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Estimates of the Strouhal number from numerical models of convection

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Abstract. We determine the Strouhal number (hereafter St), which is essentially a nondimensional measure of the correlation time, from numerical calculations of convection. We use two independent methods to estimate St. Firstly, we apply the minimal tau-approximation (MTA) on the equation of the time derivative of the Reynolds stress. A relaxation time is obtained from which St can be estimated by normalising with a typical turnover time. Secondly, we calculate the correlation and turnover times separately, the former from the autocorrelation of velocity and the latter by following test particles embedded in the flow. We find that the Strouhal number is in general of the order of 0.1 to 1, i.e. rather large in comparison to the typical assumption in the mean-field theories that $\mathrm{St} \ll 1$. However, there is a clear decreasing trend as function of the Rayleigh number and increasing rotation. Furthermore, for the present range of parameters the decrease of St does not show signs of saturation, indicating that in stellar convection zones, where the Rayleigh numbers are much larger, the Strouhal number may indeed be significantly smaller.

Key words: convection – dynamo theory

1. Introduction

The mean-field theories of hydromagnetic dynamos and angular momentum transport need knowledge of turbulent correlations, namely the electromotive force and Reynolds stresses, respectively. The explicit calculation of these is virtually impossible for any astrophysical conditions due to the lack of necessary computational cababilities. Therefore the small-scale effects are usually parametrised by transport coefficients which relate the turbulent correlations to the mean quantities (e.g. Steenbeck, Krause & Rädler 1969; Rüdiger 1989). However, in order to calculate the transport coefficients, knowledge of the small-scale velocities and magnetic fields are still needed. Thus, simplifying assumptions such as the first order smoothing approximation (hereafter FOSA) have been used (Steenbeck et al. 1969). Although the results obtained with FOSA are in agreement with observations in many cases, the basic assumptions of the theory have rarely been thoroughly studied (however, see Petrovay & Zsargó 1998; Käpylä et al. 2004b).

The validity condition for FOSA is that either the Reynolds or the Strouhal number is small. The former condition is hardly ever met in astrophysics, but of the latter in the context of convection and dynamo theory no systematic study seems to exist. A further notion associated with the Strouhal number is that even though St may not be small in the sense that $\mathrm{St} \ll 1$, it is still possible to calculate the transport coefficients from a cumulative series expansion if $\mathrm{St} < 1$ (e.g. Knobloch 1978; Nicklaus & Stix 1988).

Recently it has been shown numerically that St can *exceed* unity for forced turbulence (Brandenburg, Käpylä & Mohammed 2004; Brandenburg & Subramanian 2004) in the contexts of passive scalar diffusion and mean-field dynamos, respectively. In these studies, instead of FOSA, the minimal tau-approximation (hereafter MTA) is found to be in better agreement with the calculations. In MTA, instead of the turbulent correlation itself, the time derivative is investigated, and the higher order terms are parameterised by a term which is just the original turbulent correlation divided by a relaxation time (Blackman & Field 2003). Furthermore, interpreting the relaxation time as the correlation time of the turbulence the Strouhal number can be estimated.

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The forced turbulence results raise the question of the value of St for convection, which seems to be rather badly known. For example, solar surface observations indicate that the lifetime and turnover time of granules is approximately the same, yielding $\mathrm{St} \approx 1$ (e.g. Stix 2002). However, this result may not be relevant for the solar dynamo which is working in the deeper layers. In the present study we estimate the Strouhal number for numerical convection by two independent methods. Firstly we apply the MTA on the equation of the Reynolds stresses, and secondly we calculate the correlation and turnover times separately directly from the flow. The correlation time is estimated from the autocorrelation of velocity and the turnover time by following test particles embedded in the flow. The computational model is a Cartesian box situated at a latitude Θ on a star. The model is described in detail in Käpylä, Korpi & Tuominen (2004a).

The remainder of the paper is organised as follows: in Sect. 2 the two methods used to estimate the Strouhal number are discussed. Sects. 3 and 4 give the results and conclusions, respectively.

2. The Strouhal number

2.1. Minimal tau-approximation

We apply the MTA on the time derivative of the Reynolds stress, $Q_{ij} = \langle u_i' u_j' \rangle$, where the brackets denote horizontal averaging and primes the fluctuation. We arrive at the equation

$$\frac{\partial Q_{ij}}{\partial t} = \Psi_{ijk}\Omega_k - \frac{Q_{ij}}{\tau_{\rm rel}} \;, \tag{1}$$

where

$$\Psi_{ijk} = -2\left(\epsilon_{ikl}\langle u_j' u_l' \rangle + \epsilon_{jkl}\langle u_i' u_l' \rangle\right) \;, \tag{2}$$

where ϵ_{ikl} is the Levi-Civita symbol and Ω the rotation vector. Taking into account only the lowest order effect for the vertical Λ-effect, which is proportional to the component $Q_{\rm yz}$ in the local convection model, the relaxation time turns out to be

$$\tau_{\rm rel} = \frac{Q_{yz}}{2\,\Omega\cos\Theta(Q_{zz} - Q_{yy})} \;. \tag{3}$$

Interpreting $\tau_{\rm rel}$ as the correlation time of the turbulence, the Strouhal number can be calculated identically as in the forced turbulence studies (Brandenburg et al. 2004; Brandenburg & Subramanian 2004)

$$\mathrm{St}^{(\rm rel)} = k_{\rm f} u_{\rm rms} \tau_{\rm rel} \;, \tag{4}$$

where $k_{\rm f}$ is the wavenumber of the energy carrying scale and $u_{\rm rms}$ the average rms-velocity in the convectively unstable region. For convection calculations $k_{\rm f}$ corresponds essentially to the largest possible scale permitted by the box dimensions.

