. □

From the proof above one does not obtain a bijection $\mathcal{N} \to X$ and in general it is not possible, but in the case $X = \mathbb{R}$ it is possible. Let $b\colon \mathbb{N} \to \mathbb{Z}$ be any bijection. In the proof above choose $C_\varnothing = \mathbb{R}$,

- (i) For $n \in \mathbb{N}$, $C_{\langle n \rangle} = [b(n), b(n) + 1]$.
- (ii) Assume $C_s = [a_1, a_2]$ is defined and $k \in \mathbb{N}$. Then let

$$C_{s^\frown k} = [a_1 + \frac{a_2 - a_1}{k+2}, a_1 + \frac{a_2 - a_1}{k+1}].$$

Then f as defined in the proof above becomes a continuous bijection (not a homeomorphism of course, since $\mathcal{N}$ is not locally connected).

2. Projections of Borel sets

2.1. DEFINITION. The smallest σ-algebra of a Polish space X, which contains all closed sets is called the collection of *Borel sets*.

The collection of the Borel sets is closed under countable intersections and unions, complements and inverse images by continuous (or generally Borel) functions. However the continuous images of Borel sets need not be Borel sets. Let us study the class of sets that are continuous images of Borel sets.

2.2. DEFINITION. A subset A of a Polish space X is *analytic* if it is a continuous image of the Baire space $\mathcal{N}$.

REMARK. The next theorem together with the next section give four alternative ways to define analytic sets.

2.3. THEOREM. *Let X be a Polish space. The following are equivalent for any $A \subset X$.*

(*i*) *A is the continuous image of* $\mathcal{N}$*, the Baire space.*

(*ii*) *A is the continuous image of a Borel set* $B \subset Y$*, for some Polish space* Y*.*

(*iii*) *A is the projection of a Borel set* $B \subset X \times Y$ *for some Polish space* Y*.*

(*iv*) *A is the projection of a closed set* $C \subset X \times \mathcal{N}$*.*

Proof. The implications $(iv) \to (iii) \to (ii)$ are trivial. We shall prove that every closed set (in any Polish space) is a continuous image of the Baire space and that every Borel set in X is the projection of a closed set in $X \times \mathcal{N}$. This will immediately lead $(ii) \to (i)$. After this it remains to show $(i) \to (iv)$. But if A is the continuous image of $\mathcal{N}$, then $A = \{f(x) \mid x \in \mathcal{N}\}$, which is the procection of the closed set $\{(f(x), x) \mid x \in \mathcal{N}\}$.

In order to prove that a closed $C \subset X$ set of a Polish space X is a continuous image of $\mathcal{N}$, recall that C is itself a Polish space (completeness is preserved by closed subspaces) and by theorem 1.3 there exists a continuous surjective map $\mathcal{N} \to C$.

Let us now prove that every Borel set in X is the projection of a closed set in $X \times \mathcal{N}$. Let P be the collection of all subsets of X that are such projections. It suffices to show that P contains all closed sets, all open sets, and is closed under countable unions and intersections.

Clearly every closed set of X is in P: if $C \subset X$ is a closed set, then $C = \mathrm{pr} C \times \{a\}$ for any $a \in \mathcal{N}$. Every open set is a countable union of closed sets, so open sets are also in P. It remains to show that P is closed under countable unions and intersections.

Let A_n be the projection of a closed $F_n \subset X \times \mathcal{N}$, i.e. $A_n = \{x \in X \mid \exists a \in \mathcal{N}((x, a) \in F_n)\}$ for each $n \in \mathbb{N}$. Let us first consider unions. For that purpose let $\Gamma \colon \mathcal{N} \to \mathcal{N} \times \mathcal{N}$ be a continuous onto map. For instance such that $\Gamma_1(a)(n) = a(2n)$ and $\Gamma_2(a)(n) = a(2n + 1)$.

