A proof of the Mazur-Ulam theorem

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Throughout this note we let $E$ and $F$ denote real normed spaces. A map $f: E \to F$ is an isometry if $\|fx - fy\| = \|x - y\|$ for all $x, y \in E$, and $f$ is affine if

$$(1) \quad f((1-t)a+tb) = (1-t)fa + tfb$$

for all $a, b \in E$ and $0 \leq t \leq 1$. Equivalently, $f$ is affine if the map $T: E \to F$, defined by $Tx = fx - f(0)$, is linear.

An isometry need not be affine. To see this, let $E$ be the real line $\mathbb{R}$, let $F$ be the plane with the norm $\|x\| = \max(|x_1|, |x_2|)$, and let $\varphi: \mathbb{R} \to \mathbb{R}$ be any function such that $|\varphi(s) - \varphi(t)| \leq |s-t|$ for all $s, t \in \mathbb{R}$, for example, $\varphi(t) = |t|$ or $\varphi(t) = \sin t$. Setting $f(s) = (s, \varphi(s))$ we get an isometry $f: E \to F$, which is usually not affine.

An isometry $f: E \to F$ is affine if

$$(2) \quad f((a + b)/2) = (fa + fb)/2$$

for all $a, b \in E$, that is, $f$ preserves midpoints of line segments. Indeed, iteration of (2) gives (1) for all dyadic rational $t$ between 0 and 1. Since an isometry is continuous, we obtain (1) for all $0 \leq t \leq 1$.

There are two important cases when every isometry is affine:

(i) $F$ is strictly convex, that is, no sphere contains a line segment. Then (2) follows from the fact that the spheres with centers at $fa$ and $fb$ with radii $\|a-b\|/2$ meet only at $(fa + fb)/2$. Every inner product space is strictly convex, and so are the spaces $l_p$ for $1 < p < \infty$.

(ii) $f$ is bijective (equivalently surjective). This result was proved by S. Mazur and S. Ulam [MU] in 1932, and their proof is also given in the books [Ba, p. 166] and [BL, 14.1].

The purpose of this note is to give a simple proof for the theorem of Mazur and Ulam. It is based on the ideas of A. Vogt [Vo], and it makes use of reflections in points.

For $z \in E$, the reflection of $E$ in $z$ is the map $\psi: E \to E$ defined by $\psi x = 2z - x$. Then $\psi \psi$ is the identity, and hence $\psi$ is bijective with $\psi^{-1} = \psi$. Moreover, $\psi$ is an isometry, and $z$ is the only fixpoint of $\psi$. The equations

$$(3) \quad \|\psi x - z\| = \|x - z\|, \quad \|\psi x - z\| = 2\|x - z\|$$

hold for all $x \in E$. 

1
Theorem of Mazur-Ulam. Every bijective isometry $f: E \to F$ between normed spaces is affine.

Proof. Let $a, b \in E$ and set $z = (a + b)/2$. Let $W$ be the family of all bijective isometries $g: E \to E$ keeping the points $a$ and $b$ fixed, and set $\lambda = \sup\{\|gz - z\|: g \in W\}$. For $g \in W$ we have $\|gz - a\| = \|gz - ga\| = \|z - a\|$, and hence $\|gz - z\| \leq \|gz - a\| + \|a - z\| = 2\|a - z\|$, which yields $\lambda < \infty$.

Let $\psi$ be the reflection of $E$ in $z$. If $g \in W$, then also $g^* = \psi g^{-1} \psi g \in W$, and hence $\|g^*z - z\| \leq \lambda$. Since $g^{-1}$ is an isometry, this and (3) imply that

$$2\|gz - z\| = \|\psi g - gz\| = \|g^{-1} \psi g - z\| = \|\psi g^{-1} \psi g - z\| = \|g^* z - z\| \leq \lambda$$

for all $g \in W$, and hence $2\lambda \leq \lambda$. Thus $\lambda = 0$, which means that $gz = z$ for all $g \in W$.

Let $f: E \to F$ be a bijective isometry. Setting $z' = (fa + fb)/2$ we must show that $fz = z'$. Let $\psi'$ be the reflection of $F$ in $z'$. Then the map $h = \psi f^{-1} \psi' f$ is in $W$, and hence $hz = z$. This implies that $\psi' f z = f z$. Since $z'$ is the only fixpoint of $\psi'$, we obtain $fz = z'$ as desired.

References