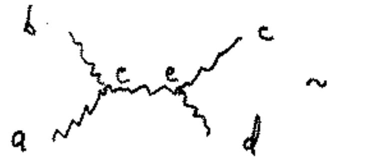
Gluon tree amplitudes

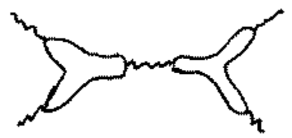
$$T_a T_b T_c - T_a T_c T_b$$

Color:  $\sim f_{abc} = -i \frac{1}{T_F} \mathrm{Tr}(T_c T_a T_b - \mathrm{Tr}\, T_c T_b T_a)$

$$\mathrm{Tr}\, T_c \,\Big|\, [T_a, T_b] = i f_{abd} T_d$$

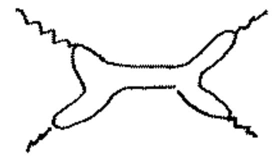
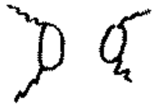
$$\mathrm{Tr}(T_c T_a T_b - T_c T_b T_a) = i f_{abd} \underbrace{\mathrm{Tr} T_c T_d}_{T_F \delta_{cd}}$$

 $\sim \mathrm{Tr}(T_a T_b T_e - T_b T_a T_e)\, \mathrm{Tr}(T_e T_c T_d - T_e T_d T_c)$

$=$  (To derive write for any herm matrix $M = m_0 1 + \sum_1^{N^2-1} m_a T_a$)

But here  $T^e_{ij} T^e_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$

$=$ 

$=$  $+$  $+$ perms

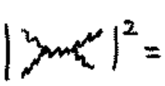
$\Rightarrow$ general tree color structure is ( $= p_i \; \varepsilon_i \; a_i$ color)

$$A_n((p_1, \varepsilon_1, a_1), \dots, (p_n\, \varepsilon_n\, a_n))$$

$$= g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T_{a_{\sigma(1)}} \cdots T_{a_{\sigma(n)}})\, A_n(\sigma(p_1\, \varepsilon_1), \dots, \sigma(p_n\, \varepsilon_n))$$

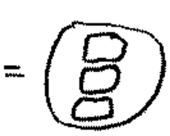
distinct cyclic orderings

contrib. only from particular cyclic ordering of gluons

Ex Directly $\Big|$  $\Big|^2 =$  $= \mathrm{Tr} F_a F_b \cdot \mathrm{Tr} F_a F_b = N_c \delta_{ab} N_c \delta_{ab} = N_c^2 (N_c^2 - 1)$

$(F_a)_{cd} = -i f_{abc}$

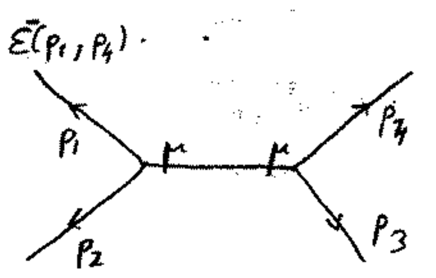
or as above

$\Big|$  $+ \cdots \Big|^2 =$  $=$  $=$  $\sim N_c^4 + \cdots$

Example

Dixon, TASI 96

$p_1 = \lambda_1 \tilde{\lambda}_1$ etc



$\varepsilon^- \sim \frac{\lambda \tilde{\mu}}{[\tilde{\lambda}\tilde{\mu}]}$

$\varepsilon^+ \sim \frac{\mu \tilde{\lambda}}{\langle \mu \lambda \rangle}$

$\varepsilon_2^- \equiv \varepsilon^-(p_2, p_4)$

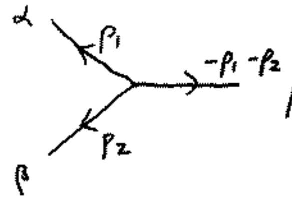
$$\varepsilon^{(-)}(p_1 p_4) = \frac{\lambda_1 \tilde{\lambda}_4}{[\tilde{\lambda}_1 \tilde{\lambda}_4]} \qquad \varepsilon^{(-)}(2,4) = \frac{\lambda_2 \tilde{\lambda}_4}{[2,4]} \quad \leftarrow p_4 \text{ gauge vector}$$