2.2. Determination of the timescales

The disadvantage of the MTA-approach is that it can only be applied to the case where rotation and the Reynolds stresses are statistically nonzero. In order to circumvent this problem, we have devised an independent way to estimate St which works also if rotation is not present and without any recourse to the mean-field theory. The method essentially consists of the calculation of the correlation and turnover times separately and directly from the flow.

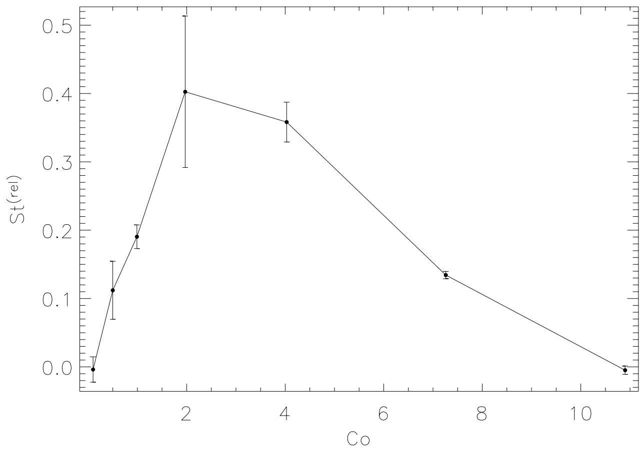

Fig. 1. The Strouhal number from MTA.

The correlation time is estimated from the autocorrelation of velocity

$$C[u_i(\boldsymbol{x},t_0), u_i(\boldsymbol{x},t)] = \frac{u_i(\boldsymbol{x},t_0)u_i(\boldsymbol{x},t)}{\sqrt{u_i^2(\boldsymbol{x},t_0)u_i^2(\boldsymbol{x},t)}} \;, \tag{5}$$

where $\boldsymbol{x}$ is the position vector and t_0 and t denote the times from which the snapshots were taken. The correlation time $\tau_{\rm c}$ is defined as the time after which the correlation drops below a threshold value. In the present study the threshold is set to 0.5. We use the vertical velocity component to determine the correlation time.

The turnover time, $t_{\rm to}$, is estimated from the trajectories of test particles which are advected by the flow. The turnover time can be defined as the time which passes between two consecutive crossovers of some fixed reference level into the same direction. However, this definition implicitly assumes that the vertical scale of convective motions is of the order of the depth of the convectively unstable layer. Thus, turnovers happening far away in comparison to the scale of convection are not registered at all. Another possibility is to define the turnover time as the time which elapses between two consecutive changes of direction (into the same direction). This latter definition registers all turnovers and we shall present results using that in the remainder of the paper. Differences between the results obtained with the two definitions are discussed further in Käpylä et al. (2004b).

Once the correlation and turnover times have been determined, the Strouhal number is simply their ratio

$$\mathrm{St} = \frac{\tau_{\rm c}}{t_{\rm to}} \;. \tag{6}$$

3. Results

The minimal tau-approximation can be applied to the Reynolds stresses only if there is appreciable rotation, and away from the poles where the stresses vanish due to symmetry in the present geometry (e.g. Käpylä et al. 2004a). A