$$
\begin{aligned}
x \in \bigcup_{n=0}^{\infty} A_n &\iff \exists n \in \mathbb{N}\, x \in A_n \\
&\iff \exists n \in \mathbb{N}\, \exists a \in \mathcal{N} (x, a) \in F_n, \quad \text{next pick } c \text{ s.t. } \Gamma_1(c) = a \text{ and } F_2(c)(0) = n \\
&\iff \exists c \in \mathcal{N} (x, \Gamma_1(c)) \in F_{\Gamma_2(c)(0)}.
\end{aligned}
$$

Thus

$$\bigcup_{n=0}^{\infty} A_n = \mathrm{pr}\{(x, c) \mid (x, \Gamma_1(c)) \in F_{\Gamma_2(c)(0)}\}$$

It remains to show that $H = \{(x, c) \mid (x, \Gamma_1(c)) \in F_{\Gamma_2(c)(0)}\} \subset \mathcal{N} \times \mathcal{N}$ is closed. Let us prove that the complement is open. Assumme $(x_0, c_0) \notin H$. Then we can find a neighbourhood $U \subset \mathcal{N}$ of $\Gamma_1(c_0)$ such that $(x_0, u) \notin F_{\Gamma_2(c_0)(0)}$ for all $u \in U$. Then

$V = \Gamma^{-1}[U\times\mathcal{N}]$ is a neightbourhood of c_0. Let $V' = V\cap\underbrace{\{c \in \mathcal{N} \mid \Gamma_2(c)(0) = \Gamma_2(c_0)(0)\}}_{\text{open}}$. Also because $F_{\Gamma_2(c_0)(0)}$ is closed, there is a neighbourhood $W \subset \mathcal{N}\times\mathcal{N}$ of (x_0, c_0) such that $W \subset \mathcal{N}\times\mathcal{N}\setminus F_{\Gamma_2(c_0)(0)}$. Take $W' = W\cap\mathcal{N}\times V'$. Then it is a neighbourhood of (x_0, c_0) such that for all $(x, c) \in W$ we have $F_{\Gamma_2(c)(0)} = F_{\Gamma_2(c_0)(0)}$ and $(x, c) \notin F_{\Gamma_2(c_0)(0)}$, i.e. $(x, c) \notin H$.

Then let us look at intersections. Let $\rho\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ be a bijection. and $\Gamma\colon \mathcal{N}^{\mathbb{N}}$ a continuous onto map such that $\Gamma(a)(n)(m) = a(\rho(n,m))$.

$$\begin{aligned} x \in \bigcap_{n=0}^{\infty} A_n &\iff \forall n \in \mathbb{N}\, x \in A_n \\ &\iff \forall n \in \mathbb{N}\, \exists a \in \mathcal{N} (x, a) \in F_n, \\ &\iff \exists c \in \mathcal{N} \forall n \in \mathbb{N} (x, \Gamma(c)(n)) \in F_n. \end{aligned}$$

Thus $\bigcap_{n=0}^{\infty} A_n = \operatorname{pr}\bigcap_{n\in\mathbb{N}}\{(x,c) \mid (x,\Gamma(c)(n)) \in F_n\}$. So it remains to show that $H = \{(x, c) \mid (x, \Gamma(c)(n)) \in F_n\}$ is closed for each n. If (x_0, c_0) is not in H, then there exists a neighbourhood $W \subset \mathcal{N}\times\mathcal{N}\setminus F_n$ of (x_0, c_0) and a neighbourhood $U \subset \mathcal{N}$ of $\Gamma(c_0)(n)$ such that $\mathcal{N}\times U \subset \mathcal{N}\times\mathcal{N}\setminus F_n$. Then $V = \Gamma^{-1}\{(d_k)_{k\in\mathbb{N}} \in \mathcal{N}^{\mathbb{N}} \mid d_n \in U\}$ is a neighbourhood of c_0. It is easy to see now that $W\cap\mathcal{N}\times V$ is a neighbourhood of (x_0, c_0) outside H. □

2.4. Corollary. *All Borel sets are analytic.* □

In section 4 we will see that the converse is not true.

