GROMOV HYPERBOLIC SPACES *

Jussi Väisälä

3.5.2004

Matematiikan laitos
Helsingin yliopisto
PL 4, Yliopistonkatu 5
00014 Helsinki, Finland
`jvaisala@cc.helsinki.fi`

Abstract: A mini monograph on Gromov hyperbolic spaces, which need not be geodesic or proper.
Keywords: Gromov hyperbolic space, Gromov boundary, quasimöbius map.

Contents

1 Introduction

The theory of Gromov hyperbolic spaces, introduced by M. Gromov in the eighties, has been considered in the books [CDP], [GdH], [Sh], [Bow], [BH], [BBI], [Ro] and in several papers, but it is often assumed that the spaces are geodesic and usually also proper (closed bounded sets are compact). A notable exception is the paper [BS] of M. Bonk and O. Schramm. The purpose of the present article is to give a fairly detailed treatment of the basic theory of more general hyperbolic spaces. However, we often (but not always) assume that the space is *intrinsic*, which means that the distance between two points is always equal to the infimum of the lengths of all arcs joining these points.

We do not assume that the reader has any previous knowledge on hyperbolic spaces.

*MSC 2000 Subject Classification: 53C23

A motivation for this article was my work [Vä5], where I generalize some results of M. Bonk, J. Heinonen and P. Koskela [BHK] for domains in Banach spaces with the quasihyperbolic metric. These metric spaces are intrinsic, but they need not be geodesic, and they are proper only in the finite-dimensional case.

The main idea in this article is that geodesics are replaced by *h-short arcs*. An arc α with endpoints x and y is h-short with $h \geq 0$ if its length $l(\alpha)$ is at most $|x - y| + h$. Geodesic rays to a boundary point will be replaced by certain sequences of h-short arcs, called *roads*, and geodesic lines between boundary points will be replaced by another kind of arc sequences, called *biroads*.

Alternatively, we could sometimes make use of the result of Bonk and Schramm [BS, 4.1] stating that every δ-hyperbolic metric space can be isometrically embedded into a δ-hyperbolic geodesic space. The proof of this embedding theorem involves transfinite induction, and I have preferred direct and more elementary proofs.

Some results and proofs are rather obvious modifications of the classical case where the space is geodesic and proper. Presumably, a part of the theory belongs to the folklore. On the other hand, some concepts are genuinely more complicated than in the classical case. For example, the center of an h-short triangle consists of three arcs and not of three points, and the roads and biroads mentioned abover are clumsier than geodesic rays and lines.

Certain ideas of the paper seem to be new also in the classical case. In 3.12 we give a simple converse of the stability theorem. (A stronger result with a harder proof has been given by Bonk [Bo].) In Section 5 we consider a function $d_{p,\varepsilon}$ (where $p \in X$ and $\varepsilon > 0$), not only as a metric of the Gromov boundary ∂X but as a "metametric" of the Gromov closure $X^* = X \cup \partial X$ of a hyperbolic space X; then $d_{p,\varepsilon}(x, x) > 0$ for $x \in X$. This enables us to extend each quasi-isometry $f\colon X \to Y$ between hyperbolic spaces to maps $f^*\colon X^* \to Y^*$ that are quasimöbius rel ∂X, not only in ∂X.

Hyperbolic spaces play an important role in group theory, but connections with group theory are not considered in this article. See [KB] for a recent survey.

Acknowledgement. I thank Juha Heinonen for calling my attention to this area and for encouragement.

2 Hyperbolic spaces

2.1. *Summary.* I start each section with a brief summary. In Section 2 we give the definition and the basic properties of hyperbolic spaces. The definition is given in terms of the Gromov product. An alternative characterization for intrinsic hyperbolic spaces in terms of slim triangles is also given.

2.2. *Notation and terminology.* By a *space* we mean a metric space. The distance between points x and y is usually written as $|x - y|$. An *arc* in a space X is a subset homeomorphic to a real interval. Unless otherwise stated, this interval is assumed to be closed. Then the arc is compact and has two endpoints. We write $\alpha\colon x \curvearrowright y$ if α is an arc with endpoints x and y. If needed, this notation also gives an orientation for α from x to y. Occasionally, we consider a singleton $\{x\}$ as an arc $\alpha\colon x \curvearrowright x$.

A space X is *intrinsic* if

$$|x - y| = \inf\{l(\alpha) \mid \alpha\colon x \curvearrowright y\},$$

for all $x, y \in X$. Intrinsic spaces are often called *length spaces* or *path-metric spaces* in the literature.

Let $h \geq 0$. We say that an arc $\alpha\colon x \curvearrowright y$ is h-*short* if

$$l(\alpha) \leq |x - y| + h.$$

Thus α is a geodesic iff it is 0-short. We see that X is intrinsic iff for each pair $x, y \in X$ and for each $h > 0$ there is an h-short arc $\alpha\colon x \curvearrowright y$.

The basic notation is fairly standard. We let R and N denote the sets of real numbers and positive integers, respectively. Balls and spheres are written as

$$B(a, r) = \{x\colon |x - a| < r\},\ \bar{B}(a, r) = \{x\colon |x - a| \leq r\},$$
$$S(a, r) = \{x\colon |x - a| = r\}.$$

More generally, if $\varnothing \neq A \subset X$, we set

$$\bar{B}(A, r) = \{x \in X : d(x, A) \leq r\}.$$

The distance between nonempty sets $A, A' \subset X$ is $d(A, A')$, and the diameter of a set A is $d(A)$. The *Hausdorff distance* between A and A' is defined by

$$d_H(A, A') = \inf\,\{r\colon A' \subset \bar{B}(A, r),\ A \subset \bar{B}(A', r)\}.$$

For an arc α, we let $\alpha[u, v]$ denote the closed subarc of α between points $u, v \in \alpha$, and for half open subarcs we write $\alpha[u, v) = \alpha[u, v] \setminus \{v\}$. For real numbers s, t we set $s \wedge t = \min\{s, t\},\ s \vee t = \max\{s, t\}$. To simplify notation we often omit parentheses writing $fx = f(x)$ etc.

2.3. *Convention.* Throughout the article, we let X denote a metric space.

2.4. Lemma. *Every subarc of an h-short arc is h-short.*

Proof. Suppose that $\alpha\colon x \curvearrowright y$ is h-short and that $u, v \in \alpha$ with $u \in \alpha[x, v]$. Then

$$\begin{aligned} |x - u| + l(\alpha[u, v]) + |v - y| &\leq l(\alpha) \leq |x - y| + h \\ &\leq |x - u| + |u - v| + |v - y| + h, \end{aligned}$$

which yields $l(\alpha[u, v]) \leq |u - v| + h$. □

2.5. *Remark.* Assume that $\alpha\colon x \curvearrowright y$ is an h-short arc of length $L = l(\alpha)$, and let $\varphi\colon [0, L] \to \alpha$ be its arclength parametrization. Then

$$|s - t| - h \leq |\varphi(s) - \varphi(t)| \leq |s - t|$$

for all $s, t \in [0, L]$. Thus the arclength parametrization of an h-short arc is an h-*rough geodesic* in the sense of Bonk and Schramm [BS]. The converse is not true, because an h-rough geodesic is defined by $|s - t| - h \leq |\varphi(s) - \varphi(t)| \leq |s - t| + h$, and it need not even be continuous.

2.6. *Arcs or paths?* It is usually possible to work alternatively with arcs or paths (maps of an interval). Whenever possible, I prefer arcs because of shorter notation. However, paths are unavoidable when studying quasi-isometries, which need not be continuous or injective.

2.7. *The Gromov product.* For $x, y, p \in X$ we define the *Gromov product* $(x|y)_p$ by

$$2(x|y)_p = |x-p| + |y-p| - |x-y|.$$

A geometric interpretation of the Gromov product is obtained by mapping the triple (x, y, p) isometrically onto a triple (x', y', p') in the euclidean plane. The circle inscribed to the triangle $x'y'p'$ meets the sides $[p', x']$ and $[p', y']$ at points x^* and y^*, respectively, and we have $(x|y)_p = |x^* - p| = |y^* - p|$.

A useful property of the Gromov product is that in hyperbolic spaces, it is roughly equal to the distance between p and an h-short arc $\alpha\colon x \curvearrowright y$; see 2.33.

We next give some elementary properties of the Gromov product.

2.8. Lemma. (1) $(x|y)_p = (y|x)_p,\ (x|y)_y = (x|y)_x = 0$.

(2) $|x-y| = (x|z)_y + (y|z)_x$.

(3) $0 \le (x|y)_p \le |x-p| \wedge |y-p|$.

(4) $|(x|y)_p - (x|y)_q| \le |p-q|$.

(5) $|(x|y)_p - (x|z)_p| \le |y-z|$.

(6) *If* $\alpha\colon p \curvearrowright y$ *is* h*-short and if* $x \in \alpha$*, then*

$$|x-p| - h/2 \le (x|y)_p \le |x-p|.$$

Proof. (1) is trivial, and (2) follows by direct computation. The first inequality of (3) is the triangle inequality. Furthermore,

$$2(x|y)_p \le |x-p| + |y-p| - (|x-p| - |y-p|) = 2|x-p|,$$

and similarly $2(x|y)_p \le 2|y-p|$, so (3) is true. The proof of (4) is equally easy:

$$2|(x|y)_p - (x|y)_q| = \big||x-p| - |x-q| + |y-p| - |y-q|\big| \le 2|p-q|,$$

and also (5) follows from the triangle inequality.

The second inequality of (6) follows from (3). Since α is h-short, we have $|p-x|+|x-y| \le l(\alpha) \le |p-y| + h$, which implies the first inequality of (6). □

2.9. Lemma. *Suppose that* $\alpha\colon y \curvearrowright z$ *is* h*-short and that* $x \in X$*. Then* $(y|z)_x \le d(x,\alpha) + h/2$*. In particular,* $(y|z)_p \le h/2$ *for all* $p \in \alpha$*.*

Proof. Let $p \in \alpha$. Since $|z-p| + |y-p| \le l(\alpha) \le |y-z| + h$, the triangle inequality gives

$$2|x-p| \ge |x-z| - |z-p| + |x-y| - |y-p| \ge 2(y|z)_x - h,$$

and the lemma follows. □

2.10. *Definition.* Let $\delta \ge 0$. A space X is *(Gromov)* δ*-hyperbolic* if

$$(x|z)_p \ge (x|y)_p \wedge (y|z)_p - \delta \tag{2.11}$$

for all $x, y, z, p \in X$. A space is *Gromov hyperbolic* or briefly *hyperbolic* if it is δ-hyperbolic for some $\delta \ge 0$.

Alternatively, the definition can be written as

$$|x-z|+|y-p| \le (|x-y|+|z-p|) \vee (|x-p|+|y-z|) + 2\delta. \tag{2.12}$$

A third formulation of (2.11) is given in (4.6). We shall occasionally make use of the inequality

$$(x|u)_p \ge (x|y)_p \wedge (y|z)_p \wedge (z|u)_p - 2\delta, \tag{2.13}$$

which is obtained by iterating (2.11).

2.14. *Examples.* The real line is 0-hyperbolic. A classical example of a hyperbolic space is the Poincaré half space $x_n > 0$ in R^n with the hyperbolic metric defined by the element of length $|dx|/x_n$. This space is δ-hyperbolic with $\delta = \log 3$ [CDP, 4.3]. More generally, uniform domains with the quasihyperbolic metric are hyperbolic; see [BHK, 1.11] for domains in R^n and [Vä5] for arbitrary Banach spaces.

Every bounded space is trivially hyperbolic, but only unbounded hyperbolic spaces are interesting.

In the rest of this section we study h-short arcs in hyperbolic spaces. The following useful result is usually (for $h = 0$) mentioned together with the so-called tripod map; see [GdH, pp. 38,41]. However, tripods are not needed in this article.

2.15. Tripod lemma. *Suppose that $\alpha_i\colon a \curvearrowright b_i,\ i = 1, 2$, are h-short arcs in a δ-hyperbolic space. Let $x_1 \in \alpha_1$ be a point with $|x_1 - a| \le (b_1|b_2)_a$, and let $x_2, x_2' \in \alpha_2$ be points with $|x_2 - a| = |x_1 - a|$ and $l(\alpha_2[a, x_2']) = l(\alpha_1[a, x_1])$. Then*

$$|x_1 - x_2| \le 4\delta + h, \quad |x_1 - x_2'| \le 4\delta + 2h.$$

Proof. Set $t = |x_1 - a| = |x_2 - a|$. By 2.8(6) we have $(x_i|b_i)_a \ge t - h/2$. Hence

$$t - |x_1 - x_2|/2 = (x_1|x_2)_a \ge (x_1|b_1)_a \wedge (b_1|b_2)_a \wedge (b_2|x_2)_a - 2\delta \ge t - h/2 - 2\delta,$$

which implies the first inequality.

Let l_i denote the length metric of $\alpha_i,\ i = 1, 2$, that is, $l_i(u, v) = l(\alpha_i[u, v])$. We have

$$|x_2 - x_2'| \le l_2(x_2, x_2') = |l_2(a, x_2) - l_2(a, x_2')| = |l_2(a, x_2) - l_1(a, x_1)|.$$

Since $t \le l_i(a, x_i) \le t + h$ for $i = 1, 2$, we obtain $|x_2 - x_2'| \le h$, and the second inequality follows. □

2.16. *Length maps.* Suppose that α and β are rectifiable arcs with $l(\alpha) \le l(\beta)$. A map $f\colon \alpha \to \beta$ is a *length map* if

$$l(f\alpha[u, v]) = l(\alpha[u, v])$$

for all $u, v \in \alpha$.

Suppose that $\alpha_1, \alpha_2\colon a \curvearrowright b$ are h-short arcs in a δ-hyperbolic space with common endpoints and that $l(\alpha_1) \le l(\alpha_2)$. Let $f\colon \alpha_1 \to \alpha_2$ be the length map fixing a. Then $|fx - x| \le 4\delta + 2h$ for all $x \in \alpha_1$ by the tripod lemma 2.15. The following two lemmas give related results for somewhat more general situations.

2.17. Ribbon lemma. *Let X be an intrinsic δ-hyperbolic space, let $\alpha_i\colon a_i \curvearrowright b_i$ be h-short arcs in X, $i = 1,2$, let $l(\alpha_1) \le l(\alpha_2)$, $|a_1 - a_2| \le \mu$, $d(b_1, \alpha_2) \le \mu$, and let $f\colon \alpha_1 \to \alpha_2$ be the length map with $fa_1 = a_2$. Then $|fx - x| \le 8\delta + 5\mu + 5h$ for all $x \in \alpha_1$.*

Proof. Let again l_i denote the length metric of α_i, $i = 1,2$. Choose a point $y \in \alpha_2$ with $|b_1 - y| \le \mu$. Then

$$\begin{aligned} l_2(y, fb_1) &\le |l_2(a_2,y) - l_2(a_2, fb_1)| = |l_2(a_2,y) - l_1(a_1,b_1)| \\ &\le \big||a_2 - y| - |a_1 - b_1|\big| + h \le 2\mu + h. \end{aligned}$$

Let $x \in \alpha_1$ and set $s = l_1(a_1, x)$, $L = l(\alpha_1)$. If $s \ge L - \mu - h$, then $l_2(fx, fb_1) = l_1(x, b_1) \le \mu + h$ and

$$\begin{aligned} |fx - x| &\le |fx - fb_1| + |fb_1 - y| + |y - b_1| + |b_1 - x| \\ &\le (\mu + h) + (2\mu + h) + \mu + (\mu + h) = 5\mu + 3h. \end{aligned}$$

Assume that $s \le L - \mu - h$. Choose an h-short arc $\alpha_0\colon a_1 \curvearrowright y$. Since

$$|a_1 - y| \ge |a_1 - b_1| - |y - b_1| \ge L - h - \mu \ge s,$$

there is a point $x_0 \in \alpha_0$ with $l_0(a_1, x_0) = s$ where l_0 is the length metric of α_0. We have

$$2(b_1|y)_{a_1} = |a_1 - b_1| + |a_1 - y| - |b_1 - y| \ge L - h + s - \mu \ge 2s.$$

Hence $|x - x_0| \le 4\delta + 2h$ by 2.15. Set $t = l(\alpha_0) - s = l_0(y, x_0)$. If $t \ge |a_2 - y| - \mu$, then $s = l(\alpha_0) - t \le |a_1 - y| + h - |a_2 - y| + \mu \le 2\mu + h$ and

$$|fx - x| \le |fx - a_2| + |a_2 - a_1| + |a_1 - x| \le 2s + \mu \le 5\mu + 2h.$$

Assume that $t \le |a_2 - y| - \mu$. There is a point $x_2 \in \alpha_2[a_2, y]$ with $l_2(x_2, y) = t$. We have

$$2(a_1|a_2)_y = |a_1 - y| + |a_2 - y| - |a_1 - a_2| \ge 2|a_2 - y| - 2\mu \ge 2t,$$

whence $|x_0 - x_2| \le 4\delta + 2h$ by 2.15. Hence

$$\begin{aligned} |fx - x| &\le |fx - x_2| + |x_2 - x_0| + |x_0 - x| \\ &\le |s + t - l_2(a_2, y)| + (4\delta + 2h) + (4\delta + 2h). \end{aligned}$$

Here

$$|s + t - l_2(a_2, y)| = |l(\alpha_0) - l_2(a_2, y)| \le \big||a_1 - y| - |a_2 - y|\big| + h \le \mu + h,$$

and we obtain the desired estimate $|fx - x| \le 8\delta + \mu + 5h$. $\square$

2.18. Second ribbon lemma. *Let X be an intrinsic δ-hyperbolic space and let $\alpha_i\colon a_i \curvearrowright b_i$ be h-short arcs in X, $i = 1,2$, with $|a_1 - a_2| \le \mu$, $|b_1 - b_2| \le \mu$. Then the Hausdorff distance $d_H(\alpha_1, \alpha_2)$ is at most $8\delta + 5\mu + 5h$.*

Proof. Let $x \in \alpha_1$. We must find a point $y \in \alpha_2$ with $|x-y| \le 8\delta+5\mu+5h$. If $l(\alpha_1) \le l(\alpha_2)$, this is given by 2.17. Assume that $l(\alpha_2) < l(\alpha_1)$ and let $f\colon \alpha_2 \to \alpha_1$ be the length map with $fa_2 = a_1$. If $x \in f\alpha_2$, we may choose $y = f^{-1}x$ by 2.17. Assume that $x \in \alpha_1[fb_2, b_1]$ and let l_i denote the length metric of α_i. We have

$$\begin{aligned} |x-b_1| &\le l_1(fb_2,b_1) = l_1(a_1,b_1) - l_1(a_1,fb_2) \\ &= l_1(a_1,b_1) - l_2(a_2,b_2) \le |a_1-b_1| + h - |a_2-b_2| \le 2\mu + h. \end{aligned}$$

Hence $|x-b_2| \le 3\mu + h$, and we may choose $y = b_2$. □

2.19. Lemma. *Let $\alpha_i\colon p \curvearrowright a_i,\ i=1,2$, be h-short arcs in a δ-hyperbolic space, let $q \ge 0$ and let $y_i \in \alpha_i$ be points with $|p-y_i| \ge (a_1|a_2)_p - q$. Then*

$$|(y_1|y_2)_p - (a_1|a_2)_p| \le 6\delta + q + 3h.$$

Proof. We write $(x|y) = (x|y)_p$ for $x, y \in X$. Set $t = (a_1|a_2)$. Since $|p-y_i| - h/2 \le (y_i|a_i) \le |p-y_i|$ by 2.8(6), we obtain

$$\begin{aligned} (y_1|y_2) &\ge (y_1|a_1) \wedge (a_1|a_2) \wedge (a_2|y_2) - 2\delta \\ &\ge |w-y_1| \wedge t \wedge |p-y_2| - h/2 - 2\delta \ge t - 2\delta - q - h/2. \end{aligned}$$