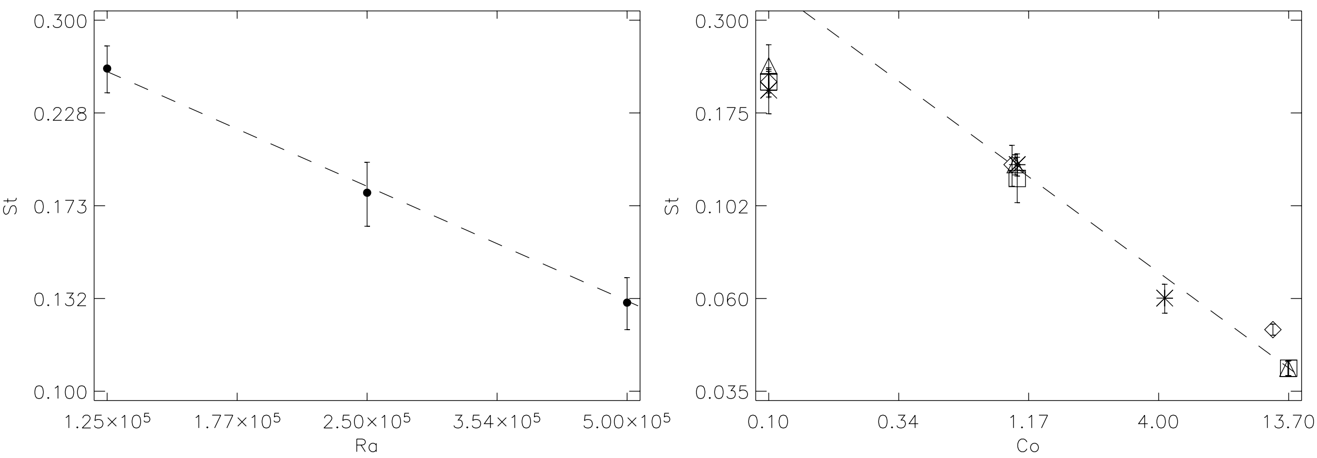

Fig. 2. The Strouhal number as function of the Rayleigh (left) and Coriolis (right) numbers. The dashed lines give the power laws $\mathrm{St} \propto \mathrm{Ra}^{-0.49}$ and $\mathrm{St} \propto \mathrm{Co}^{-0.45}$

typical result is shown in Fig. 1. The analysed calculations are those made at a latitude 30 degrees south from Käpylä et al. (2004a). The stress is taken from the middle of the convectively unstable layer which generally quite well describes the situation in the whole box. The error bars denote the modified mean error of the stress (see eq. (30) of Käpylä et al. 2004a). As function of the Coriolis number the Strouhal number is essentially determined by the value of $\tau_{\rm rel}$, which varies in a very similar manner as the vertical Λ-effect. The reason for this behaviour is that $u_{\rm rms}$ and the diagonal Reynolds stresses Q_{yy} and Q_{zz} vary only little as function of Co leading to the fact that the functional form for the relaxation time and the Λ-effect is essentially the same, Q_{yz}/Ω. The values of $\mathrm{St}^{(\rm rel)}$ vary from about 0.4 for Co = 2 to essentially zero for very slow and very rapid rotation. However, the decrease as function of Co can be at least partly due to the fact that higher order terms in eq. (2) were omitted. These terms could be taken into account by a factor $f(\Omega)$ in eq. (2), which is thought to be a decreasing function of rotation (e.g. Kitchatinov & Rüdiger 1993), and which would mean that the quantity plotted in Fig. (1) is actually $f(\Omega)\mathrm{St}$. This would mean that the Strouhal numbers for rapid rotation are probably underestimated by the present results.

Whereas the applicability of the MTA-approach is limited to cases with rotation, the Strouhal number should have a finite value also without rotation. Thus, we set out to extract the correlation and turnover times separately from the flow as described in Sect. 2.2. We find that the the correlation time decreases consistent with $\tau_{\rm c} \propto \mathrm{Ra}^{-0.45}$ as function of the Rayleigh number. The turnover time, however, changes only marginally. Thus the Strouhal number follows approximately a power law $\mathrm{St} \propto \mathrm{Ra}^{-0.49}$ (see left panel of Fig. 2). Although the parameter range in the present study is quite limited, the result is still promising in the sense that if the same trend carries over to stellar parameters, the Strouhal number may be much smaller there.

A similar trend is found when rotation is increased. The correlation time decreases rapidly as function of the Coriolis number whereas the changes in the turnover time are only minor. However, one aspect not to be overlooked here is the trend seen in the turnover time. The simple estimate, the depth of the convectively unstable region divided by the average velocity, increases as function of rotation due to the fact that overall velocities tend to diminish as rotation becomes more rapid. However, the turnover time calculated from the test particle trajectories shows an opposite trend due to the fact that the spatial scale of convection is reduced even more than the overall velocities. Thus we find that for moderate and rapid rotation, the Strouhal number approximately follows a power law $\mathrm{St} \propto \mathrm{Ra}^{-0.45}$ (see right panel of Fig. 2). The explanation is most probably the strong Coriolis forces which tend to disrupt any coherent flow structures.

4. Conclusions

We estimate the Strouhal number from numerical models of convection with two independent methods. Firstly, we apply the minimal tau-approximation of the equation of the Reynolds stress. Secondly, we calculate the correlation and turnover times directly from the flow without any recourse to a mean-field theory.

We find that the Strouhal number from the MTA reaches values of maximally ≈ 0.5. St has a maximum for intermediate rotation, where the Reynolds stress itself also peaks. The Strouhal number follows closely the same trend as the vertical Λ-effect (see e.g. Käpylä et al. 2004a) as function of rotation due to the similar functional form.

The correlation time is seen to decrease consistent with power law $\tau_{\rm c} \propto \mathrm{Ra}^{-0.45}$ as function of the Rayleigh number and approximately with $\tau_{\rm c} \propto \mathrm{Co}^{-0.45}$ for moderate and rapid rotation. These results indicate that although the values of St in this study are generally of the order of 0.1 to 1, the Strouhal number in stellar convection zones, where Ra is most certainly, and Co at least probably, much larger than in the present study, may be significantly smaller.

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