Do p. 10 a little better, why is $\bar{Q}_{\dot{2}} |p, h\rangle = \sqrt{4E}\, |p, h-\tfrac{1}{2}\rangle$?

"since Q is a spinor"

Helicity h was defined by taking Λ on p. 10 as rotation in 1,2 plane and $\hat{p} = (E, 0, 0, E)$

$$U(e^{i\theta J_3})|\hat{p} h\rangle = e^{i\theta h}\, |\hat{p}\, h\rangle$$

$$J_3 |\hat{p}\, h\rangle = h\, |\hat{p}\, h\rangle$$

What is $J_3\, \bar{Q}_{\dot{2}} |h\rangle$?

More generally, if $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ infinites. Lorentz

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

$\underbrace{U(\Lambda)}_{\substack{\text{operator } \mathcal{O} \to U(\Lambda)\mathcal{O}U^{-1}(\Lambda) \\ \text{under } \Lambda}} = e^{i\frac{1}{2}\omega_{\mu\nu} M^{\mu\nu}}$, $M^{\mu\nu} = (M^{01}, M^{02}, M^{03}, M^{12}, M^{23}, M^{31})$

$(K^1\ K^2\ K^3\ J^3\ J^1\ J^2)$

boosts rotations

then
$$\begin{cases} [M^{\mu\nu}, Q_a] = -i(\sigma^{\mu\nu})_a{}^b Q_b & \sigma^{\mu\nu} \equiv \frac{1}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \\ [M^{\mu\nu}, \bar{Q}_{\dot{a}}] = i\, \bar{\sigma}^{\mu\nu}{}_{\dot{a}}{}^{\dot{b}} \bar{Q}_{\dot{b}} & \bar{\sigma}^{\mu\nu} = \frac{1}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \end{cases}$$

Analogues of $S^{-1}(\Lambda)\gamma^\nu \underset{\sim}{S(\Lambda)} = \Lambda^\nu{}_\rho \gamma^\rho$ $\quad \sigma^\mu = (1, \vec{\sigma})\quad \bar{\sigma}^\mu = (1, -\vec{\sigma})$

$$\exp\{-\tfrac{1}{4} i\omega_{\mu\nu} \sigma^{\mu\nu}\} \quad \sigma^{\mu\nu} = \tfrac{i}{2}[\gamma^\mu, \gamma^\nu]$$

and one should get sth like

$$\bar{Q}_{\dot{2}} M_{12} - M_{12} \bar{Q}_{\dot{2}} = -\tfrac{1}{2} \bar{Q}_{\dot{b}}\, \sigma_3{}^{\dot{b}}{}_{\dot{2}} = \tfrac{1}{2} \bar{Q}_{\dot{2}} \quad \Big|\, |h\rangle$$

$$\bar{Q}_{\dot{2}} M_{12} |h\rangle - M_{12} \bar{Q}_{\dot{2}} |h\rangle = \tfrac{1}{2} \bar{Q}_{\dot{2}} |h\rangle$$

$$\Rightarrow J_3\, \bar{Q}_{\dot{2}} |h\rangle = \underbrace{(h - \tfrac{1}{2})}_{!}\, \bar{Q}_{\dot{2}} |h\rangle$$

Twistor space

Fourier: $\tilde{f}(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip\cdot x} f(p)$

$p \to x$

Now $p = \lambda_a \tilde{\lambda}_{\dot{a}} \Rightarrow \lambda_a \; \mu^{\dot{a}}$ = "twistor space"

Fourier only $\frac{1}{2}$ of p !

$$\tilde{f}(\mu^{\dot{a}}) = \int \frac{d^2\tilde{\lambda}}{(2\pi)^2} e^{i\mu^{\dot{a}}\tilde{\lambda}_{\dot{a}}} f(\tilde{\lambda})$$

What is a plane wave $e^{ip\cdot y}$ in twistor space?

usual: $\int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip\cdot x} e^{ip\cdot y} = \delta(x+y)$

now $\int \frac{d^2\tilde{\lambda}}{(2\pi)^2} e^{i\mu^{\dot{a}}\tilde{\lambda}_{\dot{a}}} e^{ix^{a\dot{a}}\lambda_a\tilde{\lambda}_{\dot{a}}} = \delta(\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a)$

$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0$, $\dot{a} = 1,2$ 2d subspace of Minkowski