$$\varepsilon^{(+)}(3\,1) = \frac{\lambda_1 \tilde{\lambda}_3}{\langle 13 \rangle} \qquad \varepsilon^{(+)}(4,1) = \frac{\lambda_1 \tilde{\lambda}_4}{\langle 14 \rangle} \quad \leftarrow p_1 \text{ -- '' --}$$

$$\varepsilon_1^- \varepsilon_2^- = \varepsilon_1^- \varepsilon_3^+ = \varepsilon_1^- \varepsilon_4^+ = 0$$

$$\boxed{\varepsilon_2^- \cdot \varepsilon_3^+ \neq} \qquad \varepsilon_2^- \cdot \varepsilon_4^+ = 0$$

the only non-zero! $\quad \varepsilon_3^+ \cdot \varepsilon_4^+ = 0$



$$= i \left[g_{\alpha\beta} (p_1 - p_2)_\mu + g_{\beta\mu} (p_2 + p_1 + p_2)_\alpha + g_{\mu\alpha} (-p_1 - p_2 - p_1)_\beta \right]$$

$$\Rightarrow i \left[\varepsilon_{2\mu}^- (2p_2 + p_1) \cdot \varepsilon_1^- + \varepsilon_{1\mu}^- (-2p_1 - p_2) \cdot \varepsilon_2^- \right]$$

$$* \frac{-i}{s_{12}} *$$ 

$$\Rightarrow A(1^- 2^- 3^+ 4^+) = -\frac{i}{s_{12}} \quad \varepsilon_2^- \cdot \varepsilon_3^+ \quad \varepsilon_1^- \cdot p_2 \quad \varepsilon_4^+ \cdot p_3$$

$s_{12} = 2 p_1 \cdot p_2 = \lambda_1 \tilde{\lambda}_1 \lambda_2 \tilde{\lambda}_2 = \langle 12 \rangle [12]$

$= s_{34} = \langle 34 \rangle [34]$

$$\frac{\lambda_2 \tilde{\lambda}_4}{[24]} \frac{\lambda_1 \tilde{\lambda}_3}{\langle 13 \rangle} \qquad \frac{\lambda_1 \tilde{\lambda}_4}{[14]} \lambda_2 \tilde{\lambda}_2 \qquad \frac{\lambda_1 \tilde{\lambda}_4}{\langle 14 \rangle} \lambda_3 \tilde{\lambda}_3$$

$$= \frac{-i}{\cancel{\langle 12 \rangle}[12]} \qquad = \frac{\langle 21 \rangle [43]}{\cancel{[24]} \cancel{\langle 13 \rangle}} \qquad = \frac{\cancel{\langle 12 \rangle} \cancel{[42]}}{[14]} \qquad = \frac{\cancel{\langle 13 \rangle} [43]}{\langle 14 \rangle}$$

$$= (-i) \frac{\langle 21 \rangle}{[12]} \frac{[43]^2 (-)}{[14] \langle 14 \rangle} = -i \frac{\langle 12 \rangle [34]^2}{[12][14]\langle 14 \rangle}$$

$$\lambda_s \mid \lambda_1 \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3 + \lambda_4 \tilde{\lambda}_4 = 0 \mid \tilde{\lambda}_r$$

$$\langle ss \rangle = [rr] = 0 \Big) \; \langle s1 \rangle [1r] + \langle s2 \rangle [2r] + \langle s3 \rangle [3r] + \langle s4 \rangle [4r] = 0$$

$$\langle 21 \rangle [14] + \langle 23 \rangle [34] = 0$$

$$A(1^- 2^- 3^+ 4^+) =$$

$$= -i \underbrace{\frac{\langle 12\rangle [34]^2}{[12][14]\cancel{\langle 14\rangle}} \cdot \frac{\langle 23\rangle\langle 34\rangle \cancel{\langle 41\rangle}}{\langle 12\rangle^3}}_{i\,\frac{[34][34]\langle 34\rangle\langle 23\rangle}{[12]\langle 12\rangle\langle 12\rangle[14]}} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$\frac{1}{2}s = p_1\cdot p_2 = p_3\cdot p_4$

