Introduction to differential geometry

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Preface

These are lecture notes for the course "Introduction to differential geometry" at the Department of Mathematics and Statistics at the University of Helsinki.

The material has been compiled from several sources, for example from books [AMR], [Br], [L2], [MT] and [Wa]. The main source has been the book [L2] by J.M. Lee.

0 Some basic notions in topology

0.1 Topological space

Let X be an arbitrary set and

$$\mathcal{P}(X) = \{A \colon A \subset X\}$$

its *power set*. A collection $\mathcal{T} \subset \mathcal{P}(X)$ is a *topology* on X if

1. $\mathcal{T}$ contains the union of any family of its members:

$$U_\alpha \in \mathcal{T} \Rightarrow \bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T},$$

   where $\mathcal{A}$ is an arbitrary set of indices;

2. $\mathcal{T}$ contains the intersection of any finite family of its members:

$$U_1, \dots, U_k \in \mathcal{T} \Rightarrow \bigcap_{i=1}^{k} U_i \in \mathcal{T};$$

3. $\emptyset \in \mathcal{T},\ X \in \mathcal{T}$.

The pair $(X, \mathcal{T})$, or just X for short, is a *topological space*. The elements of $\mathcal{T}$ are called *open sets*. A set $F \subset X$ is *closed* if the complement $X \setminus F$ is an open set.

Example 0.2. 1. Let (X, d) be a metric space. That is $d \colon X \times X \to \mathbb{R}$ satisfies the axioms of a *metric*:

$$\begin{aligned}
&d(x,y) \ge 0 \quad \forall x,y \in X \\
&d(x,y) = 0 \iff x = y \\
&d(x,y) = d(y,x) \quad \forall x,y \in X \\
&d(x,y) \le d(x,z) + d(z,y) \quad \forall x,y,z \in X \quad \text{(triangle inequality, } \triangle\text{-ineq.).}
\end{aligned}$$

Then the metric d defines a (metric) topology $\mathcal{T}_d$ on X:
$U \in \mathcal{T}_d \iff \forall x \in U \ \exists r > 0$ s.t. $B(x,r) = \{y \in X : d(x,y) < r\} \subset U$.

2. Special case: The Euclidean space $\mathbb{R}^n$ equipped with the metric $d(x,y) = |x - y|$.

3. A topological space $(X, \mathcal{T})$ is *metrizable* if $\exists$ a metric d s.t. $\mathcal{T} = \mathcal{T}_d$.

A set U is a *neighborhood* of a point $x \in X$ if $x \in U \in \mathcal{T}$ (i.e. U is open and contains x). Fact: A set $A \subset X$ is open $\iff$ $\forall x \in A \ \exists$ a neighborhood U of x s.t. $U \subset A$.

A topological space $(X, \mathcal{T})$ is *Hausdorff* if every pair of distinct points has disjoint neighborhoods. (That is, $\forall x, y \in X, \ x \neq y$, there exist $U \in \mathcal{T}, \ V \in \mathcal{T}$ s.t. $x \in U, \ y \in V$, and $U \cap V = \emptyset$.)

Example 0.3. 1. Every metrizable topological space is Hausdorff. (Exerc.)

2. <u>Example.</u> Identify points $(x, 0)$ and $(x, 1)$ of the set $\mathbb{R}^n \times \{0\} \cup \mathbb{R}^n \times \{1\}$ whenever $x \neq 0$. We obtain a space X that has "two origins". Equip X with a topology by saying that $U \subset X$ is open $\iff$ the preimage of U under the identification is open. Then the points $a = (0,0)$ and $b = (0,1)$ have no disjoint neighborhoods and hence X is not Hausdorff.

We say that a sequence $(x_i), \ i \in \mathbb{N}$, of points in X converges to a point $x \in X$ (denoted by $x_i \to x$) if for every neighborhood U of x there exists $i_0 \in \mathbb{N}$ s.t. $x_i \in U \ \forall i \geq i_0$. Fact: if X is Hausdorff and $x_i \to x$ and $x_i \to y$ then $x = y$.

Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{B} \subset \mathcal{P}(X)$ is a *basis* for the topology $\mathcal{T}$ (or a basis for X) if

1. $\mathcal{B} \subset \mathcal{T}$,

2. every $U \in \mathcal{T}, \ U \neq \emptyset$, can be written as a union of some elements of $\mathcal{B}$.

Example 0.4. Let (X, d) be a metric space. Then

$$\mathcal{B} = \{B(x,r) : x \in X, \ r > 0\}$$

is a basis for $\mathcal{T}_d$.

The following notion is important for this course: We say that $(X, \mathcal{T})$ is N_2 ("second countable") if there exists a *countable basis* $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ for $\mathcal{T}$.

Example 0.5. The Euclidean space $\mathbb{R}^n$ equipped with the usual topology is N_2. We can choose, for example, $\mathcal{B} = \{B(q,r) : q \in \mathbb{Q}^n, \ r \in \mathbb{Q}_+\}$.

Let X and Y be topological spaces. We say that a mapping $f : X \to Y$ is *continuous* at a point $x \in X$ if for every neighborhood V of $f(x)$ there exists a neighborhood U of x s.t. $fU \subset V$. A mapping f is continuous in X if it is continuous at every point of X.
Fact: $f : X \to Y$ is continuous in X $\iff$ the preimage $f^{-1}U = \{x \in X : f(x) \in U\}$ is open for every open $U \subset Y$.

A mapping $f : X \to Y$ is a *homeomorphism* if

1. f has an inverse,

2. f is continuous, and

3. the inverse f^{-1} is continuous.

Let X be a set, $(Y, \mathcal{T}')$ a topological space, and $f\colon X \to Y$ a mapping. Then the collection

$$\mathcal{T} = \{f^{-1}U \colon U \in \mathcal{T}'\}$$

is a topology on X (induced by the mapping f). Note: The mapping f is then trivially continuous.

If $(X, \mathcal{T})$ is a topological space and $A \subset X$, then the topology induced by the *inclusion* $i\colon A \to X$, $i(x) = x$, is called the *relative topology* of $(X, \mathcal{T})$ on A (denoted by $\mathcal{T}|A$). Hence

$$\mathcal{T}|A = \{U \cap A \colon U \in \mathcal{T}\}.$$

In other words, the set $V \subset A$ is open in A (i.e. $V \in \mathcal{T}|A$) $\iff$ $V = U \cap A$ for some open $U \subset X$ ($U \in \mathcal{T}$).

Both being Hausdorff and N_2 are *hereditary*:

Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then

1. $(X, \mathcal{T})$ is Hausdorff $\Rightarrow$ $(A, \mathcal{T}|A)$ is Hausdorff,

2. $(X, \mathcal{T})$ is N_2 $\Rightarrow$ $(A, \mathcal{T}|A)$ is N_2.

Let $(X_1, \mathcal{T}_1), \dots, (X_k, \mathcal{T}_k)$ be topological spaces. Denote

$$X = X_1 \times X_2 \times \cdots \times X_k = \{(x_1, \dots, x_k) \colon x_i \in X_i\}.$$

The collection

$$\mathcal{B} = \{U_1 \times U_2 \times \cdots \times U_k \colon U_i \subset X_i \text{ open}\}$$

is a basis of the *product topology* on X.

Remark 0.6.

1. Let $(X_i, \mathcal{T}_i)$, $i = 1, \dots, k$, be Hausdorff. Then $X = X_1 \times X_2 \times \cdots \times X_k$ equipped with the product topology is Hausdorff.

2. Let $(X_i, \mathcal{T}_i)$, $i = 1, \dots, k$, be topological spaces with countable bases (i.e. each $(X_i, \mathcal{T}_i)$ is N_2). Then $X = X_1 \times X_2 \times \cdots \times X_k$ with the product topology is N_2.

The next result is very useful in many existence results. First we recall:

Definition 0.7. Let (X, d) be a metric space. A sequence (x_i), $x_i \in X$, is a *Cauchy sequence* if $\forall \varepsilon > 0$ $\exists i_\varepsilon \in \mathbb{N}$ s.t. $d(x_i, x_j) < \varepsilon$ whenever $i, j \geq i_\varepsilon$. The metric space X is *complete* if every Cauchy sequence on X converges.

Theorem 0.8 (Banach fixed point theorem)**.** *Let X be a complete metric space and $f\colon X \to X$ a mapping. Suppose that there exists a constant $L \in [0, 1[$ s.t.*

$$d(f(x), f(y)) \leq L d(x, y) \quad \forall\, x, y \in X.$$

Then f has a unique fixed point $x_0 \in X$, i.e. $f(x_0) = x_0$.

Proof. Let $y_0 \in X$. Define recursively

$$y_{i+1} = f(y_i), \quad i = 0, 1, 2, \dots$$

We see by induction that

$$d(y_{i+1}, y_i) \leq L^i d(y_0, y_1).$$

By the triangle inequality,

$$d(y_i, y_j) \le (L^i + \cdots + L^{j-1}) d(y_0, y_1) \quad \text{if } i < j.$$

Since $0 \le L < 1$, the series

$$1 + L + L^2 + \cdots$$

converges, and therefore the remainder term $\to 0$. Hence

$$L^i + L^{i+1} + \cdots + L^{j-1} \to 0 \quad \text{if } i, j \to \infty.$$

It follows that (y_i) is a Cauchy sequence. Since X is complete, the sequence (y_i) converges, i.e.

$$y_i \to x_0 \in X.$$

Now

$$\begin{aligned} d(y_i, f(y_i)) &= d(f(y_{i-1}), f(y_i)) \\ &\le L \underbrace{d(y_{i-1}, y_i)}_{\le L^{i-1} d(y_0, y_1)} \le L^i d(y_0, y_1). \end{aligned}$$

We obtain

$$\begin{aligned} d(x_0, f(x_0)) &\le d(x_0, y_i) + d(y_i, f(y_i)) + \underbrace{d(f(y_i), f(x_0))}_{\le L d(y_i, x_0)} \\ &\le (1 + L) d(x_0, y_i) + L^i d(y_0, y_1) \xrightarrow{i \to \infty} 0. \end{aligned}$$

Hence $d(x_0, f(x_0)) = 0$, i.e. $x_0 = f(x_0)$. If x_0' is another fixed point,

$$d(x_0', x_0) = d(f(x_0'), f(x_0)) \le L d(x_0', x_0),$$

and since $L < 1$, we must have $x_0' = x_0$. □

0.9 Topological manifold

Definition 0.10. Let M be a topological space. We say that M is a *topological n-manifold*, $n \in \mathbb{N}$, if

1. M is Hausdorff,
2. M is second countable (M is N_2),
3. every point $x \in M$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Remark 0.11.

1. The condition 3 means that M is "locally homeomorphic with $\mathbb{R}^n$".
2. Condition 3 $\iff$ every $x \in M$ has a neighborhood U that is homeomorphic with the open unit ball $B^n(0,1) = \{y \in \mathbb{R}^n : |y| < 1\}$ (or equivalently with the whole $\mathbb{R}^n$).
3. Fact: If M is both a topological n-manifold and a topological m-manifold, then necessarily $m = n$. (We do not prove this. The proof uses algebraic topology (invariance of domain).)

4. Properties 1 and 2 do not follow from the condition 3. For instance, an uncountable disjoint union of $\mathbb{R}^n$s satisfies the condition 3 but is not N_2. On the other hand, the topological manifold in Example 0.3 satisfies the condition 3 but is not Hausdorff.

Let M be a topological n-manifold. We say that a pair (U, φ) is a chart on M if

(a) $U \subset M$ is open

(b) $\varphi \colon U \to \varphi U \subset \mathbb{R}^n$ is a homeomorphism and $\varphi U \subset \mathbb{R}^n$ is open.

If, in addition, $p \in U$, then (U, φ) is a chart at p.

In what follows we usually denote (U, x), $x = (x^1, \ldots, x^n)$, where $x \colon U \to xU \subset \mathbb{R}^n$ is a homeomorphism and $x^1, x^2, \ldots, x^n$ are the coordinate functions of x (that is, real-valued functions $x^i \colon U \to \mathbb{R}$).

The standard example of topological n-manifolds is, of course, $M = \mathbb{R}^n$ equipped with the usual topology. Earlier we recalled that $\mathbb{R}^n$ is Hausdorff and N_2.

The rough idea of a topological n-manifold: The conditions guarantee that M has many good properties of $\mathbb{R}^n$.
Hausdorff: for instance, the limit of a convergent sequence is unique.
N_2: an important property that is needed in partition of unity.

Example 0.12. 1. Every open set $U \subset \mathbb{R}^n$, $U \neq \emptyset$, is a topological n-manifold. (Hausdorff and N_2 are hereditary).

2. Graphs of continuous functions: Let $U \subset \mathbb{R}^n$ be open and $f \colon U \to \mathbb{R}^k$ continuous. We say that the graph of f is the following subset of $\mathbb{R}^n \times \mathbb{R}^k$

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \colon x \in U,\ y = f(x)\}$$

equipped with the relative topology. Then $\Gamma(f)$ is Hausdorff and N_2. Let $\pi_1 \colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be the projection $(x, y) \mapsto x$ and $\varphi_f \colon \Gamma(f) \to U$ be the restriction

$$\begin{aligned} \varphi_f &= \pi_1 | \Gamma(f), \\ \varphi_f(x, y) &= x, \quad (x, y) \in \Gamma(f). \end{aligned}$$

Since π_1 is continuous, then φ_f is continuous (relative topology). In addition, φ_f is a homeomophism since it has a continuous inverse mapping

$$\varphi_f^{-1}(x) = (x, f(x)).$$

Hence $\Gamma(f)$ is a topological n-manifold (homeomorphic with U).

3. The sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \colon |x| = 1\}$ is a topological n-manifold (relative topology). Reason: $\mathbb{S}^n$ can be covered by open sets that can be represented as graphs of continuous functions (hence reduces to the previous example). Example Let

$$U_{n+1}^+ = \{(x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \colon x^{n+1} > 0\}.$$

Now $U_{n+1}^+ = \Gamma(f) = (x, f(x))$, where $f \colon B^n \to \mathbb{R},\ f(x) = \sqrt{1 - |x|^2}$. Similarly the other sets U_i^+ and U_i^-,

$$\begin{aligned} U_i^+ &= \{(x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \colon x^i > 0\} \\ U_i^- &= \{(x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \colon x^i < 0\}. \end{aligned}$$

4. Let M_i be topological n_i-manifolds, $i = 1, 2, \dots, k$. Then
$$M = M_1 \times M_2 \times \cdots \times M_k$$
is a topological n-manifold, where $n = n_1 + n_2 + \cdots + n_k$. Reason: Earlier we noticed that M is Hausdorff and N_2. If $p = (p_1, \dots, p_k) \in M_1 \times M_2 \times \cdots \times M_k$, choose charts (U_i, φ_i) in M_i s.t. $p_i \in U_i, \ \forall i = 1, \dots, k$. The product mapping
$$\varphi_1 \times \cdots \times \varphi_k \colon U_1 \times \cdots \times U_k \to \ R^n$$
is a homeomorphism onto its image that is an open subset of $\mathbb{R}^n$. We do the same for all $p \in M$.
Example: n-torus
$$T^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n \text{ kpl}}.$$
5. Projective space $\mathbb{R}P^n$ (n-dimensional real projective space) is the set of all 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$ (or the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin). $\mathbb{R}P^n$ can be obtained by identifying points $x \in \mathbb{S}^n$ and $-x \in \mathbb{S}^n$. More precisely: define an equivalence relation on $\mathbb{S}^n$:
$$x \sim y \iff x = \pm y, \ x, y \in \mathbb{S}^n.$$
Then $\mathbb{R}P^n = \mathbb{S}^n/\!\sim = \{[x] \colon x \in \mathbb{S}^n\}$. Equipping $\mathbb{R}P^n$ with so-called quotient topology, $\mathbb{R}P^n$ becomes a topological n-manifold.

