BROKEN TUBES IN HILBERT SPACES

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Abstract. A broken tube is a special kind of a domain in an infinite-dimensional separable Hilbert space, and it has several properties which do not occur in a finite-dimensional space.

1 Introduction

Let E be a separable real Hilbert space with an orthogonal basis $(u_j)_{j \in \mathsf{Z}}$ where all vectors u_j have a constant norm. By a *broken tube* we mean a tubular neighborhood of the infinite broken line $\bigcup\{[u_{j-1}, u_j]\colon j \in \mathsf{Z}\}$. A detailed construction is given in Section 2. A broken tube W serves as a counterexample to illustrate several differences between the finite-dimensional and the infinite-dimensional quasiworld (study of quasiconformal and related maps). For example, W is (Gromov) hyperbolic and linearly locally connected but not uniform. By a result of M. Bonk, J. Heinonen and P. Koskela [BHK, 7.12], such domains do no exist in the n-space R^n.

Broken tubes have been mentioned in the author's papers [Vä3, 4.12], [Vä4, 3.15], [Vä5, 8.15] and [Vä7, 4.3] but without an explicit construction. In this article we give a rigorous construction of a broken tube W and a quasihyperbolic homeomorphism F of a straight tube U onto W. Moreover, we prove various properties of F and W.

2 Constructions

2.1. *Basic notation.* Throughout the article, we let E denote a separable real Hilbert space with an orthonormal basis $(e_1, e_2, \dots)$. Thus E is isomorphic to the sequence space l_2. The norm of a vector $x \in E$ is written as $|x|$. For balls and spheres we use the notation

$$B(a, r) = \{x\colon |x - a| < r\},\ \bar{B}(a, r) = \{x\colon |x - a| \le r\},$$
$$S(a, r) = \{x\colon |x - a| = r\}.$$

More generally, if $\varnothing \neq A \subset E$, we set $B(A,r) = \{x \in E : d(x,E) < r\}$. The distance between nonempty sets $A, A' \subset X$ is $d(A,A')$, and the diameter of a set A is $d(A)$. We write $\alpha\colon x \curvearrowright y$ if α is an arc with endpoints x and y, and $l(\alpha)$ is the length of α.

Setting $a_j = 2je_1,\ j \in \mathsf{Z}$, we divide the line $\alpha = \operatorname{span}(e_1)$ into segments $\alpha_j = [a_{j-1}, a_j]$. Choose another orthonormal basis $(e'_j)_{j\in\mathsf{Z}}$, indexed by all integers, and set

$$u_j = \sqrt{2}e'_j,\ \beta_j = [u_{j-1}, u_j],\ \beta = \bigcup\{\beta_j\colon j \in \mathsf{Z}\}.$$

Let $f\colon \alpha \to \beta$ be the homeomorphism which maps each α_j isometrically onto β_j with $fa_j = u_j$.

Set $s = 1/5$ and let U be the tube $B(\alpha, s)$. We want to extend f to a homeomorphism $F\colon U \to W$, where W is a neighborhood of β, called a *broken tube*, such that F is locally M-bilipschitz with some M.

2.2. *Auxiliary maps.* Let Q_0 be the rectangle $[-1,0] \times [-s,s] \subset \mathsf{R}^2$. Set $\lambda(y) = 1 - y\sqrt{3}$ for $-s \le y \le s$. Define a map $g_0\colon Q_0 \to \mathsf{R}^2$ by

$$g_0(x,y) = (\lambda(y)(x+1) - 1, y).$$

The map g_0 is a homeomorphism onto a trapezoid Q'_0 as in Figure 1. Each horizontal line segment $[-1,0] \times \{y\}$ is mapped onto a horizontal segment of length $\lambda(y)$, and $g_0(x,y) = (x,y)$ if $x = -1$ or $y = 0$.

