

Turbulent viscosity and Λ -effect from numerical turbulence models

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Homogeneous anisotropic turbulence simulations are used to determine off-diagonal components of the Reynolds stress tensor and its parameterization in terms of turbulent viscosity and Λ -effect. The turbulence is forced in an anisotropic fashion by enhancing the strength of the forcing in the vertical direction. The Coriolis force is included with a rotation axis inclined relative to the vertical direction. The system studied here is significantly simpler than that of turbulent stratified convection which has often been used to study Reynolds stresses. Certain puzzling features of the results for convection, such as sign changes or highly concentrated latitude distributions, are not present in the simpler system considered here.

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1 Introduction

The Reynolds stress, described by the correlation of fluctuating velocity components, $Q_{ij} = \overline{u_i u_j}$, is one of the most important generators of differential rotation in stars (Rüdiger 1989). These stresses have been studied with the help of 3D convection simulations (e.g. Pulkkinen et al. 1993; Chan 2001; Käpylä et al. 2004; Rüdiger et al. 2005). These results have revealed some surprising features such as the peaking of the horizontal stress Q_{xy} very close to the equator, and a positive (outward) flux for rapid rotation. Both of these results are at odds with theoretical considerations (Kitchatinov & Rüdiger 1993). Furthermore, disentangling of the diffusive (turbulent viscosity) and non-diffusive (Λ -effect) parts of the stress is difficult from convection simulations.

Here, we present preliminary results from anisotropic homogeneous, isothermal, non-stratified turbulence simulations in which diffusive and non-diffusive effects can be studied separately. Imposing a linear shear flow on top of isotropically driven turbulence allows the study of turbulent viscosity without Λ -effect. On the other hand, using a special form of forcing, anisotropic homogeneous turbulence can be generated. Rotation is added to study the Λ -effect. A simple analytical closure model, based on the minimal tau-approximation (hereafter MTA, see e.g. Blackman & Field 2002; Brandenburg et al. 2004), is used to compare with simulations in the cases with rotation.

2 The models

In the 3D simulations we solve the set of equations

$$\frac{D \ln \rho}{Dt} = -\nabla \cdot \mathbf{u}, \quad (1)$$

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$$\frac{D\mathbf{u}}{Dt} = -c_s^2 \nabla \ln \rho - 2 \boldsymbol{\Omega} \times \mathbf{u} + \mathbf{f}_{\text{force}} + \mathbf{f}_{\text{visc}}, \quad (2)$$

using an isothermal equation of state characterized by sound speed c_s in a fully periodic cube of volume $(2\pi)^3$. Here, $D/Dt = \partial/\partial t - \mathbf{u} \cdot \nabla$ is the advective derivative, ρ is the density, \mathbf{u} is the velocity, and $\boldsymbol{\Omega} = \Omega_0(-\sin \theta, 0, \cos \theta)^T$ is the rotation vector. By virtue of the periodic boundaries mass is conserved and the volume averaged density has a constant value of $\bar{\rho} = \rho_0$. In the present study we use an anisotropic forcing function in Fourier space according to

$$f_i^{(\text{force})} = (f_0 \delta_{ij} + f_1 \cos^2 \Theta_{\mathbf{k}} \hat{z}_i \hat{z}_j) f_i^{(\text{iso})}, \quad (3)$$

where f_0 is the amplitude of the isotropic part and f_1 the anisotropic one, $\Theta_{\mathbf{k}}$ is the angle between the vertical direction and the wave vector \mathbf{k} , and \hat{z} is the unit vector in the vertical direction. Details of the isotropic part of the forcing are given, e.g., in Brandenburg et al. (2004).

The viscous force is given by

$$\mathbf{f}_{\text{visc}} = \nu \left(\nabla^2 \mathbf{u} + \frac{1}{3} \nabla \nabla \cdot \mathbf{u} + 2 \mathbf{S} \cdot \nabla \ln \rho \right), \quad (4)$$

where ν is the viscosity and

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \quad (5)$$

is the rate of strain tensor. The simulations were made with the PENCIL-CODE¹.