Quotient topology:

Definition 0.13. Let $(X, \mathcal{T})$ be a topological space, $\sim$ an equivalence relation on X and $\pi \colon X \to X/\!\sim$ the canonical projection, $x \mapsto [x]$. Then the collection
$$\{U \subset X/\!\sim \ : \pi^{-1}U \in \mathcal{T}\}$$
is called the quotient topology of $X/\!\sim$.

The set $\Gamma = \{(x, x') \in X \times X \colon x \sim x'\}$ is the graph of the equivalence relation $\sim$. We say that $\sim$ is open (closed) if the projection $\pi \colon X \to X/\!\sim$ is an open (closed) mapping.

[Note: Let X and Y be topological spaces. A mapping $f \colon X \to Y$ is open (closed) if the image fA is open (closed) for every open (closed) $A \subset X$.]

Theorem 0.14. *If $X/\!\sim$ is Hausdorff, then the graph Γ of an equivalence relation $\sim$ is a closed set in $X \times X$. If $\Gamma \subset X \times X$ is closed and $\sim$ is open, then $X/\!\sim$ is Hausdorff.*

For the proof we need a lemma.

Lemma 0.15. *X is Hausdorff $\iff$ (the diagonal) $\Delta_X = \{(x, x) \in X \times X \colon x \in X\}$ is closed in $X \times X$.*

Proof. X Hausdorff $\iff \ \forall\, p, q \in X, \ p \neq q, \ \exists$ (disjoint) neighborhoods $U_p \ni p, \ U_q \ni q$ s.t. $(U_p \times U_q) \cap \Delta_X = \emptyset \iff (X \times X) \setminus \Delta_X$ open. □

Proof of Theorem 0.14. $X/\!\sim$ Hausdorff $\Rightarrow \ \Delta_{X/\sim}$ is closed, hence $\Gamma = (\pi \times \pi)^{-1}(\Delta_{X/\sim})$ is closed. Suppose then that Γ is closed and $\sim$ is open. If $X/\!\sim$ is not Hausdorff, $\exists$ distinct points $[x], [y] \in X/\!\sim$ such that $U_{[x]} \cap U_{[y]} \neq \emptyset$ for every neighborhoods $U_{[x]} \ni [x], \ U_{[y]} \ni [y]$. Let V_x, V_y be arbitrary neighborhoods of x and y. Since $\sim$ is open, $\pi(V_x), \pi(V_y)$ are neighborhoods of $[x]$ and $[y]$. Since $\pi(V_x) \cap \pi(V_y) \neq \emptyset, \ \exists\ x' \in V_x, \ y' \in V_y$ s.t. $[x'] = [y']$, i.e. $x' \sim y'$, so $(x', y') \in \Gamma$. Thus $(x, y) \in \bar{\Gamma}$ (every neighborhood of (x, y) intersects with Γ). Since Γ is closed, $(x, y) \in \Gamma$, so $[x] = [y]$. We obtained a contradiction, and therefore $X/\!\sim$ is Hausdorff. □

Theorem 0.16. *If X is N_2 and $\sim$ is an open equivalence relation on X, then $X/\!\sim$ is N_2.*

Proof Let $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ be a countable basis of X. Claim: $[\mathcal{B}] = \{[B_i] : i \in \mathbb{N}\}$ is a countable basis of $X/\!\sim$. (Here $[B_i] = \pi B_i$, $\pi\colon X \to X/\!\sim$ is the canonical projection.) Countability is clear. Moreover every $[B_i]$ is open because $\sim$ is open. Let $A \subset X/\!\sim$ be open. Then (by the definition of quotient topology) $\pi^{-1}A \subset X$ is open, and so $\pi^{-1}A = \bigcup_{j\in J} B_j$, $J \subset \mathbb{N}$. Thus $A = \bigcup_{j\in J} \pi(B_j) = \bigcup_{j\in J}[B_j]$ and $X/\!\sim$ is N_2. $\square$

0.17 Basic properties of a topological manifold

Let us recall the following definitions:

An *open cover* of a topological space X is a collection

$$\{V_\alpha : \alpha \in \mathcal{A}\}$$

of open subsets V_α of X s.t. $X = \bigcup_\alpha V_\alpha$. Here $\mathcal{A}$ is an arbitrary index set.

A topological space X is *compact* if every open cover of it has a finite subcover. That is, if $X = \bigcup_\alpha V_\alpha$, there exist $V_{\alpha_1}, \dots, V_{\alpha_k}$ such that $X = \bigcup_{i=1}^k V_{\alpha_i}$.

A topological space X is *locally compact* if every $x \in X$ has a neighborhood U whose closure $\bar{U}$ is compact. We say that a set $A \subset X$ is *precompact* or *relatively compact* ($A \Subset X$) if $\bar{A}$ is compact. [Recall: $\bar{U} = \{x \in X : U \cap V \neq \emptyset \ \forall$ neighborhoods V of $x\}$]

A topological space space X is *connected* if $\nexists$ subsets A, B s.t.

1. $X = A \cup B$
2. $A \neq \emptyset \neq B$
3. $A \cap B = \emptyset$
4. $A \subset X$ is open, $B \subset X$ is open.

In other words, X is connected if it can not be expressed as a union of two disjoint open sets.

A topological space X is *path connected* if every pair $x, y \in X$ can be connected by a path, i.e. $\exists$ a continuous mapping $\alpha\colon [0,1] \to X$ (a path) s.t. $\alpha(0) = x$ and $\alpha(1) = y$.

Note: path connected $\Rightarrow$ connected, but not conversely.

A topological space X is *locally (path) connected* at a point $x \in X$ if every neighborhood of x contains a (path) connected neighborhood of x.

Theorem 0.18. *A topological n-manifold M is locally compact and locally path connected.*

Proof. The claim follows from the conditions 1 and 3 in the definition of a topological n-manifold and from the corresponding properties of $\mathbb{R}^n$: Let $x \in M$ be arbitrary and (U, φ) a chart at x. Since $\varphi U \subset \mathbb{R}^n$ is open and $\varphi(x) \in \varphi U$, there exists a ball $B^n(\varphi(x), r) \subset \varphi U$. Since $\bar{B}^n(\varphi(x), r/2)$ is compact, the set $\varphi^{-1}\bar{B}^n(\varphi(x), r/2)$ is compact and hence closed, because M is Hausdorff. Thus $\varphi^{-1}B^n(\varphi(x), r/2)$ is a neighborhood of x whose closure is compact. On the other hand, $B^n(\varphi(x), r)$ is path connected, and therefore $\varphi^{-1}B^n(\varphi(x), r)$ ($\subset U$) is a path connected neighborhood of x. $\square$

We need the following lemmata for the existence of a partition of unity:

Lemma 0.19 (Lindelöf)**.** *Let X be a topological space with a countable basis and let $A \subset X$. Then every open cover $\{V_\alpha : \alpha \in \mathcal{A}\}$ of A ($A \subset \bigcup_\alpha V_\alpha$) contains a countable subcover.*

Proof. Let $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ be a countable basis of X. For each $x \in A$ there exist indices $i \in \mathbb{N}$ and $\alpha \in \mathcal{A}$ s.t. $x \in B_i \subset V_\alpha$. Let

$$\mathcal{B}' = \{B_i \in \mathcal{B} : x \in B_i \subset V_\alpha, x \in A\}.$$

Then $\mathcal{B}'$ is a cover of A. For each $B_i \in \mathcal{B}'$ choose one V_α, for which $B_i \subset V_\alpha$, and denote it by $V_{\alpha(i)}$. Since $\mathcal{B}'$ is a cover of A and $B_i \subset V_{\alpha(i)}$ $\forall B_i \in \mathcal{B}'$, the family $\{V_{\alpha(i)}\}$ is a countable cover of A. $\square$

Theorem 0.20. *Every topological n-manifold M has a countable basis $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$, where every B_i is precompact and homeomorphic with an open ball in $\mathbb{R}^n$. In particular, M is σ-compact (i.e. a countable union of compact sets).*

Proof. (i) For every $x \in M$ there exists a chart (U, φ) at x, and therefore "chart neighborhoods" U form an open cover of M. By Lemma 0.19 there exists a countable cover $\{U_i : i \in \mathbb{N}\}$ of M s.t. (U_i, φ_i) is a chart.
(ii) Denote $\tilde{U}_i = \varphi U_i$ ($\subset \mathbb{R}^n$ open) and

$$\tilde{\mathcal{B}}_i = \{B^n(x,r) : x \in \mathbb{Q}^n, r \in \mathbb{Q}_+, \bar{B}^n(x,r) \subset \tilde{U}_i\}.$$

Then every such $\bar{B}^n(x,r) \subset \tilde{U}_i$ is compact and $\tilde{\mathcal{B}}_i$ is a countable basis of $\tilde{U}_i$. Since $\varphi_i : U_i \to \tilde{U}_i$ is a homeomorphism, the family

$$\mathcal{B}_i = \{\varphi_i^{-1} B : B \in \tilde{\mathcal{B}}_i\}$$

is a countable basis of U_i and every $\overline{\varphi_i^{-1} B}$ is a compact subset of U_i. Now $\mathcal{B} = \bigcup_i \mathcal{B}_i$ satisfies the requirements of the theorem. Since $M = \bigcup_{B \in \mathcal{B}} \overline{B}$ and each $\overline{B}$ is compact, M is σ-compact. $\square$

1 Review on differential calculus in $\mathbb{R}^n$

1.1 Differentiability

Definition 1.2. Let $G \subset \mathbb{R}^n$ be open. A mapping $f : G \to \mathbb{R}^m$ is *differentiable* at $x \in G$ if there exists a linear map $A(x) \in L(\mathbb{R}^n, \mathbb{R}^m)$ s.t.

$$f(x+h) = f(x) + A(x)h + |h|\varepsilon(x,h),$$

where $\varepsilon(x,h) \xrightarrow{h \to 0} 0$. The (unique) linear map $A(x)$ is called the differential of f at x and denoted by $A(x) = f'(x) = Df(x)$.

It can be shown that the matrix of $f'(x)$ (w.r.t. standard bases) is

$$\begin{pmatrix} D_1 f_1(x) & \cdots & D_n f_1(x) \\ \vdots & \ddots & \vdots \\ D_1 f_m(x) & \cdots & D_n f_m(x) \end{pmatrix},$$

where $f = (f_1, \ldots, f_m)$.

Definition 1.3. A mapping $f : G \to \mathbb{R}^m$ is *continuously differentiable* at $x_0 \in G$ if there exists a neighborhood $U \subset G$ of x_0 s.t.

1. f is differentiable at every $x \in U$ and
2. $f' : U \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous at x_0.

<u>Note</u>: Above the topology of $L(\mathbb{R}^n, \mathbb{R}^m)$ is determined by a norm. Since $L(\mathbb{R}^n, \mathbb{R}^m)$ is finite dimensional, all norms determine the same topology.
We use the operator norm $|L| = \sup\{|Lh| : |h| = 1\}$ for linear maps $L \in L(\mathbb{R}^n, \mathbb{R}^m)$.

<u>Fact</u>: A mapping f is continuously differentiable in $G \iff \exists$ continuous partial derivatives $D_j f_i$ in G for all $i = 1, \dots, m,\ j = 1, \dots, n$.

<u>In general</u>: Let $k \in \mathbb{N} \cup \{0\}$. We say that f is k times continuously differentiable in G, denoted by $f \in C^k(G)$, if all partial derivatives

$$\frac{\partial^{|\alpha|} f_i}{\partial^\alpha x},\ i = 1, \dots, m,$$

are continuous in G for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, with $|\alpha| = \alpha_1 + \dots + \alpha_n \leq k$. Here

$$\frac{\partial^{|\alpha|} f_i}{\partial^\alpha x} = \frac{\partial^{|\alpha|} f_i}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}}.$$

If $f \in C^k(G)$ for all $k \in \mathbb{N}$, we denote $f \in C^\infty(G)$.

Definition 1.4. Let $G \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open. A mapping $f \colon G \to V$ is a C^∞-diffeomorphism if $f \in C^\infty(G)$ and $\exists f^{-1} \in C^\infty(V)$.

Inverse mapping theorem.

Theorem 1.5 (Inverse mapping theorem)**.** *Let $G \subset \mathbb{R}^n$ be open and $f \colon G \to \mathbb{R}^n,\ f \in C^1(G)$. Suppose that at a point $a \in G$*

$$J_f(a) = \det f'(a) \neq 0.$$

Then there exist neighborhoods $U \ni a$, $V \ni f(a)$, and the inverse mapping $g = f^{-1} \colon V \to U$. Moreover, $g \in C^1(V)$ and $g'(f(x)) = f'(x)^{-1},\ x \in U$.

<u>Recall</u>: $\det f'(a) \neq 0 \iff$ the linear map $f'(a) \colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible, i.e. the inverse mapping $f'(a)^{-1}$ exists.

We need the following two lemmata for the proof.

Let us denote by $GL(n, \mathbb{R})$ the space of all invertible linear maps $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ (equivalently, the space of all (real) $n \times n$-matrices $A,\ \det A \neq 0$).

Lemma 1.6.

1. *If $A \in GL(n, \mathbb{R})$ and $B \in L(\mathbb{R}^n, \mathbb{R}^n)$ s.t.*

$$|B - A||A^{-1}| < 1,$$

then $B \in GL(n, \mathbb{R})$.

2. *$GL(n, \mathbb{R}^n)$ is an open subset of $L(\mathbb{R}^n, \mathbb{R}^n)$ and the map $A \mapsto A^{-1}$ is continuous in $GL(n, \mathbb{R})$.*

Proof. Exerc. [see e.g. Rudin [Ru].] □

Lemma 1.7 (Mean value theorem)**.** *Let $G \subset \mathbb{R}^m$ be open and $J \subset G$ a closed line segment, whose end points are a and b. Let $f \colon G \to \mathbb{R}^n$ be a mapping that is differentiable at every point of J. Then for every $v \in \mathbb{R}^n$ there exists $x_v \in J$ s.t.*

$$v \cdot (f(b) - f(a)) = v \cdot (f'(x_v)(b - a)).$$

In particular, if $|f'(x)| \leq M$ for all $x \in J$, then

$$|f(b) - f(a)| \leq M|b - a|.$$

Proof. Exerc. $\square$

Proof of Theorem 1.5. (i) Write $L = f'(a)$ and choose $\lambda > 0$ s.t. $2\lambda|L^{-1}| = 1$. Since f' is continuous at a, there exists a ball $U = B^n(a, \varepsilon)$ s.t.

$$|f'(x) - L| < \lambda \quad \forall x \in U.$$

We will prove that $f|U$ is injective. Define, for every $y \in \mathbb{R}^n$ a mapping φ $(= \varphi_y)$

$$\varphi(x) = x + L^{-1}\big(y - f(x)\big) = x + L^{-1}(y) - L^{-1}\big(f(x)\big), \quad x \in G. \tag{1.8}$$

We observe: $f(x) = y \iff \varphi(x) = x$.
By the chain rule

$$\begin{aligned} \varphi'(x) &= I - L^{-1}f'(x) \qquad (I = \text{ identity}) \\ &= L^{-1}\big(L - f'(x)\big), \end{aligned}$$

hence

$$|\varphi'(x)| \le \underbrace{|L^{-1}|}_{=\frac{1}{2\lambda}} \underbrace{|L - f'(x)|}_{<\lambda} < \frac{1}{2}, \; x \in U.$$

By the mean value theorem (Lemma 1.7)

$$|\varphi(x_2) - \varphi(x_1)| \le \frac{1}{2}|x_2 - x_1|, \quad x_1, x_2 \in U.$$

Hence φ has *at most one fixed point* in U. [Indeed, if $\varphi(x_1) = x_1 \in U$ and $\varphi(x_2) = x_2 \in U$, then

$$|x_1 - x_2| = |\varphi(x_1) - \varphi(x_2)| \le \frac{1}{2}|x_1 - x_2|,$$

and therefore $x_1 = x_2$.] The same holds for every $y \in \mathbb{R}^n$, hence $f|U$ is injective.

