Quasihyperbolic geodesics in convex domains

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Abstract

We show that quasihyperbolic geodesics exist in convex domains in reflexive Banach spaces and that quasihyperbolic geodesics are quasiconvex in the norm metric in convex domains in all normed spaces.

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1 Introduction

1.1. Let E be a real normed vector space with $\dim E \geq 2$ and let $G \subsetneq E$ be a domain. The *quasihyperbolic length* of a rectifiable arc $\gamma \subset G$ or a path γ in G is the number

$$l_k(\gamma) = \int_\gamma \frac{|dx|}{\delta(x)},$$

where $\delta(x) = d(x, E \setminus G)$. For $a, b \in G$, the *quasihyperbolic distance* $k(a,b) = k_G(a,b)$ is defined by

$$k(a,b) = \inf l_k(\gamma)$$

over all arcs γ joining a and b in G. An arc γ from a to b is a *quasihyperbolic geodesic* if $l_k(\gamma) = k(a,b)$. Each subarc of a geodesic is obviously a geodesic.

The quasihyperbolic metric of a domain in R^n was introduced by F.W. Gehring and B. Palka [GP] in 1976. It is well known that a quasihyperbolic geodesic between any two points exists if $\dim E < \infty$; see [GO, Lemma 1]. This is not true in arbitrary spaces; a counterexample (due to P. Alestalo) is given in [Vä2, 3.5]. However, we show in 2.1 that quasihyperbolic geodesics exist in convex domains in reflexive Banach spaces.

In section 3 we show that quasihyperbolic geodesics in convex domains are c_0-quasiconvex (definition in 3.1) in the norm metric with a universal constant c_0. A simpler proof with a better estimate for c_0 in inner product spaces is given in Section 4. This result is related to the theorems of Gehring-Hayman [GH] and Heinonen-Rohde [HR], but it seems to be new also in $\mathsf{R}^n,\ n \geq 3$.

1.2. *Notation.* Throughout the paper, E is a real normed space of dimension ≥ 2 and $G \subsetneq E$ is a domain. The norm of a vector x is written as $|x|$. We let $[a,b]$ denote the line segment with endpoints $a, b \in E$. We write $\alpha\colon a \curvearrowright b$ if α is an arc from a to b. If needed, this notation also gives an orientation for α from a to b. The length of an arc α is $l(\alpha)$. We let $\alpha[x,y]$ denote the subarc of an arc α between points $x, y \in \alpha$. For open and closed balls and for spheres we use the customary notation $B(x,r),\ \bar{B}(x,r),\ S(x,r)$. The diameter of a set A is $d(A)$ and the distance between nonempty sets A, B is $d(A,B)$. For real numbers s,t we set $s \wedge t = \min\{s,t\},\ s \vee t = \max\{s,t\}$.

2 Existence of quasihyperbolic geodesics

2.1. Theorem. *Let E be a reflexive Banach space, let $G \subsetneq E$ be a convex domain and let $a, b \in G,\ a \neq b$. Then there is a quasihyperbolic geodesic $\gamma\colon a \curvearrowright b$.*

Proof. Fact 1. The function $\delta = \delta_G\colon E \to \mathsf{R}$ is upper semicontinuous in the weak topology of E.

If G is a half space, then $\delta|\bar{G}$ is affine, and hence δ is continuous in the weak topology. In the general case, G is the intersection of the family $\mathcal{A}$ of all half spaces containing G; see [CC, I.3.1.4]. Since $\delta = \inf\{\delta_H\colon H \in \mathcal{A}\}$, Fact 1 follows.

Choose a sequence of arcs $\gamma_i\colon a \curvearrowright b$ such that $l_k(\gamma_i) \to k(a,b)$ and set $M = \sup\{l_k(\gamma_i)\colon i \in \mathsf{N}\}$. By the standard estimate $k(x,a) \geq \log(\delta(x)/\delta(a))$ we obtain $\delta(x) \leq \delta(a)e^M$ for all $x \in \gamma_i$. Setting $\lambda_i = l(\gamma_i)$ we thus have $\lambda_i \leq M_1 = M\delta(a)e^M$ for all i. Passing to a subsequence we may therefore assume that $\lambda_i \to \lambda \geq |a-b|$ and that $\lambda_i \leq 2\lambda$ for all i.

Let $\varphi_i\colon [0,\lambda_i] \to \gamma_i$ be the length parametrization of γ_i with $\varphi_i(0) = a$. Setting $K_i = \lambda_i/\lambda$ and $\psi_i(t) = \varphi_i(K_i t)$ we obtain another parametrization $\psi_i\colon [0,\lambda] \to \gamma_i$, which is K_i-Lipschitz with $K_i \to 1$.