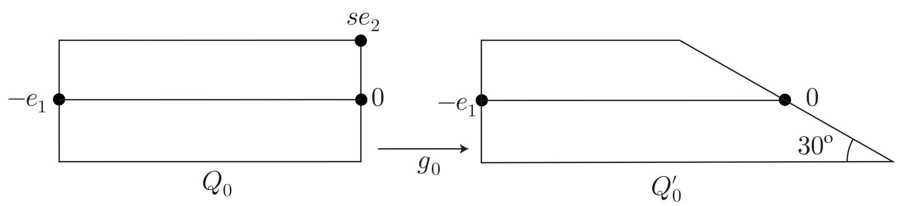

Figure 1

The map g_0 is a diffeomorphism with derivative matrix

$$A = \begin{pmatrix} \lambda(y) & -(x+1)\sqrt{3} \\ 0 & 1 \end{pmatrix}.$$

The jacobian of g_0 is $J = \lambda = \lambda(y)$, and the (Hilbert-Schmidt) matrix norm of A satisfies $\|A\|^2 = 1 + \lambda^2 + 3(x+1)^2 \le \lambda^2 + 4$. For the local dilatation $H = H(x,y)$ we have

$$H \le H + 1/H = \|A\|^2/J \le \lambda + 4/\lambda;$$

see [Vä1, p. 45]. It follows that the operator norms of A and A^{-1} satisfy

$$|A| \le \|A\| \le \sqrt{(1+s\sqrt{3})^2+4} = 2.4109\ldots$$
$$|A^{-1}| = \sqrt{H/J} \le \sqrt{1+4/(1-s\sqrt{3})^2} = 3.2192\ldots$$

By convexity, the map g_0 is M_0-bilipschitz with $M_0 = 3.22$.

As the next step we extend g_0 by reflection in the right-hand sides of Q_0 and Q'_0 to a homeomorphism $g_1 \colon Q_1 \to Q'_1$ where Q_1 is the rectangle $[-1,1] \times [-s,s]$ and Q'_1 is the union of two trapezoids as in Figure 2. We set $e^* = g_1 e_1 = -e_1/2 + \sqrt{3}e_2/2$. The map g_1 is locally M_0-bilipschitz. Observe that the restriction of g_1 to $[0, e_1] \cup [e_1 - se_2, e_1 + se_2]$ is a rotation of angle 120°.

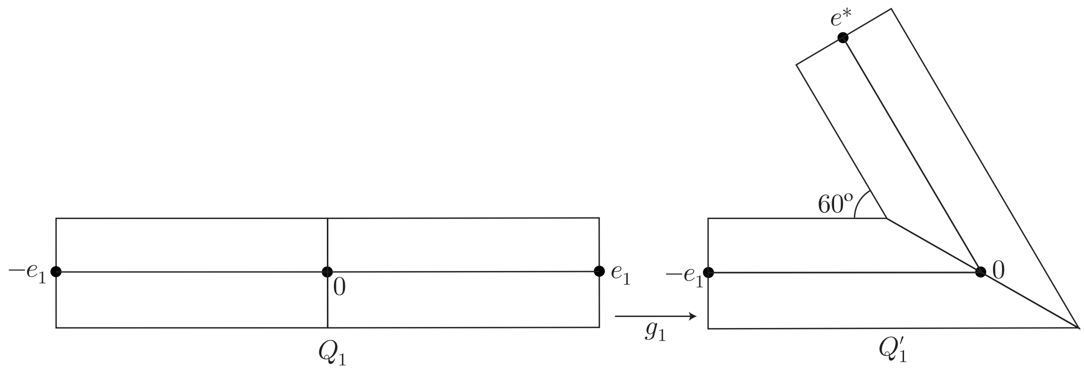

Figure 2

Recalling 2.1 we set

$$b_j = (a_j + a_{j+1})/2 = (2j+1)e_1, \; v_j = fb_j = (u_j + u_{j+1})/2, \; j \in \mathsf{Z}.$$

Let $P \colon E \to \alpha$ be the orthogonal projection. Write

$$U_j = U \cap P^{-1}[a_{j-1}, a_j], \; V_j = U \cap P^{-1}[b_{j-1}, b_j], \; D_j = U \cap P^{-1}\{b_j\}.$$

The sets U_j and V_j are cylinders of height 2, and the bases of V_j are D_{j-1} and D_j. Moreover,

$$U = \bigcup\{U_j : j \in \mathsf{Z}\} = \bigcup\{V_j : j \in \mathsf{Z}\}, \; D_j = V_j \cap V_{j+1}.$$

We identify R^2 with the subspace span (e_1, e_2) of E and set $E_2 = (\mathsf{R}^2)^\perp$. As the next step we extend g_1 to a homeomorphism $g_2 = g_1 \times \mathrm{id} \colon Q_1 \times$

$E_2 \to Q_1' \times E_2$. We have $\bar{V}_0 \cap \mathsf{R}^2 = Q_1$ and $V_0 \subset Q_1 \times E_2$. Hence g_2 defines a homeomorphism $g\colon V_0 \to W_0$ where $W_0 = g_2 V_0$. Observe that $g|D_{-1} \cup [-e_1, 0] = \mathrm{id}$ and that $g|D_0 \cup [0, e_1]$ is an isometry. Moreover, g is locally M_0-bilipschitz.