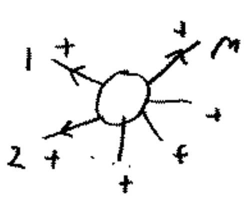
$\frac{1}{2}t = p_1\cdot p_4 = p_2\cdot p_3$

$\frac{1}{2}u = p_1\cdot p_3 = p_2\cdot p_4$

$$\frac{\langle 23\rangle[34]}{\langle 12\rangle[14]} = 1 \; ! \qquad |A|^2 = \frac{(p_1\cdot p_2)^4}{p_1\cdot p_2\; p_2\cdot p_3\; p_3\cdot p_4\; p_4\cdot p_1} = \frac{s^4}{t^2u^2}$$

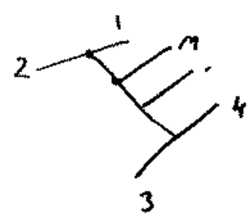
$$\boxed{A(1^- 2^- 3^+ 4^+) = i\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}}$$

<u>Note</u>



some ref. null gauge vector $\neq p_i$

Choose $\varepsilon_i^+ = \frac{\mu\,\tilde\lambda_i}{\langle \mu \lambda_i\rangle}$ $\quad p_i = \lambda_i \tilde\lambda_i$



$\Rightarrow$ all scalar prod's vanish ($\langle\mu\mu\rangle = 0$)

at most $m-2$ vertices $\succ$ $g_{\mu\nu} p_\alpha$

$\Rightarrow$ each vertex $\to$ one momentum

$\Rightarrow$ $m-2$ momenta to contract with m ε_i^+'s

$\Rightarrow$ at least one $\varepsilon_k^+\cdot\varepsilon_l^+ = 0 \Rightarrow A_m(++\cdots++) = 0$!

Change $1^+ \Rightarrow 1^-$ and take $\varepsilon_1^- = \frac{\lambda_1\tilde\lambda_m}{[\tilde 1\,\tilde m]}$

gauge vector p_m

$\varepsilon_2^+ = \frac{\lambda_1\tilde\lambda_2}{\langle 12\rangle}$

$\vdots$

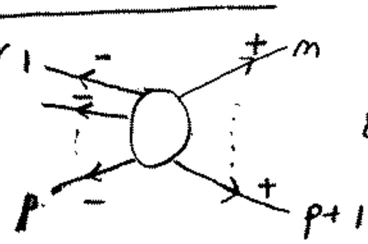
gauge vector p_1 $\quad \varepsilon_m^+ = \frac{\lambda_1\tilde\lambda_m}{\langle 1m\rangle}$

again all $\varepsilon\cdot\varepsilon = 0$!

$A_m(-++\cdots+) = 0$!

General: $(p+1)$ = gauge vector

$\varepsilon_i^{(-)} = \frac{\lambda_i\tilde\lambda_{p+1}}{[i\;p+1]}$



p_1 gauge vector only

$\varepsilon_i^{(+)} = \frac{\lambda_1\tilde\lambda_i}{\langle 1i\rangle}$

only $(p-1)\cdot(m-p-1)$ $\varepsilon^{(-)}\cdot\varepsilon^{(+)'} \neq 0$

Even after color stri~~pping~~ lots of amps which interfere upon squaring!