Let A be a countable dense subset of $[0,\lambda]$ containing $\{0,\lambda\}$. We have $|\psi_i(t) - a| \leq M_1$ for all $t \in [0,\lambda]$ and for all i. Since E is reflexive, each bounded sequence in E has a subsequence converging weakly to an element of E; see [Yo, Th. V.2.1]. Using a diagonal argument we may therefore assume that $\psi_i(t) \to \psi(t) \in E$ weakly for each $t \in A$.

For $s, t \in A$ we have

$$|\psi(s) - \psi(t)| \leq \liminf_{i\to\infty} |\psi_i(s) - \psi_i(t)| \leq \liminf_{i\to\infty} K_i|s-t| = |s-t|$$

by [Yo, Th. V.1.1]. As E is complete, the map ψ extends to a 1-Lipschitz map $g\colon [0,\lambda] \to E$ with $g(0) = a$ and $g(\lambda) = b$.

Let $u \in (0,\lambda]$ be a number such that $g[0,u] \subset G$. By Fact 1 we have $\delta(g(t)) \geq \limsup_{i\to\infty} \delta(\psi_i(t))$ for each $t \in A$. Since g and the maps ψ_i are 2-Lipschitz, this holds for all $t \in [0,\lambda]$. As g is 1-Lipschitz, we have $|g'(t)| \leq 1$ for almost every $t \in [0,u]$. By Fatou's lemma we get

$$\begin{aligned} l_k(g|[0,u]) &= \int_0^u \frac{|g'(t)|}{\delta(g(t))} dt \leq \int_0^u \liminf_{i\to\infty} \frac{1}{\delta(\psi_i(t))} dt \leq \liminf_{i\to\infty} \int_0^u \frac{dt}{\delta(\psi_i(t))} \\ &= \liminf_{i\to\infty} \frac{1}{K_i} \int_0^{K_i u} \frac{ds}{\delta(\varphi_i(s))} = \liminf_{i\to\infty} \frac{l_k(\varphi_i|[0,K_i u])}{K_i} \leq k(a,b). \end{aligned}$$

This implies that $\operatorname{im} g \subset G$ and that $l_k(g) \leq k(a,b)$, whence g defines a quasihyperbolic geodesic from a to b. $\square$

2.2. *Remark.* I do not know whether Theorem 2.1 holds for all Banach spaces. A related result for dual spaces is obtained as follows. Let E be a separable Banach space. Then each bounded sequence in the dual space E^* has a subsequence converging to an element of E^* in the weak* topology. If $G \subset E^*$ is a convex domain such that δ_G is upper semicontinuous in the weak* topology, the proof of 2.1 shows that points of G can be joined by a quasihyperbolic geodesic. This condition need not be true for an arbitrary convex domain, but it is true for a domain that is the intersection of a family of half spaces of the form $\{f \in E^*\colon fx > t\}$ where $x \in E$ and $t \in \mathsf{R}$. Consequently, quasihyperbolic geodesics exist in such domains in E^*.

3 Metric properties of geodesics

3.1. *Quasiconvexity.* An arc γ in a metric space (X,d) is of *c-bounded turning* if $d(\gamma[x,y]) \leq cd(x,y)$ for all $x,y \in \gamma$, and γ is *c-quasiconvex* if it satisfies the stronger condition $l(\gamma[x,y]) \leq cd(x,y)$. In particular, an arc is 1-quasiconvex iff it is a geodesic.

This section is devoted to the proof of the following result.

3.2. Theorem. *Let G be a convex domain in a normed space E and let $\gamma \subset G$ be a quasihyperbolic geodesic. Then γ is c_0-quasiconvex in the norm metric with a universal constant c_0.*

3.3. *Remarks.* 1. A simpler proof for the theorem with a better estimate for c_0 in inner product spaces is given in Section 4.

2. By the classical theorem of Gehring and Hayman [GH], a hyperbolic geodesic in a simply connected domain $G \subset \mathsf{R}^2$ is c_0-quasiconvex in the inner metric of the domain. An

extension to R^n was given by Heinonen and Rohde [HR], who proved that if a domain $G \subset \mathsf{R}^n$ is a K-quasiconformal image of a c-uniform domain, then each quasihyperbolic geodesic in G is c_1-quasiconvex in the inner metric with $c_1(c, K, n)$.

