# QCD & Strings

This is an enormously large topic with no end in sight.

These lectures will discuss various theoretical concepts needed to read current papers.

# Some theory background

## A. Reps of Poincare:

$$\Lambda, a \Rightarrow U(\overset{\text{Lorentz}}{\Lambda}, \overset{\text{transl.}}{a}) \qquad U(\Lambda_1\Lambda_2) = U(\Lambda_1)U(\Lambda_2)$$

(1) $P^\mu |p,\sigma\rangle = p^\mu |p,\sigma\rangle$ — some add'l state label

$$U(1,a)|p,\sigma\rangle = e^{-ip\cdot a}|p,\sigma\rangle$$

(2) $\underset{a=0}{U(\Lambda)}|p,\sigma\rangle = \sum C_{\sigma\sigma'}(\Lambda,p)|\Lambda p,\sigma'\rangle$

(3) Define $|p,\sigma\rangle$ in terms of a standard vector $\hat{p}$:

$$|p,\sigma\rangle = N_p\, U(L(p))|\hat{p},\sigma\rangle$$

$$p^\mu = L^\mu{}_\nu(p)\,\hat{p}^\nu$$

$$\begin{cases} \hat{p} = (m,\vec{0}) & p^2 = m^2 \qquad L = \text{boost} \\ \phantom{\hat{p}} = (1,0,0,1) & p^2 = 0 \Leftarrow \text{massless case!} \end{cases}$$

(4) Then $U(\Lambda)|p,\sigma\rangle = N_p\, U(\Lambda)\, U(L(p))\, |\hat{p}\,\sigma\rangle$

$$= N_p\, U(\underbrace{L(\Lambda p)L^{-1}(\Lambda p)}_{=1}\ \Lambda\ L(p))|\hat{p}\,\sigma\rangle$$

$$= N_p\, U(L(\Lambda p))\, U(\underbrace{\underbrace{L^{-1}(\Lambda p)\,\Lambda\,\underbrace{L(p)}_{p}}_{\Lambda p}}_{\hat{p}})|\hat{p}\,\sigma\rangle$$

$$U(W)|\hat{p}\sigma\rangle = \sum_{\sigma'} D(W)_{\sigma\sigma'}|\hat{p}\sigma'\rangle$$

$$W = L^{-1}(\Lambda p)\,\Lambda\, L(p) \in \text{little group of } \hat{p}$$

$$W^\mu{}_\nu\,\hat{p}^\nu = \hat{p}^\mu$$

$$\boxed{U(\Lambda)|p\,\sigma\rangle = N_p \sum_{\sigma'} D_{\sigma\sigma'}(L^{-1}(\Lambda p)\Lambda L(p))\,\frac{1}{N_{\Lambda p}}|\Lambda p,\sigma'\rangle}$$

All this is well known for $p^2 = m^2$: little group = $O(3)$

What about $(1,0,0,1)$? What $\Lambda$'s leave this invariant?

Clearly at least $O(2)$, rotations in $x, y$. But there is more to that (see, e.g., Weinberg I, (2.5.26)):

$$W^\mu{}_\nu(b,c,\theta) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \begin{pmatrix} 1+d & b & c & -d \\ b & 1 & 0 & -b \\ c & 0 & 1 & -c \\ d & b & c & 1-d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & s\theta & 0 \\ 0 & -s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$d = \frac{1}{2}(b^2+c^2)$$

$\boxed{\text{This is 2d Euclidian group; two translations } (b,c) + \text{rotation } \theta}$

Why this?