It remains to show that

$$(y_1|y_2) \le t + 6\delta + q + 3h. \tag{2.20}$$

We have

$$\begin{aligned} t &\ge (a_1|y_1) \wedge (y_1|y_2) \wedge (y_2|a_2) - 2\delta \\ &\ge |p-y_1| \wedge (y_1|y_2) \wedge |p-y_2| - h/2 - 2\delta. \end{aligned}$$

We may assume that $|p-y_1| \le |p-y_2|$. If $|p-y_1| > t+h/2+2\delta$, then $(y_1|y_2) \le t+h/2+2\delta$, and (2.20) holds. Assume that $|p-y_1| \le t+h/2+2\delta$. Let $z_i \in \alpha_i$ be points with $|p-z_i| = t$. Since α_1 is h-short, we obtain

$$|y_1 - z_1| \le \big||p-y_1| - t\big| + h \le (h/2+2\delta) \vee q + h \le 2\delta + q + 2h.$$

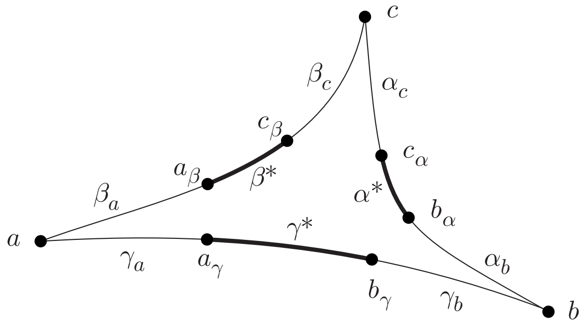
As $|z_1 - z_2| \le 4\delta + h$ by 2.15, we have $|y_1 - z_2| \le 6\delta + q + 3h$, whence $(y_1|y_2) \le (z_2|y_2) + 6\delta + q + 3h \le |p-z_2| + 6\delta + q + 3h$, and (2.20) follows. □

2.21. *Triangles.* By a *triangle* in X we mean a triple of arcs $\alpha\colon b \curvearrowright c,\ \beta\colon a \curvearrowright c,\ \gamma\colon a \curvearrowright b$. The points a, b, c are the *vertices* and the arcs α, β, γ are the *sides* of the triangle $\Delta = (\alpha, \beta, \gamma)$. A triangle is *$h$-short* if each side is h-short. We set

$$r_a = (b|c)_a,\ r_b = (a|c)_b,\ r_c = (a|b)_c,\ |\Delta| = \alpha \cup \beta \cup \gamma.$$

From 2.8(2) we get

$$|a-b| = r_a + r_b,\ |a-c| = r_a + r_c,\ |b-c| = r_b + r_c. \tag{2.22}$$



Let a_γ and b_γ be the points of γ such that setting $\gamma_a = \gamma[a, a_\gamma]$ and $\gamma_b = \gamma[b_\gamma, b]$ we have $l(\gamma_a) = r_a$ and $l(\gamma_b) = r_b$. Then γ is the union of successive subarcs $\gamma_a, \gamma^*, \gamma_b$ where $\gamma^* = \gamma[a_\gamma, b_\gamma]$ is called the *center of the side* γ in the triangle Δ. The arc γ^* may degenerate to a point; this happens iff γ is a geodesic. We say that the subdivision

$$\gamma = \gamma_a \cup \gamma^* \cup \gamma_b$$

is *the subdivision of* γ *induced by the triangle* Δ (or by the point c). The sides α and β are divided similarly. The *center of the triangle* Δ is the set

$$Z(\Delta) = \alpha^* \cup \beta^* \cup \gamma^*.$$

Observe that for each vertex v of Δ we have

$$|\Delta| \cap S(v, r_v) \subset Z(\Delta). \tag{2.23}$$

2.24. Lemma. *Suppose that X is δ-hyperbolic and that Δ is an h-short triangle in X. Then*
(1) $l(\tau^*) \le h$ *for each side τ of Δ,*
(2) $d(Z(\Delta)) \le 4\delta + 4h$.

Proof. Let Δ be as in 2.21. By (2.22) we get

$$|a - b| + l(\gamma^*) = r_a + l(\gamma^*) + r_b = l(\gamma) \le |a - b| + h,$$

which implies (1). Furthermore, $|a_\beta - a_\gamma| \le 4\delta + 2h$ by the tripod lemma 2.15, and (2) follows from (1). □

2.25. Lemma. *Suppose that Δ is an h-short triangle. Then:*
(1) $d(v, Z(\Delta)) \ge r_v - h$ *for each vertex v of Δ.*
(2) *If $x \in X$ and if $d(x, \tau) \le t$ for each side τ of Δ, then $d(x, Z(\Delta)) \le 3t + h/2$.*

Proof. Let Δ be as in 2.21. We have $d(a, \tau^*) \ge r_a - h$ for $\tau = \beta, \gamma$. By 2.9 we have $d(a, \alpha) \ge r_a - h/2$, and (1) follows.

To prove (2), choose a point $y \in \alpha$ with $|x - y| \le t$. It suffices to show that $d(y, \alpha^*) \le 2t + h/2$. We may assume that $y \in \alpha_b$. By 2.9 we obtain

$$r_b \le d(b, \beta) + h/2 \le |b - y| + |y - x| + d(x, \beta) + h/2 \le |b - y| + 2t + h/2.$$

Moreover, $|b - y| + |y - b_\alpha| \le l(\alpha_b) = r_b$, whence $d(y, \alpha^*) \le |y - b_\alpha| \le 2t + h/2$ as desired. □

2.26. *Slim triangles and the Rips condition.* Let $\delta \geq 0$. A triangle Δ in a space X is δ*-slim* if each side τ of Δ is contained in $\bar{B}(|\Delta| \setminus \tau, \delta)$.

Let $\mathcal{A}$ be a family of arcs in X such that

(1) If $\alpha \in \mathcal{A}$, then every subarc of α is in $\mathcal{A}$.

(2) For each $x, y \in X,\ x \neq y$, and $h > 0$ there is an h-short member $\alpha\colon x \curvearrowright y$ of $\mathcal{A}$.

Observe that (2) implies that X is intrinsic. For example, the family of all arcs in an intrinsic space satisfies (1) and (2). In [Vä5] I consider the case where the space is a domain in a Banach space with the quasihyperbolic metric and $\mathcal{A}$ is the family of all c-quasigeodesics with a fixed c.

We say that X is a $(\delta, h, \mathcal{A})$*-Rips space* if every h-short triangle in X with sides in $\mathcal{A}$ is δ-slim. In the case where $\mathcal{A}$ is the family of all arcs in X, we simply say that X is (δ, h)*-Rips*.

2.27. *Hyperbolicity and the Rips condition.* We shall show that for intrinsic spaces, the $(\delta, h, \mathcal{A})$-Rips condition is quantitatively equivalent to δ-hyperbolicity. We formulate this in 2.34 and 2.35, but we remark that from 2.15 and 2.24 it easily follows that an intrinsic δ-hyperbolic space is $(\delta', h, \mathcal{A})$-Rips with $\delta' = 4\delta + 2h$ for each h and $\mathcal{A}$, which is a slightly weaker result than 2.35.

Indeed, let $\Delta = (\alpha, \beta, \gamma)$ be an h-short triangle and let $x \in \alpha\colon b \curvearrowright c$. If $|x - b| \leq r_b$, then $d(x, \gamma) \leq 4\delta + h$ by the tripod lemma 2.15. Similarly $|x - c| \leq r_c$ implies that $d(x, \beta) \leq 4\delta + h$. In the remaining case we have $x \in \alpha^*$, and then $d(x, \gamma) \leq d(b_\alpha, \gamma) + l(\alpha^*) \leq 4\delta + 2h$ by 2.15 and 2.24.

As a by-product, the following proof gives Lemma 2.33, which will be very useful in applications.

2.28. Lemma. *Suppose that X is $(\delta, h, \mathcal{A})$-Rips and that $\alpha, \beta\colon x \curvearrowright y$ are h-short members of $\mathcal{A}$. Then $\alpha \subset \bar{B}(\beta, \delta)$.*

Proof. This follows from the definition by dividing β into two subarcs. □

We introduce an auxiliary notion.

2.29. *Definition.* A space X has *property* $P(\delta, h, \mathcal{A})$ if

$$(a|b)_p \wedge (a|c)_p \leq \delta$$

whenever $\alpha\colon b \curvearrowright c$ is h-short, $\alpha \in \mathcal{A},\ a \in X$ and $p \in \alpha$.

2.30. Lemma. *A δ-hyperbolic space has property $P(\delta + h/2, h, \mathcal{A})$ for every $h > 0$ and for every $\mathcal{A}$.*

Proof. With the notation of 2.29 we have

$$(a|b)_p \wedge (a|c)_p \leq (b|c)_p + \delta.$$

Here $(b|c)_p \leq h/2$ by 2.9, and the lemma follows. □

2.31. Lemma. *A $(\delta, h, \mathcal{A})$-Rips space has property $P(\delta + h/2, h, \mathcal{A})$.*

Proof. With the notation of 2.29, choose h-short members $\beta\colon a \curvearrowright c$ and $\gamma\colon a \curvearrowright b$ of $\mathcal{A}$. By the Rips condition, there is a point $q \in \beta \cup \gamma$ with $|q - p| \le \delta$. We may assume that $q \in \beta$. Then

$$\begin{aligned}|a-p| + |p-c| &\le |a-q| + |q-p| + |p-q| + |q-c| \\ &\le l(\beta) + 2\delta \le |a-c| + h + 2\delta,\end{aligned}$$

whence $(a|c)_p \le \delta + h/2$. □

2.32. Lemma. *Suppose that X has property $P(\delta, h, \mathcal{A})$ and that $\alpha\colon b \curvearrowright c$ is an h-short member of $\mathcal{A}$. Then*

$$d(p, \alpha) \le (b|c)_p + 2\delta$$

for each $p \in X$.

Proof. The arc α is the union of the closed sets $A = \{x \in \alpha\colon (p|b)_x \le \delta\}$ and $B = \{p \in \alpha\colon (p|c)_x \le \delta\}$. Since $b \in A,\ c \in B$ and since α is connected, there is a point $y \in A \cap B$. Then

$$\begin{aligned}4\delta &\ge 2(p|b)_y + 2(p|c)_y = 2|p-y| + |b-y| + |c-y| - |p-b| - |p-c| \\ &\ge 2d(p,\alpha) + |b-c| - |p-b| - |p-c| = 2d(p,\alpha) - 2(b|c)_p. \ \square\end{aligned}$$

Combining Lemmas 2.9, 2.30 and 2.32 we get the following useful result, which shows that in a hyperbolic space, the Gromov product $(x|y)_p$ is roughly equal to the distance $d(p, \alpha)$ for any h-short arc $\alpha\colon x \curvearrowright y$.

2.33. Standard estimate. *Suppose that X is δ-hyperbolic, that $p \in X$ and that $\alpha\colon x \curvearrowright y$ is h-short. Then*

$$d(p,\alpha) - 2\delta - h \le (x|y)_p \le d(p,\alpha) + h/2.$$

The second inequality is true in every space. □

2.34. Theorem. *If X is $(\delta, h, \mathcal{A})$-Rips, then X is δ'-hyperbolic with $\delta' = 3\delta + 3h/2$.*

Proof. Let $a, b, c, p \in X$. Choose h-short members $\alpha\colon b \curvearrowright c,\ \beta\colon a \curvearrowright c, \gamma\colon a \curvearrowright b$ of $\mathcal{A}$. Since $\alpha \subset \bar{B}(\beta \cup \gamma, \delta)$ by the Rips condition, we obtain by 2.9

$$(a|c)_p \wedge (a|b)_p \le d(p,\beta) \wedge d(p,\gamma) + h/2 \le d(p,\alpha) + \delta + h/2.$$

Since X has property $P(\delta + h/2, h, \mathcal{A})$ by 2.31, Lemma 2.32 gives $d(p,\alpha) \le (b|c)_p + 2\delta + h$. Consequently,

$$(a|c)_p \wedge (a|b)_p \le (b|c)_p + 3\delta + 3h/2,$$

and the theorem follows. □

2.35. Theorem. *If X is δ-hyperbolic, then X is $(\delta', h, \mathcal{A})$-Rips with $\delta' = 3\delta + 3h/2$ for each $h > 0$ and for each $\mathcal{A}$.*

Proof. Suppose that $\Delta = (\alpha, \beta, \gamma)$ is an h-short triangle in X and let $x \in \alpha$. We must show that

$$d(x, \beta \cup \gamma) \le 3\delta + 3h/2. \tag{2.36}$$

Let a, b, c be the vertices of Δ as in 2.21. By 2.30, the space X has property $P(\delta + h/2, h, \mathcal{A})$, and hence 2.32 gives

$$d(x, \gamma) \leq (a|b)_x + 2\delta + h,\ d(x, \beta) \leq (a|c)_x + 2\delta + h.$$

Consequently,

$$\begin{aligned} d(x, \beta \cup \gamma) = d(x, \beta) \wedge d(x, \gamma) &\leq (a|b)_x \wedge (a|c)_x + 2\delta + h \\ &\leq (b|c)_x + 3\delta + h. \end{aligned}$$

Since $(b|c)_x \leq h/2$ by 2.9, this yields (2.36). $\square$

2.37. *Remark.* Theorems 2.34 and 2.35 show that the properties δ-hyperbolic and $(\delta, h, \mathcal{A})$-Rips are quantitatively equivalent. Moreover, the property $(\delta, h, \mathcal{A})$-Rips is quantitatively independent of the family $\mathcal{A}$.

2.38. *Notes.* The classical versions of the results of Section 2 in geodesic spaces can be found in most of the books mentioned in the introduction. In the geodesic case, the center of a side of a triangle degenerates to one point, and thus the center $Z(\Delta)$ of a geodesic triangle Δ contains at most three points.

3 Geodesic stability

3.1. *Summary.* We study quasigeodesics and more general arcs and paths in an intrinsic hyperbolic space. We show that two such arcs or paths joining given points a and b run close to each other even if $|a - b|$ is large. This property of hyperbolic spaces is called *geodesic stability*. We also show that conversely, this property implies that the space is hyperbolic. As applications we show that a quasi-isometry between intrinsic spaces preserves hyperbolicity and study the behavior of the Gromov product in a quasi-isometry.

3.2. *Terminology.* Let $\lambda \geq 1$ and $\mu \geq 0$. We say that a map $f\colon X \to Y$ between metric spaces is a (λ, μ)-*quasi-isometry* if

(3.3) $$\lambda^{-1}|x - y| - \mu \leq |fx - fy| \leq \lambda|x - y| + \mu$$

for all $x, y \in X$. The map f need not be continuous. In the case where $f\colon I \to Y$ is a map of a real interval I, we say that such a map is a (λ, μ)-*quasi-isometric path*.

These and related maps appear with various names in the literature. For example, [BS] calls a map satisfying (3.3) is a rough quasi-isometry. In my earlier papers on the free quasiworld I replaced the left side of (3.3) by $\lambda^{-1}(|x - y| - \mu)$ and called such maps μ-coarsely or μ-roughly λ-bilipschitz. My reason was that if f is bijective, then f^{-1} satisfies exactly the same condition.

For $\mu = 0$, (3.3) reduces to the λ-*bilipschitz* condition

$$\lambda^{-1}|x - y| \leq |fx - fy| \leq \lambda|x - y|.$$

A bilipschitz map between metric spaces is always an embedding.

An arc $\alpha\colon a \curvearrowright b$ in a space X is a λ-*quasigeodesic*, $\lambda \geq 1$, if

$$l(\alpha[u, v]) \leq \lambda|u - v|$$

for all $u, v \in \alpha$. Then the arclength parametrization $\varphi\colon [0, l(\alpha)] \to \alpha$ satisfies the inequalities

$$\lambda^{-1}|s-t| \le |\varphi(s) - \varphi(t)| \le |s-t|,$$

and thus φ is λ-bilipschitz.

3.4. *Remark.* The quasi-isometry condition (3.3) is often implied by seemingly different conditions. For example, suppose that $f\colon X \to Y$ is a bijective map between intrinsic spaces such that f and f^{-1} are uniformly continuous. Then f is a quasi-isometry. To see this, let $x, y \in X$ and choose a number $q > 0$ such that $|fu - fv| \le 1$ whenever $u, v \in X$ and $|u - v| \le q$. Let $h > 0$ and choose an h-short arc $\gamma\colon x \curvearrowright y$. Let $k \ge 0$ be the unique integer with $kq \le l(\gamma) < (k+1)q$. Choose successive points $x = x_0, \dots, x_{k+1} = y$ such that $l(\gamma[x_{j-1}, x_j]) \le q$ for all j. Then $|fx_{j-1} - fx_j| \le 1$, whence

$$|fx - fy| \le k + 1 \le l(\gamma)/q + 1 \le |x - y|/q + h/q + 1.$$

As $h \to 0$, we get $|fx - fy| \le |x - y|/q + 1$, which is the first inequality of (3.3). Treating similarly the inverse map f^{-1} we obtain the second part of (3.3).

More generally, the result holds for roughly quasiconvex spaces, which means that each pair x, y can be joined by an arc γ with $l(\gamma) \le c_1|x - y| + c_2$.

To prove the stability theorem 3.7 we need two lemmas. The first lemma is valid in every metric space.

3.5. Lemma. *Suppose that $\gamma\colon x \curvearrowright y$ is an h-short arc in a space X. Let $r > 0,\ s \ge 0$, and suppose that $Q \subset X$ is a set such that $\{x, y\} \subset Q \subset \bar{B}(\gamma, r)$ and such that $d(Q_1, Q_2) \le s$ whenever $Q = Q_1 \cup Q_2,\ x \in Q_1,\ y \in Q_2$. Then $\gamma \subset \bar{B}(Q, 2r + s + h)$.*

Proof. Assume that the lemma is false. Set $t = 2r + s + h$. There is $\varepsilon > 0$ and a point $z \in \gamma$ such that $d(z, Q) = t + 4\varepsilon$. Since $l(\gamma[x, z]) \ge |x - z| \ge t + 4\varepsilon$, there is $x_1 \in \gamma[x, z]$ with $l(\gamma[x_1, z]) = t/2 + 2\varepsilon$. Similarly, there is $x_2 \in \gamma[z, y]$ with $l(\gamma[z, x_2]) = t/2 + 2\varepsilon$. Set

$$\gamma' = \gamma[x_1, x_2],\ \gamma_1 = \gamma[x, x_1],\ \gamma_2 = \gamma[x_2, y],\ U' = B(\gamma', r + \varepsilon), U_i = B(\gamma_i, r + \varepsilon),$$

$i = 1, 2$.

