5d Einstein gravity + KK compactification:

$(x^0, \vec{x}, x^4 \equiv y)$ — don't put simply $g_{M4} = 0$!

Ansatz: hep-ph/0210992

$$g_{MN} = \left(\begin{array}{c|c} g_{\mu\nu} + \varphi A_\mu A_\nu & \phi A_\mu \\ \hline \phi A_\mu & \phi \end{array}\right)$$

← keep the scalar! lots of them will appear in 10d ⇒ "landscape"

$\eta = - + + + +$

Take: $g_{\mu\nu}(x)$ $A_\mu(x)$ $\phi(x)$ no y-dep!!

$$\Rightarrow g^{MN} = \left(\begin{array}{c|c} g^{\mu\nu} & -A^\mu \\ \hline -A^\mu & \frac{1}{\varphi} + A^2 \end{array}\right) \qquad g_5 = g_4 \phi$$

$$g_{MK}\, g^{KN} = \left(\begin{array}{c|c} g_{\mu\alpha} + \phi A_\mu A_\alpha & \phi A_\mu \\ \hline \phi A_\alpha & \phi \end{array}\right)\left(\begin{array}{c|c} g^{\alpha\nu} & -A^\nu \\ \hline -A^\nu & \frac{1}{\varphi} + A^2 \end{array}\right)$$

$$= \left(\begin{array}{c|c} g_\mu{}^\nu + \phi A_\mu A^\nu - \phi A_\mu A^\nu & -(g_{\mu\alpha} + \phi A_\mu A_\alpha) A^\nu + A_\alpha (1 + \phi A^2) \\ \hline \phi A^\nu - \phi A^\nu & -\phi A^2 + 1 + \phi A^2 \end{array}\right) = \delta_K^N$$

$$ds^2 = (g_{\mu\nu} + \varphi A_\mu A_\nu)\, dx^\mu dx^\nu + 2\varphi A_\mu\, dx^\mu dy + \varphi\, dy^2$$

$$= g_{\mu\nu}\, dx^\mu dx^\nu + \varphi\, (dy + \underbrace{A_\mu dx^\mu}) ^2 \quad \Rightarrow F_{\mu\nu} \text{ also appears!}$$

note appearance of $dA = A_\mu dx^\mu$!!

Calculation (non-trivial!) gives

$$R_5 = R_4 - \frac{1}{4}\varphi F^{\mu\nu}F_{\mu\nu} - \frac{1}{\varphi}\nabla^2\phi + \frac{1}{2\phi^2}\nabla_\mu\phi \cdot \overbrace{\nabla^\mu\phi}^{\partial^\mu\phi}$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\alpha_{\mu\nu} V_\alpha \qquad \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha$$

$$\nabla_\mu \nabla^\mu \phi = \nabla_\mu \partial^\mu \phi = \partial_\mu \partial^\mu \phi + \underbrace{\Gamma^\mu_{\mu\alpha}}_{\frac{1}{\sqrt{g}}\partial_\alpha \sqrt{g}} \partial^\alpha \phi = \frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}\,\partial^\alpha \phi)$$

take $dy = a\, d\varphi$

$$S = + M_*^3 \int d^4x \overbrace{\int dy}^{2\pi a} \sqrt{g_5}\,(R_5 - 2\Lambda) \qquad \frac{1}{2} M_{Pl}^2 = M_*^3 \cdot 2\pi a \equiv \frac{1}{16\pi G_N}$$

$$= +\frac{1}{2}M_{Pl}^2 \int d^4x \sqrt{g_4}\sqrt{\varphi_m}\left(R_4 - \frac{1}{4}\varphi F_{\mu\nu}F^{\mu\nu} - \underbrace{2\nabla^2\sqrt{\varphi}}_{\text{total derivative}} - 2\Lambda\right)$$

↑ this sign must be $-$; $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = +\frac{1}{2}(\vec{E}^2 - \vec{B}^2)$