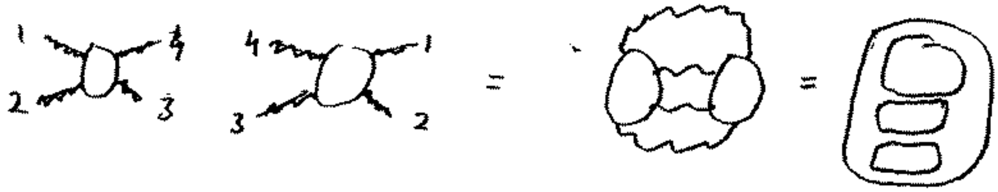
$$|\langle 12\rangle|^2 = \lambda_1^{a}\lambda_{2a}\,\tilde\lambda_1^{\dot a}\tilde\lambda_{2\dot a} = \lambda_1^{a}\tilde\lambda_1^{\dot a}\,\lambda_{2a}\tilde\lambda_{2\dot a} = p_1^{a\dot a}\,p_{2a\dot a} = 2p_1\cdot p_2$$

$$\langle 12\rangle\langle 13\rangle^* = \quad = \lambda_1^{a}\tilde\lambda_1^{\dot a}\,\lambda_{2a}\tilde\lambda_{3\dot a} = p_1^{a\dot a}\,\lambda_{2a}\tilde\lambda_{3\dot a}$$

etc

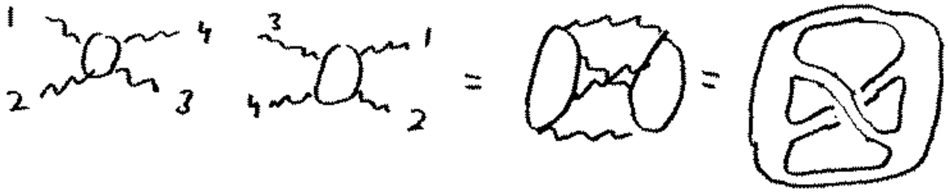
Colour contractions

$(1\ 2\ 3\ 4)\,(1\ 2\ 3\ 4)^*$



$= N_c^4$

$(1\ 2\ 3\ 4)\ (1\ 2\ 4\ 3)^*$

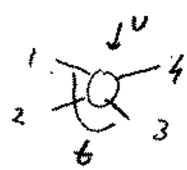


$= N_c^2$

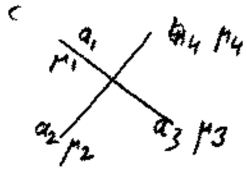
etc

Putting all pieces together $\left(\frac{1}{g_{\text{fin in}}}\sum_{\text{in}}\sum_{\text{fin}}|\ \ |^2\right)$ should give

$$|M|^2 = g^4\,\frac{N_c^2}{N_c^2-1}\,\underbrace{(s^4+t^4+u^4)\left(\frac{1}{s^2t^2}+\frac{1}{s^2u^2}+\frac{1}{t^2u^2}\right)}_{=4\left(3-\frac{st}{u^2}-\frac{su}{t^2}-\frac{tu}{s^2}\right)}$$



$s+t+u=0$

Note: 

$$= i g^2 \Big[f_{a_1 a_2 c} f_{a_3 a_4 c} \left(\varepsilon_1 \cdot \varepsilon_3 \ \varepsilon_2 \cdot \varepsilon_4 - \varepsilon_1 \cdot \varepsilon_4 \ \varepsilon_2 \cdot \varepsilon_3 \right)$$
$$+ f_{a_1 a_3 c} f_{a_2 a_4 c} \left(\varepsilon_1 \cdot \varepsilon_2 \ \varepsilon_3 \cdot \varepsilon_4 - \varepsilon_1 \cdot \varepsilon_4 \ \varepsilon_3 \cdot \varepsilon_2 \right)$$
$$+ f_{a_1 a_4 c} f_{a_3 a_2 c} \left(\varepsilon_1 \cdot \varepsilon_3 \ \varepsilon_4 \cdot \varepsilon_2 - \varepsilon_1 \cdot \varepsilon_2 \ \varepsilon_4 \cdot \varepsilon_3 \right) \Big]$$
$$= 0 \quad (\text{now})$$