(ii) Next we will prove that $V = fU$ is open. [Then we have shown that there are *neighborhoods* $U \ni a$, $V \ni f(a)$ s.t. $f|U \colon U \to V$ is bijective.]
Let $y_0 \in V$. Then $y_0 = f(x_0)$ for some $x_0 \in U$. Let $r > 0$ be so small that $\bar{B} = \bar{B}^n(x_0, r) \subset U$. We claim that $B^n(y_0, \lambda r) \subset V$ which then shows that V is open. Fix $y \in B^n(y_0, \lambda r)$, so $|y - y_0| < \lambda r$. Let $\varphi = \varphi_y$,

$$\varphi(x) = x + L^{-1}\big(y - f(x)\big).$$

We have

$$|\varphi(x_0) - x_0| = |L^{-1}(y - y_0)| \le |L^{-1}||y - y_0| < \frac{r}{2}.$$

If $x \in \bar{B}$ $(\subset U)$, then

$$\begin{aligned} |\varphi(x) - x_0| &\le |\varphi(x) - \varphi(x_0)| + |\varphi(x_0) - x_0| \\ &\le \frac{1}{2}|x - x_0| + \frac{r}{2} < r, \end{aligned}$$

and so $\varphi(x) \in B^n(x_0, r)$. Hence

$$\begin{gathered} \varphi\bar{B}^n(x_0, r) \subset \bar{B}^n(x_0, r), \\ |\varphi(x_2) - \varphi(x_1)| \le \frac{1}{2}|x_2 - x_1|, \; \forall x_1, x_2 \in \bar{B}^n(x_0, r). \end{gathered}$$

The closed ball $\bar{B}^n(x_0, r)$ is compact, hence it is complete.
By the Banach fixed point theorem the mapping φ has exactly one fixed point x in $\bar{B}^n(x_0, r)$. Hence $y = f(x) \in f\bar{B}^n(x_0, r) \subset fU = V$, and consequently V is open. We have shown: $\exists$ *neighborhoods* $U \ni a$, $V \ni f(a)$ s.t. $f|U \colon U \to V$ is bijective.

(iii) Next we prove that the inverse mapping $g = (f|U)^{-1} \colon V \to U$ is continuously differentiable, $g \in C^1(V)$. Let $y \in V$ and $y + k \in V$. Write $x = f^{-1}(y)$ and $h = f^{-1}(y + k) - x$. Then $x \in U$, $x + h = f^{-1}(y + k) \in U$, and $f(x + h) = y + k$. If $\varphi = \varphi_y$ (see (1.8)), then

$$\begin{aligned} \varphi(x+h) &= x + h + L^{-1}(y - f(x+h)) \\ \varphi(x) &= x + L^{-1}(y - f(x)), \end{aligned}$$

and hence

$$\begin{aligned} \varphi(x+h) - \varphi(x) &= h + L^{-1}\big(\underbrace{f(x)}_{=y} - \underbrace{f(x+h)}_{y+k}\big) \\ &= h - L^{-1}k. \end{aligned}$$

It follows that

$$|h - L^{-1}k| = |\varphi(x+h) - \varphi(x)| \le \frac{1}{2}|x + h - x| = \frac{1}{2}|h|,$$

hence

$$|L^{-1}k| \ge \frac{1}{2}|h|,$$

and therefore

$$|h| \le 2|L^{-1}k| \le 2|L^{-1}||k| = \frac{|k|}{\lambda}. \tag{1.9}$$

Since

$$|f'(x) - L||L^{-1}| < \frac{1}{2},$$

Lemma 1.6 implies that $f'(x)$ is invertible, i.e. $\exists\ T = f'(x)^{-1}$. We want to show that $g'(y) = T$ (recall that $g = (f|U)^{-1} \colon V \to U$). Now

$$\begin{aligned} \underbrace{g(y+k)}_{=h+x} - \underbrace{g(y)}_{=x} - Tk &= h + x - x - Tk = h - Tk \\ &= Tf'(x)h - Tk = -T\big(k - f'(x)h\big) \\ &= -T\big(\underbrace{f(x+h)}_{=y+k} - \underbrace{f(x)}_{=y} - f'(x)h\big). \end{aligned}$$

This and the estimate (1.9) imply that

$$\frac{|g(y+k) - g(y) - Tk|}{|k|} \le \frac{|T|}{\lambda} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|}. \tag{1.10}$$

If $k \to 0$, then $h \to 0$ by (1.9), and consequently the right-hand side of (1.10) tends to 0. Hence also the left-hand side of (1.10) tends to 0. We obtain

$$\frac{|g(y+k) - g(y) - Tk|}{|k|} \xrightarrow{k \to 0} 0,$$

so g is differentiable at y and

$$g'(y) = T = f'(x)^{-1} = f'\big(g(y)\big)^{-1}, \ y \in V. \tag{1.11}$$

Since g is differentiable at every $y \in V$, it is continuous in V. Moreover, $f \in C^1(U)$ by the assumption and $f'(x)^{-1}$ exists for all $x \in U$, hence $f' \colon U \to GL(n, \mathbb{R})$ is continuous. By Lemma 1.6 (b), $A \mapsto A^{-1}$ is continuous in $GL(n, \mathbb{R})$. Combining these and (1.11) we can conclude that

$$g' \colon V \to GL(n, \mathbb{R}), \ y \mapsto g'(y) = f'\big(g(y)\big)^{-1},$$

is continuous, that is $g \in C^1(V)$. □

Remark 1.12. The assumption $f \in C^1(G)$ was only used at the very end of the proof. If we merely assume that f is differentiable in G, continuously differentiable at a, and $J_f(a) \neq 0$, the corresponding inverse mapping $g = (f|U)^{-1} \colon V \to U$ is differentiable in V and continuously differentiable at $f(a)$.

Corollary 1.13. *If $G \subset \mathbb{R}^n$ is open, $f \colon G \to \mathbb{R}^n$, $f \in C^1(G)$ and $J_f(x) \neq 0$ for all $x \in G$, then f is an open mapping.*

Implicit function theorem. Let us write $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$, so

$$\begin{aligned} t \in \mathbb{R}^{m+n} \quad \Longleftrightarrow \quad t &= (t_1, \ldots, t_{m+n}) = (x_1, \ldots, x_m, y_1, \ldots, y_n) \\ &= (x, y). \end{aligned}$$

Theorem 1.14 (Implicit function theorem)**.** *Let $G \subset \mathbb{R}^{m+n}$ be open, $f \colon G \to \mathbb{R}^n$, and $(x_0, y_0) \in G$. Suppose that*

1. $f(x_0, y_0) = 0$,
2. $f \in C^1(G)$,
3. $J_u(y_0) \neq 0$, *where* $u(y) = f(x_0, y)$.

Then there are neighborhoods $X \subset \mathbb{R}^m$ of x_0 and $Y \subset \mathbb{R}^n$ of y_0 with the property that for every $x \in X$ the exists the unique $\varphi(x) \in Y$ s.t. $f\big(x, \varphi(x)\big) = 0$. The mapping $\varphi \colon X \to Y$ is continuously differentiable in X and $\varphi(x_0) = y_0$.

Proof. Define a mapping $g \colon G \to \mathbb{R}^{m+n}$,

$$g(x, y) = \big(x, f(x, y)\big).$$

Then $g(x_0, y_0) = (x_0, 0)$ and

$$\begin{aligned} &g_1(x, y) = x_1, \quad g_2(x, y) = x_2, \quad \ldots \quad g_m(x, y) = x_m \\ &g_{m+1}(x, y) = f_1(x, y), \quad g_{m+2}(x, y) = f_2(x, y), \quad \ldots \quad g_{m+n}(x, y) = f_n(x, y). \end{aligned} \tag{1.15}$$

We observe that

$$J_g(x_0,y_0) = \begin{vmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ D_1 f_1(x_0,y_0) & \cdots & D_m f_1(x_0,y_0) & D_{m+1} f_1(x_0,y_0) & \cdots & D_{m+n} f_1(x_0,y_0) \\ \vdots & & \vdots & \vdots & & \vdots \\ D_1 f_n(x_0,y_0) & \cdots & D_m f_n(x_0,y_0) & D_{m+1} f_n(x_0,y_0) & \cdots & D_{m+n} f_n(x_0,y_0) \end{vmatrix}$$

$$= \begin{vmatrix} D_{m+1} f_1(x_0,y_0) & \cdots & D_{m+n} f_1(x_0,y_0) \\ \vdots & & \vdots \\ D_{m+1} f_n(x_0,y_0) & \cdots & D_{m+n} f_n(x_0,y_0) \end{vmatrix}$$

$$= J_u(y_0) \neq 0.$$

By the inverse mapping theorem there are neighborhoods $U \ni (x_0,y_0)$ and $V \ni (x_0,0)$ s.t. $g|U \colon U \to V$ is a homeomorphism that has an inverse mapping $g^* = (g|U)^{-1} \colon V \to U$. We may assume that $V = B^{m+n}((x_0,0),r)$. By (1.15) we have

$$g_1^*(x,y) = x_1$$
$$\vdots$$
$$g_m^*(x,y) = x_m.$$

Let us denote $h = (g_{m+1}^*, \ldots, g_{m+n}^*) \colon V \to \mathbb{R}^n$ and define $\varphi \colon B^m(x_0,r) \to \mathbb{R}^n$, $\varphi(x) = h(x,0)$.
Claim: φ is the desired mapping, i.e. $f(x,\varphi(x)) = 0$.
Now

$$\begin{aligned}(x,\varphi(x)) &= (x_1,\ldots,x_m,h_1(x,0),\ldots,h_n(x,0)) \\ &= (g_1^*(x,0),\ldots,g_m^*(x,0),g_{m+1}^*(x,0),\ldots,g_{m+n}^*(x,0)) = g^*(x,0),\end{aligned}$$

hence $g(x,\varphi(x)) = g(g^*(x,0)) = (x,0)$. On the other hand, $(x,0) = g(x,\varphi(x)) = (x, f(x,\varphi(x)))$, that implies

$$f(x,\varphi(x)) = 0.$$

Moreover, since f is continuously differentiable, also g is continuously differentiable. By the inverse mapping theorem, g^* is continuously differentiable, and hence φ is continuously differentiable. Since

$$(x_0,y_0) = g^*(x_0,0) = (x_0,\varphi(x_0))$$

we have

$$\varphi(x_0) = y_0.$$

Finally choose neighborhoods $X \ni x_0$ and $Y \ni y_0$ s.t.

1. $X \times Y \subset U$
2. $\varphi X \subset Y$.

Then, for all $x \in X$, there exists $y = \varphi(x) \in Y$ s.t. $f(x, y) = 0$. It remains to prove the uniqueness. Suppose that $z \in Y$ satisfies the equation $f(x, z) = 0$, $(x, z) \in U$. Then

$$g(x, z) = (x, f(x, z)) = (x, 0) = (x, f(x, y)) = g(x, y).$$

Since $g|U$ is injective, we have $(x, z) = (x, y)$, and therefore $z = y$ □

Remark 1.16. In the above setting we have

$$\varphi'(x_0) = -\left(u'(y_0)\right)^{-1} v'(x_0),$$

where $v(x) = f(x, y_0)$.

2 Differentiable manifolds

2.1 Definitions and examples

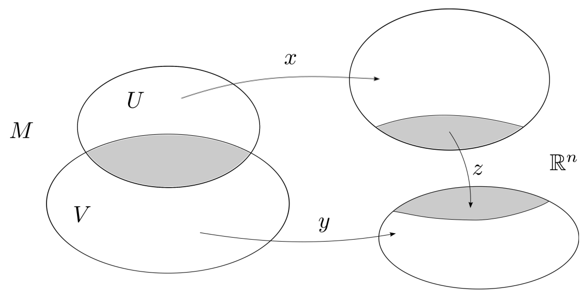
Let M be a topological n-manifold. Recall that a *chart* of M is a pair (U, x), where

1. $U \subset M$ is open,
2. $x \colon U \to xU \subset \mathbb{R}^n$ is a homeomorphism, $xU \subset \mathbb{R}^n$ open.

We say that charts (U, x) and (V, y) are C^∞-*compatible* if $U \cap V = \emptyset$ or

$$z = y \circ x^{-1} | x(U \cap V) \colon x(U \cap V) \to y(U \cap V)$$

is a C^∞-diffeomorphism.



A C^∞-*atlas*, $\mathcal{A}$, of M is a set of C^∞-compatible charts such that

$$M = \bigcup_{(U,x)\in\mathcal{A}} U.$$

A C^∞-atlas $\mathcal{A}$ is *maximal* if $\mathcal{A} = \mathcal{B}$ for all C^∞-atlases $\mathcal{B} \supset \mathcal{A}$. That is, $(U, x) \in \mathcal{A}$ if it is C^∞-compatible with every chart in $\mathcal{A}$.

Lemma 2.2. *Let M be a topological manifold. Then*

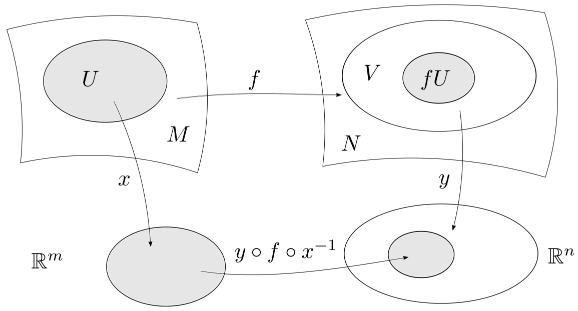
1. *every C^∞-atlas, $\mathcal{A}$, of M belongs to a unique maximal C^∞-atlas (denoted by $\bar{\mathcal{A}}$).*
2. *C^∞-atlases $\mathcal{A}$ and $\mathcal{B}$ belong to the same maximal C^∞-atlas if and only if $\mathcal{A} \cup \mathcal{B}$ is a C^∞-atlas.*

Proof. Exercise □

Definition 2.3. A *differentiable n-manifold* (or a smooth n-manifold) is a pair $(M, \mathcal{A})$, where M is a topological n-manifold and $\mathcal{A}$ is a maximal C^∞-atlas of M, also called a *differentiable structure* of M.

We abbreviate M or M^n and say that M is a C^∞-manifold, a differentiable manifold, or a smooth manifold.

Definition 2.4. Let $(M^m, \mathcal{A})$ and $(N^n, \mathcal{B})$ be C^∞-manifolds. We say that a mapping $f\colon M \to N$ is C^∞ (or smooth) if each *local representation* of f (with respect to $\mathcal{A}$ and $\mathcal{B}$) is C^∞. More precisely, if the composition $y \circ f \circ x^{-1}$ is a smooth mapping $x(U \cap f^{-1}V) \to yV$ for every charts $(U, x) \in \mathcal{A}$ and $(V, y) \in \mathcal{B}$. We say that $f\colon M \to N$ is a C^∞-diffeomorphism if f is C^∞ and it has an inverse f^{-1} that is C^∞, too.



Remark 2.5. Equivalently, $f\colon M \to N$ is C^∞ if, for every $p \in M$, there exist charts (U, x) in M and (V, y) in N such that $p \in U$, $fU \subset V$, and $y \circ f \circ x^{-1}$ is $C^\infty(xU)$.

Examples 2.6.

1. $M = \mathbb{R}^n$, $\mathcal{A} = \{id\}$, $\bar{\mathcal{A}} =$ canonical structure.