In a convex domain, the inner metric agrees with the norm metric. Since our constant is universal and since for $n \geq 3$, the domain between two parallel hyperplanes in R^n is not quasiconformally equivalent to a uniform domain, Theorem 3.2 seems to be new also in $\mathsf{R}^n,\ n \geq 3$.

The proof of 3.2 is given as a sequence of lemmas. In these lemmas we assume that $G \subsetneq E$ is a convex domain.

3.4. Lemma. *The function $\delta\colon G \to \mathsf{R}$ is concave, that is,*

$$\delta(\lambda a + \mu b) \geq \lambda\delta(a) + \mu\delta(b)$$

whenever $a, b \in G,\ 0 < \lambda < 1,\ \mu = 1 - \lambda$.

Proof. Write $z = \lambda a + \mu b,\ r = \lambda\delta(a) + \mu\delta(b)$. Let w be a point in E with $|w| < r$. It suffices to show that $z + w \in G$. Set $\alpha = \delta(a)/r,\ \beta = \delta(b)/r$. Since $|\alpha w| < \delta(a),\ |\beta w| < \delta(b)$, we have $a + \alpha w \in G,\ b + \beta w \in G$. As G is convex, the point $v = \lambda(a + \alpha w) + \mu(b + \beta w)$ lies in G. Since $v = z + w$, the lemma is proved. □

3.5. Lemma. *For each pair $a, b \in G$ we have*

$$k(a, b) \leq l_k([a, b]) \leq \frac{|a - b|}{\delta(a) \wedge \delta(b)}.$$

Proof. By 3.4 we have $\delta(x) \geq \delta(a) \wedge \delta(b)$ for all $x \in [a, b]$. □

3.6. Lemma. *Let $a \in G,\ |u| = 1,\ \Delta = \{t > 0\colon a + tu \in G\}$. Then the function $\omega\colon \Delta \to \mathsf{R}$, defined by $\omega(t) = \delta(a + tu)/t$, is strictly decreasing.*

Proof. Observe that Δ is an open interval $(0, R)$, possibly $R = \infty$. Let $0 < s < t < R$. Setting $\lambda = s/t$ we have $a + su = (1 - \lambda)a + \lambda(a + tu)$. By 3.4 this gives $\delta(a + su) > \lambda\delta(a + tu)$, whence $\omega(s) > \omega(t)$. □

3.7. Lemma. *Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in G. Then either δ is monotone on γ or there is a point $z \in \gamma$ such that δ is increasing on $\gamma[a, z]$ and decreasing on $\gamma[z, b]$.*

Proof. Assume that the lemma is false. Then there is a subarc $\beta = \gamma[u, v]$ such that $\delta(u) = \delta(v) = r$ and such that $\delta(x) < r$ for all $x \in \beta \setminus \{u, v\}$. By 3.4 we have $\delta(x) \geq r$ for all x on the line segment $\alpha = [u, v]$. Hence $l_k(\alpha) < l_k(\beta)$. As β is a geodesic, this gives a contradiction. □

3.8. Lemma. *Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in G such that δ is increasing on γ, and let $z \in \gamma$ be the first point of γ with $|a-z| = |a-b|/2$. Then $\delta(b) \le c_1\delta(z)$ where $c_1 \le 14.68$ is a universal constant.*

Proof. Set $R = |a-z| = |a-b|/2$ and $M = \delta(b)/\delta(z)$. We must find an estimate $M \le c_1$.

We may assume that $M \ge 14$. Normalize $a = 0$. Then $|z| = R$, $|b| = 2R$. Set $r_0 = \delta(a)$, $r_1 = \delta(z)$, $r_2 = \delta(b) = Mr_1$. For $0 \le t \le 2R$, let $\varphi_0(t)$ and $\varphi_1(t)$ be the first and the last point of γ with $|\varphi_0(t)| = |\varphi_1(t)| = t$. Setting $f_i(t) = \delta(\varphi_i(t))$ we get increasing functions $f_i\colon [0,2R] \to [r_0, r_2]$ with $f_0(0) = r_0,\ f_0(R) = r_1,\ f_1(2R) = r_2$. Moreover, the function f_0 is left continuous and f_1 is right continuous.

If E is an inner product space, the proof is somewhat simpler, because γ meets every sphere $S(t)$ at only one point by 4.5, and therefore $\varphi_0 = \varphi_1$ and the function $f_0 = f_1$ is continuous.