If U' meets Q, there is a point $z' \in \gamma'$ with $d(z', Q) < r + \varepsilon$. Since $r \le t/2$, we obtain the contradiction

$$t + 4\varepsilon = d(z, Q) < r + \varepsilon + |z - z'| \le r + \varepsilon + t/2 + 2\varepsilon \le t + 3\varepsilon.$$

Hence $Q \cap U' = \varnothing$. As $Q \subset \bar{B}(\gamma, r)$, we thus have $Q = Q_1 \cup Q_2$ with $Q_i = Q \cap U_i$. From the condition on Q it follows that there are points $q_i \in Q_i$ with $|q_1 - q_2| < s + \varepsilon$. Furthermore, there are $y_i \in \gamma_i$ such that $|y_i - q_i| < r + \varepsilon$. Then $|y_1 - y_2| < 2r + s + 3\varepsilon$, and we get the contradiction

$$t + 4\varepsilon \le l(\gamma') \le l(\gamma[y_1, y_2]) \le |y_1 - y_2| + h < 2r + s + 3\varepsilon + h = t + 3\varepsilon. \square$$

3.6. Projection lemma. *Suppose that X is an intrinsic (δ, h)-Rips space. Let $\gamma \subset X$ be an h-short arc and let $x_1, x_2 \in X$ and $y_1, y_2 \in \gamma$ be points such that*

(1) $|x_i - y_i| = d(x_i, \gamma) \geq R > 0$ *for* $i = 1, 2$,

(2) $|x_1 - x_2| < 2R - 4\delta - h$.

Then $|y_1 - y_2| \leq 8\delta + 2h$.

Proof. Pick $y_0 \in \gamma[y_1, y_2]$ with $|y_0 - y_1| = |y_0 - y_2|$. Choose h-short arcs $\beta_i \colon y_i \curvearrowright x_i$ and $\alpha \colon x_1 \curvearrowright x_2$. Applying twice the Rips condition we obtain $\gamma[y_1, y_2] \subset \bar{B}(\beta_1 \cup \alpha \cup \beta_2, 2\delta)$. Hence there is $z \in \beta_1 \cup \alpha \cup \beta_2$ with $|z - y_0| \leq 2\delta$.

If $z \in \alpha$, then

$$R \leq d(x_i, \gamma) \leq |x_i - z| + |z - y_0| \leq |x_i - z| + 2\delta$$

for $i = 1, 2$, and we obtain

$$2R \leq |x_1 - z| + |x_2 - z| + 4\delta \leq |x_1 - x_2| + h + 4\delta,$$

which is a contradiction by (2). Thus $z \in \beta_1 \cup \beta_2$, and we may assume that $z \in \beta_1$. Then

$$|x_1 - y_1| = d(x_1, \gamma) \leq |x_1 - z| + |z - y_0| \leq |x_1 - z| + 2\delta,$$
$$|x_1 - z| + |z - y_1| \leq l(\beta_1) \leq |x_1 - y_1| + h,$$

whence $|z - y_1| \leq 2\delta + h$. Consequently,

$$|y_1 - y_2| \leq 2|y_1 - y_0| \leq 2|y_1 - z| + 2|z - y_0| \leq 8\delta + 2h. \square$$

3.7. Stability theorem. *Suppose that X is an intrinsic δ-hyperbolic space and that $\varphi \colon [a, b] \to X$ and $\varphi' \colon [a', b'] \to X$ are (λ, μ)-quasi-isometric paths with $\varphi(a) = \varphi'(a)$ and $\varphi(b) = \varphi'(b)$. Then $d_H(\operatorname{im} \varphi, \operatorname{im} \varphi') \leq M(\delta, \lambda, \mu)$.*

Proof. Fix $h > 0$ and choose an h-short arc $\gamma \colon \varphi(a) \to \varphi(b)$. It suffices to find an estimate $d_H(\gamma, \operatorname{im} \varphi) \leq M(\lambda, \mu, \delta, h)$. The space X is (δ', h)-Rips with $\delta' = 3\delta + 3h/2$ by 2.35. We show that

$$\operatorname{im} \varphi \subset \bar{B}(\gamma, M_1), \quad \gamma \subset \bar{B}(\operatorname{im} \varphi, M_2), \tag{3.8}$$

where $M_1 = 30\lambda^4(4\delta' + h) + 8\lambda^2\mu$ and $M_2 = 2M_1 + \mu + h$.

Set $r = 5\lambda(4\delta' + h),\ R = \lambda r + \mu,\ I = [a, b]$, and define $g \colon I \to \mathsf{R}$ by $g(t) = d(\varphi(t), \gamma)$. Let $s \in I$. For the first inclusion of (3.8) we must show that $g(s) \leq M_1$.

We may assume that $g(s) > R$. Set

$$a_0 = \sup\{t \in I \colon t < s,\ g(t) \leq R\},$$
$$b_0 = \inf\{t \in I \colon t > s,\ g(t) \leq R\},$$
$$L = b_0 - a_0 - r.$$

Choose points $u_0, v_0 \in I$ such that

$$a_0 - r/2 \leq u_0 \leq a_0,\ g(u_0) \leq R,\ b_0 \leq v_0 \leq b_0 + r/2,\ g(v_0) \leq R.$$

We may assume that $s - a_0 \le b_0 - s$. Then

$$g(s) \le g(u_0) + |\varphi(s) - \varphi(u_0)| \le R + \lambda(s - u_0) + \mu \le R + \lambda(L/2 + r) + \mu.$$

It suffices to show that

$$L \le 8\lambda R + 2\lambda\mu, \tag{3.9}$$

because then

$$g(s) \le R + 4\lambda^2 R + \lambda^2\mu + \lambda r + \mu \le 6\lambda^2 R + 2\lambda^2\mu = M_1.$$

We may assume that $L > 0$. Setting $u = a_0 + r/2,\ v = b_0 - r/2$ we have $L = v - u$. Let n be the integer with $(n-1)r \le L < nr$. If $n \le 4$, then (3.9) is clearly true. Suppose that $n \ge 5$. Since $n \le L/r + 1$, we have $n \le 5L/4r$. Divide $[u, v]$ into subintervals $I_i = [t_{i-1}, t_i],\ 1 \le i \le n$, by points $t_i = u + iL/n$. For each $i \in \{0, \dots, n\}$ let $y_i \in \gamma$ be a point closest to $\varphi(t_i)$. For all i we have $|\varphi(t_i) - y_i| = g(t_i) \ge R$ and

$$|\varphi(t_{i-1}) - \varphi(t_i)| \le \lambda(t_i - t_{i-1}) + \mu < R < 2R - 4\delta' - h.$$

Hence the projection lemma 3.6 gives $|y_{i-1} - y_i| \le 8\delta' + 2h$. Since

$$|\varphi(u) - y_0| = g(u) \le g(u_0) + |\varphi(u_0) - \varphi(u)| \le R + \lambda r + \mu = 2R$$

and similarly $|\varphi(v) - y_n| \le 2R$, we get

$$|\varphi(u) - \varphi(v)| \le |\varphi(u) - y_0| + \sum_{i=1}^{n} |y_{i-1} - y_i| + |y_n - \varphi(v)| \le 4R + 2n(4\delta' + h).$$

As $|\varphi(u) - \varphi(v)| \ge L/\lambda - \mu$, this yields

$$L \le 4\lambda R + 2\lambda n(4\delta' + h) + \lambda\mu.$$

Here $n \le 5L/4r$, whence the middle term on the right is at most $L/2$, and (3.9) follows.

To prove the second part of the theorem, observe that $Q = \operatorname{im}\varphi$ satisfies the condition of 3.5 with $s = \mu$. Hence we can apply 3.5 with $s = \mu,\ r = M_1$ and obtain $\gamma \subset \bar{B}(Q, M_2)$. □

3.10. *Remarks.* 1. Theorem 3.7 holds with $M = M_1 + M_2$ for each $h > 0$. As $h \to 0$, we get the explicit bound $M = 1080\lambda^4\delta + 25\lambda^2\mu$.

2. The stability theorem 3.7 holds, in fact, in every hyperbolic space. As observed in [BS, 5.4], this follows from the embedding theorem [BS, 4.1] of Bonk and Schramm.

The following special case of the stability theorem 3.7 is frequently needed in applications:

3.11. Theorem. *Suppose that $\alpha\colon a \curvearrowright b$ is an h-short arc and that $\beta\colon a \curvearrowright b$ is a c-quasigeodesic in an intrinsic δ-hyperbolic space. Then $d_H(\alpha, \beta) \le M(\delta, c, h)$.* □

We next give a converse of 3.7.

3.12. Theorem. *Let $h > 0,\ \delta > 0$. Suppose that X is an intrinsic space such that $\tau \subset \bar{B}(\alpha, \delta)$ whenever τ and α are arcs in X with common endpoints such that α is h-short and*

$$l(\tau[u, v]) \le 3|u - v| + 4h \tag{3.13}$$

for all $u, v \in \tau$. Then X is (δ, h)-Rips.

Proof. Let $\Delta = (\alpha, \beta, \gamma)$ be an h-short triangle with vertices a, b, c as in 2.21. Let w be the point of α closest to a and let $\sigma\colon a \curvearrowright w$ be h-short. Let p be the first point of σ in α. Then $\tau = \sigma[a, p] \cup \alpha[p, b]$ is an arc from a to b. We show that τ satisfies the condition of the theorem. It suffices to show that

$$l(\sigma[u, p]) + l(\alpha[p, v]) \le 3|u - v| + 4h \tag{3.14}$$

for all $u \in \sigma[a, p],\ v \in \alpha[p, b]$.

Since

$$\begin{aligned} |a - u| + |u - p| \le l(\sigma) \le |a - w| + h &= d(a, \alpha) + h \\ &\le |a - v| + h \le |a - u| + |u - v| + h, \end{aligned}$$

we have $|u - p| \le |u - v| + h$, and thus $|p - v| \le |p - u| + |u - v| \le 2|u - v| + h$. Consequently,

$$\begin{aligned} l(\sigma[u, p]) &\le |u - p| + h \le |u - v| + 2h, \\ l(\alpha[p, v]) &\le |p - v| + h \le 2|u - v| + 2h, \end{aligned}$$

and (3.14) follows.

By the condition of the theorem we obtain $\alpha[p, b] \subset \tau \subset \bar{B}(\gamma, \delta)$. Similarly $\alpha[p, c] \subset \bar{B}(\beta, \delta)$, and the theorem is proved. $\square$

3.15. *Terminology.* We say that a map $f\colon X \to Y$ between metric spaces is *μ-roughly injective* if the diameter of each point-inverse is at most μ. A (λ, μ)-quasi-isometry is clearly $\lambda\mu$-roughly injective. A map $f\colon X \to Y$ is *μ-roughly surjective* if for each $y \in Y$ there is $x \in X$ with $|fx - y| \le \mu$. Some authors include this condition in the definition of a (rough) quasi-isometry; maps without this condition are then called (roughly) quasi-isometric embeddings or (rough) quasi-isometries into.

A map $g\colon Y \to X$ is a *μ-rough inverse* of $f\colon X \to Y$ if $|gfx - x| \le \mu$ and $|fgy - y| \le \mu$ for all $x \in X$ and $y \in Y$.

3.16. Lemma. *If $f\colon X \to Y$ has a μ-rough inverse, then f is 2μ-roughly injective and μ-roughly surjective. Conversely, if f is μ_i-roughly injective and μ_s-roughly surjective, then there is a map $g\colon Y \to X$ such that*

$$|gfx - x| \le \mu_i, \quad |fgy - y| \le \mu_s$$

for all $x \in X$ and $y \in Y$. Thus g is a μ-rough inverse of f with $\mu = \mu_i \vee \mu_s$.

Proof. Assume that $g\colon Y \to X$ is a μ-rough inverse of f. If $a, b \in X$ and $fa = fb$, then $|a - b| \le |a - gfa| + |gfb - b| \le 2\mu$, so φ is 2μ-roughly injective. Furthermore, if $y \in Y$, then $|y - fgy| \le \mu$, whence f is μ-roughly surjective.

In the converse part of the lemma, define $g\colon Y \to X$ as follows: For each $y \in Y$ choose a point $y' \in fX$ with $|y - y'| \le \mu_s$. If $y \in fX$, we let $y' = y$. Choose a point $x' \in f^{-1}\{y'\}$ and set $gy = x'$. Then $fgy = y'$, whence $|fgy - y| \le \mu_s$. If $x \in X$, then $gfx \in f^{-1}fx$, which yields $|gfx - x| \le \mu_i$ by μ_i-rough injectivity. □

3.17. Lemma. (1) *If $X \xrightarrow{f_1} Y \xrightarrow{f_2} Z$ and if f_i is a (λ_i, μ_i)-quasi-isometry, then $f_2 \circ f_1$ is a (λ, μ)-quasi-isometry with $\lambda = \lambda_1\lambda_2$ and $\mu = \lambda_2\mu_1 + \mu_2$.*

(2) *If $f\colon X \to Y$ is a μ-roughly surjective (λ, μ)-quasi-isometry, then f has a $\lambda\mu$-rough inverse, which is a $(\lambda, 3\lambda\mu)$-quasi-isometry.*

Proof. (1) follows by direct computation. In (2), the map f is $\lambda\mu$-roughly injective and μ-roughly surjective, so it has a $\lambda\mu$-rough inverse $g\colon Y \to X$ satisfying $|fgy - y| \le \mu$ for all $y \in Y$ by 3.16. If $y, y' \in Y$, then

$$\lambda^{-1}|gy - gy'| - \mu \le |fgy - fgy'| \le \lambda|gy - gy'| + \mu,$$
$$|y - y'| - 2\mu \le |fgy - fgy'| \le |y - y'| + 2\mu,$$

which yield

$$\lambda^{-1}|y - y'| - 3\lambda^{-1}\mu \le |gy - gy'| \le \lambda|y - y'| + 3\lambda\mu,$$

and the lemma follows. □.

We next show that hyperbolicity is preserved by quasi-isometries.

3.18. Theorem. *Suppose that X and Y are intrinsic metric spaces and that $f\colon X \to Y$ is a μ-roughly surjective (λ, μ)-quasi-isometry. If X is δ-hyperbolic, then Y is δ'-hyperbolic with $\delta' = \delta'(\delta, \lambda, \mu)$.*

Proof. We show that Y satisfies the condition of 3.12 with $h = 3/4$ and $\delta = M(\delta, \lambda, \mu)$. Alternatively, one can easily show by stability that Y satisfies a Rips condition.

Lemma 3.17(2) gives a $\lambda\mu$-rough inverse $g\colon Y \to X$ of f, and g is a $(\lambda, 3\lambda\mu)$-quasi-isometry. Assume that $\gamma, \tau\colon a \curvearrowright b$ are arcs in Y with common endpoints such that γ is h-short and τ satisfies (3.13). It suffices to find a number $M(\delta, \lambda, \mu)$ such that $\tau \subset \bar{B}(\gamma, M)$.

Let φ and ψ be the arclength parametrizations of γ and τ, respectively. Then φ is $(1, h)$-quasi-isometric and ψ is $(3, 4h/3)$-quasi-isometric. As $h = 3/4$, both paths are $(3, 1)$-quasi-isometric. By 3.17(1), the paths $g\circ\varphi, g\circ\psi$ are (λ', μ')-quasi-isometric with $\lambda' = 3\lambda,\ \mu' = \lambda + 3\lambda\mu$. By the stability theorem 3.7, there is $M_0(\delta, \lambda, \mu)$ such that $\operatorname{im}(g \circ \psi) \subset \bar{B}(\operatorname{im}(g \circ \varphi), M_0)$.

Let $y \in \tau$. Since $gy \in \operatorname{im}(g \circ \psi)$, there is a point $x_1 \in \operatorname{im}(g \circ \varphi)$ with $|gy - x_1| \le M_0$. Choose a point $y_1 \in \gamma$ with $gy_1 = x_1$. It suffices to show that $|y - y_1| \le M(\delta, \lambda, \mu)$. As $|fgz - z| \le \lambda\mu$ for all $z \in Y$, we obtain

$$\begin{aligned}|y - y_1| &\le |y - fgy| + |fgy - fx_1| + |fx_1 - y_1| \le \lambda\mu + \lambda|gy - x_1| + \mu + \lambda\mu \\ &\le 2\lambda\mu + \lambda M_0 + \mu = M(\delta, \lambda, \mu). \ \square\end{aligned}$$

As another application of stability we study the behavior of the Gromov product $(x|y)_w$ in a quasi-isometry.

3.19. Theorem. *Suppose that X and Y are intrinsic δ-hyperbolic spaces and that $f\colon X \to Y$ is (λ, μ)-quasi-isometry. Let $x, y, z, p \in X$ and write $x' = fx$ etc. for images. Set $s = (x|y)_p - (x|z)_p$ and $s' = (x'|y')_{p'} - (x'|z')_{p'}$. Then there is a number $C = C(\delta, \lambda, \mu) > 0$ with the following properties:*

(1) $\lambda^{-1}(x|y)_p - C \le (x'|y')_{p'} \le \lambda(x|y)_p + C$,
(2) $\lambda^{-1}|s| - C \le |s'| \le \lambda|s| + C$,
(3) *If* $s \ge 0$, *then* $\lambda^{-1}s - C \le s' \le \lambda s + C$.

Proof. Setting $z = p$ in (2) we see that (1) follows from (2). To prove (2) and (3), let $h > 0$ and let Δ_y, Δ_z be h-short triangles in X with vertices p, x, y and p, x, z, respectively, such that Δ_y and Δ_z have a common side $\alpha\colon p \curvearrowright x$. We may assume that $s \ge 0$. Choose points $u_y, u_z \in \alpha$ such that

$$|u_y - p| = (x|y)_p,\ |u_z - p| = (x|z)_p,\ u_z \in \alpha[p, u_y].$$

Since α is h-short, we have

$$s \le |u_y - u_z| \le s + h. \tag{3.20}$$

Next choose h-short triangles Δ''_y, Δ''_z in Y with vertices p', x', y' and p', x', z', respectively, and with a common side $\alpha''\colon p' \curvearrowright x'$. Let $u''_y, u''_z \in \alpha''$ be points with

$$|u''_y - p'| = (x'|y')_{p'},\ |u''_z - p'| = (x'|z')_{p'}.$$

As above, we get $|s'| \le |u''_y - u''_z| \le |s'| + h$.

Let $C_1, C_2, \dots$ denote positive constants depending only on $(\delta, \lambda, \mu, h)$. To get constants independent of h we may put $h = 1$ or, for better estimates, let $h \to 0$.

The space X is (δ', h)-Rips with $\delta' = 3\delta + 3h/2$ by 2.35. If τ is a side of Δ_y or Δ_z, its arclength parametrization is $(1, h)$-quasi-isometric, and $f \circ \varphi$ is $(\lambda, \lambda h + \mu)$-quasi-isometric. By the stability theorem 3.7, the image $\tau' = f\tau = \mathrm{im}\,(f \circ \varphi)$ lies in a neighborhood $\bar{B}(\tau'', M)$ of the corresponding side τ'' of Δ''_y or Δ''_z with $M = M(\delta, \lambda, \mu, h)$. By 2.24 we have $d(Z(\Delta_y)) \le 4\delta + 3h$, and thus $d(fZ(\Delta_y)) \le 4\lambda\delta + 3\lambda h + \mu$. Since $u_y \in Z(\Delta_y)$ by (2.23), the point $u'_y = fu_y$ lies within distance $C_1 = M + 4\lambda\delta + 3\lambda h + \mu$ from the sides of Δ''_y. By 2.25(2) this implies that $d(u'_y, Z(\Delta''_y)) \le 3C_1 + h/2$. Moreover, $u''_y \in Z(\Delta''_y),\ d(Z(\Delta''_y)) \le 4\delta + 3h$, and we obtain

$$|u'_y - u''_y| \le 3C_1 + h/2 + d(Z(\Delta''_y)) \le 3C_1 + 4\delta + 4h = C_2, \tag{3.21}$$

and similarly

$$|u'_z - u''_z| \le C_2. \tag{3.22}$$

These estimates and (3.20) imply that

$$|s'| \le |u''_y - u''_z| \le |u'_y - u'_z| + 2C_2 \le \lambda|u_y - u_z| + \mu + 2C_2 \le \lambda s + \lambda h + \mu + 2C_2,$$

which is the second inequality of (2).