2. $M = \mathbb{R}$, $\mathcal{A} = \{id\}$, $\mathcal{B} = \{x \overset{h}{\mapsto} x^3\}$. Now $\bar{\mathcal{A}} \neq \bar{\mathcal{B}}$ since $id \circ h^{-1}$ is not C^∞ at the origin. However, $(\mathbb{R}, \bar{\mathcal{A}})$ and $(\mathbb{R}, \bar{\mathcal{B}})$ are diffeomorphic by the mapping $f\colon (\mathbb{R}, \bar{\mathcal{A}}) \to (\mathbb{R}, \bar{\mathcal{B}})$, $f(y) = y^{1/3}$. Note: f is diffeomorphic with respect to structures $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ since id is the local representation of f.

$$\begin{tikzcd}
(\mathbb{R}, \bar{\mathcal{A}}) \arrow[r, bend left, "f"] \arrow[d, "id"'] & (\mathbb{R}, \bar{\mathcal{B}}) \arrow[d, "x \mapsto x^3"] \\
\mathbb{R} \arrow[r, bend left, "id"] & \mathbb{R}
\end{tikzcd}$$

3. If M is a differentiable manifold and $U \subset M$ is open, then U is a differentiable manifold in a natural way.

4. *Finite dimensional vector spaces.* Let V be an n-dimensional (real) vector space. Every norm on V determines a topology on V. This topology is independent of the choice of the norm since any two norms on V are equivalent (V finite dimensional). Let $E_1, \ldots, E_n$ be a basis of V and $E\colon \mathbb{R}^n \to V$ the isomorphism

$$E(x) = \sum_{i=1}^{n} x^1 E_i, \quad x = (x^1, \ldots, x^n).$$

Then E is a homeomorphism (V equipped with the norm topology) and the (global) chart (V, E^{-1}) determines a smooth structure on V. Furthermore, these smooth structures are independent of the choice of the basis $E^1, \dots, E_n$.

5. *Matrices.* Let $M(n \times m, \mathbb{R})$ be the set of all (real) $n \times m$-matrices. It is a nm-dimensional vector space and thus it is a smooth nm-manifold. A matrix $A = (a_{ij}) \in M(n \times m, \mathbb{R}),\ i = 1, \dots, n,\ j = 1, \dots, m$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

can be identified in a natural way with the point

$$(a_{11}, a_{12}, \dots, a_{1m}, a_{21}, \dots, a_{2m}, \dots, a_{n1}, \dots, a_{nm}) \in \mathbb{R}^{nm}$$

giving a global chart. If $n = m$, we abbreviate $M(n, \mathbb{R})$.

6. $GL(n, \mathbb{R}) =$ *general linear group*

$$\begin{aligned} &= \{L \colon \mathbb{R}^n \to \mathbb{R}^n \text{ linear isomorphism}\} \\ &= \{A = (a_{ij}) \colon \text{ invertible (non-singular) } n \times n\text{-matrix}\} \\ &= \{A = (a_{ij}) \colon \det A \neq 0\}. \end{aligned}$$

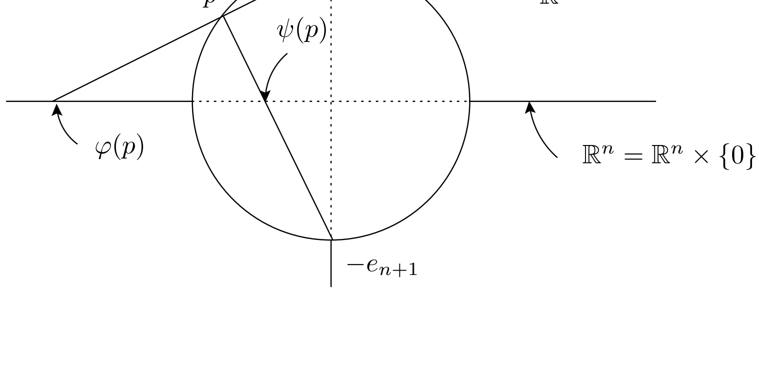
[Note: an $n \times n$-matrix A is invertible (or non-singular) if it has an inverse matrix A^{-1}.]

By the identification above, we may interprete $GL(n, \mathbb{R}) \subset M(n, \mathbb{R}) = \mathbb{R}^{n^2}$. Equip $M(n, \mathbb{R})$ with the relative topology (induced by the inclusion $GL(n, \mathbb{R}) \subset M(n, \mathbb{R}) = \mathbb{R}^{n^2}$). Now the mapping $\det \colon M(n, \mathbb{R}) \to \mathbb{R}$ is continuous (a polynomial of a_{ij} of degree n), and therefore $G(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ is open (as a preimage of an open set $\mathbb{R} \setminus \{0\}$ under a continuous mapping).

7. *Sphere* $\mathbb{S}^n = \{p \in \mathbb{R}^{n+1} \colon |p| = 1\}$. Let $e_1, \dots, e_{n+1}$ be the standard basis of $\mathbb{R}^{n+1}$, let

$$\begin{aligned} \varphi &\colon \mathbb{S}^n \setminus \{e_{n+1}\} \to \mathbb{R}^n \\ \psi &\colon \mathbb{S}^n \setminus \{-e_{n+1}\} \to \mathbb{R}^n \end{aligned}$$

be the stereographic projections, and $\mathcal{A} = \{\varphi, \psi\}$. Details are left as an exercise.



8. *Projective space* $\mathbb{R}P^n$. The real n-dimensional projective space $\mathbb{R}P^n$ is the set of all 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$, i.e. the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin. It can also be obtained by identifying points $x \in \mathbb{S}^n$ and $-x \in \mathbb{S}^n$. More precisely, define an equivalence relation

$$x \sim y \iff x = \pm y,\ x, y \in \mathbb{S}^n.$$

Then $\mathbb{R}P^n = \mathbb{S}^n/\sim = \{[x] \colon x \in \mathbb{S}^n\}$. Equip $\mathbb{R}P^n$ with so called quotient topology to obtain $\mathbb{R}P^n$ as a topological n-manifold. Details are left as an exercise.

9. *Product manifolds.* Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be differentiable manifolds and let $p_1 \colon M \times N \to M$ and $p_2 \colon M \times N \to N$ be the projections. Then

$$\mathcal{C} = \{(U \times V, (x \circ p_1, y \circ p_2)) \colon (U, x) \in \mathcal{A}, (V, y) \in \mathcal{B}\}$$

is a C^∞-atlas on $M \times N$. Example

   (a) Cylinder $\mathbb{R}^1 \times \mathbb{S}^1$

   (b) Torus $\mathbb{S}^1 \times \mathbb{S}^1 = T^2$.

10. *Lie groups.* A Lie group is a group G which is also a differentiable manifold such that the group operations are C^∞, i.e.

$$(g, h) \mapsto gh^{-1}$$

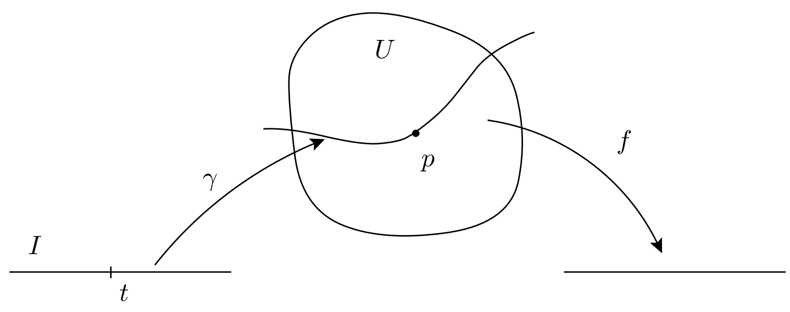
is a C^∞-mapping $G \times G \to G$. For example, $GL(n, \mathbb{R})$ is a Lie group with composition as the group operation.

Remark 2.7. 1. Replacing C^∞ by, for example, C^k, C^ω (= real analytic), or complex analytic (in which case, $n = 2m$) we may equip M with other structures.

2. There are topological n-manifolds that do not admit differentiable structures. (Kervaire, $n = 10$, in the 60's; Freedman, Donaldson, $n = 4$, in the 80's). The Euclidean space $\mathbb{R}^n$ equipped with an arbitrary atlas is diffeomorphic to the canonical structure whenever $n \neq 4$ ("Exotic" structures of $\mathbb{R}^4$ were found not until in the 80's).

2.8 Tangent space

Let M be a differentiable manifold, $p \in M$, and $\gamma \colon I \to M$ a C^∞-path such that $\gamma(t) = p$ for some $t \in I$, where $I \subset \mathbb{R}$ is an open interval.



Write
$$C^\infty(p) = \{f\colon U \to \mathbb{R} \mid f \in C^\infty(U),\ U \text{ some neighborhood of } p\}.$$

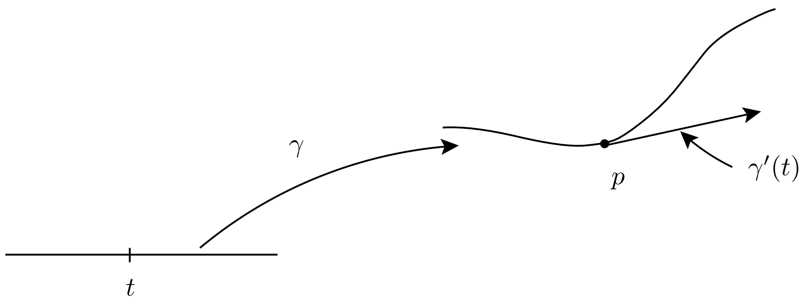
<u>Note</u>: Here U may depend on f, therefore we write $C^\infty(p)$ instead of $C^\infty(U)$.

Now the path γ defines a mapping $\dot\gamma_t\colon C^\infty(p) \to \mathbb{R}$,
$$\dot\gamma_t f = (f\circ\gamma)'(t).$$

<u>Note</u>: The real-valued function $f\circ\gamma$ is defined on some neighborhood of $t \in I$ and $(f\circ\gamma)'(t)$ is its usual derivative at t.

Interpretation: We may interprete $\dot\gamma_t f$ as "a derivative of f in the direction of γ at the point p".

Example 2.9. $M = \mathbb{R}^n$
If $\gamma = (\gamma_1, \dots, \gamma_n)\colon I \to \mathbb{R}^n$ is a smooth path and $\gamma'(t) = \big(\gamma_1'(t), \dots, \gamma_n'(t)\big) \in \mathbb{R}^n$ is the derivative of γ at t, then
$$\dot\gamma_t f = (f\circ\gamma)'(t) = f'(p)\gamma'(t) = \gamma'(t)\cdot\nabla f(p).$$



In general: The mapping $\dot\gamma_t$ satisfies:
Suppose $f, g \in C^\infty(p)$ and $a, b \in \mathbb{R}$. Then

a) $\dot\gamma_t(af + bg) = a\dot\gamma_t f + b\dot\gamma_t g$,

b) $\dot\gamma_t(fg) = g(p)\dot\gamma_t f + f(p)\dot\gamma_t g$.

We say that $\dot\gamma_t$ is a *derivation*.

Motivated by the discussion above we define:

Definition 2.10. A *tangent vector* of M at $p \in M$ is a mapping $v\colon C^\infty(p) \to \mathbb{R}$ that satisfies:

(1) $v(af + bg) = av(f) + bv(g), \quad f, g \in C^\infty(p),\ a, b \in \mathbb{R}$;

(2) $v(fg) = g(p)v(f) + f(p)v(g)$ (cf. the "Leibniz rule").

The *tangent space* at p is the ($\mathbb{R}-$)linear vector space of tangent vector at p, denoted by T_pM or M_p.

Remarks 2.11.

1. If $v, w \in T_pM$ and $c, d \in \mathbb{R}$, then $cv + dw$ is (of course) the mapping $(av + bw)\colon C^\infty(p) \to \mathbb{R}$,
$$(cv + dw)(f) = cv(f) + dw(f).$$
It is easy to see that $cv + dw$ is a tangent vector at p.

2. We abbreviate $vf = v(f)$.

3. Claim: If $v \in T_pM$ and $c \in C^\infty(p)$ is a constant function, then $vc = 0$. (Exerc.)

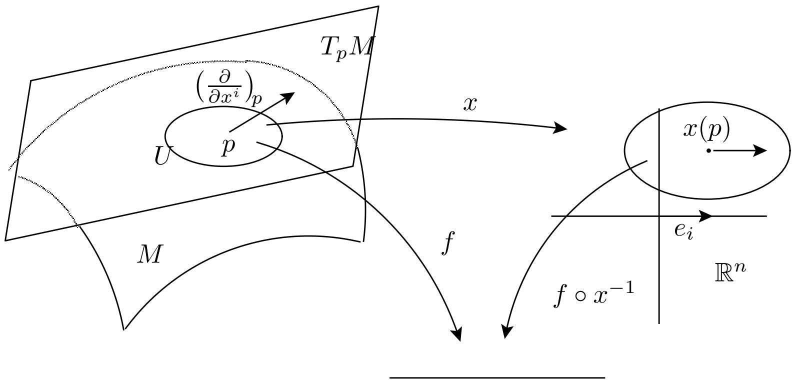
4. Let U be a neighborhood of p interpreted as a differentiable manifold itself. Since we use functions in $C^\infty(p)$ in the definition of T_pM, the spaces T_pM and T_pU can be identified in a natural way.

Let $(U, x),\ x = (x^1, x^2, \dots, x^n)$, be a chart at p. We define a tangent vector (so-called *coordinate vector*) $\left(\frac{\partial}{\partial x^i}\right)_p$ at p by setting

$$\left(\frac{\partial}{\partial x^i}\right)_p f = D_i(f \circ x^{-1})(x(p)), \quad f \in C^\infty(p).$$

Here D_i is the partial derivative with respect to i^{th} variable. We also denote

$$(\partial_i)_p = D_{x_i}(p) = \left(\frac{\partial}{\partial x^i}\right)_p.$$



Remarks 2.12. 1. It is easy to see that $(\partial_i)_p$ is a tangent vector at p.

2. If $(U, x),\ x = (x^1, \dots, x^n)$, is a chart at p, then $(\partial_i)_p x^j = \delta_{ij}$.

Next theorem shows (among others) that T_pM is n-dimensional.

Lemma 2.13. *If $f \in C^k(B),\ k \geq 1$, is a real-valued function in a ball $B = B^n(0, r) \subset \mathbb{R}^n$, then there exist functions $g_i \in C^{k-1}(B), i = 1, \dots, n$, such that $g_i(0) = D_i f(0)$ and*

$$f(y) - f(0) = \sum_{i=1}^{n} y_i g_i(y)$$

for all $y = (y_1, \dots, y_n) \in B$.

Proof. For $y \in B$ we have

$$\begin{aligned} f(y) - f(0) &= f(y) - f(y_1, \dots, y_{n-1}, 0) \\ &\quad + f(y_1, \dots, y_{n-1}, 0) - f(y_1, \dots, y_{n-2}, 0, 0) \\ &\quad + f(y_1, \dots, y_{n-2}, 0, 0) - f(y_1, \dots, y_{n-3}, 0, 0 \\ &\quad\ \vdots \\ &\quad + f(y_1, 0, \dots, 0) - f(0) \\ &= \sum_{i=1}^n \Big/_0^1 f(y_1, \dots, y_{i-1}, ty_i, 0, \dots, 0) \\ &= \sum_{i=1}^n \int_0^1 \frac{d}{dt}(f(y_1, \dots, y_{i-1}, ty_i, 0, \dots, 0))dt \\ &= \sum_{i=1}^n \int_0^1 D_i f(y_1, \dots, y_{i-1}, ty_i, 0, \dots, 0) y_i dt \\ &= \sum_{i=1}^n y_i \int_0^1 D_i f(y_1, \dots, y_{i-1}, ty_i, 0, \dots, 0) dt. \end{aligned}$$

Define

$$g_i(y) = \int_0^1 D_i f(y_1, \dots, y_{i-1}, ty_i, 0, \dots, 0) dt.$$

Then $g_i \in C^{k-1}(B)$ (since $f \in C^k(B)$) and $g_i(0) = D_i f(0)$. □

Theorem 2.14. *If* (U, x), $x = (x^1, \dots, x^n)$, *is a chart at* p *and* $v \in T_pM$, *then*

$$v = \sum_{i=1}^n vx^i (\partial_i)_p.$$

Furthermore, the vectors $(\partial_i)_p$, $i = 1, \dots, n$, *form a basis of* T_pM *and hence* $\dim T_pM = n$.