In the minimal tau-approximation one solves for the time derivative of a quantity instead of the quantity itself, i.e. in the present case

$$\dot{Q}_{ij} = \overline{\dot{u}_i u_j} + \overline{u_i \dot{u}_j}, \quad (6)$$

where dots denote time derivatives and the overbars volume averages. Inserting the Navier-Stokes equations into Eq. (6)

¹ <http://www.nordita.org/software/pencil-code/>

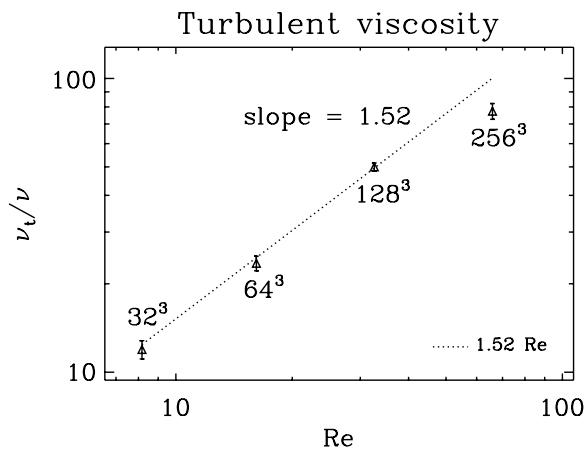


Fig. 1 The ratio of turbulent to molecular viscosity, ν_t/ν , as a function of the Reynolds number. The resolution used is denoted alongside each point. The error bars are estimated from a modified mean error of the mean; see, e.g. Eq. (30) of Käpylä et al. (2004).

and assuming a high Reynolds number so that the viscous terms can be neglected, one arrives at

$$\dot{Q}_{ij} = -2\varepsilon_{jkl}\Omega_k Q_{il} - 2\varepsilon_{ikl}\Omega_k Q_{jl} + \frac{u_i f_j + u_j f_i}{u_i f_j + u_j f_i} + T_{ij}, \quad (7)$$

where T_{ij} denotes the triple correlations. In the present case we assume that the pressure terms are subsumed in the triple correlations, which is reasonable for calculating the Λ -effect, but not valid for calculating turbulent viscosity. The basic assumption of MTA is that the triple correlations can be presented in terms of the quadratic ones

$$T_{ij} = -\tau^{-1} Q_{ij}, \quad (8)$$

where τ is a relaxation time. In the statistically steady state without rotation we have

$$Q_{ij} = Q_{ij}^{(0)} \equiv \tau (\overline{u_i f_j} + \overline{u_j f_i}), \quad (9)$$

which allows us to express the forcing in terms of the Reynolds tensor for the non-rotating case (denoted by the superscript zero). We employ the same amplitudes as those found in the 3D simulations. The only free parameter in the MTA-model is then the Strouhal number

$$St = \tau u_{\text{rms}} k_f, \quad (10)$$

where u_{rms} is the rms-value of the fluctuating component of the velocity and $k_f \approx 5$ is the mean forcing wave number. In the present study we use $St = 2$ which reproduces the same trend as a function of rotation as the 3D simulations (see Fig. 2).

3 Results

3.1 Turbulent viscosity

In order to study the turbulent viscosity, a large scale linear shear flow $\mathbf{U} = (0, Sx, 0)$ is imposed upon the system. In this system the homogeneity of the turbulence is preserved

by using the shearing box approximation (e.g. Hawley et al. 1995). The turbulent viscosity can now be computed from

$$\nu_t = -2Q_{xy}/S. \quad (11)$$

where $S = -0.1$ is used in the present study. Figure 1 shows ν_t/ν as a function of Reynolds number,

$$Re = \frac{u_{\text{rms}}}{\nu k_f}. \quad (12)$$

The absolute value of the turbulent viscosity stays almost constant as a function of Reynolds number and thus the ratio ν_t/ν increases almost linearly as a function of Re except for the largest Reynolds number run. Higher resolution simulations are needed in order to clarify the behaviour in the high Reynolds number regime.