Set $e = b/|b|$ and let $\psi\colon [0,2R] \to G$ be the segmental path $\psi(t) = te$ from a to b. Then the function $g = \delta \circ \psi\colon [0,2R] \to \mathsf{R}$ is continuous with $g(0) = r_0,\ g(2R) = r_2$.

Set

$$\begin{aligned} t_0 &= \max\{t\colon 0 \le t \le R,\ g(t) \le f_0(t)\},\\ t_1 &= \max\{t\colon g(t) \le r_1\},\\ t_2 &= \min\{t\colon R \le t \le 2R,\ g(t) \le f_1(t)\}. \end{aligned}$$

Let $0 \le t \le 2R$. By 3.4 we obtain

$$g(t) \ge \left(1 - \frac{t}{2R}\right) r_0 + \frac{t}{2R} r_2 \ge \frac{tMr_1}{2R}, \tag{3.9}$$

and hence $g(t) > r_1$ for $t > 2R/M$. As $M > 14$, this yields

$$t_1 \le R/7, \tag{3.10}$$

whence $f_0(R) < g(R)$.

The following statements are clearly true:

(1) $0 \le t_0 \le t_1 < R \le t_2 \le 2R$,

(2) $g(t_0) \le f_0(t_0),\ g(t_1) \le f_1(t_1)$,

(3) $f_0 < g$ on $(t_0, R]$, $f_1 < g$ on $[R, t_2)$.

We construct a path ω from 0 to b as the composition $\omega = \omega_1 \dots \omega_7$, where

ω_1 and ω_7 are parametrizations of the arcs $\gamma[0, \varphi_0(t_0)]$ and $\gamma[\varphi_1(t_2), b]$,

ω_2 and ω_6 are segmental paths from $\varphi_0(t_0)$ to $\psi(t_0)$ and from $\psi(t_2)$ to $\varphi_1(t_2)$,

$\omega_3 = \psi|[t_0, t_1]$, $\omega_4 = \psi|[t_1, R]$, $\omega_5 = \psi|[R, t_2]$.

Some of the paths ω_i may degenerate to points.

As γ is a quasihyperbolic geodesic, we have $l_k(\gamma) \le l_k(\omega)$, whence

$$l_k(\gamma|[\varphi_0(t_0), \varphi_1(t_2)]) \le l_k(\omega_2) + \cdots + l_k(\omega_6).$$

Let $A \subset \gamma$ be the closure of $\varphi_0(t_0, t_1]$ and define a map π of A onto $[t_0 e, t_1 e]$ by $\pi x = |x| e$. By (3) we have

$$\delta(\varphi_0(t)) = f_0(t) < g(t) = \delta(\pi(\varphi_0(t)))$$

for all $t_0 < t \le t_1$, whence $\delta(x) \le \delta(\pi x)$ for all $x \in A$. Consequently,

$$l_k(\gamma[\varphi_0(t_0), \varphi_0(t_1)]) \ge \int_A \frac{dm_1 x}{\delta(\pi x)},$$

where m_1 is the Hausdorff 1-measure. As a 1-Lipschitz map, π decreases all Hausdorff measures. Since $\pi A = [t_0 e, t_1 e]$, we obtain

$$l_k(\gamma[\varphi_0(t_0), \varphi_0(t_1)]) \ge \int_{t_0}^{t_1} \frac{dt}{\delta(\psi(t))} = l_k(\omega_3).$$

Similarly $l_k(\gamma[z, \varphi_1(t_2)]) \ge l_k(\omega_5)$. It follows that

$$l_k(\gamma[\varphi_0(t_1), z]) \le l_k(\omega_2) + l_k(\omega_4) + l_k(\omega_6).$$

We estimate the terms of this inequality.

Since $f_0(t) \le r_1$ for $t \le R$, (3.10) gives

$$l_k(\gamma[\varphi_0(t_1), z]) \ge (R - t_1)/r_1 \ge 6R/7r_1.$$

From (2), 3.5 and (3.9) we infer

$$l_k(\omega_2) \le \frac{2t_0}{t_0 M r_1 / 2R} = \frac{4R}{M r_1},$$

and similarly $l_k(\omega_6) \le 4R/Mr_1$.