The first inequality of (2) is obtained similarly:

$$\begin{aligned} |s'| &\ge |u''_y - u''_z| - h \ge |u'_y - u'_z| - h - 2C_2 \\ &\ge \lambda^{-1}|u_y - u_z| - \mu - h - 2C_2 \ge \lambda^{-1}s - \mu - h - 2C_2. \end{aligned}$$

The second inequality of (3) follows from (2). Also the first inequality of (3) follows from (2) if $s' \geq 0$. Assume that $s' < 0$. It suffices to find an estimate

$$|s'| \leq C_3, \tag{3.23}$$

because then $s' + C_3 \geq 0$, and (2) gives

$$\lambda^{-1} s \leq C_3 + C \leq s' + 2C_3 + C.$$

Since $|u''_y - p'| - |u''_z - p'| = s' < 0$, we may assume that $u''_y \in \alpha''[p', u''_z]$. We may also assume that $|s'| > C_2$. Then (3.22) gives

$$|u'_z - p'| \geq |u''_z - p'| - C_2 > |u''_z - p'| - |s'| = |u''_y - p'|.$$

Hence there is a point $v \in \alpha[p, u_z]$ with $|v' - p'| = |u''_y - p'|$. Since $\alpha' \subset \bar{B}(\alpha'', M)$, we can choose a point $v'' \in \alpha''$ with $|v'' - v'| \leq M$. We show that

$$|v'' - u''_y| \leq M + h. \tag{3.24}$$

If $v'' \in \alpha''[u''_y, x']$, the h-shortness of α'' gives

$$\begin{aligned} |v'' - u''_y| + |u''_y - p'| &\leq |v'' - p'| + h \leq |v'' - v'| + |v' - p'| + h \\ &\leq M + |u''_y - p'| + h, \end{aligned}$$

which implies (3.24). If $v'' \in \alpha''[p', u''_y]$, we similarly get $|u''_y - v''| + |v'' - p'| \leq |u''_y - p'| + h$. Here $|v'' - p'| \geq |v' - p'| - |v' - v''| \geq |u''_y - p'| - M$, and (3.24) follows.

By (3.24) we have $|v' - u''_y| \leq 2M + h$. Thus (3.21) gives

$$|v' - u'_y| \leq |v' - u''_y| + |u''_y - u'_y| \leq 2M + h + C_2 = C_4,$$

whence

$$|v - u_z| \leq |v - u_y| + h \leq \lambda C_4 + \lambda\mu + h.$$

By (3.22) we obtain

$$\begin{aligned} \lambda|v - u_z| + \mu \geq |v' - u'_z| &\geq |u''_y - u''_z| - |v' - u''_y| - |u'_z - u''_z| \\ &\geq |s'| - 2M - h = |s'| - C_4. \end{aligned}$$

These estimates imply (3.23) with $C_3 = C_4 + \lambda(\lambda C_4 + \lambda\mu + h) + \mu$. $\square$.

3.25. *Notes.* This section is mainly based on [Bo] and [BS]. The simple proof of 3.12 seems to be new. A related but deeper result was proved by Bonk [Bo]. He considers the following weaker form of geodesic stability: For each $\lambda \geq 1$ there is $M > 0$ such that for each λ-quasigeodesic $\alpha\colon x \curvearrowright y$ there is a geodesic $\gamma\colon x \curvearrowright y$ such that $\alpha \subset B(\gamma, M)$. This does not mean that $\alpha \subset B(\beta, M)$ for every geodesic $\beta\colon x \curvearrowright y$. He shows that this condition implies that a geodesic space is Gromov hyperbolic.

4 Quasisymmetric and quasimöbius maps

4.1. *Summary.* We give the theory of quasisymmetric and quasimöbius maps needed in Section 5. These maps are also considered in the relative setting and in metametric spaces.

4.2. *Metametric spaces.* Let M be a set. A function $d\colon M \times M \to [0, \infty)$ is a *metametric* if

(1) $d(a, b) = d(b, a)$ for all $a, b \in M$,

(2) $d(a, c) \le d(a, b) + d(b, c)$ for all $a, b, c \in M$,

(3) $d(a, b) = 0$ implies that $a = b$.

In other words, d satisfies the axioms of a metric except that the possibility $d(a, a) > 0$ is allowed. The pair (M, d) is a *metametric space*. In the subset $\operatorname{met} M = \{x \in M\colon d(x, x) = 0\}$, d defines a metric.

It is possible that this concept has been considered (probably with another name) in the literature, but the author has not been able to find it. A trivial example of a metametric is the constant function $d(x, y) = 1$ for all $x, y \in M$.

We say that a point $a \in M$ is *small* or *large* according as $d(a, a) = 0$ or $d(a, a) > 0$.

A metametric d of M induces a Hausdorff topology in the usual way: Write $B(a, r) = \{x \in M\colon d(x, a) < r\}$ and observe that $B(a, r) = \varnothing$ if $r \le d(a, a)/2$, because if $d(a, x) < d(a, a)/2$, then $d(a, a) \le d(a, x) + d(x, a) < d(a, a)$. A set $U \subset M$ is open if for each $a \in U$ there is $r > 0$ such that $B(a, r) \subset U$. We see that each large point of M is isolated in this topology. A basis for this topology is given by balls $B(a, r)$ for small points a and by singletons $\{b\}$ for large points b.

A metametric space is metrizable. In fact, a metametric d can be changed to a metric d_1 simply by setting $d_1(x, x) = 0$ and $d_1(x, y) = d(x, y)$ for $x \ne y$. Then d and d_1 define the same topology. By this trick one could avoid the use of metametrics, but this would be artifical and unnatural, for example, with the metametric $d_{p,\varepsilon}$, to be considered in Section 5.

Some familiar results on metric spaces fail to be true for metametric spaces. For example, if a is a large point, then the constant sequence (x_i) with $x_i = a$ converges to a (because all points x_i lie in each neighborhood of a), but $d(x_i, a)$ does not tend to 0. A map satisfying the (ε, δ)-condition is continuous, but the converse is only true at small points.

Let (M', d') be another metametric space. We say that a map $f\colon M \to M'$ is *positive* if $d'(fx, fy) > 0$ whenever $d(x, y) > 0$. In other words, the inverse image $f^{-1}\{y\}$ of each small point $y \in fX$ consists of a single small point of X. A map between metric spaces is positive iff it is injective. A map $f\colon M \to M'$ is λ*-bilipschitz* if

$$\lambda^{-1} d(x, y) \le d'(fx, fy) \le \lambda d(x, y)$$

for all $x, y \in M$. A bilipschitz map is always positive and continuous, but it need not be injective.

4.3. *Quasisymmetry.* Let (M, d) be a metametric space. We say that a triple $T = (x, y, z)$ of points in M is *positive* if $d(x, z) > 0$. The *ratio* of a positive triple $T = (x, y, z)$ is the number

$$|T| = d(x, y)/d(x, z) \ge 0.$$

Suppose that (M', d') is another metametric space. A positive map $f\colon M \to M'$ maps every positive triple $T = (x, y, z)$ in M to a positive triple $fT = (fx, fy, fz)$ in M'. Let $\eta\colon [0, \infty) \to [0, \infty)$ be a homeomorphism. We say that f is η*-quasisymmetric* if

(1) f is positive,

(2) $|fT| \le \eta(|T|)$ for each positive triple T.

A λ-bilipschitz map is η-quasisymmetric with $\eta(t) = \lambda^2 t$.

We next consider relative quasisymmetry. Let M and M' be as above and let $A \subset M$. We say that a triple $T = (x, y, z)$ in M is a *triple in* (M, A) if $x \in A$ or $\{y, z\} \subset A$. A map $f\colon M \to M'$ is *positive rel* A if $d(fx, fy) > 0$ whenever $d(x, y) > 0$ and $x \in A$. This implies that f maps every positive triple in (M, A) to a positive triple in M'. We say that f is η*-quasisymmetric rel* A if

(1) f is positive rel A,

(2) $|fT| \le \eta(|T|)$ for each positive triple T in (M, A).

We see that a map $f\colon M \to M'$ is η-quasisymmetric iff f is η-quasisymmetric rel M. If f is η-quasisymmetric rel A, the restriction $f|A$ is η-quasisymmetric.

We next show that in order that f be quasisymmetric rel A it suffices to verify (2) for each triple $T = (x, y, z)$ with $x \in A$. However, η must be replaced by another function.

4.4. Lemma. *Let $\eta\colon [0, \infty) \to [0, \infty)$ be a homeomorphism, let (M, d) and (M', d') be metametric spaces, and let $A \subset M$. Suppose that $f\colon M \to M'$ is positive rel A and that $|fT| \le \eta(|T|)$ for each triple $T = (x, y, z)$ in (M, A) with $x \in A$. Then f is η_1-quasisymmetric rel A with η_1 depending only on η.*

Proof. Let $T = (x, y, z)$ be a positive triple in M with $\{y, z\} \subset A$. For $T_1 = (z, y, x)$ we have

$$|T_1| = \frac{d(z, y)}{d(z, x)} \le \frac{d(z, x) + d(x, y)}{d(z, x)} = 1 + |T|,$$

and similarly $|fT| \le 1 + |fT_1|$. Hence $|fT| \le \eta_0(|T|)$ with $\eta_0(t) = 1 + \eta(1 + t)$.

To complete the proof we show that $|fT| \le 2\eta(2|T|)$ for small $|T|$. Assume that

$$|T| \le \tfrac{1}{2} \wedge (1 + \eta^{-1}(\tfrac{1}{2})^{-1})^{-1}.$$

Then $T_2 = (y, x, z)$ is positive, and

$$|T_2| = \frac{d(y, x)}{d(y, z)} \le \frac{d(y, x)}{d(z, x) - d(y, x)} = \frac{1}{1/|T| - 1} \le \eta^{-1}(\tfrac{1}{2}) \wedge 2|T|.$$

Hence $|fT_2| \le 1/2$, and

$$|fT| \le \frac{d'(fx, fy)}{d'(fy, fz) - d'(fx, fy)} = \frac{|fT_2|}{1 - |fT_2|} \le 2|fT_2| \le 2\eta(|T_2|) \le 2\eta(2|T|). \ \square$$

4.5. *Cross differences and cross ratios.* Let $Q = (x, y, z, w)$ be a quadruple of points in a metametric space (M, d). The *cross difference* of Q is the real number

$$\operatorname{cd} Q = \operatorname{cd}(Q, d) = d(x, y) + d(z, w) - d(x, z) - d(y, w).$$

The quadruple Q is *positive* if $d(x, z) > 0$ and $d(y, w) > 0$, and then the *cross ratio* of Q is the number

$$\operatorname{cr} Q = \operatorname{cr}(Q, d) = \frac{d(x, y)d(z, w)}{d(x, z)d(y, w)} \ge 0.$$

Permutating the points x, y, z, w we get at most 6 different numbers for $\operatorname{cr} Q$ (three numbers and their reciprocals). The reader should be warned that in the literature, at least 5 of them are called the cross ratio of (x, y, z, w).

In a metric space, the cross difference (or half of it) can be considered as a four-point version of the Gromov product, because we have

$$\operatorname{cd}(x, p, y, p) = 2(x|y)_p$$

for all $x, y, p \in M$. It is easy to see that a metric space X is δ-hyperbolic iff

$$\operatorname{cd}(x, y, z, p) \wedge \operatorname{cd}(x, y, p, z) \le 2\delta \tag{4.6}$$

for all $x, y, z, p \in X$.

It is possible to consider the cross ratio also in the extended space $\dot{M} = M \cup \{\infty\}$, but in the present article we only consider quasimöbius maps between bounded spaces.

Let X be a metric space. A direct computation shows that

$$\tfrac{1}{2}\operatorname{cd}(x, y, z, w) = -(x|y)_p - (z|w)_p + (x|z)_p + (y|w)_p \tag{4.7}$$

for all $x, y, z, w, p \in X$. Consequently, *the right-hand side is independent of the point* p. This is the key fact behind the quantitative quasimöbius invariance of the metametric $d_{p,\varepsilon}$ of the Gromov closure X^* of a hyperbolic space X, to be considered in Section 5.

4.8. *Quasimöbius maps.* Let (M, d) and (M', d') be metametric spaces. Observe that a positive map $f\colon M \to M'$ maps every positive quadruple Q in M to a positive quadruple fQ in M'. Let $\eta\colon [0, \infty) \to [0, \infty)$ be a homeomorphism. A map $f\colon M \to M'$ is said to be η*-quasimöbius* if

(1) f is positive,

(2) $\operatorname{cr} fQ \le \eta(\operatorname{cr} Q)$ for each positive quadruple Q in M.

Let $A \subset M$. A map $f\colon M \to M'$ is η*-quasimöbius rel* A if f is positive rel A and satisfies (2) for each positive quadruple $Q = (x, y, z, w)$ with $\{x, w\} \subset A$. This implies that (2) also holds for quadruples Q with $\{y, z\} \subset A$, because $\operatorname{cr}(x, y, z, w) = \operatorname{cr}(y, x, w, z)$.

4.9. *Properties.* We list some properties of quasisymmetric an quasimöbius maps.

1. Let $f\colon M \to M'$ be bijective, let $A \subset M$, and assume that f maps each small point of A to a small point. If f is η-quasisymmetric or η-quasimöbius rel A, then $f^{-1}\colon M' \to M$ is η'-quasisymmetric or η'-quasimöbius rel fA with $\eta' = \eta^{-1}(t^{-1})^{-1}$.

The condition for small points is needed to guarantee that f^{-1} is positive rel fA, but it holds automatically except in some trivial cases where M and A contain just a few points. For example, assume that $x \in A$ is small and that there are points $y \in M$ and $z \in A$ such that the points x, y, z are all distinct. Then the quadruple $Q = (x, x, y, z)$ is positive and $\operatorname{cr} Q = 0$. If f is quasimöbius rel A, then $\operatorname{cr} fQ = 0$, which implies that $d'(fx, fx) = 0$.

2. If $f\colon M \to M'$ is η-quasisymmetric (rel A), then f is θ-quasimöbius (rel A) with $\theta = \theta_\eta$. The proof in [Vä4, 6.25] for metric spaces holds almost verbatim in the metametric case.

4.10. *Notes.* Quasisymmetric maps in metric spaces were introduced in [TV], quasimöbius maps in [Vä1], and the relative case in [Vä2]. Lemma 4.4 is from [Vä4, 6.17], where its proof contains misprints. The metametric case has not been previously considered. It will turn out to be relevant in 5.35 and 5.38.

5 The Gromov boundary and closure

5.1. *Summary.* We associate to each δ-hyperbolic space a set ∂X, called the *Gromov boundary* of X. For each $p \in X$ and for small $\varepsilon > 0$ we define a metametric $d_{p,\varepsilon}$ in the *Gromov closure* $X^* = X \cup \partial X$, and $d_{p,\varepsilon}|\partial X$ is a metric of ∂X. The space ∂X with $d_{p,\varepsilon}$ is complete but, in general, not compact as in the case of proper spaces. The identity map $(X^*, d_{p,\varepsilon}) \to (X^*, d_{q,\varepsilon'})$ is η-quasimöbius with η depending only on ε'/ε. Each quasi-isometry $f\colon X \to Y$ between hyperbolic spaces extends to a map $f^*\colon X^* \to Y^*$, which is quasimöbius in the metametrics $d_{p,\varepsilon}$ and $d_{q,\varepsilon}$.

5.2. *Gromov sequences.* Let X be a metric space. We fix a base point $p \in X$; the pair (X, p) is then a *pointed space*. We shall write briefly

$$(x|y) = (x|y)_p$$

for $x, y \in X$. For a sequence of points (x_i) of points in X we use the notation

$$\bar{x} = (x_i) = (x_i)_{i \in \mathsf{N}} = (x_1, x_2, \dots).$$

We say that a sequence $\bar{x}$ in X is a *Gromov sequence* if $(x_i|x_j) \to \infty$ as $i \to \infty$ and $j \to \infty$. This implies that $|x_i - p| = (x_i|x_i) \to \infty$. Since $|(x|y)_p - (x|y)_q| \le |p - q|$, this concept is independent of the choice of the base point. In the literature, the Gromov sequences are usually called sequences converging at infinity or tending to infinity.

Convention. In the rest of this section we assume that (X, p) is a pointed δ-hyperbolic space, but now X need not be intrinsic.

We say that two Gromov sequences $\bar{x}$ and $\bar{y}$ in X are *equivalent* and write $\bar{x} \sim \bar{y}$ if $(x_i|y_i) \to \infty$ as $i \to \infty$. Since $(x_i|z_i) \ge (x_i|y_i) \wedge (y_i|z_i) - \delta$, we see that this is indeed an equivalence relation. The following observations are sometimes useful:

5.3. Lemma. (1) *A Gromov sequence is equivalent to each of its subsequences.*

(2) *If $\bar{x} \sim \bar{y}$, then $(x_i|y_j) \to \infty$ as $i, j \to \infty$.*

(3) *If $\bar{x} \sim \bar{y}$, then the sequence $(x_1, y_1, x_2, y_2, \dots)$ is a Gromov sequence equivalent to $\bar{x}$ and $\bar{y}$.*

(4) *If $\bar{u}$ and $\bar{v}$ are nonequivalent Gromov sequences, then the set of all numbers $(u_i|v_j)$ is bounded.*

(5) *If $\bar{x}$ is a Gromov sequence and if $\bar{y}$ is a sequence such that $(x_i|y_i) \to \infty$, then $\bar{y}$ is a Gromov sequence equivalent to $\bar{x}$.*

(6) *If $\bar{x}$ is a Gromov sequence and if $\bar{y}$ is a sequence such that*

$$\limsup_{i \to \infty} \frac{|x_i - y_i|}{|x_i - p|} < 1,$$

then $\bar{y}$ is a Gromov sequence equivalent to $\bar{x}$.

Proof. (1) is obvious, (2) follows from the inequality $(x_i|y_j) \ge (x_i|y_i) \wedge (y_i|y_j) - \delta$, and (3) follows from (1) and (2). If (4) is false, there are subsequences $\bar{u}'$ of $\bar{u}$ and $\bar{v}'$ of $\bar{v}$ such that $(u_i'|v_i') \to \infty$, and then $\bar{u}' \sim \bar{v}'$, which implies that $\bar{u} \sim \bar{v}$.

Part (5) follows from the inequality $(y_i|y_j) \geq (y_i|x_i) \wedge (x_i|x_j) \wedge (x_j|y_j) - 2\delta$, and (6) follows from (5) and from the estimate

$$\begin{aligned} 2(x_i|y_i) &= |x_i - p| + |y_i - p| - |x_i - y_i| \\ &\geq |x_i - p|(1 - |x_i - y_i|/|x_i - p|) \to \infty. \ \square \end{aligned}$$

5.4. *More definitions.* We let $\hat{x}$ denote the equivalence class containing the Gromov sequence $\bar{x}$. The set of all equivalence classes

$$\partial X = \{\hat{x} \colon \bar{x} \text{ is a Gromov sequence in } X\}$$

is the *Gromov boundary* of X, and the set

$$X^* = X \cup \partial X$$

is the *Gromov closure* of X. We may use the notation $\partial^* X$ for the Gromov boundary if there is a danger of misunderstanding.

5.5. *Remark on rays.* A *geodesic ray* in a space X is an isometric image of the half line $[0, \infty)$. In the classical case (X geodesic and proper) one can alternatively define a boundary point as an equivalence class of geodesic rays [GdH, p.119], and the geodesic rays are widely used as a tool. In the general case, joining a point in $x \in X$ to a point $a \in \partial X$ is somewhat problematic, because (1) geodesics do not exist and (2) the Ascoli theorem is not available. We shall return to this problem in 6.2.