2.3. *Construction of F.* We want to extend the map $f\colon \alpha \to \beta$ to an embedding $F\colon U \to E$. For each $j \in \mathsf{Z}$ we choose a bijective isometry $\psi_j\colon E \to E$ which carries the points $-e_1, 0, e^*$ into v_{j-1}, u_j, v_j. Observe that $\psi_j(-2e_1) = u_{j-1}$ and that

$$fx = \psi_j g(x - 2je_1) \tag{2.4}$$

for all $x \in [b_{j-1}, b_j]$.

The restriction $F_j = F|V_j$ will be of the form $F_j = \psi_j g \varphi_j$ where $\varphi_j\colon E \to E$ is an isometry with $\varphi_j x = x - 2je_j$ for $x \in \alpha$, which implies that

$$\varphi_j V_j = V_0,\ \varphi_j D_{j-1} = D_{-1},\ \varphi_j D_j = D_0.$$

We start by setting $\varphi_0 = \mathrm{id}$. Then $F_0 = \psi_0 g$ and $F_0 = f$ on $[b_{-1}, b_0] = [-e_1, e_1]$.

Assume that $k > 0$ and that the isometries φ_j have been defined for $j = 0, \dots, k-1$. By (2.4) we have $F_j = f$ on $[b_{j-1}, b_j]$. The map $F_{k-1} = \psi_{k-1} g \varphi_{k-1}$ is an isometry in $[a_{k-1}, b_{k-1}] \cup D_{k-1}$, and it has a unique extension to a bijective isometry $h_k\colon E \to E$. Define $\varphi_k = \psi_k^{-1} h_k$. Since $h_k b_{k-1} = v_{k-1}$, we have $\varphi_k b_{k-1} = -e_1$. Similarly $h_k a_{k-1} = u_{k-1}$ and $\varphi_k a_{k-1} = -2e_1$. Hence $\varphi_k|\alpha$ is the translation $\varphi_k x = x - 2ke_1$ and $F_k = f$ on $[b_{k-1}, b_k]$.

Assume that $x \in D_{k-1} = V_{k-1} \cap V_k$. Since $\varphi_k D_{k-1} = D_{-1}$, we have $g\varphi_k x = \varphi_k x$, whence $F_k x = \psi_k g \varphi_k x = h_k x = F_{k-1} x$. Hence the extension F is defined in the half tube $\{x \in U\colon Px \geq -1\}$.

The extension to cylinders V_k with $k < 0$ is similar, but the expression for φ_k is slightly longer. Assume that φ_j is defined for $j \geq k+1$. Let now $h_k\colon E \to E$ be the isometric extension of $F_{k+1}|D_k \cup [b_k, a_{k+1}]$ and let $h\colon E \to E$ be the isometric extension of $g|D_0 \cup [0, e_1]$. We set $\varphi_k = h^{-1} \psi_k^{-1} h_k$. Since $h_k b_k = v_k = \psi_k e^*$, we have $\varphi_k b_k = e_1$. Moreover, $h_k a_k = u_k = \psi_k 0$, whence $\varphi_k a_k = 0$. It follows that $\varphi_k|\alpha$ is the translation $\varphi_k x = x - 2ke_1$. For $x \in D_k$ we have $\psi_k^{-1} h_k x \in gD_0$, whence $F_k x = \psi_k g h^{-1} \psi_k^{-1} h_k x = h_k x = F_{k+1} x$. The construction of F is now completed.

Write

$$W = FU,\ W_j = FU_j,\ V_j' = FV_j.$$

The domain W is the *broken tube* with *axis* β and *radius* $s = 1/5$.

3 Properties

In this section we give various properties of the broken tube W and the homeomorphism $F\colon U \to W$, defined in Section 2. Many of these have been mentioned in my earlier papers.