We still need an upper bound for $l_k(\omega_4)$. For $t_1 \le t \le R$ we have

$$g(t) \ge \left(1 - \frac{t - t_1}{2R - t_1}\right) r_1 + \frac{t - t_1}{2R - t_1} M r_1 \ge \frac{2R - t_1 M + t(M-1)}{2R} r_1.$$

Hence

$$\begin{aligned} l_k(\omega_4) &\le \frac{2R}{r_1} \int_{t_1}^{R} \frac{dt}{2R - t_1 M + t(M-1)} = \frac{2R}{r_1(M-1)} \log \frac{2R - t_1 M + RM - R}{2R - t_1} \\ &\le \frac{2R}{r_1(M-1)} \log \frac{R + RM}{2R - t_1}. \end{aligned}$$

By (3.10) this gives

$$l_k(\omega_4) \le \frac{2R}{r_1(M-1)} \log \frac{7 + 7M}{13}.$$

Combining the estimates yields

$$\frac{6}{7} \leq \frac{8}{M} + \frac{2}{M-1} \log \frac{7+7M}{13}.$$

This inequality is not true for $M = 14.68$. As the right-hand side is decreasing in M, the lemma follows. □

3.11. Lemma. *Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in G such that δ is monotone on γ. Then $l(\gamma) \leq 3c_1|a-b|$ where $c_1 \leq 14.68$ is the constant of Lemma 3.8.*

Proof. We may assume that δ is increasing on γ. Set $R = |a-b|$ and $R_j = 2^{-j}R$, $j = 0, 1, 2, \ldots$. Let a_j be the first point of γ with $|a_j - a| = R_j$ and set $\gamma_j = \gamma[a_{j+1}, a_j]$. Setting $\beta_j = [a_{j+1}, a_j]$ we have $l(\beta_j) \leq 3R_{j+1}$. For $\delta_j = \delta(a_j)$ we obtain $\delta_{j+1} \leq \delta_j \leq c_1\delta_{j+1}$ by 3.8. By 3.5 this yields $l_k(\beta_j) \leq l(\beta_j)/\delta_{j+1} \leq 3c_1R_{j+1}/\delta_j$. As δ is increasing on γ, we have $l_k(\gamma_j) \geq l(\gamma_j)/\delta_j$. Moreover, $l_k(\gamma_j) \leq l_k(\beta_j)$, since γ_j is a geodesic. Combining the inequalities we get $l(\gamma_j) \leq 3c_1R_{j+1}$, whence

$$l(\gamma) \leq 3c_1 \sum_{j\geq 0} R_{j+1} = 3c_1R. \quad \square$$

3.12. Lemma. *Let $a, b \in G$, let $0 < r < R = |a-b|$, and set*

$$q = \frac{1}{r} \sup\{\delta(x)\colon x \in S(a,r) \cap G\}.$$

If $y \in S(a,r)$, $\lambda > 1$, $\delta(y) \geq qr/\lambda$, then there is an arc $\beta\colon a \curvearrowright y$ such that $\beta \subset \bar{B}(a,r) \cap G$ and

$$l_k(\beta) + \frac{1}{q} \log \frac{R}{r} \leq k(a,b) + \frac{2\lambda}{q}.$$

Proof. We may assume that $a = 0$. Let $h > 0$ and choose an arc $\gamma\colon a \curvearrowright b$ with $l_k(\gamma) \leq k(a,b) + h$. For $0 \leq t \leq r$ set $y_t = ty/|y|$, and let z_t be the first point of γ with $|z_t| = t$. Set $f(t) = \delta(z_t)$, $g(t) = \delta(y_t)$, and

$$s = \sup\{t \in [0,r]\colon g(t) \leq f(t)\}.$$

Then g is continuous and f is left continuous, whence $g(s) \leq f(s)$. Let $\beta\colon a \curvearrowright y$ be an arc contained in $\gamma[a, z_s] \cup [z_s, y_s] \cup [y_s, y]$.

As δ is concave by 3.4, we have

$$g(s) \geq s\delta(y)/r \geq qs/\lambda.$$

Hence Lemma 3.5 gives

$$l_k([z_s, y_s]) \leq \frac{\lambda|z_s - y_s|}{qs} \leq \frac{2\lambda}{q}. \tag{3.13}$$

Set $A = \operatorname{cl}\{z_t \colon s \le t \le r\}$ and define $\pi \colon A \to [y_s, y]$ by $\pi x = |x| y / r$. Since $g(t) > f(t)$ for $t \in (s, r]$, we can argue as in the proof of 3.8 and get

$$(3.14) \qquad l_k(\gamma[z_s, z_r]) \ge \int_A \frac{dm_1 x}{\delta(\pi x)} \ge \int_s^r \frac{dt}{g(t)} = l_k([y_s, y]).$$