As in Fourier space

$$\begin{cases} \int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu x^\mu} & \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \leftrightarrow -ip_\mu \\ \int d^4x \, e^{ip^\mu x_\mu} & \frac{\partial}{\partial p^\mu} \leftrightarrow ix_\mu \end{cases}$$

in twistor space

$$\begin{cases} \tilde{\lambda}_{\dot{a}} \to i\frac{\partial}{\partial \mu^{\dot{a}}} \\ -i\frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \to \mu_{\dot{a}} \end{cases}$$

Correction to p.9: We have $\begin{cases} \lambda^1 = \lambda_2 & \lambda^2 = -\lambda_1 \\ \lambda^{\dot{1}} = \lambda_{\dot{2}} & \lambda^{\dot{2}} = -\lambda_{\dot{1}} \end{cases}$

Thus if $p_{a\dot{a}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then $p^{a\dot{a}} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}$

so that

$p_{a\dot{a}} p^{a\dot{a}} = p_{a\dot{a}} (p^T)^{\dot{a}a} = \mathrm{Tr}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} +D & -B \\ -C & A \end{pmatrix}$

$= \mathrm{Tr}\begin{pmatrix} AD-BC & 0 \\ 0 & AD-BC \end{pmatrix} = 2 \det p_{a\dot{a}} = 2 p_\mu p^\mu$

Generalises to $\underset{\substack{\uparrow \\ \text{note!}}}{2} p\cdot q = p_{a\dot{a}} q^{a\dot{a}} = \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}} p_{a\dot{a}} q_{b\dot{b}}$

Helicity amplitudes

$\varepsilon_i^{(h)}(p)$

$p_i \quad \varepsilon_i(p) \quad h_i$

$p_i^2 = 0$

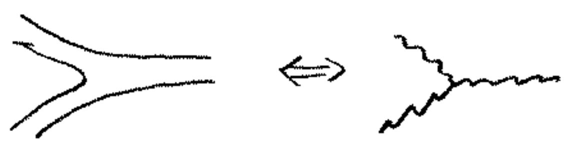
Deep theorem:

If a massless vector field theory with polarisation vector $\varepsilon(p)$ is invariant under

$$\varepsilon^\mu \to \varepsilon^\mu + \eta(p)\, p^\mu \qquad p^2 = 0$$

then it is a gauge theory

Important for strings since the vertex operators satisfy just this condition; theory of massless states of the (<u>open</u>) string is a gauge theory



For <u>closed</u> strings massless spectrum starts with

$G_{\mu\nu}, B_{\mu\nu}, \phi \Rightarrow$ gravity $\varepsilon^{\mu\nu} \to \varepsilon^{\mu\nu} + \eta^\mu(p)\, p^\nu + \eta^\nu(p)\, p^\mu$

Could one write the amplitude as $A(p_i, \varepsilon_i^{(h_i)}(p)) \to A(p_i, \overset{\pm}{h_i})$,

i.e., is there a way of finding $\varepsilon^{(h)}(p)$ up to ηp given h?

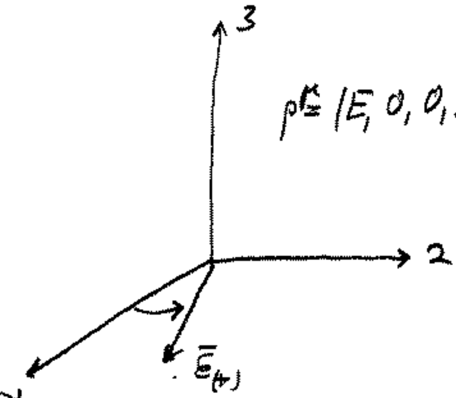
Simple in spinor formulation:

More on helicity:

$\cos(\omega t - kz) - i\sin(\omega t - kz)$

ED: $\bar{E} = E_0(\varepsilon_1 \pm i\varepsilon_2)\, e^{-i\omega t + ikz}$

$p^\mu = (E, 0, 0, E)$



$\Rightarrow E_x = \mathrm{Re}\,\bar{E}_x = E_0 \cos(\omega t - kz)$

$E_y = \mathrm{Re}\,\bar{E}_y = E_0 (\pm) \sin(\omega t - kz)$

$\varepsilon_{(+)}$ at $z=0$ $(+)$ $\begin{cases} E_x = E_0 \cos\omega t \\ E_y = +E_0 \sin\omega t \end{cases}$

rotates clockwise seen in direction of motion $\Rightarrow$ RH

General properties) $\varepsilon^* \to \varepsilon^+$ in spinor formulation

$\boxed{\varepsilon^*_{(+)} = \varepsilon_{(-)} \quad \varepsilon^\mu_{(+)} \varepsilon^{(+)}_\mu = 0, \quad \varepsilon^\mu_{(+)} \varepsilon^*_{\mu(+)} = -1 \quad \varepsilon(p)\cdot p = 0}$

$\Rightarrow \varepsilon^\mu_{(+)} = \left(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0\right) \longrightarrow \varepsilon_1^{(+)} = -\frac{1}{\sqrt{2}} \quad i\varepsilon_2^{(+)} = \frac{1}{\sqrt{2}}$ $\quad \begin{cases} \varepsilon_1^+ + i\varepsilon_2^+ = 0 \\ \varepsilon_1^+ - i\varepsilon_2^+ = -\sqrt{2} \end{cases}$

$$\varepsilon^{(+)}_{a\dot{a}} = \begin{pmatrix} 0 & -\sqrt{2} \\ 0 & 0 \end{pmatrix}$$

A polarization vector can only be determined up to $\eta p^\mu = \eta E(1,0,0,1)$

$$\varepsilon^{(-)}_{a\dot{a}} = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}$$

$p^0 + p^3 = 0$

Equally good ε's: $\varepsilon^+ + \eta p = \begin{pmatrix} 0 & -\sqrt{2} \\ 0 & 2\eta E \end{pmatrix} \quad \varepsilon^- + \eta p = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 2\eta E \end{pmatrix}$

This is in a special frame $p^\mu = (E, 0, 0, E)$. In the spinor formalism one can write in a general frame, $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$,

motivated below

arbitrary spinor

"gauge spinor"

$$\varepsilon^{(-)}_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{\frac{1}{\sqrt{2}} \tilde{\lambda}^{\dot{c}} \tilde{\mu}_{\dot{c}}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}\tilde{\mu}]} \qquad \varepsilon^{(+)}_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\frac{1}{\sqrt{2}} \mu^c \lambda_c} \qquad p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

(or, as well

$$\varepsilon^{(-)}_{a\dot{b}} = \frac{\lambda_a \tilde{\mu}_{\dot{b}}}{\tilde{\lambda}^{\dot{c}} \tilde{\mu}_{\dot{c}}}$$
)

I seem to need these!?

[Idea: the gauge spinor μ_a is chosen to minimize computation]

Conditions:

(1) $2\varepsilon\cdot p = \varepsilon_{a\dot a}\, p^{a\dot a} = N\, \lambda_a \tilde\mu_{\dot a}\, \lambda^a \tilde\lambda^{\dot a} = 0 \quad (\lambda_a\lambda^a = 0)$

(2) $2\varepsilon\cdot\varepsilon = \; = N\, \lambda_a \tilde\mu_{\dot a}\, \underbrace{\lambda^a \tilde\mu^{\dot a}}_{*} = 0$

Same for μ

$\tilde\lambda_{\dot a} = (\lambda^*)_{\dot a}$

(3) $$2\varepsilon\cdot\varepsilon^* = \frac{\lambda_a\tilde\mu_{\dot a}}{\frac{1}{\sqrt2}\tilde\lambda^{\dot c}\tilde\mu_{\dot c}} \cdot \underbrace{\frac{\tilde\lambda^{\dot a}\mu^a}{\frac{1}{\sqrt2}\lambda^c\mu_c}}_{\text{one needs Penrose to control indices \& signs here!}} = -1$$

needed!