3. The Suslin operation $\mathcal{A}$

Recall that when $x \in \mathcal{N}$, we write $x \restriction n$ to mean $x \restriction \{0, \ldots, n-1\}$. Note that $x \restriction n \in \mathrm{Seq}$ for any n.

3.1. Definition. Let $\{A_s \mid s \in \mathrm{Seq}\}$ be a family of sets indexed by elements of Seq. We define the *Suslin operation* $\mathcal{A}$ as follows.

$$\mathcal{A}\{A_s \mid s \in \mathrm{Seq}\} = \bigcup_{a\in\mathcal{N}} \bigcap_{n=0}^{\infty} A_{a\restriction n}.$$

Remark. Note that for any $a \in \mathcal{N}$ we have

$$\bigcap_{n=0}^{\infty} A_{a\restriction n} = \bigcap_{n=0}^{\infty} A_{a\restriction 0} \cap A_{a\restriction 1} \cap\cdots\cap A_{a\restriction n}$$

Thus redefining $A'_{a\restriction n} = A_{a\restriction 0} \cap A_{a\restriction 1} \cap\cdots\cap A_{a\restriction n}$ we get

$$\mathcal{A}\{A_s \mid s \in \mathrm{Seq}\} = \mathcal{A}\{A'_s \mid s \in \mathrm{Seq}\}$$

and a useful property: if $s < t$, then $A'_s \supset A'_t$. Hence we can restrict our attention to families which satisfy this property.

3.2. DEFINITION. A set is *Suslin* if it is the result of the operation $\mathcal{A}$ applied to a family of closed sets.

It is an equivalent condition for a set A to be analytic:

3.3. THEOREM. *Let X be a Polish space. A set $A \subset X$ is analytic if and only if it is Suslin.*

PROOF. Assume that $A = \mathcal{A}\{F_s \mid s \in \mathrm{Seq}\}$, where each F_s is a closed subset of X. Then

$$\begin{aligned} x \in A &\iff \exists a \in \mathcal{N} x \in \bigcap_{n=0}^{\infty} F_{a\restriction n} \\ &\iff \exists a \in \mathcal{N} (x,a) \text{ satisfies } x \in F_{a\restriction n} \text{ for all } n \in \mathbb{N} \\ &\iff \exists a \in \mathcal{N} (x,a) \in \bigcap_{n=0}^{\infty} \underbrace{\{(x,a) \mid x \in F_{a\restriction n}\}}_{\text{Borel!}} \end{aligned}$$

Thus A is the projection of the set $\bigcap_{n=0}^{\infty}\{(x,a) \mid x \in F_{a\restriction n}\} \subset X \times \mathcal{N}$, so by theorem 2.3 it is enough to show that $B_n = \{(x,a) \mid x \in F_{a\restriction n}\}$ is a Borel set. For any $n \in \mathbb{N}$ there is at most countably many such $b \in \mathcal{N}$ that all $F_{b\restriction n}$ are different (because there is only countably many different sequences of length n). Thus

$$B_n = \bigcup_{a \in \mathcal{N}} \{(x,a) \mid x \in F_{a\restriction n}\} = \bigcup_{a \in \mathcal{N}} \underbrace{F_{a\restriction n}}_{\text{closed}} \times \underbrace{O_{a\restriction n}}_{\text{open}}$$

and the union is really a countable union of Borel sets. Hence B_n and $\bigcap_{n=0}^{\infty} B_n$ are Borel.

Assume now that A is analytic and that there exists a continuous surjective map $f\colon \mathcal{N} \to A$. For every $a \in \mathcal{N}$ we have

$$\bigcap_{n=0}^{\infty} f[O_{a\restriction n}] = \bigcap_{n=0}^{\infty} \underbrace{\overline{f[O_{a\restriction n}]}}_{\text{closure}} = \{f(a)\}.$$

(Reason: by continuity for all $\varepsilon > 0$ there is n such that $f[O_{a\restriction n}] \subset B(f(a),\varepsilon)$ etc..) Hence

$$A = f[\mathcal{N}] = \bigcup_{a \in \mathcal{N}} \bigcap_{n=0}^{\infty} \overline{f[O_{a\restriction n}]} = \mathcal{A}\{\overline{f[O_{a\restriction n}]} \mid a \restriction n \in \mathrm{Seq}\}$$

□

4. Analytic set which is not a Borel set

Because complement of a Borel set is Borel, it is sufficient to find an analytic set A whose complement is not analytic (and hence not Borel by corollary 2.4).