[Only reps inv. under translations seem to be physically relevant!

$$\underset{W}{\begin{pmatrix} a & e & f & g \\ b & 1 & 0 & h \\ c & 0 & 1 & i \\ d & j & k & l \end{pmatrix}} \cdot \underset{\hat{p}}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}} = \begin{pmatrix} a+g \\ b+h \\ c+i \\ d+l \end{pmatrix} = \underset{\hat{p}}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$\underset{W}{\begin{pmatrix} \phantom{a} \end{pmatrix}} \underset{\hat{t}}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}} = \underset{W\hat{t}}{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = a - d = 1$$

$$\Rightarrow a = 1 + d$$

$$W\hat{t} \cdot \hat{p} = W\hat{t} \cdot W\hat{p} = \hat{t} \cdot \hat{p} = 1$$

$$(W\hat{t})^2 = \hat{t}^2 = 1 = a^2 - b^2 - c^2 - d^2 = (1+d)^2 - b^2 - c^2 - d^2 = 1$$

$$\Downarrow\ d = \frac{1}{2}(b^2+c^2)$$

$$\Rightarrow W = \begin{pmatrix} 1+d & e & f & -d \\ b & 1 & 0 & -b \\ c & 0 & 1 & -c \\ d & j & k & 1-d \end{pmatrix}$$

$\hookrightarrow$ get this from $W^T g W = g$

It seems the reps. corresponding to the translations of the Euclidian group are experimentally excluded (Weinberg I, pp. 71-72) so what remains is

$$U(\Lambda)|p\ \sigma\rangle = \frac{N_p}{N_{\Lambda p}}\, e^{i\sigma \underbrace{\theta(\Lambda,p)}}\, |p\ \sigma\rangle \qquad \left(N_p = \frac{1}{\sqrt{p^0}}\right)$$

rotation angle in

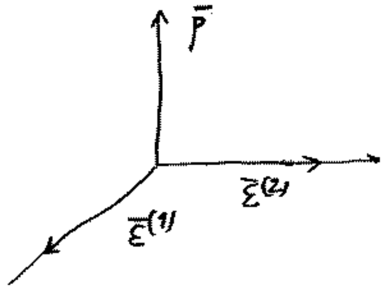
$W = L^{-1}(\Lambda p)\,\Lambda\, L(p) \qquad W\hat{p} = \hat{p}$

Even more arguments $\left( P|p\,\sigma\rangle = \eta_\sigma\, e^{\pm i\pi\sigma}\, |(p^0, -\vec{p}), -\sigma\rangle \right)$
WI (2.6.22)

are needed to restrict $\sigma = 0\ \pm\frac{1}{2}\ \pm 1\ \pm 2 \cdots$

[What would massless reps. transforming non-trivially under translations ($b, c \neq 0$) correspond to? ]

## Photon, gluon polarisation




$\bar{p} = (0, 0, p) \qquad \bar{p}\cdot\bar{\varepsilon}^{(r)} = 0$

$p^\mu = (p, 0, 0, p) \quad \varepsilon^\mu_{(1)} = (0, 1, 0, 0) \quad p\cdot\varepsilon_{(r)} = 0$

Pure photon state $\quad \bar{\varepsilon} = c_1\bar{\varepsilon}^{(1)} + c_2\bar{\varepsilon}^{(2)}$

Density matrix

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(c_1^* \; c_2^*) = \begin{pmatrix} c_1c_1^* & c_1c_2^* \\ c_1^*c_2 & c_2c_2^* \end{pmatrix}$$

$$\equiv \frac{1}{2}(1 + \bar{P}\cdot\bar{\sigma}) \qquad Tr\rho^2 = \frac{1}{2}(1+\bar{P}^2) \quad Tr\rho = 1$$

$$\bar{P} = \left(2\,\mathrm{Re}\,c_1c_2^*,\ 2\,\mathrm{Im}\,c_2c_1^*,\ |c_1|^2 - |c_2|^2\right) = \text{real 3-vector}$$ (no geom. significance)

For spin $\frac{1}{2}$ $\quad \bar{S} = \frac{1}{2}\hbar\bar{\sigma}$ and $\langle\bar{S}\rangle = Tr\,\rho\,\bar{\sigma} = \bar{P}$