3.2 Λ -effect

In anisotropic turbulence under the influence of rotation non-zero off-diagonal Reynolds stresses are generated according to Rüdiger (1989),

$$Q_{ij} = \Lambda_{ijk}\Omega_k + \text{diffusive terms}. \quad (13)$$

Considering the computational domain as a small rectangular part of a sphere, we note that Q_{xy} describes horizontal (latitudinal), and Q_{yz} vertical (radial) transport of angular momentum. In the present study the vertical (z) direction is taken to be the preferred one. Furthermore, we use $f_1 = 0.2$ and $f_0 = 10^{-6}$ which results in turbulence that is dominated by the z -component which, bearing in mind the geometry of the system, is also the case for convection. The ratio of vertical to horizontal turbulence intensity is $\overline{u_z^2}/\overline{u_H^2} \approx 2.2$, where $\overline{u_H^2} = (\overline{u_x^2} + \overline{u_y^2})/2$. A resolution of 32^3 was used in the present calculations with $Re \approx 7$. The Reynolds number dependence of the Λ -effect is weak if Re is sufficiently large (not shown). This issue will be studied further in a future publication.

The left hand panels of Fig. 2 show the results from direct 3D calculations. The rotational influence is quantified by the Coriolis number

$$Co = 2\Omega_0(u_{\text{rms}}k_f)^{-1}. \quad (14)$$

Note that in comparison to the commonly used definition in convection simulations, our Coriolis numbers are smaller by a factor of 2π . The main feature of the horizontal stress is that it is always positive and peaks around 30 degrees latitude for all Coriolis numbers studied so far. This result is in stark contrast to the convection calculations where Q_{xy} peaks always very near the equator (e.g. Chan 2001; Käpylä et al. 2004; Hupfer et al. 2005).

The vertical stress is predominantly negative with a maximum at the equator. Although there seems to be a regime in which the vertical stress is positive for intermediate rotation rates, the sign remains negative for the most rapid rotation cases studied so far. This feature is also at odds with convection simulations, which exhibit positive Q_{yz} for rapid enough rotation (Käpylä et al. 2004; Chan 2007, private communication).