Proof. For $u \in U$ we write $x(u) = y = (y^1, \dots, y^n) \in \mathbb{R}^n$, so $x^i(u) = y^i$. We may assume that $x(p) = 0 \in \mathbb{R}^n$. Let $f \in C^\infty(p)$. Since $f \circ x^{-1}$ is C^∞, there exist (by Lemma 2.13) a ball $B = B^n(0, r) \subset xU$ and functions $g_i \in C^\infty(B)$ such that

$$(f \circ x^{-1})(y) = (f \circ x^{-1})(0) + \sum_{i=1}^n y_i g_i(y) \quad \forall y \in B$$

and $g_i(0) = D_i(f \circ x^{-1})(0) = (\partial_i)_p f$. Thus

$$f(u) = f(p) + \sum_{i=1}^n x^i(u) h_i(u),$$

where $h_i = g_i \circ x$ and

$$h_i(p) = g_i(0) = (\partial_i)_p f.$$

Hence

$$\begin{aligned} vf &= \underbrace{v(f(p))}_{=0} + \sum_{i=1}^n \underbrace{x^i(p)}_{=0} vh_i + \sum_{i=1}^n (vx^i) h_i(p) \\ &= \sum_{i=1}^n vx^i (\partial_i)_p f. \end{aligned}$$

This holds for every $f \in C^\infty(p)$, and therefore

$$v = \sum_{i=1}^{n} vx^i(\partial_i)_p.$$

Hence the vectors $(\partial_i)_p,\ i = 1, \ldots, n$, span T_pM. To prove the linear independence of these vectors, suppose that

$$w = \sum_{i=1}^{n} b_i(\partial_i)_p = 0.$$

Then

$$0 = wx^j = \sum_{i=1}^{n} b_i \underbrace{(\partial_i)_p x^j}_{=\delta_{ij}} = b_j$$

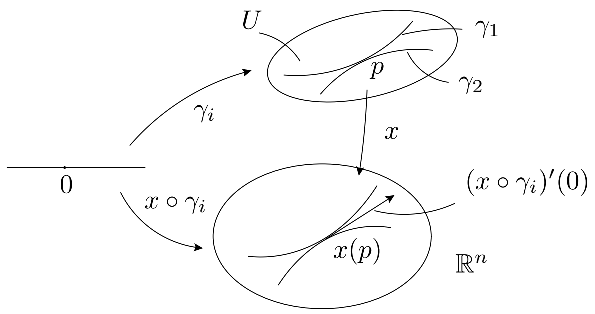
for all $j = 1, \ldots, n,$, and so vectors $(\partial_i)_p,\ i = 1, \ldots, n$, are linearly independent. □

Remark 2.15. Our definition for tangent vectors is useful only for C^∞-manifolds. Reason: If M is a C^k-manifold, then the functions h_i in the proof of Theorem 2.14 are not necessarily C^k-smooth (only C^{k-1}-smoothness is granted).

Another definition that works also for C^k-manifolds, $k \geq 1$, is the following: Let M be a C^k-manifold and $p \in M$. Let $\gamma_i \colon I_i \to M$ be C^1-paths, $0 \in I_i \subset \mathbb{R}$ open intervals, and $\gamma_i(0) = p,\ i = 1, 2$. Define an equivalence relation $\gamma_1 \sim \gamma_2 \iff$ for every chart (U, x) at p we have

$$(x \circ \gamma_1)'(0) = (x \circ \gamma_2)'(0)$$

Def.: Equivalence classes = tangent vectors at p. In the case of a C^∞-manifold this definition coincides with the earlier one ($[\gamma] = \dot\gamma_0$).



2.16 Tangent map

Definition 2.17. Let M^m and N^n be differentiable manifolds and let $f \colon M \to N$ be a C^∞ map. The *tangent map* of f at p is a linear map $f_* \colon T_pM \to T_{f(p)}N$ defined by

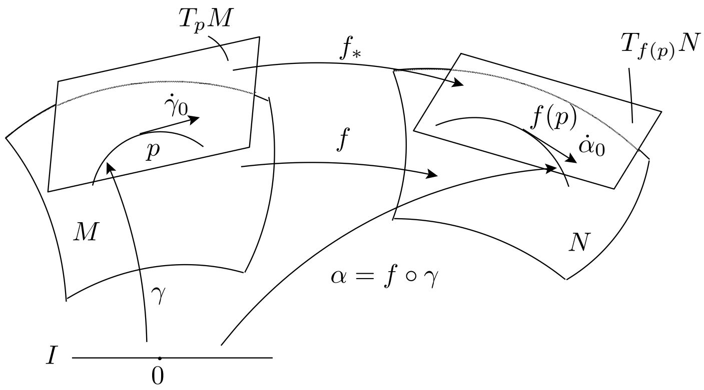
$$(f_*v)g = v(g \circ f), \quad \forall g \in C^\infty(f(p)),\ v \in T_pM.$$

We also write f_{*p} or T_pf.

Remarks 2.18.
1. It is easily seen that f_*v is a tangent vector at $f(p)$ for all $v \in T_pM$ and that f_* is linear.
2. If $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, then $f_{*p} = f'(p)$ (see the canonical identification $T_p\mathbb{R}^n = \mathbb{R}^n$ below).

3. "Chain rule": Let M, N, and L be differentiable manifolds and let $f\colon M\to N$ and $g\colon N\to L$ be C^∞-maps. Then
$$(g\circ f)_{*p} = g_{*f(p)}\circ f_{*p}$$
for all $p\in M$. (Exerc.)

4. An interpretation of a tangent map using paths:
Let $v\in T_pM$ and let $\gamma\colon I\to M$ be a C^∞-path such that $\gamma(0)=p$ and $\dot\gamma_0=v$. Let $f\colon M\to N$ be a C^∞-map and $\alpha=f\circ\gamma\colon I\to N$. Then $f_*v=\dot\alpha_0$. (Exerc.)



Let $x=(x^1,\dots,x^m)$ be a chart at $p\in M^m$ and $y=(y^1,\dots,y^n)$ a chart at $f(p)\in N^n$. What is the matrix of $f_*\colon T_pM\to T_{f(p)}N$ with respect to bases $\big(\frac{\partial}{\partial x^i}\big)_p, i=1,\dots,m$, and $\big(\frac{\partial}{\partial y^j}\big)_{f(p)}, j=1,\dots,n$,? By Theorem 2.14,
$$f_*\Big(\frac{\partial}{\partial x^j}\Big)_p = \sum_{i=1}^n f_*\Big(\frac{\partial}{\partial x^j}\Big)_p y^i \Big(\frac{\partial}{\partial y^i}\Big)_{f(p)},\quad 1\le j\le m.$$
Thus we obtain an $n\times m$ matrix (a_{ij}),
$$a_{ij} = f_*\Big(\frac{\partial}{\partial x^j}\Big)_p y^i = \frac{\partial}{\partial x^j}(y^i\circ f).$$
This is called the Jacobian matrix of f at p (with respect to given bases). As a matrix it is the same as the matrix of the linear map $g'(x(p))$, $g=y\circ f\circ x^{-1}$, with respect to standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$.

Recall that $f\colon M^m\to N^n$ is a diffeomorphism if f and its inverse f^{-1} are C^∞. A mapping $f\colon M\to N$ is a *local diffeomorphism* at $p\in M$ if there are neighborhoods U of p and V of $f(p)$ such that $f\colon U\to V$ is a diffeomorphism.
Note: Then necessarily $m=n$. (Exerc.)

Theorem 2.19. *Let $f\colon M\to N$ be C^∞ and $p\in M$. Then f is a local diffeomorphism at p if and only if $f_*\colon T_pM\to T_{f(p)}N$ is an isomorphism.*

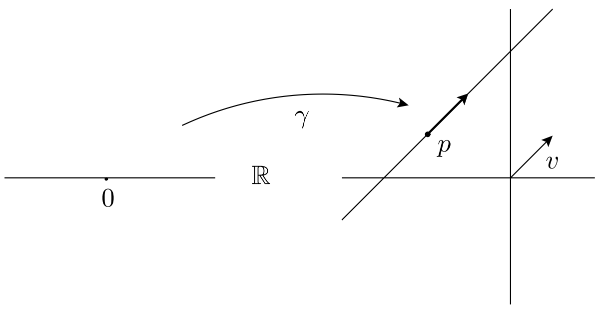
Proof. Apply the inverse function theorem (of $\mathbb{R}^n$). Details are omitted, □

Tangent space of an n-dimensional vector space. Let V be an n-dimensional (real) vector space. Recall that any (linear) isomorphism $x\colon V\to\mathbb{R}^n$ induces the same C^∞-structure on V. We may identify V and T_pV in a natural way for any $p \in V$: If $p \in V$, then there exists a canonical isomorphism $i\colon V \to T_pV$. Indeed, let $v \in V$ and $\gamma\colon \mathbb{R} \to V$ the path

$$\gamma(t) = p + tv.$$

We set

$$i(v) = \dot{\gamma}_0.$$



<u>Example</u>: $V = \mathbb{R}^n$, $\quad T_p\mathbb{R}^n \cong \mathbb{R}^n$ canonically.

If $f\colon M \to \mathbb{R}$ is C^∞ and $p \in M$, we define the *differential* of f, $df\colon T_pM \to \mathbb{R}$, by setting

$$df\, v = vf, \quad v \in T_pM.$$

(Also denoted by df_p.)
By the isomorphism $i\colon \mathbb{R} \to T_{f(p)}\mathbb{R}$ as above, we obtain $df = i^{-1} \circ f_*$. Usually we identify $df = f_*$.
<u>Note</u>: Since $df\colon T_pM \to \mathbb{R}$ is linear, $df \in T_pM^*$ (= the dual of T_pM).

$$\begin{tikzcd}
T_pM \arrow[rr, "f_*"] \arrow[dr, bend right, "df"'] & & T_{f(p)}\mathbb{R} \\
& \mathbb{R} \arrow[ur, bend right, "i"'] &
\end{tikzcd}$$

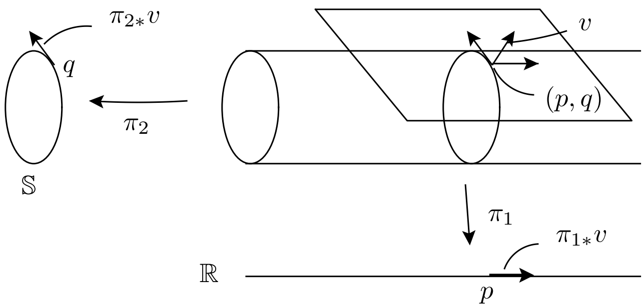
Tangent space of a product manifold. Let M and N be differentiable manifolds and let

$$\pi_1\colon M \times N \to M,$$
$$\pi_2\colon M \times N \to N$$

be the projections. Using these projections we may identify $T_{(p,q)}(M \times N)$ and $T_pM \oplus T_qN$ in a natural way: Define a canonical isomorphism

$$\tau\colon T_{(p,q)}(M \times N) \to T_pM \oplus T_qN,$$
$$\tau v = \underbrace{\pi_{1*}v}_{\in T_pM} + \underbrace{\pi_{2*}v}_{\in T_pN}, \quad v \in T_{(p,q)}(M \times N).$$

<u>Example</u>: $M = \mathbb{R}$, $N = \mathbb{S}^1$



Let $f\colon M \times N \to L$ be a C^∞-mapping, where L is a differentiable manifold. For every $(p,q) \in M \times N$ we define mappings

$$f_p\colon N \to L, \quad f^q\colon M \to L,$$
$$f_p(q) = f^q(p) = f(p,q).$$

Thus, for $v \in T_pM$ and $w \in T_qN$, we have

$$f_*(v+w) = (f^q)_*v + (f_p)_*w. \quad \text{(Exerc.)}$$

2.20 Tangent bundle

Let M be a differentiable manifold. We define the *tangent bundle* TM over M as a disjoint union of all tangent spaces of M, i.e.

$$TM = \bigsqcup_{p\in M} T_pM.$$

Points in TM are thus pairs (p,v), where $p \in M$ and $v \in T_pM$. We usually abbreviate $v = (p,v)$, because the condition $v \in T_pM$ determines $p \in M$ uniquely.

Let $\pi\colon TM \to M$ be the projection

$$\pi(v) = p, \quad \text{if } v \in T_pM.$$

The tangent bundle TM has a canonical structure of a differentiable manifold.

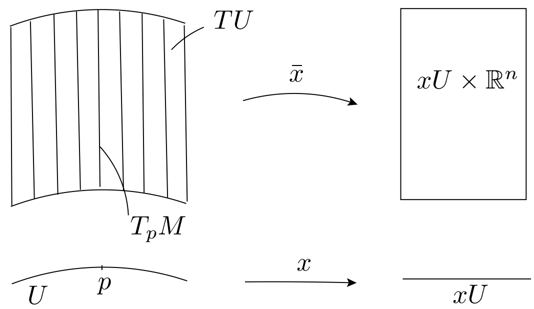
Theorem 2.21. *Let M be a differentiable n-manifold. The tangent bundle TM over M can be equipped with a natural topology and a C^∞-structure of a smooth $2n$-manifold such that the projection $\pi\colon TM \to M$ is smooth.*

Proof. (Idea): Let (U,x), $x = (x^1,\ldots,x^n)$, be a chart on M. Define a one-to-one mapping

$$\bar{x}\colon TU \to xU \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$$

as follows. [Here $TU = \bigsqcup_{p\in U} T_pU = \bigsqcup_{p\in U} T_pM$.] If $p \in U$ and $v \in T_p$, we set

$$\bar{x}(v) = \big(\underbrace{x^1(p),\ldots,x^n(p)}_{\in\mathbb{R}^n}, \underbrace{vx^1,\ldots,vx^n}_{\in\mathbb{R}^n}\big)$$



First we transport the topology of $\mathbb{R}^n \times \mathbb{R}^n$ into TM by using maps $\bar{x}$ and then we verify that pairs $(TU, \bar{x})$ form an atlas of TM. We obtain a C^∞-structure for TM. [Details are left as an exercise.] □

In the sequel the tangent bundle over M means TM equipped with this C^∞-structure. It is an example of a vector bundle over M.

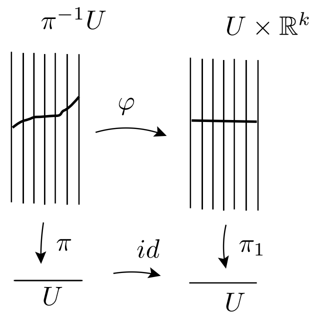
Let $\pi\colon TM \to M$ be the projection ($\pi(v) = p$ for $v \in T_pM$). Then $\pi^{-1}(p) = T_pM$ is a *fibre* over p. If $A \subset M$, then a map $s\colon A \to TM$, with $\pi \circ s = id$, is a *section* of TM in A (or a *vector field*).

Smooth vector bundles. Let M be a differentiable manifold. A smooth *vector bundle of rank* k over M is a pair (E, π), where E is a smooth manifold and $\pi\colon E \to M$ is a smooth surjective mapping (projection) such that:

(a) for every $p \in M$, the set $E_p = \pi^{-1}(p) \subset E$ is a k-dimensional real vector space (= a fiber of E over p);

(b) for every $p \in M$ there exist a neighborhood $U \ni p$ and a diffeomorphism $\varphi\colon \pi^{-1}U \to U \times \mathbb{R}^k$ (= local trivialization of E over U) such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$



[above $\pi_1\colon U \times \mathbb{R}^k \to U$ is the projection] and that $\varphi|E_q\colon E_q \to \{q\} \times \mathbb{R}^k$ is a linear isomorphism for every $q \in U$.