Helicity states:

$$\begin{cases} c_1 = \frac{1}{\sqrt{2}} \quad c_2 = \frac{i}{\sqrt{2}} \quad \bar{P} = \bar{e}_z \quad h = + \quad RH \quad \varepsilon^\mu_+ = \left(0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0\right) \\ \phantom{c_1} = \frac{1}{\sqrt{2}} \quad \phantom{c_2} = -\frac{i}{\sqrt{2}} \quad \phantom{\bar{P}} = -\bar{e}_z \quad \phantom{h} = - \quad LH \quad \varepsilon^\mu_- = \left(0, \frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0\right) \end{cases}$$

$$= \varepsilon^\mu_-(p)\big|_{p=(p,0,0,p)}$$

$\varepsilon_{(+)}\cdot\varepsilon^*_{(+)} = -1 \qquad \varepsilon_{(-)}\cdot\varepsilon^*_{(+)} = 0$

$$\sum_{h=\pm}\varepsilon^{(h)}_\mu\,\varepsilon^{(h)*}_\nu = -g_{\mu\nu} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}$$

$n^2 = 0$ another light-like vector

$$\left\{\begin{matrix} * \; n^\mu \\ * \; p^\mu \end{matrix}\right. = 0$$

GT: $\quad \varepsilon^\mu(p) \to \varepsilon^\mu(p) + \lambda p^\mu \qquad$ In $U(1)$ theory charge cons. $\partial^\mu J_\mu = 0$

means $\quad M_\mu(p)\,p^\mu = 0$


We shall presently write $p^\mu$, $\varepsilon^\mu$ in terms of spinors, $\mu \to a\dot{a}$

$$R_{\mu\nu}(p,n) = -g_{\mu\nu} + \frac{p\cdot n\,(p_\mu n_\nu + p_\nu n_\mu) - p^2 n_\mu n_\nu - n^2 p_\mu p_\nu}{(p\cdot n)^2 - p^2 n^2}$$ projects any vector

$v^\mu \perp p \;\&\; n$: $\quad p\cdot Rv = n\cdot Rv = 0 \qquad R_\mu{}^\mu = -2$

# Spinor reps of Lorentz $\Lambda(4)$

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \qquad g_{\alpha\beta}\, x'^\alpha x'^\beta = \underline{g_{\alpha\beta}\,\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu}\; x^\mu x^\nu$$

$$= g_{\mu\nu} = \Lambda^\alpha{}_\mu\, g_{\alpha\beta}\, \Lambda^\beta{}_\nu$$

$$\Rightarrow g = \Lambda^T g\, \Lambda$$

Map $\quad x^\mu \Rightarrow X = x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix} = X^+$

$$\sigma^\mu = (1, \bar{\sigma})$$

$$\det X = x_0^2 - \bar{x}^2 = x_\mu x^\mu$$

If $\; M \in SL(2,C)$ and $X \to X' = M X M^+ \qquad X'^+ = M X^+ M^+$

then $\det X' = \det M \det X \det M = \det X$

$$x'_\mu \sigma^\mu = \Lambda_\mu{}^\nu x_\nu\, \sigma^\mu = M\, x_\nu \sigma^\nu M^+$$

$$\Rightarrow M \sigma^\nu M^+ = \Lambda_\mu{}^\nu \sigma^\mu$$

$$\boxed{\Lambda \Rightarrow (\pm) M \;\text{ is a } (\tfrac{1}{2}, 0) \text{ spinor rep for } \Lambda(4)}$$

$$D^{(\frac{1}{2},0)}(\Lambda) = e^{\frac{1}{2}(\bar{y} + i\bar{\theta})\cdot\bar{\sigma}} = M = \varepsilon^{-1}(M^T)^{-1}\varepsilon$$

$\bar{y}, \bar{\theta}$ are boost & rot in $\Lambda$

define $\quad D^{(0,\frac{1}{2})}(\Lambda) = e^{\frac{1}{2}(-\bar{y} + i\bar{\theta})\cdot\bar{\sigma}} = (M^+)^{-1} = \varepsilon M^* \varepsilon^{-1}$

equiv. reps $\qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Note :

Now call vectors transforming by $M$ $\lambda_a$ $\quad a = 1, 2$

" " " " $M^*$ $\tilde\lambda_{\dot a}$ $\quad \dot a = 1, 2$

$$\lambda_a \to \lambda_a' = M_a{}^b \lambda_b \qquad \tilde\lambda_{\dot a} \to \tilde\lambda_{\dot a}' = M^*{}_{\dot a}{}^{\dot b} \lambda_{\dot b}$$