Similarly: $\underbrace{h = -}$ particles are here

$$A(1^- 2^+ 3^- 4^+) = i \, \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \qquad |A|^2 = \frac{u^4}{s^2 t^2}$$

Be more careful about permutations:

$$M^{a_1 \cdots a_4} \overset{p_i h_i}{(1\,2\,3\,4)} \equiv \sum_{\text{all perm's}}^{4!} \underbrace{\mathrm{Tr}(F_{a_1} \cdots F_{a_4})}\, L(1234)$$

$$\begin{cases} = \text{cyclically invariant} \to 3! \\ = (-)^n \,\mathrm{Tr}\, F_{a_n} \cdots F_{a_1} \quad \to \frac{1}{2} 3! = 3 \end{cases}$$

$$= \sum_{\substack{\text{non-cyclic} \\ \text{non-refl.}}\text{perms}}^{\frac{1}{2}3! = 3} \mathrm{Tr}\, F_{a_1} \cdots F_{a_4} \ \underbrace{C(1234)}$$

$$= \sum_{\text{cyclic}}^{n} L(1234) + (-)^n \sum_{\text{cyclic}}^{n} L(4321)$$

$\Rightarrow$ need $C^{++--}(p_1 p_3 p_2 p_4)$ $C^{+-+-}(p_1 p_2 p_3 p_4)$ keep 1 fixed, cyclically $\begin{cases} 2\,3\,4 \\ 3\,4\,2 \\ 4\,2\,3 \end{cases}$

$$M^{++--}(1234) = 4i \frac{g^2}{N_c} \langle 34 \rangle^4 \left[\frac{\mathrm{Tr}\, F_{a_1} F_{a_2} F_{a_3} F_{a_4}}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} + \frac{(a_1 a_3 a_4 a_2)}{\langle 13 \rangle \langle 34 \rangle \langle 42 \rangle \langle 21 \rangle} + \frac{(a_1 a_4 a_2 a_3)}{\langle 14 \rangle \langle 42 \rangle \langle 23 \rangle \langle 31 \rangle} \right]$$

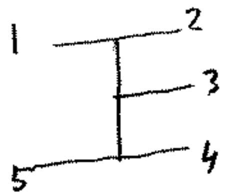
reminder (cf. p. 9)

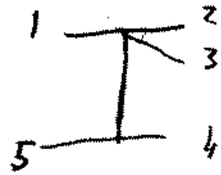
$$\langle 12 \rangle = \lambda_1^a \lambda_{2a} = (p_2^x - i p_2^y) \sqrt{\frac{p_1^0 - p_1^3}{p_2^0 - p_2^3}} - (p_1^x - i p_1^y) \sqrt{\frac{p_2^0 - p_2^3}{p_1^0 - p_1^3}}$$

numerically good!

$$\begin{cases} p^\mu = (p^0, p^x, p^y, p^3) \\ \lambda_1 = \sqrt{p_0 + p_3}\, e^{i\varphi_1} = -\lambda^2 \qquad (p_0 + p_3)(p_0 - p_3) = p_x^2 + p_y^2 \\ \lambda_2 = \sqrt{p_0 - p_3}\, e^{i\varphi_2} = \lambda^1 \\ p_3 = -p^3 \quad \sin(\varphi_2 - \varphi_1) = \frac{p_2}{p_1} = \frac{+p^y}{p^x} \end{cases}$$

<u>5 gluons</u> much like 4 gluons





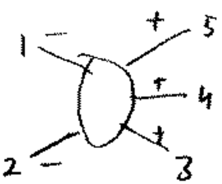
again only one, MHV, amp

$\pm + + + + = 0$ MHV: $- - + + + \approx - - - + +$, $- - - - + = 0$ etc.