The result is now in the "string frame" with a seemingly x-dependent G_N:

$$\int d^4x\, \sqrt{g_4}\, \frac{\sqrt{\varphi(x)}}{16\pi G_N}\, R_4 \cdots$$

i.e., a Brans-Dicke type gravity theory

$$S_{B\text{-}D} = \int d^4x\, \sqrt{g_4}\, \Big[f(\varphi)\, R + \underbrace{\tfrac{1}{2}\, \nabla^\mu \phi\, \nabla_\mu \phi - V(\varphi)}_{\text{massless scalar}} \Big]$$

coupled to gravity

⌈ Expect $\Box \phi = \lambda\, T^\mu{}_\mu \underset{\substack{\text{estimate} \\ \Box \sim \frac{1}{R_0^2}}}{\Rightarrow} \phi \sim \lambda R_0^2 \underbrace{\rho_0}_{\sim \rho_c \sim \frac{H^2}{G}} \sim \lambda \cdot \frac{(R_0 H)^2}{G} \sim \frac{\lambda}{G}$

so $G \sim \frac{1}{\phi}$ $\quad \frac{1}{G} \sim \phi$

In detail, one form of B-D is "Jordan conformal frame"

$$S_{B\text{-}D} = \frac{1}{16\pi} \int d^4x\, \sqrt{g_4} \Big[\phi R + \frac{\omega}{\phi}\, \nabla^\mu \phi\, \nabla_\mu \phi + \mathcal{L}_{matter} \Big]$$

$$\Rightarrow \begin{cases} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \frac{1}{\phi} T_{\mu\nu} + \phi\text{-terms} \\ \Box \phi = \frac{8\pi}{2\omega + 3}\, T^\mu{}_\mu \end{cases}$$

$(\phi = 1/G)$

$\Rightarrow$ usual GR if $\omega \to \infty$

($\omega > 5 \cdot 10^4$ from data)

Extremely popular with varying G (and α, etc) ⌋

But one can also perform a conformal transformation and transform away $\sqrt{\varphi}$:

$\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu} \quad \Rightarrow \sqrt{g} \to \sqrt{\omega^{2d}} = \omega^d \sqrt{g}$

$\tilde{g}^{\mu\nu} = \omega^{-2} g^{\mu\nu} \qquad R = g^{\mu\nu} g^{\mu\nu} \partial\partial g_{\mu\nu} \to \omega^{-2} R$ + corr's

$$\sqrt{\varphi}\sqrt{g}\, R \to \sqrt{\varphi}\, \omega^{d-2} R = \sqrt{\varphi}\, \omega^2 R = R$$

$$\text{if } \omega^2 = \frac{1}{\sqrt{\varphi(x)}}$$

Conformal transformation formulas have been discussed many times:

$$\begin{cases} R \to \omega^{-2} R - 2(d-1)\,\omega^{-3}\,\nabla^2\omega + (d-4)\text{-term} & d=4 \\ \nabla^2\varphi \to \omega^{-2}\nabla^2\varphi + (d-2)\,\omega^{-3}\,\nabla^\alpha\omega\,\nabla_\alpha\varphi & \omega^2 = \varphi^{-1/2} \\ \sqrt{g_4} \to \omega^4\sqrt{g_4} = \varphi^{-1}\sqrt{g_4} & \omega = \varphi^{-1/4} \end{cases}$$

$$S \to \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{g_4}\; \varphi^{-\frac{1}{2}} \Big\{ \varphi^{\frac{1}{2}} R - 6\varphi^{3/4} \underbrace{\nabla^2 \varphi^{-1/4}} - \frac{1}{4}\varphi\cdot \underset{\substack{\uparrow \\ \tilde{g}^{\mu\alpha}\tilde{g}^{\nu\alpha}}}{\varphi} F_{\mu\nu}F^{\mu\nu} - 2\Lambda \Big\}$$

$$-\frac{1}{4}\partial^\mu(\varphi^{-\frac{5}{4}}\partial_\mu\varphi)$$
$$= \frac{5}{16}\varphi^{-\frac{9}{4}}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{4}\varphi^{-\frac{5}{4}}\partial^2\varphi$$

$$S = \frac{1}{2} M_{Pl}^2 \int d^4x\sqrt{g_4}\Big\{ R - \frac{1}{4}\varphi^{\frac{3}{2}} F_{\mu\nu}F^{\mu\nu} + \frac{3}{2}\underbrace{\varphi^{-1}\partial^2\varphi} - \frac{15}{8}\varphi^{-2}\partial^\mu\varphi\partial_\mu\varphi - \varphi^{-1/2}\cdot 2\Lambda\Big\}$$

$$\underbrace{\hat{=} -\partial^\mu\varphi^{-1}\cdot\partial_\mu\varphi = \varphi^{-2}}\,\partial^\mu\varphi\cdot\partial_\mu\varphi$$

$$\hat{=} -\frac{3}{8}\varphi^{-2}\partial^\mu\varphi\partial_\mu\varphi$$

$\Longrightarrow$ EOM (string frame!)

$$\boxed{\begin{cases} R_{\mu\nu} = \frac{1}{2}\varphi\, F^\alpha{}_\mu F_{\alpha\nu} \\ \nabla^\mu F_{\mu\nu} = -\frac{3}{2\varphi} F_{\mu\nu}\nabla^\mu\varphi \\ \nabla^2\varphi^{1/2} = \frac{1}{4}\varphi^{3/2} F^{\mu\nu}F_{\mu\nu} \end{cases}}$$

$\varphi = $ const demands $F^2 = 0$!!