Upper indices transform $\sim$ equiv. reps $M^{T,-1}$ & $M^{\dagger,-1}$

$$\lambda^a \to \lambda^{a\prime} = (M^{T,-1})^a{}_b \lambda^b \qquad \dots$$

so that

$$\lambda^a \psi_a \to \lambda^b \underbrace{(M^{T,-1})^a{}_b M_a{}^c}_{(M^{-1}M)_b{}^c} \psi_c = \lambda^b \psi_b$$

is invariant. But for $2\times 2$ matrices; det $= 1$,

if $\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \to \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$, $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \to \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$

then $\lambda_2\psi_1 - \lambda_1\psi_2 \to \underbrace{(ad-bc)}_{=1}(\lambda_2\psi_1 - \lambda_1\psi_2) =$ invariant,

i.e., $-\begin{vmatrix}\lambda_1 & \lambda_2\\ \psi_1 & \psi_2\end{vmatrix}$ $\begin{cases} = \lambda^1\psi_1 + \lambda^2\psi_2 = \lambda^a\psi_a & \text{if } \boxed{\lambda^a = \varepsilon^{ab}\lambda_b} \; \boxed{\varepsilon^{ab} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}} \\ = -\lambda_1\psi^1 - \lambda_2\psi^2 = -\lambda_a\psi^a & \lambda^1 = \lambda_2 \quad \lambda^2 = -\lambda_1 \end{cases}$

To get conversely $\lambda_1 = -\lambda^2$ $\lambda_2 = \lambda^1$ one has the choice of defining $\varepsilon_{ab} = \pm\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Choose $\varepsilon_{ab} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ so that

$$\boxed{\lambda_a = -\varepsilon_{ab}\lambda^b = \lambda^b \varepsilon_{ba}}$$

Consistent?
$$\varepsilon_{ab} = (\underbrace{-\varepsilon_{ak})(-\varepsilon_{bl}}_{\varepsilon_{lb}})\varepsilon^{kl} = \varepsilon_{ak}\,\varepsilon_{bl}\,\varepsilon^{kl} = \varepsilon_{ab} \quad \text{Yes!}$$

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \delta_b{}^k$$

$$\left[\varepsilon_{ak}\,\varepsilon^{bk} = \delta_a^b \equiv \varepsilon_a{}^b \equiv -\varepsilon^b{}_a\right]$$

Exercise: Derive the "Schouten identity":

Consider 3 spinors $\lambda$ $\psi$ $\chi$. For sure

$$\begin{vmatrix} \lambda_a & \psi_a & \chi_a \\ \lambda_1 & \psi_1 & \chi_1 \\ \lambda_2 & \psi_2 & \chi_2 \end{vmatrix} = +\lambda_a\,\psi\chi + \psi_a\,\chi\lambda + \chi_a\,\lambda\psi = 0 \quad a=1,2$$

$$\psi\chi = \psi_2\chi_1 - \psi_1\chi_2 = \varepsilon^{bc}\psi_b\chi_c$$

Now factor out $\lambda_\alpha\psi_\beta\chi_\gamma$:

write always $\lambda_a = \lambda_\alpha\,\delta^\alpha_a \quad \psi_b = \psi_\beta\,\delta^\beta_b \quad \chi_c = \chi_\gamma\,\delta^\gamma_c$

$$\lambda_a\,\varepsilon^{bc}\psi_b\chi_c + \psi_a\,\varepsilon^{bc}\chi_b\lambda_c + \chi_a\,\varepsilon^{bc}\lambda_b\psi_c$$

$$\lambda_\alpha\psi_\beta\chi_\gamma\left[\delta^\alpha_a\,\varepsilon^{bc}\delta^\beta_b\delta^\gamma_c + \delta^\beta_a\,\varepsilon^{bc}\delta^\gamma_b\delta^\alpha_c + \delta^\gamma_a\,\varepsilon^{bc}\delta^\alpha_b\delta^\beta_c\right] = 0$$