Recall that $s = 1/5$ and that $M_0 = 3.22$. The following list of properties follows almost directly from the construction:

3.1. Properties. (1) *W is bounded with $d(W) = d_0 \le 2 + 2M_0 s < 3.29$.*

(2) *Each W_j is convex.*

(3) *Let L_j be the line containing $\beta_j = [u_{j-1}, u_j]$. Then $W_j \subset B(L_j, s)$.*

(4) *The image of each line $L \subset U$, parallel to e_1, is a broken line consisting of line segments $W_j \cap FL$, parallel to L_j. The angle between adjacent segments is $60°$. Moreover, $d(W_j \cap FL, L_j) = d(L, \alpha)$.*

(5) *If $|i - j| \ge 2$, then $d(W_i, W_j) = \sqrt{2} - 2s = 1.014\ldots > 1$.*

(6) *F is locally M_0-bilipschitz.*

(7) *F is M_0-Lipschitz.*

A metric space is *c-quasiconvex* if each pair of points x, y can be joined by an arc γ with $l(\gamma) \le c|x - y|$.

3.2. Property. *Each set $W_j \cup W_{j+1}$ is 2-quasiconvex.*

Proof. Let $x \in W_j$, $y \in W_{j+1}$. Let L_x and L_y be the lines through x and y, parallel to β_j and β_{j+1}, respectively. Let P_2 be the orthogonal projection of E onto the 2-dimensional plane containing $\beta_j \cup \beta_{j+1}$. There are points $x' \in L_x$ and $y' \in L_y$ with $P_2 x' = P_2 y'$. Now $\gamma = [x, x'] \cup [x', y]$ lies in $W_j \cup W_{j+1}$, and the angle between $[x, x']$ and $[x', y]$ is at least $60°$. Hence $l(\gamma) \le 2|x - y|$. □

By 3.1(6), Property 3.2 implies:

3.3. Property. *The map $F|U_j \cup U_{j+1}$ is $2M_0$-bilipschitz for each $j \in \mathsf{Z}$.*

For the next result we recall that the *quasihyperbolic metric* $k = k_G$ of a domain $G \subsetneq E$ is defined by the element of length $|dx|/d(x, \partial G)$. A homeomorphism $f\colon G \to G'$ onto a domain G' is *M-quasihyperbolic* if it is M-bilipschitz in the quasihyperbolic metrics of G and G'. For domains in R^n, an M-quasihyperbolic map is K-quasiconformal with $K = M^{2n-2}$.

By [Vä5, 5.16], a locally M-bilipschitz map is M^2-quasihyperbolic, and 3.1(6) implies:

3.4. Property. *The map $F\colon U \to W$ is M_1-quasihyperbolic with $M_1 = M_0^2 < 10.4$.*

We compose F with some other maps. Let H be the half space $\{x \in E \colon Px > 0$. In [Vä3, 4.11] we constructed a $(\pi/2)$-quasihyperbolic map of H onto a tube of radius $\pi/2$ by rotating the map log of the right half plane around the real axis. Composing this map with a similarity we obtain a $(\pi/2)$-quasihyperbolic map $h \colon H \to U$. Each ray from 0 is mapped onto a line parallel to e_1, and $Phx = (\pi/10) \log |x|$ for $x \in H$.

The composed map $F_1 = F \circ h \colon H \to W$ is M_2-quasihyperbolic with $M_2 = \pi M_1/2 < 16.3$.

We next consider boundary properties of some maps. For this purpose it is convenient to work in the extended space $\dot{E} = E \cup \{\infty\}$. This is a Hausdorff space where the neighborhoods of ∞ are complements of closed bounded sets in E. Boundaries and closures of domains are henceforth taken in $\dot{E}$.

If $f \colon G \to E$ is a map, the *cluster set* of f at a point $b \in \partial G$ is the set of all $y \in \dot{E}$ for which there is a sequence (x_i) in G converging to b such that $fx_i \to y$. For maps into R^n, the cluster set is never empty, because $\dot{\mathsf{R}}^n$ is compact. However, the following result is obvious from the construction:

3.5. Property. *The cluster set of the quasihyperbolic map* $F_1 \colon H \to W$ *is empty at the boundary points* 0 *and* ∞.

Composing F_1 with a möbius map we obtain a quasihyperbolic map F_2 of a ball onto W such that the cluster set of F_2 is empty at two boundary points. Using tower maps (see [Vä5, 8.13]) one can find quasihyperbolic maps of a ball with an empty cluster set at each point of an infinite set. However, the following question remains open:

3.6. *Open question.* Does there exist a quasihyperbolic map $f \colon G \to G'$ such that the cluster set is empty at every boundary point?

The answer is negative if G or G' is uniform; see [Vä4, 3.2].

For domains in $\dot{\mathsf{R}}^n$, the number of boundary components is a topological invariant. This is no longer true in $\dot{E}$. For example, a ball is homeomorphic to the domain between two concentric spheres. However, these domains are not quasihyperbolically equivalent; see [Vä4, 3.13].