Let $x \in G \setminus B(a, r)$ and set $x_r = rx/|x|$. Since $\delta(x_r) \le qr$, the concavity of δ gives $\delta(x) \le q|x|$. Consequently, $l_k(\gamma[z_r, b]) \ge \frac{1}{q} \log \frac{R}{r}$. By (3.13) and (3.14) this gives

$$l_k(\beta) + \frac{1}{q} \log \frac{R}{r} \le l_k(\gamma) + \frac{2\lambda}{q} \le k(a, b) + h + \frac{2\lambda}{q},$$

and the lemma follows. □

3.15. Lemma. *If $\gamma \colon a_1 \curvearrowright a_2$ is a quasihyperbolic geodesic, then $|a_1 - z| + |z - a_2| \le c_2 |a_1 - a_2|$ for all $z \in \gamma$ with a universal constant $c_2 \le 2e^{14/3} + 1 < 214$.*

Proof. Set $r = |a_1 - a_2|$ and $q_i r = \sup\{\delta(x) \colon x \in S(a_i, r) \cap G\}$, $i = 1, 2$. We may assume that $q_1 \le q_2$. Let $x \in S(a_2, r) \cap G$ be a point with $|x - a_1| \ge r$, and let $x_1 \in [a_1, x]$ be the point with $|x_1 - a_1| = r$. By concavity of δ we have $\delta(x_1) \ge r\delta(x)/|x - a_1|$, whence $\delta(x) \le 2q_1 r$.

Suppose that $\delta(x') > 2q_1 r$ for some $x' \in S(a_2, r) \cap G$ with $|x' - a_1| < r$. Let $u \in G \setminus B(a_1, r)$ and let $u_1 \in [x', u] \cap S(a_1, r)$. Again by concavity we get

$$q_1 r \ge \delta(u_1) \ge \left(1 - \frac{|u_1 - x'|}{|u - x'|}\right) \delta(x') + \frac{|u_1 - x'|}{|u - x'|} \delta(u) > \left(1 - \frac{2r}{|u - x'|}\right) 2q_1 r,$$

whence $|u - x'| < 4r$. Thus $G \subset B(x', 4r)$, and the inequality of the lemma holds with $c_2 = 10$. We may therefore assume that $q_2 \le 2q_1$.

Let $z \in \gamma$ and set $R_i = |z - a_i|$, $i = 1, 2$. If $R_1 \wedge R_2 \le r$, then $R_1 + R_2 \le 3r$. Hence we may assume that $R_i > r$ for both i. Let $\lambda > 1$ and let $\beta_i \colon a_i \curvearrowright y_i \in S(a_i, r)$ be the arc given by Lemma 3.12 for $a = a_i$, $b = z$. Then

$$(3.16) \qquad l_k(\beta_i) + \frac{1}{q_i} \log \frac{R_i}{r} \le k(a_i, z) + \frac{2\lambda}{q_i}.$$

Since $\delta(y_i) \ge q_i r/\lambda \ge q_1 r/\lambda$, Lemma 3.5 gives

$$l_k([y_1, y_2]) \le |y_1 - y_2| \lambda / q_1 r \le 3\lambda / q_1,$$

which implies that $k(a_1, a_2) \le l_k(\beta_1) + l_k(\beta_2) + 3\lambda/q_1$. On the other hand, $k(a_1, a_2) = k(a_1, z) + k(a_2, z)$. Hence (3.16) gives

$$\frac{1}{q_1} \log \frac{R_1}{r} + \frac{1}{q_2} \log \frac{R_2}{r} \le \frac{3\lambda}{q_1} + \frac{2\lambda}{q_1} + \frac{2\lambda}{q_2}.$$

Since $q_1 \le q_2 \le 2q_1$, we obtain $\log(R_1^2 R_2 / r^3) \le 14\lambda$. As $\lambda \to 1$, this yields $R_1 \wedge R_2 \le e^{14/3} r$. This implies the lemma because $|R_1 - R_2| \le r$. □

3.17. *Proof of Theorem 3.2.* If δ is monotone on γ, the theorem follows from 3.11. In the general case we apply Lemma 3.7 and divide γ into two subarcs on which δ is monotone, and the theorem follows from 3.11 and 3.15 with $c_0 = 3c_1c_2$. $\square$

3.18. *Remark.* The proof gives the numerical bound $c_0 < 9500$, which is presumably very far from the best possible. The example in 5.1 gives the lower bound $c_0 \geq 2$. In inner product spaces we obtain in the next section the better bound $c_0 < 36$.