We want to define the Gromov product $(a|b)$ for all $a, b \in X^*$. Suppose that $a, b \in \partial X$ and choose Gromov sequences $\bar{x} \in a,\ \bar{y} \in b$. The numbers $(x_i|y_j)$ need not converge to a limit but they converge to a rough limit in the following sense:

5.6. Lemma. *Let $a, b \in \partial X,\ a \neq b$, and let $\bar{x}, \bar{x}' \in a,\ \bar{y}, \bar{y}' \in b,\ z \in X$. Then*

$$\begin{aligned} \limsup_{i,j\to\infty} (x'_i|y'_j) &\leq \liminf_{i,j\to\infty} (x_i|y_j) + 2\delta < \infty, \\ \limsup_{i\to\infty} (x'_i|z) &\leq \liminf_{i\to\infty} (x_i|z) + \delta < \infty. \end{aligned}$$

Proof. We prove the first part of the lemma; the proof of the second part is similar but simpler. Set $s = \liminf_{i,j\to\infty} (x_i|y_j)$. Then $s < \infty$, because otherwise $\bar{x} \sim \bar{y}$ and thus $a = b$. Since $(x_i|x'_k) \to \infty$ and $(y_j|y'_l) \to \infty$ as $i, j, k, l \to \infty$, there is $m \in \mathsf{N}$ such that $(x_i|x'_k) \geq s + 3\delta$ and $(y_j|y'_l) \geq s + 3\delta$ for all $i, j, k, l \geq m$. For these indices we have

$$(x_i|y_j) + 2\delta \geq (x_i|x'_k) \wedge (x'_k|y'_l) \wedge (y'_l|y_j) \geq (s + 3\delta) \wedge (x'_k|y'_l).$$

As $i, j \to \infty$, this implies that $s + 2\delta \geq (s + 3\delta) \wedge (x'_k|y'_l)$, whence $(x'_k|y'_l) \leq s + 2\delta$ for $k, l \geq m$. The lemma follows. $\square$

5.7. *Definitions.* Given $a, b \in \partial X$, we could try four possible definitions for $(a|b)$. Choose Gromov sequences $\bar{x} \in a,\ \bar{y} \in b$, form the liminf and the limsup of $(x_i|y_j)$, and then the supremum and the infimum over all members of a and b. By 5.6, these four numbers lie in an interval of length 2δ. We choose the smallest of these numbers and define

(5.8) $$(a|b) = \inf \{\liminf_{i,j\to\infty}(x_i|y_j)\colon \bar{x} \in a, \bar{y} \in b\}.$$

The same definition is used in [CDP] and [Sh], but [GdH] and [BH] have sup instead of inf. However, with sup I cannot extend the basic inequality $(a|c) \geq (a|b) \wedge (b|c) - \delta$ to points $a, b, c \in \partial X$ unless δ is replaced by 2δ.

Observe that for $a \in \partial X$ we have $(a|a) = \infty$ and that $(a|b) < \infty$ for $a \neq b$.

For $a \in \partial X$ and $y \in X$ we set

$$(a|y) = (y|a) = \inf \{\liminf_{i\to\infty}(x_i|y)\colon \bar{x} \in a\}.$$

Then $(a|y) \leq |y - p| < \infty$ by 2.8(2).

5.9. *Notation.* For sequences $\bar{x}$ and $\bar{y}$ in X and for $z \in X$ we set

$$\mathrm{li}\,(\bar{x}|\bar{y}) = \liminf_{i\to\infty}(x_i|y_i), \quad \mathrm{ls}\,(\bar{x}|\bar{y}) = \limsup_{i\to\infty}(x_i|y_i),$$
$$\mathrm{li}\,(\bar{x}|z) = \liminf_{i\to\infty}(x_i|z), \quad \mathrm{ls}\,(\bar{x}|z) = \limsup_{i\to\infty}(\bar{x}|z).$$

5.10. Lemma. *If* $a, b \in \partial X$, *then* $(a|b) = \inf \{\mathrm{li}\,(\bar{x}|\bar{y})\colon \bar{x} \in a,\ \bar{y} \in b\}$.

Proof. Let s denote the right-hand side. Trivially $(a|b) \leq s$. We may assume that $a \neq b$ and thus $(a|b) < \infty$. Choose $t > (a|b)$. There are sequences $\bar{x} \in a$ and $\bar{y} \in b$ with $\liminf_{i,j\to\infty}(x_i|y_j) < t$. Hence there are increasing sequences of integers (i_k) and (j_k) such that the sequence $k \mapsto (x_{i_k}|y_{j_k})$ tends to a limit $t' \leq t$. Now the subsequence (x_{i_k}) is in a and similarly $(y_{j_k}) \in b$ by 5.3(1), whence $s \leq t' \leq t$, and the lemma follows. □.

5.11. Lemma. *Suppose that* $\bar{x} \in a \in \partial X$, $\bar{y} \in b \in \partial X,\ z \in X$. *Then*

$$(a|b) \leq \mathrm{li}\,(\bar{x}|\bar{y}) \leq \mathrm{ls}\,(\bar{x}|\bar{y}) \leq (a|b) + 2\delta,$$
$$(a|z) \leq \mathrm{li}\,(\bar{x}|z) \leq \mathrm{ls}\,(\bar{x}|z) \leq (a|z) + \delta,$$

Proof. The case $a = b$ is clear, and we may assume that $a \neq b$. The first inequality in both cases follows from the definition of $(a|b)$ and $(a|z)$. The last inequalities follow from 5.6. □

5.12. Proposition. *If* $a, b, c \in X^*$, *then* $(a|c) \geq (a|b) \wedge (b|c) - \delta$.

Proof. We prove the case $a, b, c \in \partial X$. Let $\bar{x} \in a,\ \bar{y} \in b,\ \bar{z} \in c$. Then $(x_i|z_i) \geq (x_i|y_i) \wedge (y_i|z_i) - \delta$ for each $i \in \mathsf{N}$, and hence

$$\mathrm{li}\,(\bar{x}|\bar{z}) \geq \liminf_{i\to\infty}((x_i|y_i) \wedge (y_i|z_i)) - \delta = \mathrm{li}\,(\bar{x}|\bar{y}) \wedge \mathrm{li}\,(\bar{y}|\bar{z}) - \delta \geq (a|b) \wedge (b|c) - \delta.$$

By 5.10 this implies the lemma. □

5.13. *The functions ϱ_ε and d_ε.* Let $0 < \varepsilon \le 1$. For $a, b \in X^*$ we write

$$\varrho_\varepsilon(a,b) = \varrho_{p,\varepsilon}(a,b) = e^{-\varepsilon(a|b)}$$

with the agreement $e^{-\infty} = 0$. Then $\varrho_\varepsilon(a,b) = \varrho_\varepsilon(b,a)$, and $\varrho_\varepsilon(a,b) = 0$ if and only if $a = b \in \partial X$. Furthermore, for $a, b, c \in X^*$ we have

$$-(\log \varrho_\varepsilon(a,c))/\varepsilon = (a|c) \ge (a|b) \wedge (b|c) - \delta,$$

and hence

$$e^{-\delta\varepsilon}\varrho_\varepsilon(a,c) \le \varrho_\varepsilon(a,b) \vee \varrho_\varepsilon(b,c). \tag{5.14}$$

We set

$$d_\varepsilon(a,b) = d_{p,\varepsilon}(a,b) = \inf \sum_{j=1}^{n} \varrho_\varepsilon(a_{j-1}, a_j) \tag{5.15}$$

over all finite sequences $a = a_0, \dots, a_n = b$ in X^*.

5.16. Proposition. *Suppose that $\varepsilon\delta \le 1/5$. Then the function d_ε is a metametric in X^*, and the corresponding metric space* $\operatorname{met} X$ *is ∂X. Moreover,*

$$\varrho_\varepsilon(a,b)/2 \le d_\varepsilon(a,b) \le \varrho_\varepsilon(a,b) \tag{5.17}$$

for all $a, b \in X^*$.

Proof. Clearly d_ε satisfies the conditions (1) and (2) of a metametric in 4.2 and the second inequality of (5.17). Since $\varrho_\varepsilon(a,b) = 0$ iff $a = b \in \partial X$, it suffices to prove the first inequality of (5.17). I follow [GdH, 7.10].

Since $e^{\varepsilon\delta} \le e^{1/5} < 5/4$, (5.14) gives

$$\tfrac{4}{5}\varrho_\varepsilon(a,c) \le \varrho_\varepsilon(a,b) \vee \varrho_\varepsilon(b,c) \tag{5.18}$$

for all $a, b, c \in X^*$. Let $a = a_0, \dots, a_n = b \in X^*$ and set $R = \sum_{j=1}^{n} \varrho_\varepsilon(a_{j-1}, a_j)$. It suffices to show that $\varrho_\varepsilon(a,b)/2 \le R$. This is trivially true if $n = 1$, and we proceed by induction on n. Let k be the largest integer with $\sum_{j=1}^{k} \varrho_\varepsilon(a_{j-1}, a_j) \le R/2$. Then $0 \le k \le n-1$ and $\sum_{j=k+2}^{n} \varrho_\varepsilon(a_{j-1}, a_j) \le R/2$.

Case 1. $1 \le k \le n-2$. By the induction hypothesis we have

$$\varrho_\varepsilon(a, a_k) \le R, \quad \varrho_\varepsilon(a_{k+1}, b) \le R.$$

Moreover, $\varrho_\varepsilon(a_k, a_{k+1}) \le R$. Applying twice the estimate (5.18) we obtain

$$\varrho_\varepsilon(a,b)/2 \le \tfrac{16}{25}\varrho_\varepsilon(a,b) \le \varrho_\varepsilon(a,a_k) \vee \varrho_\varepsilon(a_k, a_{k+1}) \vee \varrho_\varepsilon(a_{k+1}, b) \le R.$$

Case 2. $k = 0$. Arguing as in Case 1 we get

$$\tfrac{4}{5}\varrho_\varepsilon(a,b) \le \varrho_\varepsilon(a,a_1) \vee \varrho_\varepsilon(a_1, b) \le R.$$

Case 3. $k = n-1$ This case is similar to Case 2. □

5.19. *Convention.* From now on I always assume that $\varepsilon \le 1 \wedge (1/5\delta)$. Then $\varrho_\varepsilon/2 \le d_\varepsilon \le \varrho_\varepsilon$.

5.20. *Remarks.* 1. In the literature, the distance $d_\varepsilon(a,b)$ is usually only considered for points $a, b \in \partial X$. It is defined by the formula (5.15), where all points a_j lie in ∂X. This gives a number $d'_\varepsilon(a,b)$, and we have $d_\varepsilon \le d'_\varepsilon \le \varrho_\varepsilon \le 2d_\varepsilon$ by 5.16.

2. The condition $\varepsilon \le 1$ is mainly for convenience; most considerations are valid whenever $\varepsilon\delta \le 1/5$.

3. If $a \in X$, then $(a|x) \le |a-p| = (a|a)$ for all $x \in X^*$. Hence $\varrho_\varepsilon(a,x) \ge \varrho_\varepsilon(a,a)$, which implies that $d_\varepsilon(a,a) = \varrho_\varepsilon(a,a) = e^{-\varepsilon|a-p|}$.

4. The points of X are large in the metametric d_ε and the points of ∂X are small.

5. We have $d_\varepsilon(p,a) = \varrho_\varepsilon(p,a) = 1$ for all $a \in X^*$, whence X^* is *bounded* with diameter $1 \le d(X^*) \le 2$.

6. A sequence in X is Gromov iff it is Cauchy in the metametric d_ε. Hence we may consider X^* as the completion of X. A proof for the completeness is given in 5.31.

5.21. Lemma. *Let $\bar{x}$ be a sequence in a hyperbolic space X and let $a \in \partial X$. Then the following conditions are equivalent:*

(1) $(x_i|a) \to \infty$,

(2) $d_\varepsilon(x_i, a) \to 0$,

(3) $\bar{x}$ *is a Gromov sequence and* $\bar{x} \in a$.

Proof. Clearly (1) is equivalent to the condition $\varrho_\varepsilon(x_i, a) \to 0$, and the equivalence (1) ⇔ (2) follows from (5.18).

(3) ⇒ (1): Let $M > 0$. Since $\bar{x}$ is a Gromov sequence, there is an integer m such that $(x_i|x_j) \ge M$ for $i, j \ge m$. For $i \ge m$ we thus have $\liminf_{j\to\infty}(x_i|x_j) \ge M$. By 5.11 this yields $M \le (x_i|a) + \delta$, whence $(x_i|a) \to \infty$.

(1) ⇒ (3): Since $(x_i|x_j) \ge (x_i|a) \wedge (x_j|a) - \delta \to \infty$ by 5.12, $\bar{x}$ is a Gromov sequence. Setting $b = \hat{x}$ we have $(x_i|b) \to \infty$ by the part (3) ⇒ (1) of the lemma. It follows that $(a|b) \ge (a|x_i) \wedge (b|x_i) - \delta \to \infty$, whence $(a|b) = \infty$ and thus $a = b$. □

5.22. Corollary. *If $\bar{x} \in a \in \partial X$ and $\bar{y} \in b \in \partial X$, then $d_\varepsilon(x_i, y_i) \to d_\varepsilon(a,b)$. If $z \in X$, then $d_\varepsilon(x_i, z) \to d_\varepsilon(a, z)$.* □

5.23. *Remark.* Considering X^* as the completion of X in the metametric d_ε, we could extend d_ε from X to X^* without defining the Gromov product for boundary points.

We next give an improvement of Lemma 5.10. The result is not needed later in this article.

5.24. Lemma. *If $a, b \in \partial X$, then there are sequences $\bar{x} \in a$ and $\bar{y} \in b$ such that $(x_i|y_i) \to (a|b)$. If $z \in X$, there is $\bar{x} \in a$ such that $(x_i|z) \to (a|z)$.*

Proof. We only prove the first part of the lemma. We may assume that $(a|b) = t < \infty$. From 5.10 it follows that for each $n \in \mathsf{N}$ there are sequences $\bar{x}^n \in a$ and $\bar{y}^n \in b$ such that $\mathrm{li}\,(\bar{x}^n|\bar{y}^n) < t + 1/n$. Passing to subsequences and using 5.3(1) we may assume that $(x_i^n|y_i^n) < t + 1/n$ for all i and n. By 5.21 we have $(x_i^n|a) \to \infty$ and $(y_i^n|b) \to \infty$ as $i \to \infty$. For each n we can therefore choose an index $i(n)$ with $(x_{i(n)}^n|a) > n$ and $(y_{i(n)}^n|b) > n$. Define sequences $\bar{x}'$ and $\bar{y}'$ by $x'_n = x_{i(n)}^n$, $y'_n = y_{i(n)}^n$. By 5.21 we have $\bar{x}' \in a$, $\bar{y}' \in b$. Moreover, $(x'_n|y'_n) < t + 1/n$ for all n, whence $\mathrm{ls}\,(\bar{x}'|\bar{y}') \le t$. On the other hand, $\mathrm{li}\,(\bar{x}'|\bar{y}') \ge (a|b) = t$, and the lemma follows. □

5.25. *The role of p and ε.* We study how the metametric $d_{p,\varepsilon}$ of X^* depends on p and ε. Suppose that $0 < \varepsilon, \varepsilon' \leq 1 \wedge (1/5\delta)$ and set $\alpha = \varepsilon'/\varepsilon$. The definition of $\varrho_{p,\varepsilon}$ gives $\varrho_{p,\varepsilon'} = \varrho_{p,\varepsilon}^{\alpha}$. By (5.17) this yields

$$d_{p,\varepsilon}(x,y)^{\alpha}/2 \leq d_{p,\varepsilon'}(x,y) \leq 2^{\alpha} d_{p,\varepsilon}(x,y)^{\alpha} \tag{5.26}$$

for all $x, y \in X^*$. This means that the identity map $(X, d_{p,\varepsilon}) \to (X, d_{p,\varepsilon'})$ is a *snowflake map* in the sense of [BS, p. 281]. In particular, this map is η-quasisymmetric with $\eta(t) = 2^{\alpha+1}t^{\alpha}$.

Next let $p, q \in X$ and set $r = |p - q|,\ \lambda = e^r$. By 2.8(4) we have $|(x|y)_p - (x|y)_q| \leq r$. As $\varepsilon \leq 1$, we obtain $\varrho_{p,\varepsilon}/\lambda \leq \varrho_{q,\varepsilon} \leq \lambda\varrho_{p,\varepsilon}$. By (5.17) this yields

$$d_{p,\varepsilon}(x,y)/2\lambda \leq d_{q,\varepsilon}(x,y) \leq 2\lambda d_{p,\varepsilon}(x,y) \tag{5.27}$$

for all $x, y \in X^*$. In other words, the identity map $(X^*, d_{p,\varepsilon}) \to (X^*, d_{q,\varepsilon})$ is 2λ-bilipschitz.

It follows that this identity map is η-quasisymmetric with $\eta(t) = 4\lambda^2 t$. The dependence of η on $|p - q|$ cannot be avoided. Similar estimates show that the identity map $(X^*, d_{p,\varepsilon}) \to (X^*, d_{q,\varepsilon'})$ is η-quasisymmetric with $\eta(t) = 2^{\alpha+3}\lambda^2 t^{\alpha}$. The family of metrics $d_{p,\varepsilon}|\partial X$ has been called the canonical quasisymmetric gauge; see [BHK, p. 18].

The quasimöbius version is more quantitative:

5.28. Proposition. *Let X be a δ-hyperbolic space, let $0 < \varepsilon, \varepsilon' \leq 1 \wedge (1/5\delta)$ and let $p, q \in X$. Then the identity map $(X^*, d_{p,\varepsilon}) \to (X^*, d_{q,\varepsilon})$ is η-quasimöbius with $\eta(t) = 16t$, and the identity map $(X^*, d_{p,\varepsilon}) \to (X^*, d_{q,\varepsilon'})$ is θ-quasimöbius with $\theta(t) = 4^{\alpha+1}t^{\alpha}$ where $\alpha = \varepsilon'/\varepsilon$.*

Proof. Let $Q = (x, y, z, w)$ be a quadruple in X. From (4.7) it follows that the number

$$\frac{\varrho_{p,\varepsilon}(x,y)\varrho_{p,\varepsilon}(z,w)}{\varrho_{p,\varepsilon}(x,z)\varrho_{p,\varepsilon}(y,w)}$$

is independent of p. By (5.17) this implies that $\operatorname{cr}(Q, d_{q,\varepsilon}) \leq 16\operatorname{cr}(Q, d_{p,\varepsilon})$. By 5.22 this holds for all quadruples Q in X^*.

The second part follows similarly by the equality $\varrho_{q,\varepsilon'} = \varrho_{q,\varepsilon}^{\alpha}$. $\square$

5.29. *A topology of X^*.* Let $\mathcal{T}^*$ be the topology of X^* induced by the metametric d_ε; see 4.2. From (5.26) and (5.27) it follows that $\mathcal{T}^*$ is independent of p and ε. In this topology, every point of X is isolated, and ∂X is the topological boundary of X in X^*.

It is possible (and perhaps more useful) to define a topology $\mathcal{T}_1^*$ of X^* that induces the original topology of X and the same topology of ∂X as $\mathcal{T}^*$; see [BH, p. 429], [KB, p. 6]. This topology consists of all $U \in \mathcal{T}^*$ such that $U \cap X$ is open in the original topology of X.

5.30. *Completeness.* A Cauchy sequence in a metametric space (M, d) is defined as usual, but a convergent sequence need not be Cauchy. Indeed, if $d(a, a) > 0$, then the constant sequence $(a, a, \dots)$ converges to a but it is not Cauchy.