Note: writing one must flip $a \leftrightarrow b$!

$$2\varepsilon\cdot\varepsilon^* = \frac{\lambda_a\tilde\mu_{\dot b}}{\frac{1}{\sqrt2}\tilde\lambda^{\dot c}\tilde\mu_{\dot c}} \cdot \frac{\tilde\lambda^{\dot b}\mu^a}{\frac{1}{\sqrt2}\lambda^c\mu_c}$$

More on the $\frac{1}{\sqrt2}$: We may ask what is the $\tilde\mu_{\dot a}$ which gives $\varepsilon^\mu_{(\pm)} + \eta p^\mu$ when $p^\mu = (E,0,0,E) \Rightarrow \lambda_a = \begin{pmatrix}0\\ \sqrt{2E}\end{pmatrix}$

We found

$$(\varepsilon^{(-)} + \eta p)_{a\dot a} = \begin{pmatrix}0 & 0\\ -\sqrt2 & 2\eta E\end{pmatrix} = \frac{1}{-\frac{1}{\sqrt2}\underbrace{(\tilde\lambda^{\dot 1}\tilde\mu_{\dot 1} + \tilde\lambda^{\dot 2}\tilde\mu_{\dot 2})}_{}}\begin{pmatrix}0 & 0\\ \sqrt{2E}\,\tilde\mu_{\dot 1} & \sqrt{2E}\,\tilde\mu_{\dot 2}\end{pmatrix}$$

$\hookrightarrow \tilde\lambda_{\dot 2}\tilde\mu_{\dot 1} - \underbrace{\tilde\lambda_{\dot 1}}_{=0}\tilde\mu_{\dot 2} = \sqrt{2E}\,\tilde\mu_{\dot 1}$

Need $-\frac{1}{\sqrt2}$ to get this element!

$$= \frac{1}{-\frac{1}{\sqrt2}\sqrt{2E}\,\tilde\mu_{\dot 1}}\begin{pmatrix}0 & 0\\ \sqrt{2E}\,\tilde\mu_{\dot 1} & \sqrt{2E}\,\tilde\mu_{\dot 2}\end{pmatrix} = \begin{pmatrix}0 & 0\\ -\sqrt2 & -\sqrt2\,\frac{\tilde\mu_{\dot 2}}{\tilde\mu_{\dot 1}}\end{pmatrix}$$

$$\Rightarrow \boxed{\tilde\mu_{\dot a} = z\begin{pmatrix}1\\ -\eta\sqrt2 E\end{pmatrix}}$$

One also sees that $\varepsilon^{(-)}_{a\dot a} = \dfrac{\lambda_a\tilde\mu_{\dot a}}{-\frac{1}{\sqrt2}\tilde\lambda\tilde\mu}$ has $h = -1$

(since $h = +$ produced $\varepsilon^+ + \eta p = \begin{pmatrix}0 & \cdot\\ 0 & \cdot\end{pmatrix}$ which can only come from $\mu_a\tilde\lambda_{\dot a}$!)

(4) uniqueness: arbitrariness in μ = gauge tr

If $\tilde{\mu} \to \tilde{\mu} + A\tilde{\mu} + B\tilde{\lambda}$ (2 complex dofs!)

then
$$\frac{\lambda_a \tilde{\mu}_{\dot a}}{\tilde{\lambda}^{\dot a} \tilde{\mu}_{\dot a}} \to \frac{\lambda_a(\tilde{\mu}_{\dot a} + A\tilde{\mu}_{\dot a} + B\tilde{\lambda}_{\dot a})}{\tilde{\lambda}^{\dot a}(\tilde{\mu}_{\dot a} + A\tilde{\mu}_{\dot a} + \underset{\downarrow\, 0}{B\tilde{\lambda}_{\dot a}})} = \frac{(1+A)\lambda_a\tilde{\mu}_{\dot a} + B\lambda_a\tilde{\lambda}_{\dot a}}{(1+A)\tilde{\lambda}\tilde{\mu}}$$

$$\varepsilon^{(-)}_{a\dot a} \to \varepsilon^{(-)}_{a\dot a} + \frac{B}{1+A} p_{a\dot a} = \text{gauge transformation}$$

Spinor helicities

$\vec{\Sigma}\cdot\hat{p}\, u(p) = \pm u(p)$ even with $m>0$! $\rightarrow$ $\leftarrow$

$$(i\not\partial - \cancel{m}^{\,0})\psi = 0 \quad \underset{\text{Chiral rep}}{\Longrightarrow} \quad \begin{cases} i(-\partial_0\psi_- - \vec{\sigma}\cdot\vec{\partial}\psi_-) = 0 & i\bar{\sigma}^\mu\partial_\mu\psi_- = 0 \\ i(-\partial_0\psi_+ + \vec{\sigma}\cdot\vec{\partial}\psi_+) = 0 & i\sigma^\mu\partial_\mu\psi_+ = 0 \end{cases}$$