Final step: want for small fields

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi$$

= "canonical normalisation of kinetic terms"

$$-\frac{1}{2}M_{Pl}^2\,\frac{3}{8}\varphi^{-2}\partial^\mu\varphi\,\partial_\mu\varphi = -\frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi$$

if $\varphi = e^{\sqrt{\frac{8}{3}}\phi/M_{Pl}}$

$$-\frac{1}{4}\, e^{\sqrt{6}\phi/M_{Pl}}\, \underbrace{\frac{1}{2}M_{Pl}^2}\, F_{\mu\nu}^2$$

absorb this by $A_\mu \frac{M_{Pl}}{\sqrt{2}} \to A_\mu$

$$-\varphi^{-1/2}M_{Pl}^2\Lambda \Rightarrow -e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\, M_{Pl}^2\Lambda$$

$\boxed{-+++}$

$$\Rightarrow \boxed{S = \int d^4x \sqrt{g_4} \left\{ \frac{1}{2} M_{Pl}^2 R - \frac{1}{4} e^{\sqrt{6}\phi/M_{Pl}} F^{\mu\nu}F_{\mu\nu} - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - \underbrace{M_{Pl}^2 \Lambda}_{\approx T_{00}^{vac}} e^{-\sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}}} \right\}}$$

On signs:

(1) $-\frac{1}{4} F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}(F^{0i}F_{0i} + F_{ij}^2) = +\frac{1}{2}(\bar{E}^2 - \bar{B}^2)$ both for $+---$ and $-+++$

$$\underbrace{g^{\mu\alpha} g^{\nu\beta}}_{2\ g's} F_{\mu\nu} F_{\alpha\beta}$$

(2) $\frac{1}{2} \partial^\mu \phi \partial_\mu \phi \overset{+---}{=} \frac{1}{2}\left[(\partial_0\phi)^2 - (\bar{\nabla}\phi)^2\right] \Rightarrow \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi} \partial_0 \varphi - \mathcal{L} = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\bar{\nabla}\phi)^2 + V(\phi)$

$-V(\phi)$

$g^{\mu\alpha} \partial_\alpha \phi \partial_\mu \phi$
↑
$1\ g$

$\mathcal{L} = +\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V \qquad +---$

$\quad -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V \qquad -+++ \quad$ (as above)

(3) $\int d^4x \sqrt{g_4}\ \frac{1}{2} M^2 (+R - 2\Lambda) \Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} =$ "$8\pi G T_{\mu\nu}^{vac}$"

$-+++ \qquad T_{00}^{vac} = +\frac{\Lambda}{8\pi G} \sim \Lambda M_{Pl}^2$

$\Rightarrow$ with these conv's $\begin{cases} \Lambda > 0 & dS \\ \Lambda < 0 & AdS \end{cases}$

$\Rightarrow$ EOM "Einstein frame"

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{M_{Pl}^2} \left(e^{\sqrt{6}\frac{\phi}{M}} T_{\mu\nu}^{elmag} + T_{\mu\nu}^{\phi} \right) \\ \nabla^\nu F_{\mu\nu} = -\frac{\sqrt{6}}{M} F_{\mu\nu} \nabla^\nu \phi \\ \nabla^2 \phi = \sqrt{\frac{3}{8M^2}}\ e^{\sqrt{6}\frac{\phi}{M_P}} F_{\mu\nu} F^{\mu\nu} - \sqrt{\frac{2}{3}} M_P \Lambda e^{-\sqrt{\frac{2}{3}} \frac{\phi}{M}} \end{cases}$$

Time dependent vacuum energy
$\Rightarrow$ <u>quintessence</u>

All this interesting structure came simply from reducing 5d gravity to 4d à la KK neglecting y dependence. $d\tilde{s} = \sqrt{\rho}\, dy = e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}} dy$

$\Rightarrow$ Tower of masses $m_n = \frac{n}{a} e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}} \qquad n = 0, 1, 2, \ldots$