A metametric space is said to be *complete* if every Cauchy sequence is convergent. A closed subset of a complete metametric space is complete.

5.31. Proposition. *If X is hyperbolic, then (X^*, d_ε) is a complete metametric space and $(\partial X, d_\varepsilon)$ is a complete metric space.*

Proof. Since ∂X is closed in X^*, it suffices to show that X^* is complete. Assume that $\bar{a} = (a_i)$ is a Cauchy sequence in X^*. We first consider the special case where the points a_i lie in X. Since

$$-\varepsilon(a_i|a_j) = \log \varrho_\varepsilon(a_i, a_j) \leq \log 2d_\varepsilon(a_i, a_j) \to -\infty$$

as $i, j \to \infty$, the sequence $\bar{a}$ is Gromov. By 5.21 it converges to $\hat{a} \in \partial X$.

Next let $\bar{a}$ be arbitrary. For each i we can find a point $x_i \in X$ with $d_\varepsilon(x_i, a_i) < 1/i$ by 5.21. For $i < j$ we have $d_\varepsilon(x_i, x_j) \leq d_\varepsilon(a_i, a_j) + 2/i$. Hence $\bar{x}$ is a Gromov sequence. Setting $b = \hat{x}$ we have $d_\varepsilon(a_i, b) \leq 1/i + d_\varepsilon(x_i, b) \to 0$ by 5.21, whence $\bar{a}$ converges to b. □

5.32. *Boundary extension of quasi-isometries.* In the previous results of this section, the spaces are not assumed to be intrinsic. From now on, intrinsicness is required, because we want to make use of Theorem 3.19 on the change of the Gromov product in a quasi-isometry.

Suppose that X and Y are intrinsic δ-hyperbolic spaces and that $f\colon X \to Y$ is a (λ, μ)-quasi-isometry. We first choose the base points $p \in X$ and $q \in Y$ so that $q = fp$. Let $0 < \varepsilon \leq 1 \wedge (1/5\delta)$. We consider X^* and Y^* with the metametrics $d_\varepsilon = d_{p,\varepsilon}$ and $d'_\varepsilon = d_{q,\varepsilon}$; see 5.13.

We want to extend f to a map $f^*\colon X^* \to Y^*$ between the Gromov closures. The following considerations are essentially from [BS, Section 6]. Let $\bar{x}$ be a Gromov sequence in X. Then 3.19(1) yields $(fx_i|fx_j) \geq \lambda^{-1}(x_i|x_j) - C \to \infty$ as $i, j \to \infty$, whence $f\bar{x} = (fx_i)$ is a Gromov sequence. Furthermore, if $\bar{x} \sim \bar{y}$, then $f\bar{x} \sim f\bar{y}$ by 3.19(1). Consequently, f has an extension to a map $f^*\colon X^* \to Y^*$, defined by $f^*\hat{x} = \hat{z}$ where $\bar{z} = f\bar{x}$, and f^* is continuous in the topologies defined by the metametrics d_ε and d'_ε. (But f need not be continuous in the original topologies of X and Y.) Moreover, f^* defines a continuous map $\partial f\colon \partial X \to \partial Y$ between metric spaces.

The assignment $f \mapsto f^*$ has clearly the functorial properties $\mathrm{id}^* = \mathrm{id}$ and $(f \circ g)^* = f^* \circ g^*$.

We show that ∂f is injective. Suppose that $a, b \in \partial X$ with $f^*a = f^*b$. Choosing Gromov sequences $\bar{x} \in a$ and $\bar{y} \in b$ we have $(fx_i|fy_i) \to \infty$. By 3.19(1), this implies that $(x_i|y_i) \to \infty$, whence $a = b$.

It follows that f^* is a positive map in the metametrics d_ε and d'_ε.

We prove that f^* is quasisymmetric in the metametrics d_ε and d'_ε. Let (x, y, z) be a positive triple in X^* and set $t = d_\varepsilon(x, y)/d_\varepsilon(x, z)$. Writing $x' = fx$ etc. we must find an estimate

$$d'_\varepsilon(x', y') \leq \eta(t) d'_\varepsilon(x', z') \tag{5.33}$$

where $\eta(t) \to 0$ as $t \to 0$. We may assume that $x, y, z \in X$, because (5.33) can be extended to X^* by continuity.

We have $t\varrho_\varepsilon(x, z)/2 \leq \varrho_\varepsilon(x, y) \leq 2t\varrho_\varepsilon(x, z)$, whence

$$-\log 2t \leq \varepsilon(x|y) - \varepsilon(x|z) \leq -\log \tfrac{t}{2}. \tag{5.34}$$

Case 1. $t \leq 1/2$. Now $(x|y) - (x|z) \geq 0$, and 3.19(3) gives

$$\varepsilon(x'|y') - \varepsilon(x'|z') \geq -\lambda^{-1} \log 2t - \varepsilon C$$

with $C = C(\delta, \lambda, \mu)$. Since $\varepsilon \leq 1$, this and (5.17) imply (5.33) with $\eta(t) = 4e^C t^{1/\lambda}$.

Case 2. $t > 1/2$. Now (5.34) gives

$$\varepsilon|(x|y) - (x|z)| \leq (\log 2t) \vee |\log \tfrac{t}{2}|,$$

and by 3.19(2) we get

$$\varepsilon(x'|z') - \varepsilon(x'|y') \le \lambda\big((\log 2t) \vee |\log \tfrac{t}{2}|\big) + \varepsilon C.$$

Consequently, (5.33) holds with

$$\eta(t) = \begin{cases} 4e^C t^{1/\lambda} & \text{for } 0 < t \le 1/2, \\ 2^{2\lambda+1}e^C & \text{for } 1/2 < t < 1, \\ 2^{\lambda+1}e^C t^\lambda & \text{for } t \ge 1. \end{cases}$$

We see that f^* is in fact power quasisymmetric ($\eta(t)$ of the form $c(t^\alpha \vee t^{1/\alpha})$). The function η depends on (δ, λ, μ) but not on ε.

The map $\partial f \colon \partial X \to \partial Y$ need not be surjective; see 5.37. It is known to be surjective if f is roughly surjective [BS, 6.3(4)]. We prove that ∂f is surjective if f is *weakly surjective,* by which we mean that

$$\limsup_{|y-q|\to\infty} \frac{d(y, fX)}{|y-q|} < 1.$$

The definition is independent of the choice of the base point q of Y. A roughly surjective map is trivially weakly surjective.

Let $\bar{y} \in b \in \partial Y$. Writing $r_i = d(y_i, fX)/|y_i - q|$ we have $\limsup_{i\to\infty} r_i = r < 1$. Choose a number s with $r < s < 1$. Replacing $\bar{y}$ by a subsequence we may assume that $r_i < s$ for all i. We can find points $x_i \in X$ with $|y_i - fx_i| < s|y_i - q|$ for all i. Then $f\bar{x} = (fx_i)$ is a Gromov sequence equivalent to $\bar{y}$ by 5.3(6). Since

$$(fx_i|fx_j) \le \lambda(x_i|x_j) + C$$

by 3.19, also $\bar{x}$ is a Gromov sequence. Moreover, $\partial f\hat{x} = b$, whence ∂f is surjective. In fact, ∂f is a homeomorphism, which follows from quasisymmetry and can also be easily proved directly.

We summarize these results in the following theorem. A more quantitative result is given in 5.38 in terms of quasimöbius maps.

5.35. Theorem. *Suppose that X and Y are pointed intrinsic δ-hyperbolic spaces and that $f \colon X \to Y$ is a base point preserving (λ, μ)-quasi-isometry. Then f has an extension $f^* \colon X^* \to Y^*$, which is continuous in the metametrics d_ε and d'_ε, where $0 < \varepsilon \le 1 \wedge (1/5\delta)$. Moreover, f^* defines an injective map $\partial f \colon \partial X \to \partial Y$.*

The map f^* is η-quasisymmetric in d_ε and d'_ε and hence η-quasisymmetric rel ∂X with η depending only on δ, λ, μ. If f is weakly surjective, then ∂f is a homeomorphism onto ∂Y.

5.36. *Remark.* A quasi-isometry $f \colon X \to Y$ need not be injective. In the metametrics d_ε and d'_ε it is nevertheless quasisymmetric. This phenomenon cannot occur in metric spaces, where a quasisymmetric map is injective by definition.

5.37. *Examples.* 1. Let X be the Poincaré 2-disk with its hyperbolic metric and let Y be the 3-disk. The natural embedding $f \colon X \to Y$ is an isometry. The induced boundary map ∂f is the inclusion of the circle ∂X into the sphere ∂Y. It is not surjective.

2. Let Y be the Poincaré half plane and let $X \subset Y$ be the half disk $\{x \in Y : |x| < 1\}$ where $|x|$ is the euclidean norm, equipped with the metric inherited from Y. Then $\partial_G Y$ is the extended real line, $\partial_G X = [-1, 1]$, and the inclusion $f : X \to Y$ induces the inclusion $\partial f : \partial_G X \to \partial_G Y$, which is not surjective.

In these examples, X and Y are geodesic locally compact hyperbolic spaces. In the first example f is not open, and in the second example X is not proper. This is natural in view of the following result, which can be proved by standard path lifting arguments.

Suppose that $f : X \to Y$ is a locally λ-bilipschitz open map between metric spaces, where X is proper and Y rectifiably connected. Then f is surjective.

We next give a base point invariant quasimöbius version of Theorem 5.35.

5.38. Theorem. *Suppose that X and Y are intrinsic δ-hyperbolic spaces and that $f : X \to Y$ is a (λ, μ)-quasi-isometry. Let $p \in X$ and $q \in Y$. Then f has an extension $f^* : X^* \to Y^*$, which is continuous in the metametrics $d_{p,\varepsilon}$ and $d_{q,\varepsilon}$, where $0 < \varepsilon \le 1 \wedge (1/5\delta)$. Moreover, f^* defines an injective map $\partial f : \partial X \to \partial Y$.*

The map f^* is η-quasimöbius in $d_{p,\varepsilon}$ and $d_{q,\varepsilon}$ and hence η-quasimöbius rel ∂X with η depending on δ, λ, μ but not on the base points p and q and not on ε. If f is weakly surjective, then ∂f is a homeomorphism onto ∂Y.

Proof. Set $q' = fp$. The map f defines a base point preserving map $f_1 : (X, p) \to (Y, q')$, and the extension f_1^* is η-quasisymmetric with $\eta = \eta_{\delta,\lambda,\mu}$ by 5.35. By 4.9.2, the map f_1^* is θ-quasimöbius with $\theta = \theta_\eta$ in $d_{p,\varepsilon}$ and $d_{q',\varepsilon}$. From 5.28 it follows that f^* is 16θ-quasimöbius. □

6 Roads and biroads

6.1. *Summary.* In a proper geodesic hyperbolic space, one can join a point of the space to a boundary point by a geodesic ray, and two boundary points by a geodesic line. In an arbitrary intrinsic hyperbolic space, geodesic rays will be replaced by certain sequences of arcs, called *roads*, and geodesic lines by another kind of sequences of arcs, called *biroads*. Unfortunately, this makes the theory more complicated than in the classical case.

6.2. *Roads.* In the theory of proper geodesic hyperbolic spaces, *geodesic rays* have turned out to be useful. For example, one can define a boundary point as an equivalence class of geodesic rays. In a general intrinsic space they are no longer available. One can join points of X to points of ∂X by quasi-isometric rays; see [BS, 5.2], [KB, 2.16] and Remark 6.4 below. However, I prefer to work with certain sequences of h-short arcs, called *roads*.

Similarly, *geodesic lines* will be replaced by another kind of arc sequences, called *biroads* and considered in 6.10.

Let X be a metric space and let $\mu \ge 0,\ h \ge 0$. A (μ, h)-*road* in X is a sequence $\bar{\alpha}$ of arcs $\alpha_i : y_i \curvearrowright u_i$ with the following properties:

(1) Each α_i is h-short.

(2) The sequence of lengths $l(\alpha_i)$ is increasing and tends to ∞.

(3) For $i \le j$, the length map $g_{ij} : \alpha_i \to \alpha_j$ with $g_{ij} y_i = y_j$ satisfies $|g_{ij}x - x| \le \mu$ for all $x \in \alpha_i$.

Observe that (3) implies that $|y_i - y_j| \leq \mu$ for all i and j. If $y_i = y_j = y$ for all i and j, we say that $\bar{\alpha}$ is a *road from* y. The *locus* $|\bar{\alpha}|$ of a road $\bar{\alpha}$ is the union of all arcs α_i.

In the case $\mu = 0,\ h = 0$ we have a geodesic ray. More precisely, the locus $|\bar{\alpha}|$ is a geodesic ray, and each α_i is an initial subarc.

The indexing set for a road is usually N, but occasionally it is convenient to use a subset $\{i \in \mathsf{N} \colon i \geq k\}$ for some $k \geq 1$. For example, if $\bar{\alpha}$ is a road indexed by N and if $z \in \alpha_k$, we can define a *subroad* $\bar{\beta}$ of $\bar{\alpha},\ \beta_i \colon z_i \curvearrowright u_i,\ i \geq k$, by

$$z_i = g_{ki} z, \quad \beta_i = \alpha_i[z_i, u_i].$$

6.3. Lemma. *Suppose that $\bar{\alpha}$ is a (μ, h)-road, $\alpha_i \colon y_i \curvearrowright u_i$. Then (u_i) is a Gromov sequence.*

Proof. Let $i \leq j$ and let $g_{ij} \colon \alpha_i \to \alpha_j$ be the length map as above. By 2.8(6) we obtain

$$(g_{ij} u_i | u_j)_{y_j} \geq |g_{ij} u_i - y_j| - h/2 \geq l(\alpha_i) - 3h/2.$$

Since $|y_j - y_1| \leq \mu$ and $|g_{ij} u_i - u_i| \leq \mu$, this and 2.8 imply that $(u_i | u_j)_{y_1} \geq l(\alpha_i) - 3h/2 - 2\mu \to \infty$ as $i \to \infty$. $\square$

6.4. *Remark.* Let $\bar{\alpha}$ be a (μ, h)-road, $\alpha_i \colon y_i \curvearrowright u_i$, and set $L_i = l(\alpha_i),\ L_0 = 0$. Let $\varphi_i \colon [0, L_i] \to \alpha_i$ be the arclength parametrization of α_i with $\varphi_i(0) = y_i$. Define a map $\varphi \colon [0, \infty) \to X$ by $\varphi(t) = \varphi_i(t)$ for $L_{i-1} \leq t < L_i$.

Let $0 \leq s \leq t$ and choose indices $i \leq j$ with $L_{i-1} \leq s < L_i,\ L_{j-1} \leq j < L_j$. Then

$$|s - t| - h \leq |\varphi_j(s) - \varphi_j(t)| \leq |s - t|.$$

Since $|\varphi_i(s) - \varphi_j(s)| \leq \mu$, we see that φ satisfies the rough isometry condition

$$|s - t| - \mu - h \leq |\varphi(s) - \varphi(t)| \leq |s - t| + \mu$$

for all $s, t \geq 0$.

However, it is usually easier to work with the sequence $\bar{\alpha}$ than with the function φ.

6.5. *Roads in a hyperbolic space.* Suppose that X is a δ-hyperbolic space and that $\bar{\alpha}$ is a (μ, h)-road in $X,\ \alpha_i \colon y_i \curvearrowright u_i$. By 6.3 the sequence (u_i) is Gromov and thus defines an element $b = \hat{u}$ of the Gromov boundary ∂X. We write $\bar{\alpha} \colon \bar{y} \curvearrowright b$ and say that $\bar{\alpha}$ *joins* the sequence $\bar{y} = (y_i)$ to b. If $y_i = y$ for all i, we write $\bar{\alpha} \colon y \curvearrowright b$.

We show that if X is intrinsic, then each pair $y \in X,\ b \in \partial X$ can be joined by a road. This result will be given in 6.7, and it follows immediately from the following more precise result.

6.6. Lemma. *Let X be a δ-hyperbolic space, let $y \in X$, let $\bar{u}$ be a Gromov sequence in X and let $\alpha_m \colon y \curvearrowright u_m$ be a sequence of h-short arcs. Then there is a $(4\delta + 2h, h)$-road $\bar{\beta} \colon y \curvearrowright \hat{u}$ such that each β_i is a subarc of some $\alpha_{m(i)}$ with $l(\beta_i) = i$.*

Proof. Since $(u_j | u_k)_y \to \infty$, we can choose for each $i \in \mathsf{N}$ an integer $m(i)$ such that $(u_j | u_k)_y \geq i$ for $j, k \geq m(i)$ and such that $m(1) < m(2) < \ldots$ We have $|y - u_{m(i)}| = (u_{m(i)} | u_{m(i)})_y \geq i$, which implies that there is a subarc $\beta_i = \alpha_{m(i)}[y, v_i]$ of length i. We show that $\bar{\beta}$ is the desired road.

By 2.8(6) we have

$$(v_i|u_{m(i)})_y \geq |y - v_i| - h/2 \geq i - 3h/2.$$

This implies that $\bar{v}$ is a Gromov sequence equivalent to $\bar{u}$; see 5.3(5). Let $i \leq j$ and let $g\colon \beta_i \to \beta_j$ be the length map fixing y. For $x \in \beta_i$ we have $|x - y| \leq i \leq (u_{m(i)}|u_{m(j)})_y$. By 2.15 this yields $|gx - x| \leq 4\delta + 2h$, whence $\bar{\beta}$ is a $(4\delta + 2h, h)$-road. □

6.7. Theorem. *Let X be an intrinsic δ-hyperbolic space and let $y \in X,\ a \in \partial X,\ h > 0$. Then there is a $(4\delta + 2h, h)$-road $\bar{\beta}\colon y \curvearrowright a$.* □

If $\alpha\colon y \curvearrowright b \in \partial X$ is a geodesic ray in a hyperbolic space X, then $(x|b) \to \infty$ as $x \in \alpha$ tends to the end b. We next give a version of this result for roads.

6.8. Lemma. *Let $\bar{\alpha}\colon \bar{y} \curvearrowright b \in \partial X$ be a (μ, h)-road in a hyperbolic pointed space (X, p) and let $M > 0$. Then there is a subroad (see 6.2) $\bar{\beta}$ of $\bar{\alpha}$ such that $(x|b) \geq M$ for all $x \in |\bar{\beta}|$.*

Proof. The result is clearly independent of p, and we may assume that $p = y_1$. Write $\alpha_i\colon y_i \curvearrowright u_i$. As $\bar{u}$ is a Gromov sequence by 6.3, there is k such that $(u_k|u_j) \geq M+\delta+4\mu+3h/2$ for all $j \geq k$. Let $\bar{\beta}\colon \bar{z} \curvearrowright b$ be the subroad of $\bar{\alpha}$ defined by $\beta_i = \alpha_i[g_{ki}u_k, u_i],\ i > k$. Suppose that $x \in \beta_i$ for some $i > k$. It suffices to show that $(x|b) \geq M$.

Let $j \geq i$. Since α_j is h-short, we have $|g_{ij}x - y_j| \geq |g_{kj}u_k - y_j| - h$. By 2.8(6) this yields $(g_{ij}x|u_j)_{y_j} \geq (g_{kj}u_k|u_j)_{y_j} - 3h/2$, which implies that

$$(x|u_j) \geq (u_k|u_j) - 4\mu - 3h/2 \geq M + \delta.$$

As $j \to \infty$, we obtain $(x|b) \geq M$ by 5.11. □

If α and β are geodesic rays in a geodesic δ-hyperbolic space X converging to the same point $b \in \partial X$, then α and β run eventually close to each other. More precisely, there are subrays $\alpha_1 \subset \alpha$ and $\beta_1 \subset \beta$ such that the bijective length map $f\colon \alpha_1 \to \beta_1$ satisfies $|fx - x| \leq 16\delta$ for all $x \in \alpha_1$; see [GdH, 7.2]. Similar results can be obtained for (μ, h)-roads in an intrinsic hyperbolic space. We prove the following result, which seems to be sufficient in several applications. See also 6.25.