3.7. *Open question.* Does there exist a quasihyperbolic map of a ball onto a domain with nonconnected boundary?

On the other hand, the following result can be proved with the aid of the map F:

3.8. Proposition. *There is a quasihyperbolic map* $F^* \colon G \to G'$ *between domains in* E *such that* ∂G *is connected and* $\partial G'$ *has an infinite number of components.*

Proof. Choose an infinite family of rays $A_i \subset H,\ i \in \mathsf{N}$, from the origin such that $G = H \setminus \bigcup\{A_i \colon i \in \mathsf{N}\}$ is a domain. Then F defines a homeomorphism $F^* \colon G \to G' = W \setminus \{FA_i \colon i \in \mathsf{N}\}$, and F^* is M^*-quasihyperbolic with $M^* = 4M_1^2$ by [Vä5, 5.12]. Clearly G and G' have the desired properties. □

We recall some metric properties of domains $G \subsetneq E$. Let $\delta > 0$. A domain G is *δ-hyperbolic* if it is a (Gromov) δ-hyperbolic space in the quasihyperbolic metric. This means that

$$(x|z)_p \geq (x|y)_p \wedge (y|z)_p - \delta$$

for all $x, y, z, p \in G$, where $(\cdot|\cdot)_p$ is the Gromov product, defined by

$$2(x|y)_p = k(x,p) + k(y,p) - k(x,y).$$

Let $c \geq 1$. A domain G is a *c-John domain* if each pair $x, y \in G$ can be joined by an arc $\gamma \subset G$ satisfying the cigar condition

$$l(\gamma[x,z]) \wedge l(\gamma[z,y]) \leq cd(z, \partial G) \tag{3.9}$$

for all $z \in \gamma$. The domain is *c-uniform* if, in addition, the arc γ satisfies the turning condition

$$l(\gamma) \leq c|x-y|. \tag{3.10}$$

Furthermore, G is *c-linearly locally connected* or briefly c-LLC if the following conditions hold for all $x \in G$ and $r > 0$:

(LLC_1) Each pair of points in $G \cap B(x,r)$ can be joined by an arc in $G \cap B(x, cr)$.

(LLC_2) Each pair of points in $G \setminus \bar{B}(x,r)$ can be joined by an arc in $G \setminus \bar{B}(x, r/c)$.

A c-uniform domain is trivially c-John. Moreover, c-John $\Rightarrow$ $3c$-LLC_2 and c-uniform $\Rightarrow$ $3c$-LLC. See, for example, [Vä7, 4.2]. A c-uniform domain is $\delta(c)$-hyperbolic by [Vä7, 2.11].

3.11. Property. *W is a hyperbolic domain.*

Proof. The half space H is hyperbolic and $F_1 \colon H \to W$ is quasihyperbolic. Hence W is hyperbolic by [Vä6, 3.16]. □

3.12. Property. *If $\gamma \subset W$ is an arc joining u_i and u_j, then $l(\gamma) \geq 2|i-j|/M_0$. Hence W is not quasiconvex.*

Proof. Since F is locally M_0-bilipschitz, we have

$$2|i-j| = |a_i - a_j| \leq l(F^{-1}\gamma) \leq M_0 l(\gamma). \ \square$$

3.13. Property. *W is not a John domain.*

Proof. Assume that W is c-John. Let $j > 0$ and choose an arc $\gamma\colon u_0 \curvearrowright u_j$ in W satisfying (3.9). We have $l(\gamma) \geq 2j/M_0$ by 3.12. Let $z \in \gamma$ be the point bisecting the length of γ. Then $j/M_0 \leq cd(z, \partial W) \leq cd_0$, which is a contradiction for large j. □

From 3.13 and also from 3.12 we infer:

3.14. Property. *W is not a uniform domain.*

3.15. Property. *W is LLC.*

Proof. LLC_1: Let $x \in W,\ r > 0$, and let $a, b \in W \cap B(x, r)$. If $r \geq 1/2$, we can join a and b in $W \subset B(x, d_0)$, and (LLC_1) holds with $c = d_0/r < 6.58$. Assume that $r < 1/2$. From 3.1(5) it follows that $a, b \in W_j \cup W_{j+1}$ for some j. By 3.2 we can find an arc $\gamma\colon a \curvearrowright b$ in W with $l(\gamma) \leq 2|a - b| < 4r$. Then $\gamma \subset B(x, 3r)$, and (LLC_1) holds with $c = 3$.

