A proof of the Mazur-Ulam theorem

Jussi Väisälä

Throughout this note we let E and F denote real normed spaces. A map $f\colon E \to F$ is an *isometry* if $\|fx - fy\| = \|x - y\|$ for all $x, y \in E$, and f is *affine* if

$$f((1-t)a + tb) = (1-t)fa + tfb \tag{1}$$

for all $a, b \in E$ and $0 \le t \le 1$. Equivalently, f is affine if the map $T\colon E \to F$, defined by $Tx = fx - f(0)$, is linear.

An isometry need not be affine. To see this, let E be the real line $\mathbf{R}$, let F be the plane with the norm $\|x\| = \max(|x_1|, |x_2|)$, and let $\varphi\colon \mathbf{R} \to \mathbf{R}$ be any function such that $|\varphi(s) - \varphi(t)| \le |s - t|$ for all $s, t \in \mathbf{R}$, for example, $\varphi(t) = |t|$ or $\varphi(t) = \sin t$. Setting $f(s) = (s, \varphi(s))$ we get an isometry $f\colon E \to F$, which is usually not affine.

An isometry $f\colon E \to F$ is affine if

$$f((a+b)/2) = (fa + fb)/2 \tag{2}$$

for all $a, b \in E$, that is, f preserves midpoints of line segments. Indeed, iteration of (2) gives (1) for all dyadic rational t between 0 and 1. Since an isometry is continuous, we obtain (1) for all $0 \le t \le 1$.

There are two important cases when every isometry is affine:

(i) F is strictly convex, that is, no sphere contains a line segment. Then (2) follows from the fact that the spheres with centers at fa and fb with radii $\|a - b\|/2$ meet only at $(fa + fb)/2$. Every inner product space is strictly convex, and so are the spaces l_p for $1 < p < \infty$.

(ii) f is bijective (equivalently surjective). This result was proved by S. Mazur and S. Ulam [MU] in 1932, and their proof is also given in the books [Ba, p. 166] and [BL, 14.1].

The purpose of this note is to give a simple proof for the theorem of Mazur and Ulam. It is based on the ideas of A. Vogt [Vo], and it makes use of reflections in points.

For $z \in E$, the *reflection of* E *in* z is the map $\psi\colon E \to E$ defined by $\psi x = 2z - x$. Then $\psi\psi$ is the identity, and hence ψ is bijective with $\psi^{-1} = \psi$. Moreover, ψ is an isometry, and z is the only fixpoint of ψ. The equations

$$\|\psi x - z\| = \|x - z\|, \quad \|\psi x - x\| = 2\|x - z\| \tag{3}$$

hold for all $x \in E$.

Theorem of Mazur-Ulam. *Every bijective isometry $f: E \to F$ between normed spaces is affine.*

Proof. Let $a, b \in E$ and set $z = (a+b)/2$. Let W be the family of all bijective isometries $g: E \to E$ keeping the points a and b fixed, and set $\lambda = \sup\{\|gz - z\|: g \in W\}$. For $g \in W$ we have $\|gz - a\| = \|gz - ga\| = \|z - a\|$, and hence $\|gz - z\| \le \|gz - a\| + \|a - z\| = 2\|a - z\|$, which yields $\lambda < \infty$.

Let ψ be the reflection of E in z. If $g \in W$, then also $g^* = \psi g^{-1} \psi g \in W$, and hence $\|g^* z - z\| \le \lambda$. Since g^{-1} is an isometry, this and (3) imply that

$$2\|gz - z\| = \|\psi gz - gz\| = \|g^{-1}\psi gz - z\| = \|\psi g^{-1}\psi gz - z\| = \|g^* z - z\| \le \lambda$$

for all $g \in W$, and hence $2\lambda \le \lambda$. Thus $\lambda = 0$, which means that $gz = z$ for all $g \in W$.

Let $f: E \to F$ be a bijective isometry. Setting $z' = (fa + fb)/2$ we must show that $fz = z'$. Let ψ' be the reflection of F in z'. Then the map $h = \psi f^{-1} \psi' f$ is in W, and hence $hz = z$. This implies that $\psi' fz = fz$. Since z' is the only fixpoint of ψ', we obtain $fz = z'$ as desired.

References

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Matematiikan laitos, PL 4, Hallituskatu 5, 00014 Helsingin yliopisto, Finland
jvaisala@cc.helsinki.fi