$\lambda_1 \tilde{\lambda}_3$ $\qquad$ $\lambda_1 \tilde{\lambda}_5$

$\varepsilon_i^{(-)} \sim$ $\qquad$ $\lambda_1 \tilde{\lambda}_4$ $\qquad$ $\varepsilon_i^{(+)}$

$\lambda_2 \tilde{\lambda}_3$ $\qquad$ $\lambda_1 \tilde{\lambda}_3$



only two nonzero:

$\varepsilon_2^{(-)} \cdot \varepsilon_4^{(+)} \qquad \varepsilon_2^{(-)} \cdot \varepsilon_5^{(+)}$

 contains 1 p from $\perp$ and 5 indices

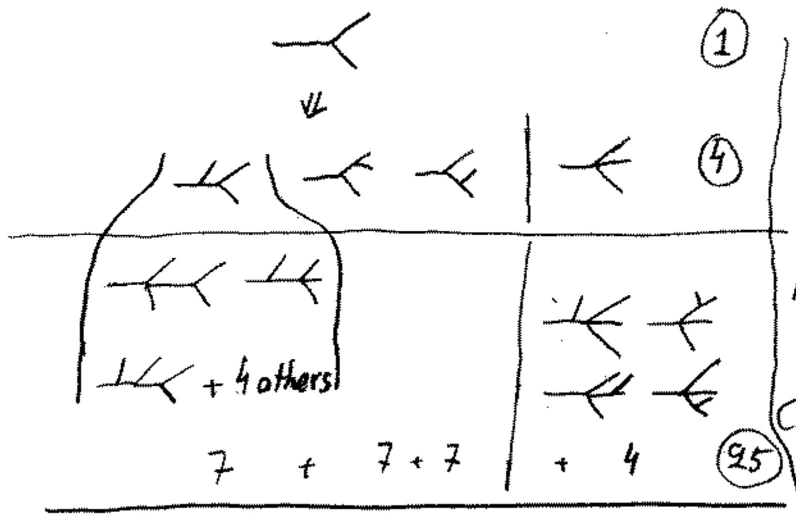
$\mu \nu \alpha \beta \gamma$

$$M^{--+++}(12345) = 4, \frac{g^3}{N_c} \langle 12 \rangle^4 \sum_{\substack{\text{non-cyclic} \\ \text{non-refl.}}}^{\frac{1}{2}(5-1)! = 12} \frac{\mathrm{Tr}\, a_1 a_2 a_3 a_4 a_5}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

$$|M^{--+++}(12345)|^2 = 2 g^6 N_c^3 (N_c^2 - 1) \; (p_1 \cdot p_2)^4 \underbrace{\sum_{\substack{\text{non-c} \\ \text{non-refl}}}^{12} \frac{1}{p_1 \cdot p_2 \; p_2 \cdot p_3 \; \cdots \; p_5 \cdot p_1}}$$

spin & color averaging $\quad \overline{\sum} |M|^2 = \frac{N_c^3 g^6}{2 (N_c^2 - 1)} (p_{12}^4 + p_{13}^4 + p_{14}^4 + p_{15}^4 + p_{23}^4 + p_{24}^4 + p_{25}^4 + p_{34}^4 + p_{35}^4 + p_{45}^4)$

n gluons, # of Feynman diags



n	N_{diag}
9	559 405
10	10 500 000
11	224 000 000
12	5 348 843 500

$$N_{diag} = \left(y\frac{\partial}{\partial x} + xy^3\frac{\partial}{\partial y}\right)^{n-3} xy^3\Big|_{x=y=1}$$

Caravaglios - Mangano - ...
hep-ph/9807570

etc, if p 3-vertices, g external legs + propagators then adding 1 gluon: p diags of type $p-1$ $g+1$ (make $\prec \Rightarrow \leftarrow$)
g — " — $p+1$ $g+2$ (add gluon to each gluon line)

at each stage # is $\left(y\partial_x + xy^3\partial_y\right) x^p y^g \Big|_{x=y=1}$

$$= p\, x^{p-1} y^{g+1} + g\, x^{p+1} y^{g+2}$$

iterate from $\succ = xy^3$

$2^6 = 64$ hel. amps
$\Downarrow$ zero amps
50
$\Downarrow$ parity
25
$\Downarrow$ permutations
2