4.1. Lemma. *There exists a universal closed set V in $\mathcal{N}^3$, i.e. a closed set V such that for every closed set $C \subset \mathcal{N}^2$ there exists $a \in \mathcal{N}$ such that*

$$C = \{(x, y) \mid (x, y, a) \in V\}.$$

Proof. Let us construct a universal open set, i.e. an open set V' such that for every open $A \subset \mathcal{N}^2$ there exists $b \in \mathcal{N}$ such that

$$A = \{(x, y) \mid (x, y, b) \in V'\}.$$

Then the universal closed set will (clearly) be its complement. Let $(G_i)_{i=0}^{\infty}$ be an enumeration of all basic open sets in $\mathcal{N}^2$. Now every open set is a union of these. Define

$$V' = \bigcup_{n\in\mathbb{N}} \{(x, y, z) \in \mathcal{N} \mid (x, y) \in G_{z(n)}\}.$$

Each set $B_n = \{(x, y, z) \in \mathcal{N} \mid (x, y) \in G_{z(n)}\}$ is open: assume that $(x', y', z') \in B_n$, then $G_{z'(n)} \times \{z \mid z'(n) = z(n)\}$ is an open neighbourhood of $(x', y', z') \subset B_n$. Hence V' is open. Now consider any open $A \subset \mathcal{N}^2$. There exists a function $b \in \mathcal{N}$ such that $A = \bigcup_{n\in\mathbb{N}} G_{b(n)}$. We have

$$A = \bigcup_{n\in\mathbb{N}} \{(x, y) \mid (x, y) \in G_{b(n)}\} = \{(x, y) \mid (x, y, b) \in V'\}.$$

Now $V = \mathcal{N}^3 \setminus V'$ is the intended universal closed set. □

4.2. Theorem. *There is an analytic set $A \subset \mathcal{N}$ whose complement is not analytic.*

Proof. Let V be the universal closed set in $\mathcal{N}^3$ constructed in the previous lemma. Then

$$U = \mathrm{pr}_2 V = \{(x, z) \in \mathcal{N} \exists y \in \mathcal{N}((x, y, z) \in V)\} \subset \mathcal{N}^2$$

is analytic being projection of a closed set (Theorem 2.3). We claim that it is a universal analytic set, i.e. for every analytic $A \subset \mathcal{N}$ there exists $b \in \mathcal{N}$ such that $A = \{x \mid (x, b) \in U\}$.

Let $A \subset \mathcal{N}$ be any analytic set. By theorem 2.3 there exists closed $C \subset \mathcal{N}^2$ such that $A = \mathrm{pr}C$. But since V was universal, there exists $a \in \mathcal{N}$ such that

$$C = \{(x, y) \mid (x, y, a) \in V\},$$

whence

$$\mathrm{pr}C = \mathrm{pr}\{(x,y) \mid (x,y,a) \in V\} = \{x \mid \underbrace{\exists b \in \mathcal{N}(x,b,a) \in V}_{\text{condition for } U!}\} = \{x \mid (x,b) \in U\}.$$

Now let $D = \{x \mid (x,x) \in U\}$. D is analytic being a homeomorphic image of the projection of U to the diagonal. But its complement can not be analytic. Namely if the complement is analytic, then $\mathcal{N} \setminus D = \{x \mid (x,a) \in U\}$ for some a; observe a contradiction taking $x = a$. □