LLC_2: We begin by proving two facts.

Fact 1. If $x \in U$ and $0 < t < s$, then $U \setminus \bar{B}(x, t)$ is connected.

For each line $L \subset U$, parallel to e_1, the set $L \setminus \bar{B}(x, t)$ is either L or a set consisting of two rays, and there is L with $L \setminus \bar{B}(x, t) = L$. Hence each pair of points in $U \setminus \bar{B}(x, t)$ can be joined in $U \setminus \bar{B}(x, t)$ by an arc consisting of at most 5 line segments.

Fact 2. If $x \in U$ and $t < 2M_0$, then $W \cap \bar{B}(Fx, t/2M_0) \subset F[U \cap \bar{B}(x, t)]$.

Set $x' = Fx$ and let $y' = fy \in W \cap \bar{B}(x', t/2M_0)$. Then $|x' - y'| < 1$. By 3.1(5) there is j such that $\{x, y\} \subset U_j \cup U_{j+1}$. By 3.3 we have $|x - y| \leq 2M_0|x' - y'| \leq t$, and Fact 2 is proved.

Now let $x' = Fx \in W$, let $r > 0$, and let $a' = Fa,\ b' = Fb$ be points in $W \setminus \bar{B}(x', r)$. Then $r < d_0$. Since F is M_0-Lipschitz, the points a and b lie in $U \setminus \bar{B}(x, r/M_0)$.

Setting $t = sr/d_0$ we have $t < s$ and $t < r/M_0$. By Fact 1 there is an arc $\gamma\colon a \curvearrowright b$ in $U \setminus \bar{B}(x, t)$. From Fact 2 it follows that $F\gamma$ joins a' and b' in $W \setminus \bar{B}(x', t/2M_0)$. Hence W is LLC_2 with $c = 10M_0d_0 < 106$. □

3.16. *Remarks.* By [BHK, 7.2], a bounded hyperbolic LLC domain in R^n is uniform. From the properties of W we see that in the space E, such a domain need not even be John or quasiconvex.

The second part of [BHK, 7.12] implies that a bounded hyperbolic LLC_2 domain $G \subset \mathsf{R}^n$ is *inner uniform*, that is, each pair $x, y \in G$ can be joined by an arc γ satisfying the cigar condition (3.9) and the inner turning condition

$$l(\gamma) \leq cl(\alpha) \tag{3.10.i}$$

for all arcs $\alpha\colon x \curvearrowright y$ in G. This result does not hold in E either. On the other hand, this implies that every bounded hyperbolic John domain $G \subset \mathsf{R}^n$ is inner uniform. I do not know whether the free version of this is true:

3.17. *Open question.* Is every hyperbolic John domain in E (or more generally, in an arbitrary Banach space) inner uniform?

We finally consider some conditions which in R^n give alternative characterizations for uniform domains. These are obtained by replacing length by diameter or distance.

A domain $G \subset E$ is *diameter c-uniform* if each pair $x, y \in G$ can be joined by an arc $\gamma \subset G$ such that

$$d(\gamma[x,z]) \wedge d(\gamma[z,y]) \le c d(z, \partial G) \tag{3.9.dia}$$

for all $z \in \gamma$ and such that

$$d(\gamma) \le c|x-y|. \tag{3.10.dia}$$

The domain G is *distance c-uniform* if the above conditions hold with (3.9.dia) replaced by

$$|z-x| \wedge |z-y| \le c d(z, \partial G). \tag{3.9.dist}$$

Trivially, we have

c-uniform $\Rightarrow$ diameter c-uniform $\Rightarrow$ distance c-uniform.

In the converse direction, distance c-uniform implies diameter c_1-uniform with $c_1 = c_1(c)$. Since a direct proof of this seems to be unpublished, we give a proof in the appendix, but I feel that this concept is not useful in infinite dimensions. Furthermore, a diameter c-uniform domain $G \subset \mathsf{R}^n$ is c_2-uniform with $c_2 = c_2(c, n)$ by [Ma, 4.5], and hence these three properties are n-quantitatively equivalent for domains in R^n. However, the following result shows that distance ($\Leftrightarrow$ diameter) uniform domains in E need not be uniform.

3.18. Property. *W is a distance uniform domain.*

Proof. Let $x' = Fx,\ y' = Fy$ be points in W. We consider two cases.