$$\varepsilon_a{}^\alpha\,\varepsilon^{\beta\gamma} + \varepsilon_a{}^\beta\,\varepsilon^{\gamma\alpha} + \varepsilon_a{}^\gamma\,\varepsilon^{\alpha\beta} = 0$$

$$\Downarrow$$

$$\varepsilon_{a\alpha}\,\varepsilon_{\beta\gamma} + \varepsilon_{a\beta}\,\varepsilon_{\gamma\alpha} + \varepsilon_{a\gamma}\,\varepsilon_{\alpha\beta} = 0$$

reletter: $\boxed{\varepsilon_{ab}\varepsilon_{cd} + \varepsilon_{ac}\,\varepsilon_{db} + \varepsilon_{ad}\,\varepsilon_{bc} = 0}$ many diff letterorders are possible

or: $\varepsilon_{ab}\,\varepsilon^{cd} + \varepsilon_a{}^c\,\varepsilon^d{}_b + \varepsilon_a{}^d\,\varepsilon_b{}^c = 0$

$$\left[\varepsilon_{ab}\,\varepsilon^{cd} = \varepsilon_a{}^c\,\varepsilon_b{}^d - \varepsilon_a{}^d\,\varepsilon_b{}^c\right]$$

Application: Contract with a tensor $F_{kcd}$ ($k$: some indices)

$$\Rightarrow\ \varepsilon_{ab}\,F_{kc}{}^c = F_{kab} - F_{kba}$$

If $F$ is odd in $a,b$ (like $F_{\mu\nu} = F_{a\dot a b\dot b}$!)

then $\boxed{F_{kab} = \frac{1}{2}\varepsilon_{ab}\,F_{kc}{}^c}$ will be important for determining $F_{\mu\nu}$ & gluon helicity!

Complex conj. reps : $M$ and $M^*$ are not equivalent

(say, for $G \in SU(2)$ $(i\sigma_2)^{-1} G^* i\sigma_2 = G$, but no $U$:
$= \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ $\qquad U^{-1} M^* U = M$ )

$$\lambda'_a = M_a{}^b \lambda_b \Rightarrow \lambda'^*_a = (M_a{}^b)^* \underbrace{\lambda_b^*}_{\text{call this } \lambda_{\dot b}}$$

! Infeld-van der Waerden 1933

$$\lambda'_{\dot a} = M^*{}_{\dot a}{}^{\dot b} \lambda_{\dot b}$$

or further $\tilde{\lambda}_{\dot b}$ or $\bar{\lambda}_{\dot b}$

also $\lambda^a = \varepsilon^{ab} \lambda_b \Rightarrow \lambda^{a*} = (\varepsilon^{ab})^* \lambda_b^*$

$$\Rightarrow \lambda^{\dot a} = \varepsilon^{\dot a \dot b} \lambda_{\dot b}$$

sometimes sign is changed here!

$$\text{thus } (\varepsilon_{ab})^* \equiv \varepsilon_{\dot a \dot b} \underbrace{= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{real!!}}$$

Exercise: Prove that

$$\varepsilon_{\substack{\mu\nu\alpha\beta \\ a\dot a\, b\dot b\, c\dot c\, d\dot d}} = i \left( \varepsilon_{ac} \varepsilon_{bd} \varepsilon_{\dot a \dot d} \varepsilon_{\dot b \dot c} - \varepsilon_{ad} \varepsilon_{bc} \varepsilon_{\dot a \dot c} \varepsilon_{\dot b \dot d} \right) \quad !$$

Work out total antisymmetry!

$\mu \to a\dot{a}$

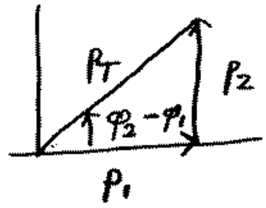
$$p_{a\dot{a}} = p_\mu \sigma^\mu{}_{a\dot{a}} = \begin{pmatrix} p_0 + p_3 & p_1 - i p_2 \\ p_1 + i p_2 & p_0 - p_3 \end{pmatrix}$$