$\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$ $\quad \gamma^0 = \begin{pmatrix}0 & +1 \\ +1 & 0\end{pmatrix}$ $\vec{\gamma} = \begin{pmatrix}0 & \vec{\sigma} \\ -\vec{\sigma} & 0\end{pmatrix}$ $\gamma_5 = \begin{pmatrix}+1 & 0 \\ 0 & -1\end{pmatrix}$

$$i\sigma^\mu\partial_\mu\psi = 0 \Rightarrow i\sigma^\mu_{a\dot a}\partial_\mu\psi^a = 0$$

$$\psi^a = \lambda^a e^{ip\cdot x} \qquad \Rightarrow \sigma^\mu_{a\dot a}\lambda^a p_\mu = \underbrace{p_{a\dot a}}_{=\lambda_a\tilde{\lambda}_{\dot a}}\lambda^a = 0$$

$m=0$ Spin 1 & $m=0$ Spin $\frac{1}{2}$ can be mapped on each other

$\Rightarrow$ $\overrightarrow{}$ $\overleftarrow{}$

$u_+ = \frac{1}{2}(1+\gamma_5)u(p)$ $\quad \frac{1}{2}(1-\gamma_5)u(p)$

see Dixon TASI lect hep-ph/9601359

$$\varepsilon^\mu_{(-)}(p,q) \equiv -\frac{\overset{\bar{u}_+(q)}{\langle q_+|}\gamma_\mu\overset{u_+(p)}{|p_+\rangle}}{\underset{\bar{u}_+(q)\,u_-(p)}{\langle q_+|p_-\rangle}}$$

$\boxed{F_{\mu\nu}}$

We had $\qquad x^\mu \Rightarrow X_{a\dot{a}} \equiv x_\mu \sigma^\mu{}_{a\dot{a}}\,, \qquad g_{\mu\nu} = \varepsilon_{ab}\,\varepsilon_{\dot{a}\dot{b}}$

What about $\quad F_{\mu\nu} = -F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu \hat{=} -i(p_\mu \varepsilon_\nu - p_\nu \varepsilon_\mu)$?

$\mu \to a\dot{a}$, $\nu \to b\dot{b}$

$$\Rightarrow \quad F_{a\dot{a}b\dot{b}} \equiv F_{ab\dot{a}\dot{b}} = -F_{ba\dot{b}\dot{a}}$$

order of undotted & dotted is irrelevant!

$$\equiv \underbrace{\tfrac{1}{2}\left(F_{ab\dot{a}\dot{b}} - F_{ab\dot{b}\dot{a}}\right)}_{\text{antis in } \dot{a}\dot{b}\text{, symm in } ab} + \underbrace{\tfrac{1}{2}\left(F_{ab\dot{b}\dot{a}} - F_{ba\dot{b}\dot{a}}\right)}_{\text{antis. in } ab\text{, symm in } \dot{a}\dot{b}}$$

same, cancel

$F_{\mu\nu} = -F_{\nu\mu}$

$$= \varepsilon_{\dot{a}\dot{b}} \underbrace{\tfrac{1}{2} F_{ab\dot{c}}{}^{\dot{c}}}_{\text{symm. in } ab} + \varepsilon_{ab} \underbrace{\tfrac{1}{2} F_c{}^c{}_{\dot{a}\dot{b}}}_{\text{symm in } \dot{a}\dot{b}}$$

$$\boxed{F_{\mu\nu} \to F_{a\dot{a}b\dot{b}} = \varepsilon_{\dot{a}\dot{b}}\,\phi_{ab} + \varepsilon_{ab}\,\psi_{\dot{a}\dot{b}}}$$

Real $F_{\mu\nu}$: $\quad F_{\mu\nu} = \varepsilon_{\dot{a}\dot{b}}\,\phi_{ab} + \varepsilon_{ab}\,\phi^*_{\dot{a}\dot{b}} \qquad (\varepsilon^*_{ab} = \varepsilon_{\dot{a}\dot{b}})$

$$\vec{E}, \vec{B} \;\hat{=}\; \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{12} & \phi_{22} \end{pmatrix}$$

6 real $\qquad\qquad$ 3 complex

$$= \begin{pmatrix} \tfrac{1}{2}\left[E_1 - iE_2 - i(B_1 - iB_2)\right] & -\tfrac{1}{2}(E_3 - iB_3) \\ \text{symm} & -\tfrac{1}{2}\left[E_1 + iE_2 - i(B_1 + iB_2)\right] \end{pmatrix} \quad !$$