Case 1. There is i such that $x, y \in G_i = \operatorname{int}(U_i \cup U_{i+1})$. As a convex bounded domain, G_i is b_1-uniform with a universal constant b_1; see, for example, [Vä5, 10.4.2]. Hence there is a b_1-uniform arc $\gamma\colon x \curvearrowright y$ in G_i. As $F|G_i$ is $2M_0$-bilipschitz by 3.3, the arc $F\gamma\colon x' \curvearrowright y'$ is $4M_0^2 b_1$-uniform in $FG_i \subset W$.

Case 2. There are i and j such that $x \in U_i,\ y \in U_j,\ |i-j| \ge 2$. Set $x_0 = Px,\ y_0 = Py,\ \gamma = [x, x_0] \cup [x_0, y_0] \cup [y_0, y],\ \gamma' = F\gamma$. Then $\gamma'\colon x' \curvearrowright y'$.

Since $|x' - y'| > 1$ by 3.1(5), we have $d(\gamma') \le d_0 \le d_0|x' - y'|$, whence γ satisfies (3.10.dia) with $c = d_0$.

Let $z' = Fz \in \gamma'$. Since F is locally M_0-bilipschitz, the map F^{-1} is M_0-Lipschitz on every line segment in W, whence $d(z, \partial U) \le M_0 d(z', \partial W)$. If $z \in [x, x_0]$ or $z \in [y, y_0]$, then

$$|z' - x'| \wedge |z' - y'| \le M_0(|z - x| \wedge |z - y|) \le M_0 d(z, \partial U) \le M_0^2 d(z', \partial W).$$

If $z \in [x_0, y_0]$, then $d(z', \partial W) \ge d(z, \partial U)/M_0 = s/M_0$, and (3.9.dist) holds with $c = d_0 M_0/s = 5 d_0 M_0$. □

Appendix: Diameter and distance uniform domains

The following proof is modified from [Vä2, 2.18].

Theorem. *Let G be a distance c-uniform domain in a normed space. Then G is c_1-diameter uniform with $c_1(c) \le 64c^4$.*

Proof. We write $\delta(x) = d(x, \partial G)$ for $x \in G$. Let $a, b \in G$ and set $R = |a - b|$. We must find an arc $\gamma\colon a \curvearrowright b$ satisfying (3.9.dia) and (3.10.dia) with a constant $c_1(c)$. We may assume that $\delta(a) \vee \delta(b) \le R$, since otherwise we can choose $\gamma = [a, b]$.

Choose an arc $\gamma_0\colon a \curvearrowright b$ satisfying (3.9.dist) and (3.10.dia). Let $z_0 \in \gamma_0$ be a point with $|z_0 - a| = |z_0 - b|$. Then

$$R/2 \le |z_0 - a| = |z_0 - b| \le cR, \tag{3.19}$$

whence

$$\delta(z_0) \ge R/2c. \tag{3.20}$$

It suffices to find an arc $\alpha\colon a \curvearrowright z_0$ satisfying

$$d(\alpha) \le c_1/2, \quad d(\alpha[a, x]) \le c_1 \delta(x) \text{ for } x \in \alpha. \tag{3.21}$$

In fact, by symmetry, there is also an arc $\beta\colon b \curvearrowright z_0$ with similar properties. Leaving out a loop from $\alpha \cup \beta$ we obtain the desired arc $\gamma\colon a \curvearrowright b$.

Setting $r = \delta(a)/4$ we have $r \le R/4 \le |z_0 - a|/2$. Hence there is a unique integer $m \ge 2$ such that

$$2^{m-1} r \le |z_0 - a| < 2^m r. \tag{3.22}$$

We define inductively points $x_j \in S(a, 2^{j-1} r) \cap G$ and arcs $\alpha_j\colon x_j \curvearrowright z_0$, $1 \le j \le m - 1$, as follows: Let $x_1 \in S(a, r)$ be arbitrary. Assume that the points

$x_1, \dots, x_j$ have been chosen. Let $\alpha_j\colon x_j \curvearrowright z_0$ be an arc given by the distance c-uniformity. Let x_{j+1} be the first point of α_j in $S(a, 2^j r)$.

Define successive arcs $\alpha'_0, \dots, \alpha'_{m-1}$ by $\alpha'_0 = [a, x_1]$, $\alpha'_{m-1} = \alpha_{m-1}$ and by $\alpha'_j = \alpha_j[x_j, x_{j+1}]$ for $1 \le j \le m-2$. The union $\alpha'_0 \cup \dots \cup \alpha'_{m-1}$ need not be an arc, but leaving out a finite number of loops we obtain an arc $\alpha = \alpha''_0 \cup \dots \cup \alpha''_{m-1}\colon a \curvearrowright z_0$, where the sets $\alpha''_j \subset \alpha'_j$ are either empty or successive subarcs of α. We show that α satisfies (3.21).