If $p^2 = 0$ (massless particle) (or $x^2 = 0$, light cone)

this can be written as

$$p_{a\dot{a}} = \underset{\substack{\uparrow \\ \text{now put * explicitly!}}}{\lambda_a \lambda^*_{\dot{a}}} = \begin{pmatrix} \lambda_1 \lambda_1^* & \lambda_1 \lambda_2^* \\ \lambda_2 \lambda_1^* & \lambda_2 \lambda_2^* \end{pmatrix} \qquad \begin{cases} p_0 = \frac{1}{2}\left(|\lambda_1|^2 + |\lambda_2|^2\right) \\ p_3 = \cdots \\ p_1 = \cdots \end{cases}$$

$$\Rightarrow \quad \lambda = \begin{pmatrix} \sqrt{p_0 + p_3}\, e^{i\varphi_1} \\ \sqrt{p_0 - p_3}\, e^{i\varphi_2} \end{pmatrix}$$




$$\underset{\text{take } \varphi_1 = 0}{=} \begin{pmatrix} \sqrt{p_0 + p_3} \\ \sqrt{p_0 - p_3}\, \dfrac{p_1 + i p_2}{p_T} \end{pmatrix} \qquad \begin{aligned} p_0^2 - p_3^2 &= p_T^2 \\ \frac{p_0 - p_3}{p_T} &= \frac{p_T}{p_0 + p_3} \end{aligned}$$

$$= \begin{pmatrix} \sqrt{p_0 + p_3} \\ \dfrac{p_1 + i p_2}{\sqrt{p_0 + p_3}} \end{pmatrix}$$

Scalar product

$$p \cdot q = p_\mu q^\mu = p_{a\dot{a}}\, q^{a\dot{a}} = \varepsilon^{ab} \varepsilon^{\dot{a}\dot{b}}\, p_{a\dot{a}}\, q_{b\dot{b}}$$

$$= \lambda_a \lambda_{\dot{a}}\, \mu^a \mu^{\dot{a}} = \lambda^a \mu_a\, \lambda^{\dot{a}} \mu_{\dot{a}} = \lambda\mu\ \lambda^* \mu^*$$

$$= \underbrace{\langle \lambda, \mu \rangle}_{\substack{\text{prod.} \\ \text{in } a}} \underset{\substack{\text{prod} \\ \text{in } \dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]} \quad \text{(Witten's notation)}$$

So in a sense $\lambda\mu \sim \sqrt{p \cdot q}$ like $\vec{\alpha} \cdot \vec{p} + \beta m = \sqrt{\vec{p}^2 + m^2}$

$=$ vector space $(\lambda a + \mu b)$ with a product $ab$

Compare SuSy algebra:

Poincare + SuSy

Lorentz + Translations

$\Lambda = e^{i\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu}}$ $T = e^{ia^\mu P_\mu}$

superspace $x^\mu$ $Q_a$ $\bar{Q}_{\dot{a}}$

Add 4 spinor generators $Q_a$ $\bar{Q}_{\dot{a}}$ which anticommute

$M_{\mu\nu}$, $P_\mu$ satisfy Poincaré algebra (Lie algebra with commutators)

$$\begin{cases} [P_\mu, P_\mu] = 0 \\ [P_\mu, M_{\alpha\beta}] = i(g_{\mu\alpha}P_\beta - g_{\mu\beta}P_\alpha) \\ [M_{\mu\nu}, M_{\alpha\beta}] = -i\, g_{\mu\alpha}M_{\nu\beta} + 3 \text{ terms} \end{cases}$$

$$\{Q_a, Q_b\} = \{\bar{Q}_{\dot{a}}, \bar{Q}_{\dot{b}}\} = 0$$

and

conv.

operators here!

$$\boxed{\{Q_a, \bar{Q}_{\dot{a}}\} = 2\sigma^\mu{}_{a\dot{a}}\, P_\mu}$$

just numbers $\in \mathbb{C}$ here

graded Lie algebra

$$\boxed{p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} = \sigma^\mu{}_{a\dot{a}}\, p_\mu}$$

$\sigma^\mu{}_{a\dot{a}}$ is the same!