The manifold E is called the *total space* and M is called the *base* of the bundle. If there exists a local trivialization of E over the whole manifold M, $\varphi\colon \pi^{-1}M \to M \times \mathbb{R}^k$, then E is a *trivial bundle.*

A *section* of E is any map $\sigma\colon M \to E$ such that $\pi \circ \sigma = id\colon M \to M$. A *smooth section* is a section that is smooth as a map $\sigma\colon M \to E$ (note that M and E are smooth manifolds). *Zero section* is a map $\zeta\colon M \to E$ such that

$$\zeta(p) = 0 \in E_p \quad \forall p \in M.$$

A *local frame* of E over an open set $U \subset M$ is a k-tuple $(\sigma_1, \dots, \sigma_k)$, where each σ_i is a smooth section of E (over U) such that $(\sigma_1(p), \sigma_2(p), \dots, \sigma_k(p))$ is a basis of E_p for all $p \in U$. If $U = M$, $(\sigma_1, \dots, \sigma_k)$ is called a *global frame.*

2.22 Submanifolds

Definition 2.23. Let M and N be differentiable manifolds and $f\colon M \to N$ a C^∞-map. We say that :

1. f is a *submersion* if $f_{*p}\colon T_pM \to T_{f(p)}N$ is surjective $\forall p \in M$.

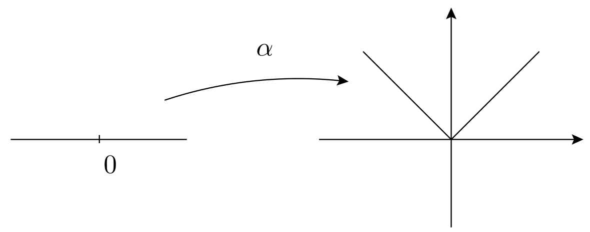
2. f is an *immersion* if $f_{*p}\colon T_pM \to T_{f(p)}N$ is injective $\forall p \in M$.
3. f is an *embedding* if f is an immersion and $f\colon M \to fM$ is a homeomorphism (note relative topology in fM).

If $M \subset N$ and the inclusion $i\colon M \hookrightarrow N,\ i(p) = p$, is an embedding, we say that M is a *submanifold* of N.

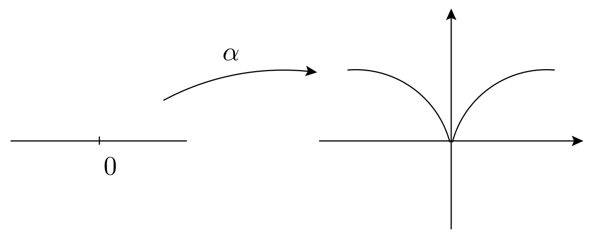
Remark 2.24. If $f\colon M^m \to N^n$ is an immersion, then $m \leq n$ and $n - m$ is the *codimension* of f.

Examples 2.25. (a) If $M_1, \dots, M_k$ are smooth manifolds, then all projections $\pi_i\colon M_1 \times \cdots \times M_k \to M_i$ are submersions.

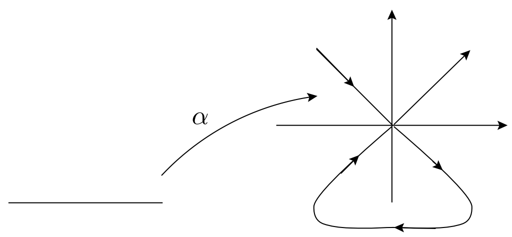
(b) $(M = \mathbb{R},\ N = \mathbb{R}^2)$ $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ \alpha(t) = (t, |t|)$ is not differentiable at $t = 0$.



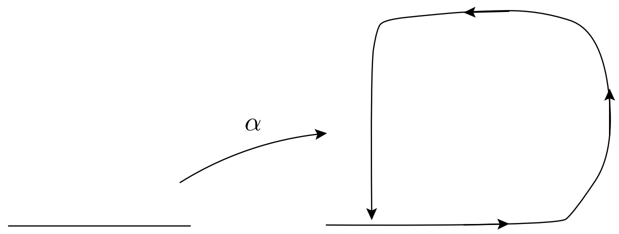
(c) $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ \alpha(t) = (t^3, t^2)$ is C^∞ but not an immersion since $\alpha'(0) = 0$.



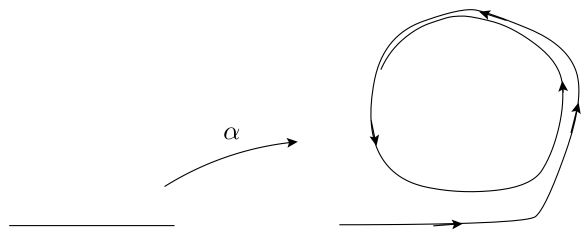
(d) $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ \alpha(t) = (t^3 - 4t, t^2 - 4)$ is C^∞ and an immersion but not an embedding ($\alpha(\pm 2) = (0, 0)$).



(e) The map α (in the picture below) has an inverse but it is not an embedding since the inverse in not continuous (in the relative topology of the image).



(f) The following α is an embedding.



Remark 2.26. The notion of a submanifold has different meanings in the literature. For instance, Bishop-Crittenden [BC] allows the case (e) in the definition of a submanifold.

Theorem 2.27. *Let $f\colon M^m \to N^n$ be an immersion. Then each point $p \in M^m$ has a neighborhood U such that $f|U\colon U \to N^n$ is an embedding.*

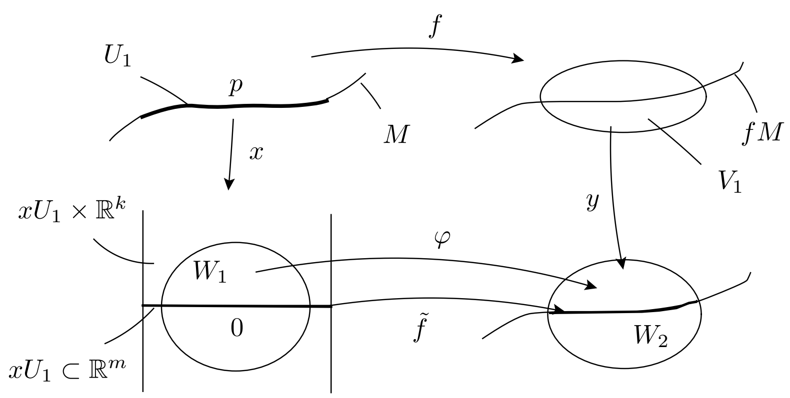
Proof. Fix $p \in M$. We have to find a neighborhood $U \ni p$ such that $f|U\colon U \to fU$ is a homeomorphism when fU is equipped with the relative topology. Let (U_1, x) and (V_1, y) be charts at points p and $f(p)$, respectively, such that $fU_1 \subset V_1$, $x(p) = 0$ $(\in \mathbb{R}^m)$, and $y(f(p)) = 0$ $(\in \mathbb{R}^n)$. Write $\tilde f = y \circ f \circ x^{-1}$, $\tilde f = (\tilde f_1, \dots, \tilde f_n)$. Since f is an immersion, $\tilde f'(0)\colon \mathbb{R}^m \to \mathbb{R}^n$ is injective. We may assume that $\tilde f'(0)\mathbb{R}^m = \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^k$, $k = n - m$ (otherwise, apply a rotation in $\mathbb{R}^n$). Then $\det \tilde f'(0) \neq 0$, when $\tilde f'(0)$ is interpreted as a linear map $\mathbb{R}^m \to \mathbb{R}^m$. Define a mapping

$$\begin{aligned}
&\varphi\colon xU_1 \times \mathbb{R}^k \to \mathbb{R}^n,\\
&\varphi(\tilde x, t) = \big(\tilde f_1(\tilde x), \tilde f_2(\tilde x), \dots, \tilde f_m(\tilde x), \tilde f_{m+1}(\tilde x) + t_1, \dots, \tilde f_{m+k}(\tilde x) + t_k)\big),\\
&\tilde x \in xU_1, \quad t = (t_1, \dots, t_k) \in \mathbb{R}^k.
\end{aligned}$$

The matrix of $\varphi'(0,0)\colon \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$ is

$$\begin{pmatrix} \frac{\partial \tilde f_i(0)}{\partial \tilde x_j} & 0 \\ * & I_k \end{pmatrix},$$

and therefore $\det \varphi'(0,0) = \det \tilde f'(0) \neq 0$. By the inverse mapping theorem, there are neighborhoods $0 \in W_1 \subset xU_1 \times \mathbb{R}^k$ and $0 \in W_2 \subset \mathbb{R}^n$ such that $\varphi|W_1\colon W_1 \to W_2$ is a diffeomorphism. Write $\tilde U = W_1 \cap xU_1$ and $U = x^{-1}\tilde U$ $(\subset U_1)$. Since $\varphi|xU_1 \times \{0\} = \tilde f$, we have $\varphi|\tilde U = \tilde f$. In particular, $f|U\colon U \to fU$ is a homeomorphism, when fU is equipped with the relative topology. □



Example 2.28. Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^∞-function such that $\nabla f(p) = \big(D_1 f(p), \dots, D_{n+1} f(p)\big) \neq 0$ for every $p \in M = \{x \in \mathbb{R}^{n+1} \colon f(x) = 0\} \neq \emptyset$. Then M is an n dimensional submanifold of $\mathbb{R}^{n+1}$.

Proof of the claim above. (Idea): Let $p \in M$ be arbitrary. Applying a transformation and a rotation if necessary we may assume that $p = 0$ and

$$\nabla f(0) = \big(0, \dots, 0, \frac{\partial f}{\partial x_{n+1}}(0)\big).$$

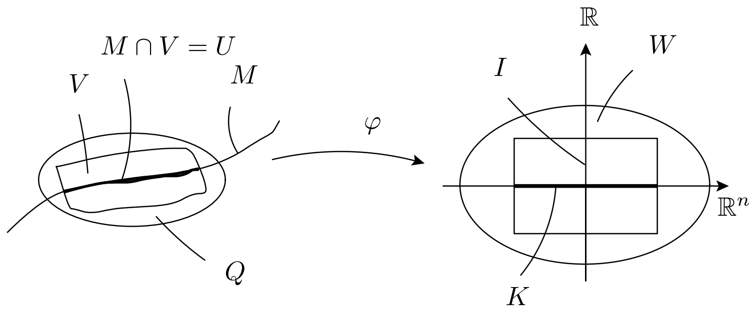
Then $\frac{\partial f}{\partial x_{n+1}}(0) \neq 0$. Define a mapping $\varphi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

$$\varphi(x) = (x_1, \dots, x_n, f(x)), \quad x = (x_1, \dots, x_n, x_{n+1}).$$

Then

$$\det \varphi'(0) = \begin{vmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{\partial f}{\partial x_{n+1}}(0) \end{vmatrix} = \frac{\partial f}{\partial x_{n+1}}(0) \neq 0.$$

By the inverse mapping theorem, there exist neighborhoods $Q \ni p$ and $W \ni \varphi(0) = (0,0) \in \mathbb{R}^n \times \mathbb{R}$ such that $\varphi\colon Q \to W$ is a diffeomorphism.



Choose an open set $K \subset \mathbb{R}^n$, $0 \in K$, and an open interval $I \subset \mathbb{R}$, $0 \in I$, such that $K \times I \subset W$. Let $V = \varphi^{-1}(K \times I) \cap Q$ and $U = V \cap M$. Then $\varphi\colon V \to K \times I$ is a diffeomorphism. Let $y = \varphi|U$. Repeat the above for all $p \in M$ and conclude that pairs (U, y) form a C^∞-atlas of M. Since the inclusion $i\colon M \hookrightarrow \mathbb{R}^{n+1}$ satisfies

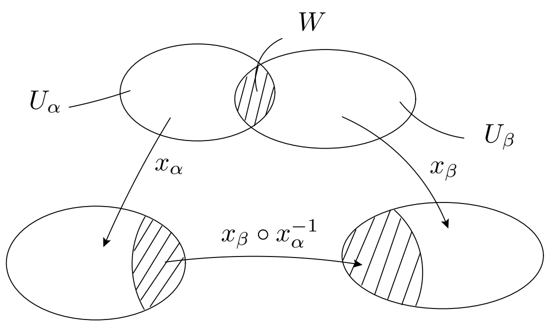
$$i|U = y^{-1} \circ \varphi|U,$$

i is an embedding. □

2.29 Orientation

Definition 2.30. A smooth manifold M is *orientable* if it admits a smooth atlas $\{(U_\alpha, x_\alpha)\}$ such that for every α and β, with $U_\alpha \cap U_\beta = W \neq \emptyset$, the Jacobian determinant of $x_\beta \circ x_\alpha^{-1}$ is positive at each point $q \in x_\alpha W$, i.e.

$$\det\big(x_\beta \circ x_\alpha^{-1}\big)'(q) > 0, \quad \forall q \in x_\alpha W. \tag{2.31}$$



In the opposite case M is *nonorientable.* If M is orientable, then an atlas satisfying (2.31) is called an *orientation* of M. Furthermore, M (equipped with such atlas) is said to be *oriented.* We say that two atlases satisfying (2.31) determine *the same orientation* if their union satisfies (2.31), too.

Remarks 2.32.
1. Warning: The notion of a smooth structure has different meanings in the literature (e.g. do Carmo [Ca2]). What goes wrong if we define orientability by saying: "M is orientable if it admits a C^∞-structure such that (2.31) holds?" (Exerc.)
2. An is orientable and connected smooth manifold has exactly two distinct orientations. (Exerc.)
3. If M and N are smooth manifolds and $f\colon M \to N$ is a diffeomorphism, then
$$M \text{ is orientable} \iff N \text{ is orientable.}$$
4. Let M and N be connected oriented smooth manifolds and $f\colon M \to N$ a diffeomorphism. Then f induces an orientation on N. If the induced orientation of N is the same as the initial one, we say that f is *sense-preserving* (or f preserves the orientation). Otherwise, f is called *sense-reversing* (or f reverses the orientation).

Examples 2.33.
1. Suppose that there exists an atlas $\{(U,x),(V,y)\}$ of M such that $U \cap V$ is connected. Then M is orientable.
Proof. The mapping $y \circ x^{-1}\colon x(U \cap V) \to y(U \cap V)$ is diffeomorphic, so
$$\det\bigl(y \circ x^{-1}\bigr)'(q) \neq 0 \quad \forall q \in x(U \cap V).$$
Since $q \mapsto \det\bigl(y \circ x^{-1}\bigr)'(q)$ is continuous and $x(U \cap V)$ is connected, the determinant can not change its sign. If the sign is positive, we are done. If the sign is negative, replace the chart $(V,y),\ y = (y_1,\dots,y_n)$, by a chart $(V,\tilde{y}),\ \tilde{y} = (-y_1,y_2,\dots,y_n)$. Then the atlas $\{(U,x),(V,\tilde{y})\}$ satisfies (2.31). $\square$
2. In particular, the sphere S^n is orientable.

3 Vector fields and their flows

3.1 Vector fields

Let M be a differentiable manifold and $A \subset M$. Recall that a mapping $X\colon A \to TM$ such that $X(p) \in T_pM$ for all $p \in M$ is called a *vector field* in A. We usually write $X_p = X(p)$. If $A \subset M$ is open and $X\colon A \to TM$ is a C^∞-vector field, we write $X \in \mathcal{T}(A)$. Clearly $\mathcal{T}(A)$ is a real vector space, where addition and multiplication by a scalar are defined pointwise: If $X, Y \in \mathcal{T}(A)$ and

$a, b \in \mathbb{R}$, then $aX + bY,\ p \mapsto aX_p + bY_p$, is a smooth vector field. Furthermore, a vector field $V \in \mathcal{T}(A)$ can be multiplied by a smooth (real-valued) function $f \in C^\infty(A)$ producing a smooth vector field $fV,\ p \mapsto f(p)V_p$.