We have $\alpha''_j \subset \alpha'_j \subset \bar{B}(a, 2^{m-2}r)$ for $0 \le j \le m-2$ and $d(\alpha''_{m-1}) \le d(\alpha_{m-1}) \le c|x_{m-1} - z_0| \le 2^{m+1}cr$, whence

$$d(\alpha) \le 2^{m+2}cr. \tag{3.23}$$

By (3.19) and (3.22) we have $2^{m-1}r \le cR$, and the first part of (3.21) follows with $c_1 = 16c^2$.

To prove the second part, let $x \in \alpha'_j$ and set $t_j = d(\alpha'_0 \cup \dots \cup \alpha'_j)$, $u(x) = t_j/\delta(x)$. It suffices to get an estimate $u(x) \le c_1$. If $j = 0$, then $u(x) \le 1$. If $1 \le j \le m-2$, then $t_j \le 2^{j+1}r$. We consider two cases.

Case 1. $|x - x_j| \le 2^{j-3}r/c$. If $j \ge 2$, then $x_j \in \alpha_{j-1}$, and the distance cigar condition for α_{j-1} yields

$$c\delta(x_j) \ge |x_j - x_{j-1}| \wedge |x_j - z_0| \ge 2^{j-2}r. \tag{3.24}$$

This is clearly also true for $j = 1$. Hence $\delta(x) \ge 2^{j-3}r/c$ and $u(x) \le 16c$.

Case 2. $|x - x_j| \ge 2^{j-3}r/c$. Since $|x - z_0| \ge 2^{m-2}r$, the distance cigar condition for α_j gives $\delta(x) \ge 2^{j-3}r/c^2$, whence $u(x) \le 16c^2$.

Finally assume that $j = m-1$. By (3.23) we have $t_{m-1} \le 2^{m+2}cr$. We consider three cases.

Case 1. $|x - x_{m-1}| \le 2^{m-4}r/c^2$. Since (3.24) holds for $j = m-1$, we have $\delta(x) \ge 2^{m-4}r/c$ and $u(x) \le 64c^2$.

Case 2. $|x - z_0| \le 2^{m-4}r/c^2$. By (3.19), (3.20) and (3.21) we get $\delta(z_0) \ge 2^{m-2}r/c^2$, whence $\delta(x) \ge 2^{m-3}r/c^3$ and $u(x) \le 32c^3$.

Case 3. $|x - x_{m-1}| \wedge |x - z_0| \ge 2^{m-4}r/c^2$. Now the distance cigar condition for α_{m-1} gives $\delta(x) \ge 2^{m-4}r/c^2$ and $u(x) \le 64c^4$. □

References

[BHK] M. Bonk, J. Heinonen and P. Koskela, Uniformizing Gromov hyperbolic spaces, Astérisque 270, 2001, 1–99.

[Ma] O. Martio, Definitions for uniform domains, Ann. Acad. Sci. Fenn. Math. 5, 1980, 197–205.

[Vä1] J. Väisälä, Lectures on n-dimensional quasiconformal mappings, Lecture Notes in Mathematics 229, Springer, 1971

[Vä2] J. Väisälä, Uniform domains, Tôhoku Math. J. 40, 1988, 101-118.

[Vä3] J. Väisälä, Free quasiconformality in Banach spaces I, Ann. Acad. Sci. Fenn. Math. 15, 1990, 255–310.

[Vä4] J. Väisälä, Free quasiconformality in Banach spaces III, Ann. Acad. Sci. Fenn. Math. 17, 1992, 393–408.

[Vä5] J. Väisälä, The free quasiworld. Freely quasiconformal and related maps in Banach spaces, Quasiconformal geometry and dynamics (Lublin, 1996), 55–118, Banach Center Publ. 48, Polish Acad. Sci., Warsaw, 1999.

[Vä6] J. Väisälä, Gromov hyperbolic spaces, Preprint, 2004, www.helsinki.fi/~jvaisala/preprints.html.

[Vä7] J. Väisälä, Gromov hyperbolic domains in Banach spaces, Preprint, 2004, www.helsinki.fi/~jvaisala/preprints.html.

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