6.9. Closeness lemma. *Let X be an intrinsic δ-hyperbolic space and let $\bar{\alpha}\colon \bar{y} \curvearrowright b$ and $\bar{\beta}\colon \bar{z} \curvearrowright b$ be (μ, h)-roads in X converging to the same point $b \in \partial X$. Then for each $x_0 \in X$ there is $R > 0$ such that $d(x, |\bar{\beta}|) \leq 7\delta + \mu + 3h$ for all $x \in |\bar{\alpha}| \setminus B(x_0, R)$.*

Proof. The result is clearly independent of x_0, and we choose $x_0 = y_1$. Write $\alpha_i\colon y_i \curvearrowright u_i$, $\beta_i\colon z_i \curvearrowright v_i$ and set $K = |y_1 - z_1|$. We show that the lemma holds with $R = K+1+7\delta+4\mu+4h$. Assume that $x \in |\bar{\alpha}| \setminus B(y_1, R)$ and choose i with $x \in \alpha_i$. Since $(u_j|v_j)_{y_1} \to \infty$, there is $m \geq i$ with $(u_m|v_m)_{y_1} \geq |x - y_1| + 3\mu$. Choose h-short arcs $\gamma\colon y_m \curvearrowright v_m$ and $\tau\colon y_m \curvearrowright z_m$.

There is a point $x_1 \in \alpha_m$ with $|x_1 - x| \leq \mu$. We have

$$|x_1 - y_m| \leq |x_1 - x| + |x - y_1| + |y_1 - y_m| \leq (u_m|v_m)_{y_1} - \mu \leq (u_m|v_m)_{y_m}.$$

By the tripod lemma 2.15 we can find a point $x_2 \in \gamma$ with $|x_2 - x_1| \leq 4\delta + h$. As X is $(3\delta, 3h/2)$-Rips by 2.35, there is a point $x_3 \in \beta_m \cup \tau$ with $|x_3 - x_2| \leq 3\delta + 3h/2 \leq 3\delta + 2h$.

We have $l(\tau) \le |y_m - z_m| + h \le K + 2\mu + h$, whence τ lies in the ball $\bar{B}(y_1, K + 3\mu + h)$. Since

$$|x_2 - y_1| \ge |x - y_1| - |x - x_1| - |x_1 - x_2| \ge R - 4\delta - \mu - h,$$

we obtain

$$d(x_2, \tau) \ge R - K - 4\delta - 4\mu - 2h = 1 + 3\delta + 2h.$$

Hence $x_3 \in \beta_m$. This implies the lemma, because $|x_3 - x| \le 7\delta + \mu + 3h$. □

6.10. *Biroads.* Let X be a metric space and let $\mu \ge 0,\ h \ge 0$. By a (μ, h)-*biroad* in X we mean a sequence $\bar{\gamma}$ of arcs $\gamma_i \colon u_i \curvearrowright v_i$ in X together with length maps $g_{ij} \colon \gamma_i \to \gamma_j$ for $i \le j$ with the following properties:

(1) Each γ_i is h-short.

(2) For some (and hence for all) $x_1 \in \gamma_1$ we have $|u_i - x_1| \to \infty,\ |v_i - x_1| \to \infty$.

(3) $g_{ii} = \mathrm{id},\ g_{ik} = g_{jk} \circ g_{ij}$ for $i \le j \le k$.

(4) $|g_{ij}x - x| \le \mu$ for all $i \le j$ and $x \in \gamma_i$.

The locus $|\bar{\gamma}|$ is again defined as the union of all arcs γ_i.

In the case $\mu = 0,\ h = 0$ we have a geodesic line. More precisely, the locus $|\bar{\gamma}|$ is a geodesic line, and the maps g_{ij} are inclusions.

Each point $y_1 \in \gamma_1$ divides a (μ, h)-biroad $\bar{\gamma}$ into two (μ, h)-roads $\bar{\alpha}$ and $\bar{\beta}$, where

$$\alpha_i = \gamma_i[g_{1i}y_1, u_i],\ \beta_i = \gamma_i[g_{1i}y_1, v_i].$$

6.11. Lemma. *Suppose that γ is a (μ, h)-biroad, $\gamma_i \colon u_i \curvearrowright v_i$. Then $\bar{u}$ and $\bar{v}$ are Gromov sequences and $(u_i|v_i)_p \le \mu + h/2$ for all $p \in \gamma_1$ and for all i.*

Proof. Dividing γ into two roads we see from 6.3 that $\bar{u}$ and $\bar{v}$ are Gromov sequences. The inequality follows from 2.8(4) and (2.9). □

6.12. *Biroads in a hyperbolic space.* Let X be a δ-hyperbolic space and let γ be a (μ, h)-biroad in $X,\ \gamma_i \colon u_i \curvearrowright v_i$. By 6.11, the sequences $\bar{u}$ and $\bar{v}$ define distinct elements $a = \hat{u}$ and $b = \hat{v}$ of the Gromov boundary ∂X. We write $\bar{\gamma} \colon a \curvearrowright b$ and say that the biroad $\bar{\gamma}$ *joins* a to b.

6.13. Lemma. *Let X be an intrinsic δ-hyperbolic space and let $a, b \in \partial X,\ a \ne b,\ h > 0$. Then there is a (μ, h)-biroad $\bar{\gamma} \colon a \curvearrowright b$ with $\mu = 12\delta + 10h$.*

Proof. Fix a base point $p \in X$. By 6.3 there are (μ_0, h)-roads $\bar{\alpha} \colon p \curvearrowright a,\ \alpha_i \colon p \curvearrowright u_i$ and $\bar{\beta} \colon p \curvearrowright b,\ \beta_i \colon p \curvearrowright v_i$ with $\mu_0 = 4\delta + 2h$. Observe that $\mu = 3\mu_0 + 4h$. Since $a \ne b$, the sequence of numbers $(u_i|v_i)$ is bounded by 5.3(4). Passing to subsequences we may assume that

$$|(u_i|v_i) - (u_j|v_j)| \le h \tag{6.14}$$

for all i and j. Moreover, we may assume that

$$l(\alpha_{i+1}) \ge l(\alpha_i) + 3h,\ \ l(\beta_{i+1}) \ge l(\beta_i) + 3h \tag{6.15}$$

for all i. We choose h-short arcs $\gamma_i \colon u_i \curvearrowright v_i$ and show that the sequence $\bar{\gamma}$ with suitable length maps g_{ij} is the desired biroad.

For each $i \in \mathsf{N}$, the h-short triangle $\Delta_i = (\alpha_i, \beta_i, \gamma_i)$ induces the subdivisions

$$\alpha_i = \alpha_i' \cup \alpha_i^* \cup \alpha_i'',\ \beta_i = \beta_i' \cup \beta_i^* \cup \beta_i'',\ \gamma_i = \gamma_i' \cup \gamma_i^* \cup \gamma_i''$$

of the sides with $p \in \alpha_i' \cap \beta_i',\ u_i \in \gamma_i'$; see 2.21. The lengths of the centers $\alpha_i^*, \beta_i^*, \gamma_i^*$ are at most h by 2.24. Moreover,

$$l(\alpha_i') = l(\beta_i') = (u_i|v_i),\ l(\alpha_i'') = l(\gamma_i'),\ l(\beta_i'') = l(\gamma_i''). \tag{6.16}$$

We write $\gamma_i^* = \gamma_i[y_i, z_i]$ with $y_i \in \gamma_i',\ z_i \in \gamma_i''$.

Let $i < j$. We show that

$$l(\gamma_i') \leq l(\gamma_j') - h. \tag{6.17}$$

By (6.16) we have

$$\begin{aligned} l(\alpha_i) &\geq l(\alpha_i') + l(\alpha_i'') = (u_i|v_i) + l(\gamma_i'), \\ l(\alpha_j) &= l(\alpha_j') + l(\alpha_j^*) + l(\alpha_j'') \leq (u_j|v_j) + h + l(\gamma_j'). \end{aligned}$$

By (6.14) and (6.15) these inequalities imply (6.17).

Similarly $l(\gamma_i'') \leq l(\gamma_j'') - h$, whence $l(\gamma_i^*) + l(\gamma_i'') \leq l(\gamma_j^*) + l(\gamma_j'')$. It follows that there is a well defined orientation preserving length map $g = g_{ij} \colon \gamma_i \to \gamma_j$ with $gy_i = y_j$. It remains to show that

$$|gx - x| \leq \mu \tag{6.18}$$

for each $x \in \gamma_i$. We consider three cases.

Case 1. $x \in \gamma_i'$. This case is rather similar to Case 2 but easier. We omit the proof, which gives (6.18) in the improved form $|gx - x| \leq 3\mu_0 + 2h = \mu - 2h$.

Case 2. $x \in \gamma_i''$. Set $s = l(\gamma_i[z_i, x])$. There is a bijective length map $\varphi_i \colon \gamma_i'' \to \beta_i''$ fixing v_i with $|\varphi_i x - x| \leq \mu_0$; see 2.15. Let $f \colon \beta_i \to \beta_j$ be the length map fixing p. Since $\bar{\beta}$ is a (μ_0, h)-road, we have $|f\varphi_i x - \varphi_i x| \leq \mu_0$. Furthermore,

$$l(\beta_j[p, f\varphi_i x]) = l(\beta_i[p, \varphi_i x]) = (u_i|v_i) + l(\beta_i^*) + s. \tag{6.19}$$

The point $w = \varphi_j z_j$ is the common endpoint of β_j^* and β_j''.

Subcase 2a. $f\varphi_i x \in \beta_j''$. Setting $z = \varphi_j^{-1} f\varphi_i x$ we have $|x - z| \leq 3\mu_0$. By (6.19) we obtain

$$\begin{aligned} l(\gamma_j[y_j, z]) &= l(\gamma_j^*) + l(\beta_j[w, f\varphi_i x]) \\ &= l(\gamma_j^*) + (u_i|v_i) + l(\beta_i^*) + s - (u_j|v_j) - l(\beta_j^*). \end{aligned}$$

On the other hand,

$$l(\gamma_j[y_j, gx]) = l(\gamma_i[y_i, x]) = l(\gamma_i^*) + s.$$

Since the length of each center arc is at most h, these estimates and (6.14) yield

$$|z - gx| \leq |l(\gamma_j^*) + l(\beta_i^*) - l(\gamma_i^*) - l(\beta_j^*)| + |(u_i|v_i) - (u_j|v_j)| \leq 3h,$$

whence $|gx - x| \leq 3\mu_0 + 3h = \mu - h$.

Subcase 2b. $f\varphi_i x \notin \beta''_j$. Set $t = l(\beta_j[f\varphi_i x, w])$. By (6.19) we have

$$(u_j|v_j) + l(\beta_j^*) - t = (u_i|v_i) + l(\beta_i^*) + s,$$

which yields $s + t \le 2h$. If $gx \in \gamma_j^*$, then $|gx - z_j| \le h$. If $gx \notin \gamma_j^*$, then

$$|gx - z_j| \le l(\gamma_j[y_j, gx]) = l(\gamma_i) + s \le h + s,$$

which is thus valid in both cases. As $w = \varphi_j z_j$, we get

$$\begin{aligned} |gx - x| &\le |gx - z_j| + |z_j - w| + |w - f\varphi_i x| + |f\varphi_i x - x| \\ &\le h + s + \mu_0 + t + 2\mu_0 = 3\mu_0 + 3h = \mu - h. \end{aligned}$$

Case 3. $x \in \gamma_i^*$. Now $|x - y_i| \le h$ and $|gx - y_j| \le h$. By Case 1 we have $|y_i - y_j| = |y_i - gy_i| \le \mu - 2h$. These estimates yield $|gx - x| \le \mu$, and the theorem is proved. □

We next give a version of the standard estimate 2.33 for biroads.

6.20. Extended standard estimate. *Let X be δ-hyperbolic, let $p \in X$ and let $\bar\alpha\colon a \curvearrowright b$ be a (μ, h)-biroad. Then*

$$d(p, |\bar\alpha|) - 4\delta - h \le (a|b)_p \le d(p, |\bar\alpha|) + \mu + h/2.$$

Proof. Write $\alpha_i\colon u_i \curvearrowright v_i$. Then 2.33 gives

$$d(p, \alpha_i) - 2\delta - h \le (u_i|v_i)_p \le d(p, \alpha_i) + h/2.$$

Since $d(p, |\bar\alpha|) \le d(p, \alpha_i) \le d(p, |\bar\alpha|) + \mu$ for large i, the lemma follows from 5.11. □

6.21. *Strings.* Working with a road or a biroad $\bar\alpha = (\alpha_i)$ is somewhat uncomfortable, because one must often choose a particular member α_i and then go from one member to another with the length maps g_{ij}. It is sometimes easier to work with an object obtained by identifying the members of $\bar\alpha$. This object is called the string of $\bar\alpha$, and it is defined as follows:

Let $\bar\alpha\colon \bar y \curvearrowright b,\ \alpha_i\colon y_i \curvearrowright u_i$ be a (μ, h)-road in a domain G and let $\mathrm{du}\,\bar\alpha$ be the disjoint union of all α_i, that is,

$$\mathrm{du}\,\bar\alpha = \{(x, i)\colon i \in \mathsf{N},\ x \in \alpha_i\}.$$

Define an equivalence relation in $\mathrm{du}\,\bar\alpha$ by setting $(x, i) \sim (y, j)$ if either $i \le j,\ y = g_{ij}x$ or $j \le i,\ x = g_{ji}y$. The set $\mathrm{str}\,\bar\alpha$ of all equivalence classes is the *string* of $\bar\alpha$. For each i, we let

$$\pi_i\colon \alpha_i \to \mathrm{str}\,\bar\alpha \tag{6.22}$$

denote the natural map, defined by $(x, i) \in \pi_i x$.

If $\xi, \zeta \in \mathrm{str}\,\bar\alpha$, we can find representatives $(x, i) \in \xi,\ (z, i) \in \zeta$ with the same index i. Since the maps g_{ij} are length maps, the number

$$l(\xi, \zeta) = l(\alpha_i[x, z])$$

depends only on ξ and ζ. The function l is a metric in $\mathrm{str}\,\bar\alpha$, and the maps π_i of (6.22) are length maps in a natural sense. The initial points y_i of α_i define an initial point y^* of $\mathrm{str}\,\bar\alpha$, and we obtain a bijective isometry $\omega\colon \operatorname{str}\bar\alpha \to [0,\infty)$ by setting $\omega(\xi) = l(y^*, \xi)$. The map ω also defines a linear order in $\operatorname{str}\bar\alpha$; then $\xi \le \zeta$ iff there are $i \in \mathsf{N}$ and $(x,i) \in \xi$, $(z,i) \in \zeta$ such that the points y_i, x, z, u_i are in this order on α_i.

The *locus* $|\xi|$ of an element $\xi \in \operatorname{str}\bar\alpha$ is the set of all $x \in G$ such that $(x,i) \in \xi$ for some i. Then $|\bar\alpha| = \bigcup\{|\xi| \colon \xi \in \operatorname{str}\bar\alpha\}$.

The *string of a* (μ,h)*-biroad* $\bar\alpha\colon a \curvearrowright b,\ \alpha_i\colon u_i \curvearrowright v_i$, in G is defined similarly. Now there is a bijective order preserving isometry $\omega\colon \operatorname{str}\bar\alpha \to \mathsf{R}$, and ω is unique up to an additive constant. We shall use obvious notation like $[\xi_1, \xi_2]$ and $[-\infty, \xi_0]$ for intervals in $\operatorname{str}\alpha$.

6.23. *Extended triangles.* It is possible to extend parts of the theory of h-short triangles (see 2.21) to the case where some of the vertices lie on the Gromov boundary ∂X. Some sides of such a generalized triangle will be (μ,h)-roads or (μ,h)-biroads with suitable μ and h. We consider only the case where all vertices lie on the boundary and prove first the following version of the Rips condition for such triangles:

6.24. Theorem. *Let X be an intrinsic δ-hyperbolic space, let $a,b,c \in \partial X$, let $\bar\alpha\colon b \curvearrowright c$, $\bar\beta\colon c \curvearrowright a$, $\bar\gamma\colon a \curvearrowright b$ be (μ,h)-biroads and let $x \in |\bar\alpha|$. Then $d(x, |\bar\beta| \cup |\bar\gamma|) \le C(\delta,\mu,h) = 46\delta + 11\mu + 22h$.*

Proof. Expressing $\bar\alpha$ as a union of two (μ,h)-roads we find by 6.9 a member $\alpha_i\colon b_1 \curvearrowright c_1$ of $\bar\alpha$ such that

$$d(x,\alpha_i) \le \mu,\ d(b_1, |\bar\gamma|) \le C_1,\ d(c_1, |\bar\beta|) \le C_1$$

where $C_1 = 7\delta + \mu + 3h$. Similarly we find an arc $\beta_j\colon c_2 \curvearrowright a_2$ such that

$$d(a_2, |\bar\gamma|) \le C_1,\ d(c_1, \beta_j) \le C_1 + \mu.$$

Choose h-short arcs $\beta'\colon c_1 \curvearrowright a_2$ and $\gamma'\colon a_2 \curvearrowright b_1$. There is a point $x_1 \in \alpha_i$ with $|x_1 - x| \le \mu$. Since X is $(3\delta + 2h, h)$-Rips by 2.35, we find a point $x_2 \in \beta' \cup \gamma'$ with $|x_2 - x_1| \le 3\delta + 2h$.

If $x_2 \in \beta'$, it follows from the second ribbon lemma 2.18 that there is $y \in \beta_j$ with $|x_2 - y| \le C_2 = 8\delta + 5(C_1 + \mu) + 5h$. Then $|y - x| \le \mu + 3\delta + 2h + C_2 = C$.

If $x_2 \in \gamma'$, we choose $k \in \mathsf{N}$ such that $d(a_2, \gamma_k) \le C_1 + \mu$ and $d(b_1, \gamma_k) \le C_1 + \mu$. By 2.18 we again find a point $y \in \gamma_k$ with $|x_2 - y| \le C_2$, and then $|x - y| \le C$. □

In order to prove a version of the tripod lemma 2.15 for extended triangles we make some preparation. Let X be a hyperbolic space, let a,b,c be distinct points in ∂X, and let

$$\bar\alpha\colon b \curvearrowright c,\ \bar\beta\colon c \curvearrowright a,\ \bar\gamma\colon a \curvearrowright b$$

be (μ,h)-biroads. Then $(\bar\alpha, \bar\beta, \bar\gamma)$ is an extended (μ,h)-triangle. For $i \in \mathsf{N}$, we let π_i denote each of the natural maps $\alpha_i \to \operatorname{str}\bar\alpha,\ \beta_i \to \operatorname{str}\bar\beta,\ \gamma_i \to \operatorname{str}\bar\gamma$. Given an element $\xi_\alpha \in \operatorname{str}\bar\alpha$, the intervals $(-\infty, \xi_\alpha]$ and $[\xi_\alpha, \infty)$ define two (μ,h)-roads converging to b and c, respectively. If, in addition, $\xi_\beta \in \operatorname{str}\bar\beta$ and $\xi_\gamma \in \operatorname{str}\bar\gamma$, there are three natural (orientation reversing) bijective length maps and their inverses between the corresponding intervals, for example, $f\colon [\xi_\beta, \infty) \to (-\infty, \xi_\gamma]$ with $f\xi_\beta = \xi_\gamma$.