Let M be a differentiable n-manifold and $A \subset M$ open. We say that vector fields $V^1, \ldots, V^n$ in A form a *local frame* (or a *frame* in A) if the vectors $V_p^1, \ldots, V_p^n$ form a basis of T_pM for every $p \in A$. In the case $A = M$ we say that vector fields $V^1, \ldots, V^n$ form a *global frame*. Furthermore, M is called *parallelizable* if it admits a smooth global frame. This is equivalent to TM being a trivial bundle.[1]

Definition 3.2. (*Einstein summation convention*) If in a term the same index appears twice, both as upper and a lower index, that term is assumed to be summed over all possible values of that index (usually from 1 to the dimension).

Example:

$$v^i \partial_i = \sum_{i=1}^n v^i \partial_i,$$

$$g_{ij} dx^i dx^j = \sum_{i,j=1}^n g_{ij} dx^i dx^j.$$

Let $(U, x),\ x = (x^1, \ldots, x^n)$, be a chart and $(\partial_i)_p = (\frac{\partial}{\partial x^i})_p,\ i = 1, \ldots, n$, the corresponding coordinate vectors at $p \in U$. Then the mappings

$$\partial_i \colon U \to TM,\ p \mapsto (\partial_i)_p = \left(\frac{\partial}{\partial x^i}\right)_p,$$

are vector fields in U, so-called *coordinate vector fields*. Since the vector fields ∂_i form a frame, so-called *coordinate frame*, in U, we can write any vector field V in U as

$$V_p = v^i(p)(\partial_i)_p, \quad p \in U,$$

where $v^i \colon U \to \mathbb{R}$. Functions v^i are called the *component functions* of V with respect to (U, x).

Lemma 3.3. *Let V be a vector field on M.Then the following are equivalent:*

(a) $V \in \mathcal{T}(M)$;

(b) the component functions of V with respect to any chart are smooth;

(c) If $U \subset M$ is open and $f \colon U \to \mathbb{R}$ is smooth, then the function $Vf \colon U \to \mathbb{R},\ (Vf)(p) = V_p f$, is smooth.

Proof. Exercise. □

Remark 3.4. In particular, coordinate vector fields are smooth by (b).

Suppose that $A \subset M$ is open and $V, W \in \mathcal{T}(A)$. If $f \in C^\infty(p)$, where $p \in A$, then $Vf \in C^\infty(p)$ and thus $W_p(Vf) \in \mathbb{R}$ (= "the derivative of Vf in the direction of W_p“). The function $A \to \mathbb{R},\ p \mapsto W_p(Vf)$, is denoted by WVf. Thus $(WVf)(p) = W_p(Vf)$. We also denote $(WV)_p f = W_p(Vf)$.

[1]Every Lie group is parallelizable; $\mathbb{S}^1, \mathbb{S}^3$, and $\mathbb{S}^7$ are the only parallelizable spheres; $\mathbb{R}P^1,\ \mathbb{R}P^3$, and $\mathbb{R}P^7$ are the only parallelizable projective spaces; a product $\mathbb{S}^n \times \mathbb{S}^m$ is parallelizable if at least one of the numbers $n > 0$ or $m > 0$ is odd.

Remark 3.5. $(WV)_p$ is <u>not</u> a derivation, so $(WV)_p \notin T_p(M)$, in general. <u>Reason</u>: Leibniz's rule (2) does not hold (choose $f = g$).

Definition 3.6. Suppose that $A \subset M$ is open and $V, W \in \mathcal{T}(A)$. We define the *Lie bracket* of V and W by setting

$$[V, W]_p f = V_p(Wf) - W_p(Vf), \quad p \in A, \ f \in C^\infty(p).$$

Theorem 3.7. *Let $A \subset M$ be open and $V, W \in \mathcal{T}(A)$. Then*

(a) $[V, W]_p \in T_p M$;

(b) $[V, W] \in \mathcal{T}(A)$ *and it satisfies*

$$[V, W]f = V(Wf) - W(Vf), \ f \in C^\infty(A); \tag{3.8}$$

(c) if v^i and w^i are the component functions of vector fields V and W, respectively, with respect to a chart $x = (x^1, \ldots, x^n)$, then

$$[V, W] = \left(v^i \partial_i w^j - w^i \partial_i v^j\right) \partial_j. \tag{3.9}$$

<u>Note</u>: The formula (3.9) can be written as

$$[V, W] = \left(V w^j - W v^j\right) \partial_j.$$

Proof. (a) We have to prove that $[V, W]_p$ satisfies conditions (1) and (2) in the definition of a tangent vector.
Condition (1) is clear.
Condition (2): Let $f, g \in C^\infty(p)$. Then

$$\begin{aligned}
[V, W]_p(fg) &= V_p(W(fg)) - W_p(V(fg)) \\
&= V_p(fWg + gWf) - W_p(fVg + gVf) \\
&= f(p)V_p(Wg) + (W_p g)(V_p f) + g(p)V_p(Wf) + (W_p f)(V_p g) \\
&\quad - f(p)W_p(Vg) - (V_p g)(W_p f) - g(p)W_p(Vf) - (V_p f)(W_p g) \\
&= f(p)(V_p(Wg) - W_p(Vg)) + g(p)(V_p(Wf) - W_p(Vf)) \\
&= f(p)[V, W]_p g + g(p)[V, W]_p f.
\end{aligned}$$

(b) Formula (3.8) follows immediately from the definition of a Lie bracket. Let $f \in C^\infty(A)$. Now functions Wf, Vf, $V(Wf)$, and $W(Vf)$ are smooth by Lemma 3.3 (c) since $V, W \in \mathcal{T}(A)$. Hence also $[V, W]f = V(Wf) - W(Vf)$ is a smooth function and therefore $[V, W] \in \mathcal{T}(A)$.

(c) If $V = v^i \partial_i$, $W = w^j \partial_j$, and f is smooth, we obtain by a direct computation that

$$\begin{aligned}
[V, W]f &= V(Wf) - W(Vf) = v^i \partial_i (w^j \partial_j f) - w^j \partial_j (v^i \partial_i f) \\
&= v^i (\partial_i w^j)(\partial_j f) + v^i w^j \partial_i (\partial_j f) - w^j (\partial_j v^i)(\partial_i f) - w^j v^i \partial_j (\partial_i f) \\
&= v^i (\partial_i w^j)(\partial_j f) - w^j (\partial_j v^i)(\partial_i f).
\end{aligned}$$

In the last step we used the fact that $\partial_j(\partial_i f) = \partial_i(\partial_j f)$ for a smooth function f. Changing the roles of indices i and j in the last sum we obtain (3.9).

□

Lemma 3.10. *The Lie bracket satisfies:*

(a) Bilinearity:

$$[a_1X_1 + a_2X_2, Y] = a_1[X_1, Y] + a_2[X_2, Y] \quad ja$$
$$[X, a_1Y_1 + a_2Y_2] = a_1[X, Y_1] + a_2[X, Y_2]$$

for $a_1, a_2 \in \mathbb{R}$*;*

(b) Antisymmetry: $[X, Y] = -[Y, X]$.

(c) Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

(d)

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

Proof. (a) Follows directly from the definition.

(b) Follows directly from the definition.

(c)

$$\begin{aligned}
[X, [Y, Z]]f &= X\big([Y, Z]f\big) - [Y, Z](Xf) \\
&= X\big(Y(Zf) - Z(Yf)\big) - Y\big(Z(Xf)\big) + Z\big(Y(Xf)\big) \\
&= X\big(Y(Zf)\big) - X\big(Z(Yf)\big) - Y\big(Z(Xf)\big) + Z\big(Y(Xf)\big)
\end{aligned}$$

$$[Y, [Z, X]]f = Y\big(Z(Xf)\big) - Y\big(X(Zf)\big) - Z\big(X(Yf)\big) + X\big(Z(Yf)\big)$$

$$[Z, [X, Y]]f = Z\big(X(Yf)\big) - Z\big(Y(Xf)\big) - X\big(Y(Zf)\big) + Y\big(X(Zf)\big).$$

Adding up both sides yields

$$[X, [Y, Z]]f + [Y, [Z, X]]f + [Z, [X, Y]]f = 0.$$

(d)

$$\begin{aligned}
[fX, gY]h &= fX(gYh) - gY(fXh) \\
&= fgX(Yh) + f(Xg)(Yh) - gfY(Xh) - g(Yf)(Xh) \\
&= fg[X, Y]h + f(Xg)Yh - g(Yf)Xh.
\end{aligned}$$

□

Lemma 3.11. *Let* (U, x)*,* $x = (x^1, \ldots, x^n)$*, be a chart and* ∂_i*,* $i = 1, \ldots, n$*, the corresponding coordinate vector fields. Then*

$$[\partial_i, \partial_j] = 0 \quad \forall i, j.$$

Proof. Let $p \in U$ and $f \in C^\infty(p)$. Then

$$\begin{aligned}
(\partial_i)_p(\partial_j f) &= (\partial_i)_p \left[\Big(D_j(f \circ x^{-1}) \Big) \circ x \right] \\
&= D_i \left[(D_j(f \circ x^{-1}) \circ x) \circ x^{-1} \right] (x(p)) = D_i D_j (f \circ x^{-1})(x(p)).
\end{aligned}$$

Since $D_i D_j g = D_j D_i g$ for a smooth function g, we obtain the claim. □

Example 3.12. Let us denote the points of $\mathbb{R}^3$ by (x, y, t) and the standard coordinate vectors of $\mathbb{R}^3$ by

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}.$$

Let $X, Y, T \in \mathcal{T}(\mathbb{R}^3)$ be the vector fields

$$\begin{aligned}X &= \frac{\partial}{\partial x},\\ Y &= \frac{\partial}{\partial y} + x\frac{\partial}{\partial t},\\ T &= \frac{\partial}{\partial t}.\end{aligned}$$

Then $[X, Y] = T$ since

$$\begin{aligned}[X, Y]f &= X(Yf) - Y(Xf)\\ &= \frac{\partial}{\partial x}(\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial t}) - (\frac{\partial}{\partial y} + x\frac{\partial}{\partial t})(\frac{\partial f}{\partial x})\\ &= \frac{\partial^2 f}{\partial x \partial y} + x\frac{\partial^2 f}{\partial x \partial t} + \frac{\partial x}{\partial x}\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial y \partial x} - x\frac{\partial^2 f}{\partial t \partial x}\\ &= \frac{\partial f}{\partial t} = Tf.\end{aligned}$$

Similarly we can compute that $[X, T] = 0,\ [Y, T] = 0$.

Let M and N be differentiable manifolds and $f\colon M \to N$ a smooth mapping. If V is a vector field on M, then $f_{*p}V_p$ is a tangent vector in $T_{f(p)}N$. This need not define a vector field on N. For instance, if f is not onto (a surjection), we can not attach such a vector to a point $q \in N \setminus fM$. On the other hand, if f is not an injection, there are points $p_1 \neq p_2$ s.t. $f(p_1) = f(p_2)$. Then it is possible that $f_{*p_1}V_{p_1} \neq f_{*p_2}V_{p_2}$, and consequently f_*V is not a vector field on N.
If there exist vector fields $V \in \mathcal{T}(M)$ and $W \in \mathcal{T}(N)$ such that $f_{*p}V_p = W_{f(p)}$ for all $p \in M$, we call vector fields V and W *f-related* and denote $W = f_*V$.

Lemma 3.13. *Suppose that $f\colon M \to N$ is a smooth mapping, $V \in \mathcal{T}(M)$ and $W \in \mathcal{T}(N)$. Then V and W are f-related if and only if for every smooth function h that is defined in some open subset of N we have*

$$V(h \circ f) = (Wh) \circ f. \tag{3.14}$$

Proof. For every $p \in M$,

$$V(h \circ f)(p) = V_p(h \circ f) = (f_{*p}V_p)h$$

and

$$((Wh) \circ f)(p) = (Wh)(f(p)) = W_{f(p)}h.$$

Thus (3.14) holds if and only if

$$f_{*p}V_p = W_{f(p)}$$

for every $p \in M$, in other words, if and only if V and W re f-related. $\square$

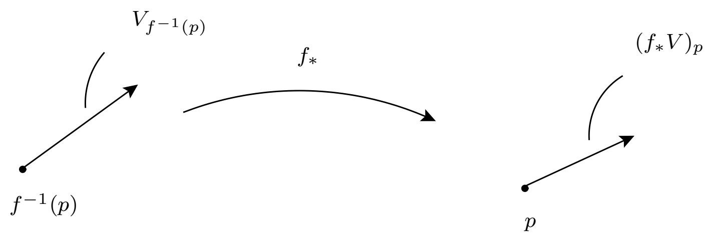
Remark 3.15. If $f\colon M \to N$ is smooth and $V \in \mathcal{T}(M)$, there need not exist a vector field $W \in \mathcal{T}(N)$ such that V and W would be f-related.

Lemma 3.16. *If $f\colon M \to N$ is a diffeomorphism and $V \in \mathcal{T}(M)$, there exists a unique vector field $W \in \mathcal{T}(N)$ such that V and W are f-related.*

Proof. If $V \in \mathcal{T}(M)$ and $f\colon M \to N$ is a diffeomorphism, define a vector field $f_*V \in \mathcal{T}(N)$ by setting

$$(f_*V)_p = f_*V_{f^{-1}(p)}, \quad p \in N. \tag{3.17}$$

Then clearly f_*V is the only smooth vector field on N that is f-related with V. □



Lemma 3.18. *Let $f\colon M \to N$ be a smooth mapping and $V^i \in \mathcal{T}(M)$, $W^i \in \mathcal{T}(N)$, $i = 1,2$, vector fields s.t. V^i and W^i are f-related. Then $[V^1, V^2]$ and $[W^1, W^2]$ are f-related. If f is a diffeomorphism,*

$$[f_*V^1, f_*V^2] = f_*[V^1, V^2].$$

Proof. Exerc.

Definition 3.19. Let $f\colon M \to M$ be a diffeomorphism and $X \in \mathcal{T}(M)$ a vector field such that $f_*X = X$, i.e. X is f-related with itself. Then we say that X is *invariant with respect to f*, or that X is *f-invariant.*

Note: The condition $f_*X = X$ means that $f_{*p}X_p = X_{f(p)}$ for all $p \in M$.

Left invariants vector fields on a Lie group. Let G be a Lie group. Then every point $g \in G$ defines diffeomorphism $L_g\colon G \to G$ and $R_g\colon G \to G$ (*left translation* and *right translation*),

$$L_g(h) = gh, \quad R_g(h) = hg.$$

A vector field X is called *left invariant* if it is invariant under every left translations, in other words, if $L_{g*}X = X$ for every $g \in G$ (more precisely, $L_{g*h}X_h = X_{gh}$ for all $g, h \in G$). A right invariant vector field is defined similarly.