4 Inner product spaces

In this section we assume that E is an inner product space. The inner product of vectors x, y is written as $x \cdot y$. We shall give a simplified proof of Theorem 3.2 in this case and obtain a better estimate for the constant c_0.

4.1. Theorem. *Let G be a convex domain in an inner product space E and let $\gamma \subset G$ be a quasihyperbolic geodesic. Then γ is c_0-quasiconvex in the norm metric with a universal constant $c_0 < 36$.*

For the proof, we first study the behavior of quasihyperbolic length in a central projection. The *central projection* onto a sphere $S(z,r) \subset E$ is the map $p\colon E \setminus \{z\} \to S(z,r)$, defined by $px = z + r(x-z)/|x-z|$.

4.2. Lemma. *Let $z \in G$, let $\varphi\colon [a,b] \to G \setminus B(z,r)$ be a nonconstant rectifiable path and let p be the central projection onto $S(z,r)$. Then*

$$l_k(p \circ \varphi) \leq l_k(\varphi),$$

where the equality holds iff $\operatorname{im}\varphi \subset S(z,r)$.

Proof. Normalize $z = 0,\ r = 1$. Then $px = x/|x|$, $|p'(x)| = 1/|x|$, and we obtain

$$l_k(p \circ \varphi) = \int_{p\circ\varphi} \frac{|dx|}{\delta(x)} \leq \int_{\varphi} \frac{|p'(x)||dx|}{\delta(px)} = \int_{\varphi} \frac{|dx|}{|x|\delta(px)};$$

see, for example, [Vä1, 5.3], whose proof is valid in all normed spaces. By 3.6 we have $\delta(px) \geq \delta(x)/|x|$ for $x \in \operatorname{im}\varphi$ with equality iff $|x| = 1$, and the lemma follows. $\square$

4.3. *Remark.* In a normed space, similar arguments give the inequality $l_k(p \circ \varphi) \leq 2l_k(\varphi)$. I have not found this useful in the problems of the present paper.

4.4. Lemma. *Let $z \in G$ and let $a, b \in \bar{B}(z,r) \cap G$. If $\gamma\colon a \curvearrowright b$ is a quasihyperbolic geodesic, then $\gamma \setminus \{a,b\} \subset B(z,r)$.*

Proof. We first show that $\gamma \subset \bar{B}(z, r)$. If this is not true, then there is a subarc $\beta = \gamma[u, v]$ with $\beta \cap \bar{B}(z, r) = \{u, v\}$. Let φ be the length parametrization of β. Then 4.2 gives the contradiction $k(u, v) \le l_k(p \circ \varphi) < l_k(\varphi) = k(u, v)$.

Assume that there is a point $y \in \gamma \setminus \{a, b\}$ with $|y - z| = r$. As G is open, there are $z_1 \in G$ and $r_1 > 0$ such that $a, b \in \bar{B}(z_1, r_1)$, $y \notin \bar{B}(z_1, r_1)$. Since $\gamma \subset \bar{B}(z_1, r_1)$ by the first part of the proof, we obtain a contradiction. □

4.5. Corollary. *The following statements hold for a quasihyperbolic geodesic $\gamma\colon a \curvearrowright b$ in G.*

(1) *For each $r \in [0, |a - b|]$ there is a unique point $x \in \gamma$ with $|x - a| = r$.*

(2) *If u, v, w are successive points of γ, then*

$$(u - v) \cdot (w - v) > 0, \quad |u - v| + |v - w| \le \sqrt{2}|u - w|.$$

(3) *γ is of 1-bounded turning.* □

4.6. *Proof of Theorem 4.1.* The proof makes use of Lemmas 3.4–3.8, but the proof for 3.8 in the present case is somewhat simpler because $\varphi_0 = \varphi_1$ by 4.5(1). In the proof of 3.11, we apply 4.5(2) to conclude that the arc γ_j lies in the ball segment $\{x \in \bar{B}(a, R_j)\colon x \cdot a_{j+1} \ge R_{j+1}^2\}$, whence $l(\beta_j) \le R_{j+1}\sqrt{3}$. Hence Lemma 3.11 holds with 3 replaced by $\sqrt{3}$. Lemmas 3.12 and 3.15 are replaced by the second inequality of 4.5(2), and we obtain the theorem as in 3.17 with $c_0 = c_1\sqrt{6} < 36$. □

5 An example

We consider the plane $E = l_1^2$ with the norm $\|x\| = |x_1| + |x_2|$.