6 gluons

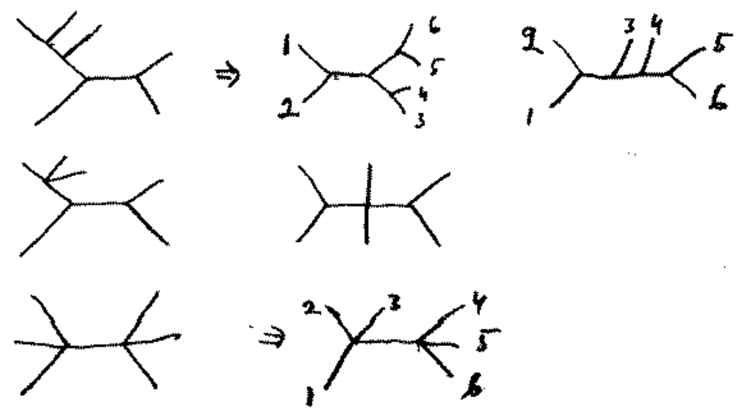
$++++++$ $\quad -+++++ = 0$

2 amps! $\begin{cases} --++++ & \text{MHV, "simple"} \\ ---+++ & \end{cases}$

$----++ \sim$ MHV

$-----+ \quad ------ = 0$

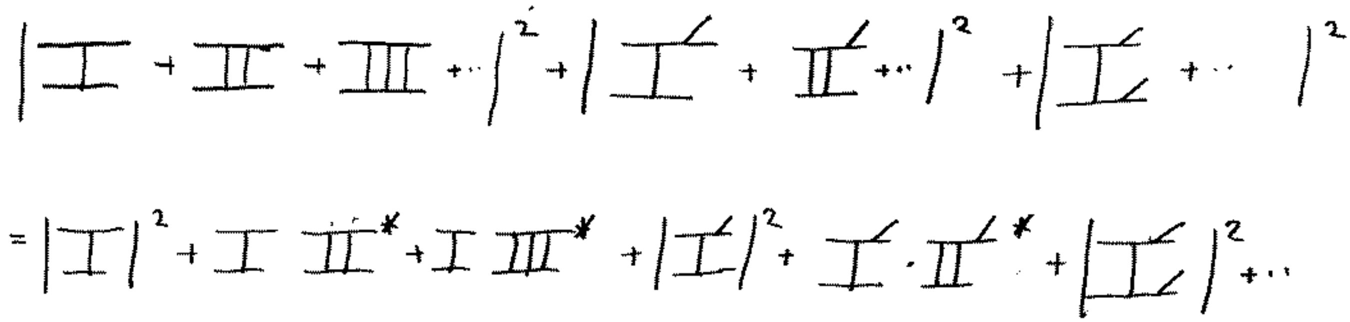
Topologies:



Much hard work leads to

$$M^{--++++}(123456) = 4i\,\frac{g^{n-2}}{N_c}\,\langle 12\rangle^4 \sum_{\substack{\text{non-cyclic}\\ \text{non-reflective perm's}}}^{\overset{=60}{\frac{1}{2}(n-1)!}} \frac{\mathrm{Tr}(T_{a_1}T_{a_2}\cdots T_{a_6})}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}$$

and much much more work (Gunion-Kalinowski, PRD34/86)2119) to an analytic form for M^{---+++} which has to be squared. Useful only numerically, but needed for NNLO jet computations:



g^4	g^6	g^8	g^6	g^8	g^8
L	NL	NNL	NL	NNL	NNL
Born	1loop virtual	2loop virtual	real	one-loop real	6-gluon emission

A general formula for $M^{---+\cdots+}$ (Kosower 1990) contains