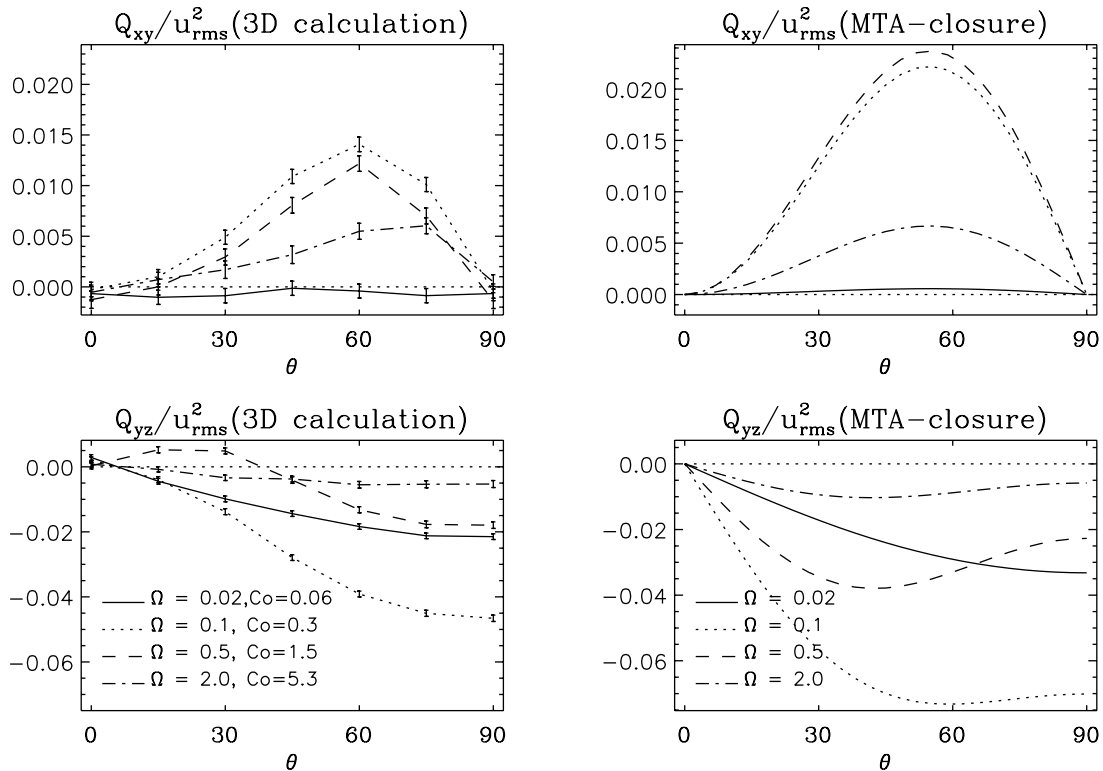


Fig. 2 The non-diffusive Reynolds stresses (i.e. Λ -effect) Q_{xy} (upper panels) and Q_{yz} (lower panels) from direct 3D simulations (left panels) and the MTA-closure (right panels) as functions of latitude and rotation rate. The error bars are defined similarly as in Fig. 1.

The MTA-model results are shown in the right panels of Fig. 2. Qualitatively the results match the numerical simulations rather well. The sign and latitude distribution of Q_{xy} is reproduced fairly well. The amplitude, however, is clearly too large. Similar conclusions can be drawn from the results for Q_{yz} , although there the latitude distribution from the closure model tends to show a persistent maximum at mid-latitudes which is not observed in the 3D simulations. Also the amplitude is too large by almost a factor of two.

A possible explanation of the discrepancies between the numerical results and the closure model is that the latter does not take isotropizing effects properly into account. This can be due to an improper treatment of the pressure terms in the closure model.

4 Conclusions

Shear flow turbulence simulations show that the ratio of turbulent to molecular viscosity increases linearly up to $Re \approx 30$ with $\nu_t/\nu \approx 1.5 Re$. For the largest Reynolds number the scaling seems somewhat shallower but the present data is not yet sufficient to substantiate this. The Λ -effect from homogeneous, anisotropic turbulence does not exhibit the puzzling features found in convection simulations. Further study is required in order to understand which of the neglected physics is responsible for the lack of these features.

The MTA-closure is able to reproduce many of the qualitative aspects of the simulation results including a maxi-

imum of the horizontal stress at about 30° latitude, with its largest value for $Co \approx 0.5$. In the model, the vertical stress can have a maximum away from the equator for $Co \gtrsim 0.2$, which is not seen in the simulations. Nevertheless, both simulations and model have the largest vertical stress for $Co \approx 0.3$. However, the model generally overestimates the magnitudes of the stresses. More detailed analysis of the simulation and closure results will be presented in a future publication.

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References

- Blackman, E.G., Field, G.B.: 2002, *PhRvL* 89, 265007
- Brandenburg, A., Käpylä, P.J., Mohammed, A.: 2004, *PhFl* 16, 1020
- Chan, K.L.: 2001, *ApJ* 548, 1102
- Hawley, J.F., Gammie, C.F., Balbus, S.A.: 1995, *ApJ* 440, 742
- Hupfer, C., Käpylä, P.J., Stix, M.: 2005, *AN* 326, 223
- Käpylä, P.J., Korpi, M.J., Tuominen, I.: 2004, *A&A* 422, 793
- Kitchatinov, L.L., Rüdiger, G.: 1993, *A&A* 276, 96
- Pulkkinen, P., Tuominen, I., Brandenburg, A., Nordlund, Å., Stein, R.-F.: 1993, *A&A* 267, 265
- Rüdiger, G.: 1989, *Differential rotation and Stellar Convection: The Sun and Solar-type Stars*, Pergamon Press, Oxford
- Rüdiger, G., Egorov, P., Ziegler, U.: 2005, *AN* 326, 315