Dual fields:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} = \begin{cases} iF_{\mu\nu} & \text{self-dual} \\ -iF_{\mu\nu} & \text{anti-self-dual}\end{cases}$$

$$\Rightarrow \quad \tilde{F}_{ab\dot{a}\dot{b}} = -i\,\varepsilon_{\dot{a}\dot{b}}\,\phi_{ab} + i\,\varepsilon_{ab}\,\psi_{\dot{a}\dot{b}} = i\,F_{ab\dot{b}\dot{a}}$$

using expl. form of $\varepsilon_{\mu\nu\alpha\beta}$ (p.8)

Reminder: Instantons are self-dual or antiself-dual solutions of Euclidian YM: $\tilde{F}_{\mu\nu} = \pm F_{\mu\nu}$

$$x^0 = -i x_4 \quad (\tau = it) \Rightarrow \quad E^i = F_{0i} = iF_{4i} = iE_i^{(E)}, \quad B^i = B_i^{(E)}$$

$$\underset{\text{self-dual}}{\tilde{F}_{01} = F^{12} = iF_{01}} \quad \xrightarrow{\text{Mink} \to \text{Eucl}} \quad \tilde{F}_{41} = F_{12} = \pm F_{41} \quad \text{no } i!$$

Classical solutions, in general, are extrema of

$$S[A_\mu] = \int d^4x \, \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu$$

$$0 \le \int d^4x \, \frac{1}{4}\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)^2 = \int d^4x \, \frac{1}{4}\left(F^2_{\mu\nu} + \tilde{F}^2_{\mu\nu} \pm 2F_{\mu\nu}\tilde{F}_{\mu\nu}\right)$$

same!

$$= \int d^4x \, \underset{>0}{\frac{1}{2}F^2_{\mu\nu}} \pm \underbrace{\frac{1}{2}\int d^4x \, F_{\mu\nu}\tilde{F}_{\mu\nu}}_{=\frac{16\pi^2}{g^2}m} \ge 0$$

important topological quantity, winding number of maps of points at 4d Euclidian infinity = surface S^3 of 4d sphere to group manifold (also S^3 for $SU(2)$)

For $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$ obtain $=0$, i.e.

$$S[A_\mu] = \int d^4x \, \frac{1}{4}F^2_{\mu\nu} = \frac{8\pi^2}{g^2}m$$

QCD CP problem: why is

$$\int d^4x \, \frac{1}{4} F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$$

missing from the action?

Now work out $\underset{a\dot{a}\,b\dot{b}}{F_{\mu\nu}} = -i(p_\mu\, \varepsilon^\pm_\nu - p_\nu\, \varepsilon^\pm_\mu)$

$$F^{(-)}_{ab\dot{a}\dot{b}} = -i\left(\lambda_a \tilde{\lambda}_{\dot{a}}\, \lambda_b \tilde{\mu}_{\dot{b}} - \lambda_b \tilde{\lambda}_{\dot{b}}\, \lambda_a \tilde{\mu}_{\dot{a}}\right)\cdot N$$

just some complex numbers!

$$= -iN\,\lambda_a\lambda_b(\tilde{\lambda}_{\dot{a}}\tilde{\mu}_{\dot{b}} - \tilde{\lambda}_{\dot{b}}\tilde{\mu}_{\dot{a}}) = \underbrace{\phi_{ab}}_{\text{symmetric}}\,\varepsilon_{\dot{a}\dot{b}} \quad \text{anti-self-dual}$$

$$F^{(+)}_{ab\dot{a}\dot{b}} = \varepsilon_{ab}\,\psi_{\dot{a}\dot{b}}$$

If $F_{a\dot{a}\,b\dot{b}} = \varepsilon_{\dot{a}\dot{b}}\,\phi_{ab}$ then $\tilde{F}_{a\dot{a}b\dot{b}} = -i\,\varepsilon_{\dot{a}\dot{b}}\,\phi = -iF$

n has opposite) $\rightarrow \Rightarrow \begin{cases} F^{(-)} \text{ is antiselfdual} \\ F^{(+)} \text{ is selfdual} \end{cases}$ see Penrose I p. 151