$$2P_0 = \frac{1}{2}\left[Q_1\bar{Q}_i + \bar{Q}_i Q_1 + (1\to 2)\right] \geq 0 \quad \langle 0|H|0\rangle \geq 0 \quad !$$

Next take the massless Poincaré states $|p, h\rangle$, $h = \sigma =$ helicity discussed on p. 1-3 and see how $Q_a$ $\bar{Q}_{\dot{a}}$ affect them. (e.g., Bailin-Love, SuSy..., 1.4). Take $Q_a|p\, h\rangle = 0$ (anyway $Q_a Q_a = 0$), $a = 1, 2$; then, taking $p^\mu = (E, 0, 0, E)$, $Q_a\bar{Q}_{\dot{a}} + \bar{Q}_{\dot{a}}Q_a = 4E\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}_{a\dot{a}}$

find $\bar{Q}_{\dot{1}}|p, h\rangle = 0$, $\boxed{\bar{Q}_{\dot{2}}|p, h\rangle = \sqrt{4E}\,|p, h-\tfrac{1}{2}\rangle}$ !!

SUSY means $|p, h\rangle$ $|p, h-\tfrac{1}{2}\rangle$ both appear !!

$|p, h=1\rangle$ $|p, h=\tfrac{1}{2}\rangle$

gluon gluino

Extended SuSy: The previous was $N=1$ SuSy, only $Q_a$

Try to add more $Q_a^i$, $i=1,\dots,N$, with some internal symm. built in $i$.

$$\{Q_a^i, Q_b^j\} = 0 \quad \Big[\text{actually even } = \varepsilon_{ab}\underbrace{\underset{-Z^{ji}}{Z^{ij}}}_{\text{center charges, if any exist}} \text{ is allowed}\Big]$$

$$\underbrace{\{Q_a^i, \bar{Q}^j_{\dot{a}}\}} = \delta^{ij}\, 2\sigma^\mu_{a\dot{a}} P_\mu$$

like $\{\vartheta_i, \vartheta_j^*\} \underset{\text{transform}}{=} U_{ik} U^*_{jl} \underbrace{\{\vartheta'_k, \vartheta'^*_l\}}_{=\delta_{kl}} = (UU^\dagger)_{ij} = \delta_{ij}$

$\vartheta_i = U_{ik}\vartheta'_k$ $\qquad$ if $UU^\dagger = 1$

$\Rightarrow$ $U(N)$ internal invariance

What are the states now? $\quad \bar{Q}^1_{\dot{2}} \quad \bar{Q}^2_{\dot{2}} \quad \bar{Q}^3_{\dot{2}} \quad \bar{Q}^4_{\dot{2}} \quad N=4$

$|p, h\rangle \qquad p^\mu = (E, 0, 0, E)$

apply one $\bar{Q}^i_{\dot{2}}$ → $|p, h-\frac{1}{2}, i=1,\dots,4\rangle$ — $N=4$ states

apply two $\bar{Q}^i_{\dot{2}}$ → $|p, h-1, \dots\rangle$ — $\binom{N}{2} = \frac{N(N-1)}{2} = 6$ states

apply three $\bar{Q}^i_{\dot{2}}$ → $|p, h-\frac{3}{2}, \dots\rangle$ — $\binom{N}{3} = \frac{N!}{3!(N-3)!} = 4$ states

apply four $\bar{Q}^i_{\dot{2}}$ → $|p, h-2, \dots\rangle$ — $\binom{N}{4} = 1$ state

**$N=4$ susy YM,**

| $h$ | | |
|---|---|---|
| $1$ | 1 state | 1 gluon |
| $\frac{1}{2}$ | 4 states | 4 gluinos |
| $0$ | 6 " | 6 scalars |
| $-\frac{1}{2}$ | 4 " | |
| $-1$ | 1 state | |

$$\begin{cases} g_B = 1\cdot 2 + 6\cdot 1 = 8 \\ g_F = 4\cdot 2 = 8 \end{cases}$$

$$p(T) = \underbrace{\left(g_B + \tfrac{7}{8} g_F\right)}_{15} \cdot N^2_{\text{color}} \frac{\pi^2}{90} T^4$$