5.1. Theorem. *Quasihyperbolic geodesics in the half plane $H = \{x \in l_1^2\colon x_2 > 0\}$ are subarcs of the polygonal arcs with vertices $(s, 0)$, (s, r), $(s + 2r, r)$, $(s + 2r, 0)$ where $s \in \mathsf{R}$, $r > 0$.*

Proof. Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in H. Now $\delta(x) = x_2$ for all $x \in H$.

Case 1. δ is monotone, say increasing, on γ. Then $a_2 \le b_2$. Set $u = (a_1, b_2)$ and $\alpha = [a, u] \cup [u, b]$. We show that $l_k(\alpha) \le l_k(\gamma)$.

Let $\varepsilon > 0$ and let $D = (x^0, \dots, x^N)$ be a finite sequence of successive points of γ with $x^0 = a$, $x^N = b$ such that $\|x^{i-1} - x^i\| \le \varepsilon$ for all $i = 1, \dots, N$. Consider the Riemann sum

$$s_D = \sum_{i=1}^{N} \|x^{i-1} - x^i\| / \delta(x^i),$$

which converges to $l_k(\gamma)$ as $\varepsilon \to 0$. We have

$$\|x^{i-1} - x^i\| = |x_1^{i-1} - x_1^i| + x_2^i - x_2^{i-1}, \quad \delta(x^i) = x_2^i \le b_2,$$

whence

$$s_D \geq |b_1 - a_1|/b_2 + \sum_{i=1}^{N} (x_2^i - x_2^{i-1})/x_2^i.$$

As $\varepsilon \to 0$, this yields

$$l_k(\gamma) \geq \frac{|b_1 - a_1|}{b_2} + \int_{a_2}^{b_2} \frac{dt}{t} = l_k(\alpha).$$

If $\gamma \neq \alpha$, then there is a subarc $\beta \subset \gamma$ such that β is not a vertical segment and $x_2 < b_2$ for all $x \in \beta$. Now the proof above gives the contradiction $l_k(\gamma) > l_k(\alpha)$. Hence $\gamma = \alpha$.

Case 2. δ is not monotone on γ. By Lemma 3.7 there is a point $y \in \gamma$ such that δ is increasing on $\gamma[a, y]$ and decreasing on $\gamma[y, b]$. Set $u = (a_1, y_2),\ v = (b_1, y_2)$. By Case 1 we have $\gamma = [a, u] \cup [u, v] \cup [v, b]$ and

$$l_k(\gamma) = \log \frac{y_2}{a_2} + \frac{|a_1 - b_1|}{y_2} + \log \frac{y_2}{b_2}.$$

The function $y_2 \mapsto l_k(\gamma)$ attains its minimum at $y_2 = |a_1 - b_1|/2$. The theorem follows. $\square$

5.2. *Remarks.* Theorem 5.1 gives the lower bound $c_0 \geq 2$ for the constant of 3.2. Moreover, we see that Lemmas 4.2 and 4.4 are not tue in l_1^2. In Corollary 4.5, part (3) and the inequality of (2) are not true but I do not know whether (1) holds in all normed spaces.

6 Open problems

(1) Is Theorem 2.1 true in all Banach spaces? In particular, do the quasihyperbolic geodesics exist in each half space of l_1 or l_∞?

(2) Find better estimates (or even sharp bounds) for the constant c_0 in 3.2 and 4.1.

(3) An arc $\gamma \subset G$ is a *c-quasigeodesic* if $l_k(\gamma[x, y]) \leq ck(x, y)$ for all $x, y \in \gamma$, that is, γ is c-quasiconvex in the quasihyperbolic metric. Is a c-quasigeodesic in a convex domain $c'(c)$-quasiconvex in the norm metric?

(4) Let us say that a domain G in a normed space E is a *c-Gehring-Hayman domain* if every quasihyperbolic geodesic is c-quasiconvex in the inner metric of G. Find a geometric characterization for these domains.

(5) Is there a free version of the theorem of Heinonen-Rohde? For example, let $D \subset E$ be a c-uniform domain and let φ be a homeomorphism of D onto a domain such that φ is M-bilipschitz in the quasihyperbolic metric or, more generally, (M, C)-roughly bilipschitz. Is G c'-Gehring-Hayman with $c'(c, M)$ or $c'(c, M, C)$?

(6) Still more generally, assume that G is Gromov δ-hyperbolic in the quasihyperbolic metric. Is G a $c'(\delta)$-Gehring-Hayman domain? According to [BHK, p. 75], such a domain in R^n is $c'(\delta, n)$-Gehring-Hayman.

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