6.25. Extended tripod lemma. *Let X be an intrinsic δ-hyperbolic space and let $(\bar\alpha, \bar\beta, \bar\gamma)$ be an extended (μ,h)-triangle as above. Then there are elements $\xi_\alpha \in \operatorname{str}\bar\alpha,\ \xi_\beta \in \operatorname{str}\bar\beta,\ \xi_\gamma \in \operatorname{str}\bar\gamma$ such that the corresponding length maps f satify the inequality*

$$d(|f\xi|, |\xi|) \le C(\delta,\mu,h)$$

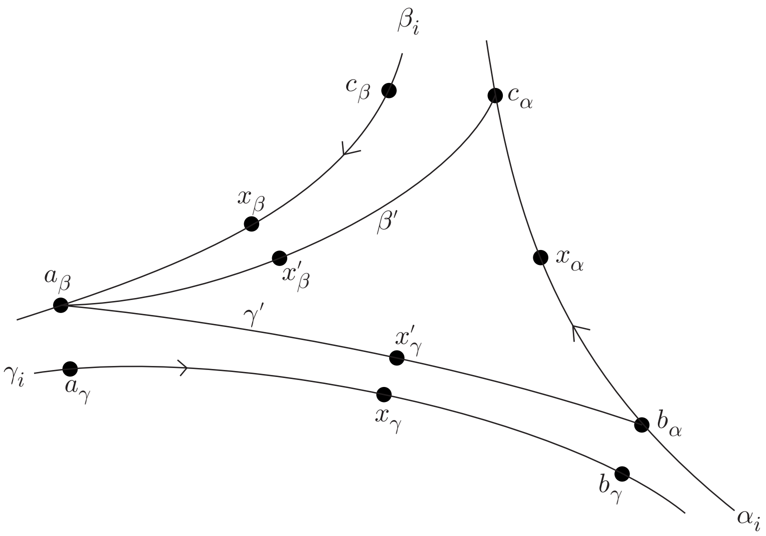
for all ξ in the domain of f.

Proof. By the closeness lemma 6.9 we find elements $\zeta(\alpha,b),\zeta(\alpha,c)\in\operatorname{str}\bar\alpha,\ \zeta(\beta,c),$ $\zeta(\beta,a)\in\operatorname{str}\bar\beta$ and $\zeta(\gamma,a),\zeta(\gamma,b)\in\operatorname{str}\bar\gamma$ such that

$$d(|\zeta(\gamma,a)|,|\zeta(\beta,a)|)\vee d(|\zeta(\gamma,b)|,|\zeta(\alpha,b)|)\vee d(|\zeta(\alpha,c)|,|\zeta(\beta,c)|)\le C_1$$

with $C_1=7\delta+2\mu+3h$. Choose an integer i such that the natural image $\pi_i\alpha_i$ covers the interval $[\zeta(\alpha,b),\zeta(\alpha,c)]$ and such that the corresponding relations hold for $\pi_i\beta_i$ and $\pi_i\gamma_i$. Choose points $b_\alpha,c_\alpha\in\alpha_i,\ c_\beta,a_\beta\in\beta_i,\ a_\gamma,b_\gamma\in\gamma_i$ such that, for example, b_α is the unique point with $\pi_ib_\alpha=\zeta(\alpha,b)$. Then

(6.26) $$|a_\beta-a_\gamma|\vee|b_\gamma-b_\alpha|\vee|c_\alpha-c_\beta|\le C_1+2\mu.$$



Choosing h-short arcs $\beta'\colon c_\alpha\curvearrowright a_\beta$ and $\gamma'\colon a_\beta\curvearrowright b_\alpha$ we obtain an h-short triangle with sides $\alpha_i[b_\alpha,c_\alpha],\beta',\gamma'$. Choose points $x_\alpha\in\alpha_i,\ x'_\beta\in\beta',\ x'_\gamma\in\gamma'$ in the center of this triangle; see 2.21. By (6.26) and by the second ribbon lemma 2.18 we find points $x_\beta\in\beta_i$ and $x_\gamma\in\gamma_i$ with $|x_\beta-x'_\beta|\vee|x_\gamma-x'_\gamma|\le C_2=8\delta+5(C_1+2\mu)+5h$. Since the diameter of the center is at most $4\delta+4h$ by 2.24, we have

(6.27) $$|x_\alpha-x_\beta|\vee|x_\beta-x_\gamma|\vee|x_\gamma-x_\alpha|\le C_3=4\delta+4h+2C_2.$$

We show that the lemma holds with $\xi_\alpha=\pi_ix_\alpha,\ \xi_\beta=\pi_ix_\beta,\ \xi_\gamma=\pi_ix_\gamma$.

Consider the intervals $[\xi_\beta,\infty)\subset\operatorname{str}\bar\beta,\ (-\infty,\xi_\gamma]\subset\operatorname{str}\bar\gamma$, and let $f\colon[\xi_\beta,\infty)\to(-\infty,\xi_\gamma]$ be the length map with $f\xi_\beta=\xi_\gamma$. Let $\xi>\xi_\beta$. It suffices to find an estimate

(6.28) $$d(|f\xi|,|\xi|)\le C(\delta,\mu,h).$$

By the closeness lemma 6.9 we can find elements $\zeta_\beta\in\operatorname{str}\bar\beta$ and $\zeta_\gamma\in\operatorname{str}\bar\gamma$ such that $\zeta_\beta>\xi,\ \zeta_\gamma<\xi_\gamma,\ d(|\zeta_\beta|,|\zeta_\gamma|)\le C_1\le C_3$. Choose integers m and n such that writing $\gamma_n\colon u_n\curvearrowright v_n$ we have

$$[\xi_\beta,\zeta_\beta]\subset\pi_m\beta_m,\ [\zeta_\gamma,\xi_\gamma]\subset\pi_n\gamma_n,\ l(\beta_m)\le l(\pi_nu_n,\xi_\gamma).$$

Let $y_\beta, z_\beta \in \beta_m,\ y_\gamma, z_\gamma \in \gamma_n$ be the unique points with $\pi_m y_\beta = \xi_\beta,\ \pi_m z_\beta = \zeta_\beta,\ \pi_n y_\gamma = \xi_\gamma,\ \pi_n z_\gamma = \zeta_\gamma$. Then $|y_\beta - y_\gamma| \vee |z_\beta - z_\gamma| \le C_3 + 2\mu$ by (6.27). Let $g\colon \beta_m[y_\beta, z_\beta] \to \gamma_n$ be the orientation reversing length map with $gy_\beta = y_\gamma$. There is a point $x \in \beta_m[y_\beta, z_\beta]$ with $\pi_n x = \xi$. By the ribbon lemma 2.17 we have $|gx - x| \le C_4 = 8\delta + 5(C_3 + 2\mu) + 5h$. Since $x \in |\xi|$ and $gx \in |f\xi|$, this implies (6.28) with $C = C_4 = 458\delta + 110\mu + 125h$. □

6.29. *Extended stability.* We next extend the stability theory of Section 3 to the case where at least one endpoint lies on the Gromov boundary, The main result is given in 6.32. We start with the easy case of two roads or biroads with common endpoints.

6.30. Lemma. *Suppose that X is an intrinsic δ-hyperbolic space. If $\bar\alpha, \bar\alpha'$ are (μ, h)-roads $\bar y \curvearrowright b \in \partial X$ or (μ, h)-biroads $a \curvearrowright b$, then $d_H(|\bar\alpha|, |\bar\alpha'|) \le 43\delta + 11\mu + 20h$.*

Proof. We prove the case of biroads. Let $x \in |\bar\alpha|$. By 6.9 we can find members $\alpha_i\colon u_i \curvearrowright v_i$ of $\bar\alpha$ and α'_j of $\bar\alpha'$ such that

$$d(u_i, \alpha'_j) \le C,\ d(v_i, \alpha'_j) \le C,\ d(x, \alpha_i) \le \mu,$$

where $C = 7\delta + 2\mu + 3h$. By Lemma 2.18 this yields $\alpha_i \subset \bar B(\alpha'_j, 8\delta + 5C + 5h)$, whence $d(x, |\bar\alpha'|) \le 43\delta + 11\mu + 20h$. □

6.31. Lemma. *Suppose that X is an intrinsic δ-hyperbolic space and that $\varphi\colon [0, \infty) \to X$ is a (λ, μ)-quasi-isometry; see* (3.3)*. Then $\varphi(t)$ converges to a point $b \in \partial X$ as $t \to \infty$.*

If $\varphi\colon \mathsf{R} \to X$ is a (λ, μ)-quasi-isometry, then $\varphi(t)$ converges to limits $a, b \in \partial X$ as $t \to -\infty$ or $t \to \infty$.

We shall write $a = \varphi(-\infty)$ and $b = \varphi(\infty)$.

Proof. It suffices to prove the first part of the lemma. Let $0 < s \le t$. We must show that $(\varphi(s)|\varphi(t)) \to \infty$ as $s \to \infty$. Let $h = 1$ and choose an h-short arc $\alpha_{st}\colon \varphi(s) \curvearrowright \varphi(t)$. By 3.7 we have $d_H(\alpha_{st}, \varphi[s, t]) \le M(\delta, \lambda, \mu)$. By the standard estimate 2.33 we get

$$(\varphi(s)|\varphi(t)) \ge d(p, \alpha_{st}) - 2\delta - h \ge d(p, \varphi[s, t]) - M - 2\delta - h.$$

For each $u \in [s, t]$ we have

$$|p - \varphi(u)| \ge |\varphi(0) - \varphi(u)| - |\varphi(0) - p| \ge s/\lambda - \mu - |\varphi(0) - p| \to \infty$$

as $s \to \infty$, and the lemma follows. □

6.32. Theorem. *(Extended stability) Suppose that X is an intrinsic δ-hyperbolic space.*

(1) *Let $\varphi\colon [0, \infty) \to X$ be a (λ, μ)-quasi-isometry and let $\bar\alpha\colon \varphi(0) \curvearrowright \varphi(\infty)$ be a (μ, h)-road. Then $d_H(|\bar\alpha|, \operatorname{im}\varphi) \le M(\delta, \lambda, \mu, h)$.*

(2) *Let $\varphi\colon \mathsf{R} \to X$ be a (λ, μ)-quasi-isometry and let $\bar\alpha\colon \varphi(-\infty) \curvearrowright \varphi(\infty)$ be a (μ, h)-biroad. Then $d_H(|\bar\alpha|, \operatorname{im}\varphi) \le M(\delta, \lambda, \mu, h)$.*

Proof. We prove part (2); the proof of (1) is rather similar. We may assume that $0 < h \le \mu$. Let M_0 be the number $M(\delta, \lambda, \mu)$ given by the stability theorem 3.7. Define a sequence of numbers $R_0 < R_1 < \dots$ by

$$R_0 = 0,\ R_{i+1} = \lambda(\lambda R_i + 2M_0 + 2\mu + h + 1).$$

Set $u_i = \varphi(-R_i),\ v_i = \varphi(R_i)$ and choose h-short arcs $\beta_i\colon u_i \curvearrowright v_i$. We show that the sequence $\bar\beta$ is a (μ_1, h)-biroad $\bar\beta\colon \varphi(-\infty) \curvearrowright \varphi(\infty)$ with $\mu_1 = \mu_1(\delta, \lambda, \mu, h)$.

We have $d_H(\beta_i, \varphi[-R_i, R_i]) \le M_0$. Hence we can choose points $y_i \in \beta_i$ with $|y_i - \varphi(0)| \le M_0$. Setting $s_i = l(\beta_i[y_i, v_i])$ we have

$$s_i \le |y_i - v_i| + h \le |y_i - \varphi(0)| + |\varphi(0) - \varphi(R_i)| + h \le M_0 + \lambda R_i + \mu + h,$$
$$s_{i+1} \ge |y_{i+1} - v_{i+1}| \ge |\varphi(0) - \varphi(R_{i+1})| - |\varphi(0) - y_{i+1}| \ge R_{i+1}/\lambda - \mu - M_0,$$

whence $s_{i+1} \ge s_i + 1$. Similarly $t_{i+1} \ge t_i + 1$ for $t_i = l(\beta_i[u_i, y_i])$. Consequently, for each pair $i \le j$ there is a unique orientation preserving length map $g_{ij}\colon \beta_i \to \beta_j$ with $g_{ij}y_i = y_j$. Since $d(u_i, \beta_j) \vee d(v_i, \beta_j) \le M_0$, it follows by the ribbon lemma 2.17 that

$$|g_{ij}x - x| \le 8\delta + 5M_0 + 5h = \mu_1(\delta, \lambda, \mu, h)$$

for all $x \in \beta_i$. Hence $\bar\beta$ is a (μ_1, h)-biroad from $\varphi(-\infty)$ to $\varphi(\infty)$.

Since $d_H(|\bar\beta|, \operatorname{im}\varphi) \le M_0$, the theorem follows from 6.30. □

6.33. *Roughly starlike spaces.* Let X be a δ-hyperbolic space and let $K \ge 0,\ \mu \ge 0,\ h \ge 0$. We say that X is (K, μ, h)*-roughly starlike* with respect to a point $y \in X$ if for each $x \in X$ there is a (μ, h)-road $\bar\alpha\colon y \curvearrowright b \in \partial X$ with $d(x, |\bar\alpha|) \le K$.

In the case $\mu = 0,\ h = 0$, the condition is the same as in [BHK, p. 18].

The space X is said to be (K, μ, h)-roughly starlike with respect to a *boundary point* $a \in \partial X$ if for each $x \in X$ there is a (μ, h)-biroad $\bar\alpha\colon a \curvearrowright b \in \partial X$ with $d(x, |\bar\alpha|) \le K$.

If X is (K, μ, h)-roughly starlike with respect to $z \in X^*$ for all $h > 0$, we say that X is (K, μ)*-roughly starlike* with respect to z.

The essential parameter of rough starlikeness is K. In fact, in intrinsic spaces we can always choose h to be arbitrarily small and μ fairly small:

6.34. Lemma. *Suppose that X is an intrinsic δ-hyperbolic space.*

(1) If X is (K_0, μ_0, h_0)-roughly starlike with respect to $y \in X$, then X is (K_1, μ_1, h)-roughly starlike with respect to y for every $h > 0$ and for $\mu_1 = 4\delta + 1,\ K_1 = K_1(K_0, \mu_0, h_0, \delta)$.

(2) If X is (K_0, μ_0, h_0)-roughly starlike with respect to $a \in \partial X$, then X is (K_2, μ_2, h)-roughly starlike with respect to a for every $h > 0$ and for $\mu_2 = 12\delta + 1,\ K_2 = K_2(K_0, \mu_0, h_0, \delta)$.

Proof. (1) We may assume that $h \le 1/2$. Let $x \in X$ and choose a (μ_0, h_0)-road $\bar\alpha\colon y \curvearrowright b$ with $d(x, |\bar\alpha|) \le K_0$. By 6.7 there is a (μ_1, h)-road $\bar\beta\colon y \curvearrowright b$. By 6.30 we have

$$d_H(|\bar\alpha|, |\bar\beta|) \le C = 43\delta + 11(\mu_1 \vee \mu_0) + 20(1 \vee h_0),$$

whence $d(x, |\bar\beta|) \le K_0 + C$.

Part (2) is proved similarly with the aid of Lemma 6.13. □

6.35. Lemma. *(Two-point starlikeness) Suppose that X is an intrinsic δ-hyperbolic space and that X is (K, μ)-roughly starlike with $\mu = 12\delta + 1$ with respect to $a_0 \in \partial X$. Let $x_1, x_2 \in X,\ h > 0$. Then there is a (μ, h)-biroad $\bar\alpha\colon a_1 \curvearrowright a_2$ such that $d(x_i, |\bar\alpha|) \le K_1(K, \delta)$ for $i = 1, 2$.*

Proof. We may assume that $h \le 1/10$. There are (μ,h)-biroads $\bar\alpha_i\colon a_0 \curvearrowright a_i,\ i=1,2,$ such that $d(x_i,|\bar\alpha_i|) \le K$. If $a_1 = a_2$, we can choose $\bar\alpha = \bar\alpha_1$ by 6.30. Assume that $a_1 \ne a_2$ and set $C = 46\delta + 11\mu + 3$. Choose points $y_i \in |\bar\alpha_i|$ with $|x_i - y_i| \le K+1,\ i=1,2.$ We may assume that $d(y_2,|\bar\alpha_1|) \wedge d(y_1,|\bar\alpha_2|) > C$, since otherwise we may take $\bar\alpha = \bar\alpha_1$ or $\bar\alpha = \bar\alpha_2$. Choose a (μ,h)-biroad $\bar\alpha_3\colon a_1 \curvearrowright a_2$. By the extended Rips condition 6.24 we have $d(y_i,|\bar\alpha_3|) \le C,\ i=1,2$, and the lemma holds with $\bar\alpha = \bar\alpha_3,\ K_1 = K + C + 1$. $\square$.

In [Vä5] we shall make use of roads and biroads to study hyperbolic domains with the quasihyperbolic metric in Banach spaces. These domains are always roughly starlike with respect to each point in the domain and in its boundary.

References

[Bo] M. Bonk, Quasi-geodesic segments and Gromov hyperbolic spaces, Geom. Dedicata 62, 1996, 281–298.

[BHK] M. Bonk, J. Heinonen and P. Koskela, Uniformizing Gromov hyperbolic spaces, Astérisque 270, 2001, 1–99.

[BS] M. Bonk and O. Schramm, Embeddings of Gromov hyperbolic spaces, GAFA 10, 2000, 266–306.

[Bow] B. H. Bowditch, Notes on Gromov's hyperbolicity criterion for path-metric spaces, Group theory from a geometrical point of view, ed. by E. Ghys et al, 64–137, World Scientific, 1991.

[BH] M.R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Springer, 1999.

[CDP] M. Coornaert, T. Delzant and A. Papadopoulos, Géométrie et théorie des groupes, Lecture Notes in Math. 1441, Springer, 1990.

[BBI] D. Burago, Y. Burago and S. Ivanov, A course in metric geometry, Graduate Studies Math 33, AMS, 2001.

[GdH] E. Ghys and P. de la Harpe (eds), Sur les groupes hyperboliques d'après Mikhael Gromov, Birkhäuser, 1990.

[KB] I. Kapovich and N. Benakli, Boundaries of hyperbolic groups, Combinatorial and geometric group theory, 39–93, Contemp. Math. 296, Amer. Math. Soc., 2002.

[Ro] J. Roe, Lectures on coarse geometry, University Lecture Series 31, Amer. Math. Soc., 2003.

[Sh] H. Short (editor), Notes on word hyperbolic groups, Group theory from a geometrical point of view, ed. by E. Ghys et al, 3–64, World Scientific, 1991.

[TV] P. Tukia and J. Väisälä, Quasisymmetric embeddings of metric spaces, Ann. Acad. Sci. Fenn. Math. 5, 1980, 97–114.

[Vä1] J. Väisälä, Quasimöbius maps, J. Analyse Math. 44, 1985, 218–234.

[Vä2] J. Väisälä, Free quasiconformality in Banach spaces II, Ann. Acad. Sci. Fenn. Math. 16, 1991, 355–379.

[Vä4] J. Väisälä, The free quasiworld. Freely quasiconformal and related maps in Banach spaces, Quasiconformal geometry and dynamics (Lublin, 1996), 55–118, Banach Center Publ. 48, Polish Acad. Sci., Warsaw, 1999.

[Vä5] J. Väisälä, Hyperbolic and uniform domains in Banach spaces, Preprint, 2004, www.helsinki.fi/~jvaisala/preprints.html.

Index