5. Analytic sets are Lebesgue measurable

5.1. LEMMA. *Let μ be a locally finite Borel measure on a Polish space X. Then μ is Borel regular.*

5.2. LEMMA. *Let μ be locally finite Borel measure on a Polish space X. For every set $A \subset X$ there exists a measurable set $B \supset A$ with the property that whenever $Z \subset B \setminus A$ is measurable, then $\mu(Z) = 0$.*

PROOF. By previous lemma μ is Borel regular. If $\mu(A) < \infty$, choose B such that $\mu(B) = \mu(A)$. If $\mu(A) = \infty$, then $A = \bigcup_{n\in\mathbb{N}} A_n$ with each A_n being of finite measure and $A_n \cap A_n = \varnothing$, when $n \neq m$. Choose $B_n \supset A_n$ with the inteded property and take $B = \bigcup_{n\in\mathbb{N}} B_n$. If $Z \subset B \setminus A$ is measurable, then it is a countable union of measurable $Z_n \subset B_n \setminus A_n$ which are null. □

5.3. THEOREM. *Let μ be a locally finite Borel measure on a Polish space X. Then every analytic set is μ-measurable.*

PROOF. Let A be analytic, $A = f[\mathcal{N}]$. Like in proof of theorem 3.3, define $A_s = f[O_s]$, we can now view A in the form

$$\mathcal{A}\{A_s \mid s \in \mathrm{Seq}\} = \mathcal{A}\{\overline{A_s} \mid s \in \mathrm{Seq}\} \qquad \bigstar$$

Furthermoe we know that $O_s = \bigcup_{k\in\mathbb{N}} O_{s^\frown k}$ and so $A_s = \bigcup_k A_{s^\frown k}$.

For each s, let $B_s \supset A_s$ be (by previous lemma) such that B_s is measurable and whenever $Z \subset B \setminus A$ is measurable, then $\mu(Z) = 0$. Because $\overline{A_s}$ is measurable, we can make $A_s \subset B_s \subset \overline{A_s}$. Then by $\bigstar$ $A = \mathcal{A}\{B_s \mid s \in \mathrm{Seq}\}$. Note that $A_\varnothing = A$ and $B = B_\varnothing \supset A_\varnothing$. It is now enough to show that $B \setminus A$ is null.

We claim that

$$B \setminus A = B \setminus \bigcup_{a\in\mathcal{N}} \bigcap_{n=0}^{\infty} B_{a\restriction n} \subset \bigcup_{s\in\mathrm{Seq}} \left(B_s \setminus \bigcup_{k=0}^{\infty} B_{s^\frown k} \right).$$

First equality is clear by $A = \mathcal{A}\{B_s \mid s \in \mathrm{Seq}\}$. For the second let $x \in B$ be such that x is not a member of the right-hand side. Then for every $s \in \mathrm{Seq}$, if $x \in B_s$, then $x \in B_{s^\frown k}$ for some k. Recalling that $B = B_\varnothing$ we can construct by induction a sequence $\langle k_0, k_1, \ldots \rangle$ such that $x \in B_{\langle k_0, k_1, \ldots, k_n \rangle}$ for each n and hence $x \in \bigcap_{n=0}^{\infty} B_{a \restriction n}$, where $a = \langle k_0, k_1, \ldots \rangle$ and thus x cannot be in the left-hand side.

Let us now show that each $B_s \setminus \bigcup_{k=0}^{\infty} B_{s^\frown k}$ is null. Because Seq is countable it will follow that $B \setminus A$ is null. But taking

$$Z = B_s \setminus \bigcup_{k=0}^{\infty} B_{s^\frown k} \subset B_s \setminus \bigcup_{k=0}^{\infty} A_{s^\frown k} = B_s \setminus A_s$$

and applying the definition of B_s, we see that Z is measurable and thus must be null. □

5.4. Corollary. *Analytic sets are Lebesgue-measurable.* □

Bibliography

[Jech] Jech, T., Set Theory. ISBN-10 3-540-44085-7 Springer-Verlag Berlin Heidelberg New York.