Theorem 3.20. *Let G be a Lie group and T_eG its tangent space at the neutral element $e \in G$. Then every vector $X_e \in T_eG$ defines a unique left invariant vector field X. In particular, G is parallelizable.*

Proof. For every $g \in G$ there is a unique left translation that maps the neutral element e to g, namely L_g. Hence if such a vector field X exists, it is defined uniquely by the formula

$$X_g = L_{g*}X_e.$$

On the other hand, this formula defines a left invarian vector field since, for every $h \in G$,

$$L_{g*h}X_h = L_{g*h}(L_{h*}X_e) = L_{g*} \circ L_{h*}X_e = L_{gh*}X_e = X_{gh}.$$

Let us prove next that the mapping $g \mapsto X_g$ is smooth. Let $f\colon U \to \mathbb{R}$ be a smooth function defined on some open set $U \subset G$. Choose a C^∞-path $\gamma\colon]-\varepsilon, \varepsilon[\to G$ such that $\dot\gamma_0 = X_e$. Then we have, for every $g \in U$

$$\begin{aligned}(Xf)(g) &= X_g f = (L_{g*}X_e)f = \dot\gamma_0(f \circ L_g)\\ &= (f \circ L_g \circ \gamma)'(0) = \frac{d}{dt} f\big(g\gamma(t)\big)_{|t=0}.\end{aligned}$$

The mapping $\varphi,\ (g,t) \mapsto \varphi(g,t) = f\big(g\gamma(t)\big)$, is smooth in $G\times]-\varepsilon,\varepsilon[$ since it is a composition of the group operation, the function f, and the path γ. Therefore

$$\frac{d}{dt} f\big(g\gamma(t)\big)_{|t=0}$$

is a smooth function of g, so $g \mapsto (Xf)(g)$ is smooth. By Lemma 3.3 (c), $X \in \mathcal{T}(G)$. Finally, let $X_e^1, \dots, X_e^n$ be a basis of T_eG. Then the corresponding left invariant vector fields $X^1, \dots, X^n$ form a global frame. Indeed, if there exists $g \in G$ s.t. $X_g^1, \dots, X_g^n$ are not linearly independent, we can write some vector X_g^i as a linear combination of the others, i.e.

$$X_g^i = \sum_{j \neq i} a_j X_g^j, \quad \text{where } a_j \in \mathbb{R}.$$

Then, by the left invariance,

$$X_e^i = \sum_{j \neq i} a_j X_e^j$$

yielding a contradiction since $X_e^1, \dots, X_e^n$ is a basis of T_eG. Thus G has a global frame, so it is parallelizable. ☐

Submanifolds and the Lie bracket. Let us recall that M is a submanifold of N if the inclusion $i\colon M \hookrightarrow N$ is an embedding. For each $p \in M$ we identify T_pM and i_*T_pM, so T_pM can be interpreted as a vector subspace of T_pN. Then a vector $T_pM \ni v = i_*v \in T_pN$ operates on $C^\infty(p)$ by

$$vf = (i_*v)f = v(f \circ i) = v(f|M).$$

[Here $C^\infty(p) = \{f \in C^\infty(U)\colon U \subset N \text{ a neighborhood of } p\}$ and $f|M$ means $f|U \cap M$.]

Theorem 3.21. *Let $M \subset N$ be a submanifold. If $X, Y \in \mathcal{T}(N)$ are such that $X_p, Y_p \in T_pM\ \forall p \in M$, then also $[X,Y]_p \in T_pM\ \forall p \in M$.*

We apply the following lemma.

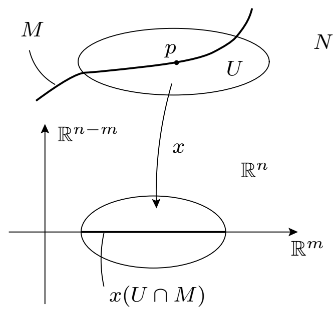
Lemma 3.22. *Let $M^m \subset N^n$ be a submanifold and $p \in M$. Then*

$$T_pM = \{v \in T_pN\colon vf = 0\ \forall f \in C^\infty(p),\ f|M = 0\}.$$

Proof. Suppose first that $v \in T_pM \subset T_pN$ (more precisely $v = i_*w$ for some $w \in T_pM$). Let $f \in C^\infty(U),\ f|U \cap M = 0$, for some neighborhood $U \subset N$ of p. Then $f \circ i = 0$, so

$$vf = (i_*w)f = w(f \circ i) = 0.$$

Suppose then that $v \in T_pN$ satisfies the condition $vf = 0$ for all $f \in C^\infty(p)$, with $f|M = 0$. We will prove that $v \in T_pM$. From the proof of Theorem 2.27 we see that there exists a chart $(U,x),\ x = (x^1, \dots, x^n)$, of N such that $x^{m+1} = \dots = x^n = 0$ in $U \cap M$ and $(x^1, \dots, x^m)$ is a chart of $U \cap M$ at p.



Write

$$v = \sum_{i=1}^{n} v^i (\partial_i)_p.$$

Now T_pM is a subspace of T_pN spanned by coordinate vectors $(\partial i)_p, i = 1, \ldots, m$. Hence $v \in T_pM$ if and only if $v^j = 0$ for all $j = m+1, \ldots, n$. Choose, for every $j = m+1, \ldots, n$ a function $f\colon U \to \mathbb{R}, f(x) = x^j$. Then $f \in C^\infty(p)$ and $f|M \cap U = 0$. We obtain

$$0 = vf = \sum_{i=1}^{n} v^i \frac{\partial x^j}{\partial x^i}(p) = v^j,$$

so $v \in T_pM$. □

Proof of Theorem 3.21. Let $p \in M$ and let $f \in C^\infty(U),\ f|U \cap M = 0$, for some neighborhood $U \subset N$ of p. Since $X_q \in T_qM$ and $Y_q \in T_qM$ for all $q \in M$, we have

$$X_q f = 0 \ \text{ and } \ Y_q f = 0 \ \forall q \in U \cap M$$

by Lemma 3.22. Hence

$$(Xf)|U \cap M = 0 \ \text{ and } \ (Yf)|U \cap M = 0.$$

Applying Lemma 3.22 to functions Xf and Yf we obtain

$$Y_p(Xf) = 0 \text{ and } X_p(Yf) = 0,$$

so

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf) = 0.$$

Thus $[X, Y]_p \in T_pM$ (again by Lemma 3.22). □

For the converse direction we so-called Frobenius theorem:

Let M be a smooth n-manifold and $k \in \{1, \ldots, n-1\}$. Suppose that for every $p \in M$ a k-dimensional subspace $\Delta_p \subset T_pM$ is given. Assume furthermore that every $p \in M$ has a neighborhood U and smooth vector fields $X^1, \ldots, X^k \in \mathcal{T}(U)$ such that the vectors $X_q^1, \ldots, X_q^k$ form a basis of Δ_q for every $q \in U$. Then we say that

$$\Delta = \bigsqcup_{p \in M} \Delta_p \quad (\subset TM)$$

is a *smooth k-dimensional (tangent) distribution on M* (or a smooth field of k-planes or a smooth subbundle of TM).

Let $\Delta \subset TM$ be a smooth tangent distribution. A submanifold $N \subset M$ is called an *integral manifold* of Δ if $T_pN = \Delta_p$ for every $p \in N$. Furthermore, we say that Δ is *integrable* if, for every $p \in M$, there exists an integral manifold N of Δ such that $p \in N$.

We say that a smooth distribution Δ is *involutive* if, for every vector fields $X, Y \in \mathcal{T}(M)$, with $X_p, Y_p \in \Delta_p\ \forall p \in M$, also $[X, Y]_p \in \Delta_p\ \forall p \in M$.

The existence of an integral manifold is characterized by the Frobenius theorem:

Theorem 3.23 (Frobenius). *Let M be a differentiable n-manifold and Δ a smooth k-dimensional tangent distributionon M, $1 \le k \le n-1$. Then*

$$\Delta \text{ is integrable} \iff \Delta \text{ is involutive.}$$

Furthermore, if Δ is integrable, there exists a chart (U, x) at every point $p \in M$ such that every

$$x^{-1}\{y \in \mathbb{R}^n \colon y \in \mathbb{R}^k \times \{c\}\},\ c \in \mathbb{R}^{n-k},$$

is an integral manifold of Δ (or $\emptyset$).

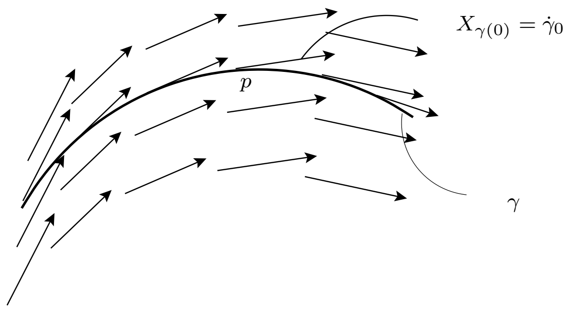
Proof. Omitted (see e.g. Lee [L2, §19]).

3.24 Integral curves

Definition 3.25. Let M be a differentiable manifold and $X \in \mathcal{T}(M)$. We say that a C^∞-polku $\gamma\colon I \to M$ is an *integral curve* of X if

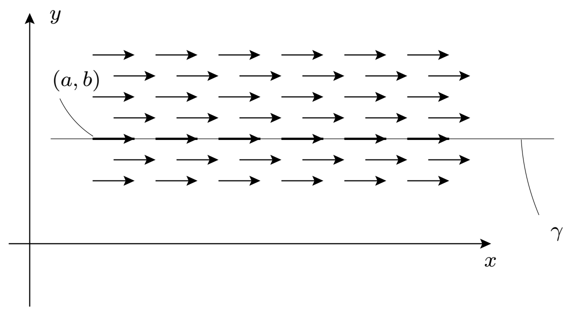
$$\dot\gamma_t = X_{\gamma(t)}\ \forall t \in I.$$

If, in addition, $I \subset \mathbb{R}$ is an open interval, $0 \in I$, and $\gamma(0) = p$, we say that γ is an integral curve of X starting at $p \in M$.



Example 3.26. Let us denote the points of $\mathbb{R}^2$ by (x, y) and let $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ be the corresponding coordinate vector fields.

(a) Let $X \in \mathcal{T}(\mathbb{R}^2)$, $X = \frac{\partial}{\partial x}$. Clearly every path $\gamma(t) = (t + a, b)$, where $a, b \in \mathbb{R}$ are constants, is an integral curve of X starting at (a, b). Thus through every point of the plane goes an integral curve of X.



(b) Let $V \in \mathcal{T}(\mathbb{R}^2)$,

$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

If $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = \big(x(t), y(t)\big)$, is a C^∞-path, then

$$\begin{aligned} & \dot{\gamma}_t = V_{\gamma(t)} \\ \iff\quad & x'(t)\left(\frac{\partial}{\partial x}\right)_{\gamma(t)} + y'(t)\left(\frac{\partial}{\partial y}\right)_{\gamma(t)} = x(t)\left(\frac{\partial}{\partial x}\right)_{\gamma(t)} + y(t)\left(\frac{\partial}{\partial y}\right)_{\gamma(t)} \\ \iff\quad & \begin{cases} x'(t) = x(t) \\ y'(t) = y(t) \end{cases} \\ \iff\quad & \begin{cases} x(t) = ae^t \\ y(t) = be^t, \end{cases} \end{aligned}$$

where $a, b \in \mathbb{R}$. We observe that the path $\gamma(t) = (ae^t, be^t)$, where $a, b, \in \mathbb{R}$, is an integral curve of V starting that $\gamma(0) = (a, b)$. Thus through every point of the plane goes an integral curve of V.

As can be seen in (b), we must solve a system of ordinary differential equations (in local coordinates) in order to find integral curves of a given vector field.

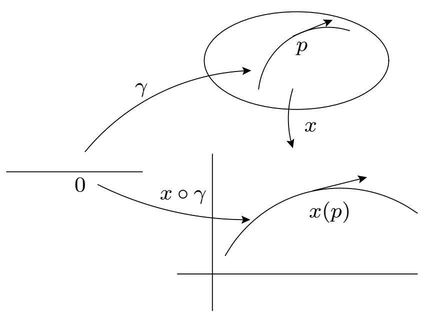
In general:

Lemma 3.27. *Let (U, x), $x = (x^1, \ldots, x^n)$, be a chart on M and let $V \in \mathcal{T}(U)$,*

$$V_p = v^i(p)(\partial_i)_p, \quad p \in U.$$

Suppose that $\gamma : I \to U$ is a C^∞-path, where $I \subset \mathbb{R}$ is an open interval and $0 \in I$. Then γ is an integral curve of V starting at $p \in U$ if and only if. for every $i = 1, \ldots, n$

$$\begin{cases} (x^i \circ \gamma)'(t) &= v^i(\gamma(t)) \quad \textit{for every } t \in I \\ (x^i \circ \gamma)(0) &= x^i(p). \end{cases} \tag{3.28}$$



Proof. (Exerc.)

Remark 3.29. Defining

$$\begin{aligned} \beta = (\beta^1, \ldots, \beta^n)\colon I \to xU \subset \mathbb{R}^n, \quad \beta = x \circ \gamma, \\ w^i = v^i \circ x^{-1}\colon xU \to \mathbb{R},\ i = 1, \ldots, n, \end{aligned}$$

the system (3.28) can be written in the form

$$\begin{cases} \frac{d}{dt}\beta^i(t) &= w^i(\beta^1(t),\dots,\beta^n(t)) \quad \forall t\in I,\ i=1,\dots,n, \\ \beta^i(0) &= x^i(p). \end{cases} \tag{3.30}$$

Let $V \in \mathcal{T}(M)$. It follows from the theory of systems of ordinary differential equations that, for every $p \in M$, there exists a *unique maximal* integral curve $\gamma^p\colon I_p \to M$ of V starting at p, i.e. if $\gamma\colon I \to M$ is an integral curve of V staring at p, then $I \subset I_p$ and $\gamma = \gamma^p|I$. We return to the proof of this later.

Lemma 3.31. *Let $V \in \mathcal{T}(M)$ and $\gamma\colon I \to M$ be an integral curve of V. Define, for every $a \in \mathbb{R}$,*

$$I + a = \{t + a\colon t \in I\}$$

and $\tilde\gamma\colon I + a \to M,\ \tilde\gamma(t) = \gamma(t-a)$. Then also $\tilde\gamma$ is an integral curve of V.

Proof. Let $t_0 \in I + a$ and $f \in C^\infty(\tilde\gamma(t_0))$. Then

$$\begin{aligned} \tilde\gamma'(t_0)f &= \frac{d}{dt}(f\circ\tilde\gamma)(t)_{|t=t_0} = \frac{d}{dt}(f\circ\gamma)(t-a)_{|t=t_0} \\ &= (f\circ\gamma)'(t_0-a) = \dot\gamma_{t_0-a}f = V_{\gamma(t_0-a)}f \\ &= V_{\tilde\gamma(t_0)}f. \quad \square \end{aligned}$$

3.32 Flows

Let M be a smooth manifold. We say that an open set $\mathcal{D} \subset \mathbb{R}\times M$ is a *flow domain* if for all $p \in M$

$$\mathcal{D} \cap (\mathbb{R}\times\{p\}) = I_p \times \{p\},$$

where $I_p \subset \mathbb{R}$ is an open interval, $0 \in I_p$. A smooth mapping $\theta\colon \mathcal{D} \to M$ is a (local) *flow* on M if it satisfies the following *group laws*

$$\begin{aligned} \theta(0,p) &= p \quad \forall p \in M, \\ I_{\theta(s,p)} &= I_p - s \quad \forall s \in I_p, \\ \theta(t,\theta(s,p)) &= \theta(t+s,p) \quad \forall s \in I_p,\ t+s \in I_p. \end{aligned}$$

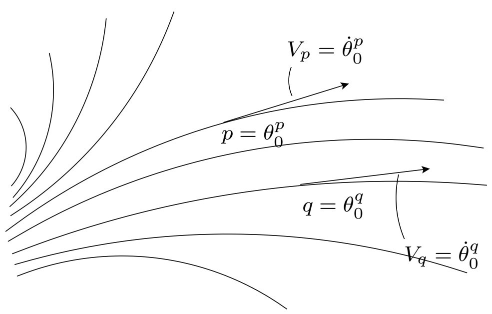
We also denote

$$\theta_t^p = \theta_t(p) = \theta^p(t) = \theta(t,p).$$

The mapping $\theta^p\colon I_p \to M,\ t \mapsto \theta_t^p$, is a C^∞-path. The vector field V,

$$V_p = \dot\theta_0^p,$$

is called the *infinitesimal generator* of θ.



Theorem 3.33 (Properties of a flow)**.** *Let $\theta\colon \mathcal{D} \to M$ be a flow. Then:*

(a) $\forall t \in \mathbb{R}$, *the set* $\mathcal{D}_t = \{p \in M \colon (t,p) \in \mathcal{D}\} \subset M$ *is open,* $\theta_t \colon \mathcal{D}_t \to \mathcal{D}_{-t}$ *is a diffeomorphism and* $\theta_t^{-1} = \theta_{-t}$.

(b) $\theta_t \circ \theta_s = \theta_{t+s}$ *(whenever the left-hand side is defined).*

(c) the infinitesimal generator V of θ is a smooth vector field.

(d) $\forall p \in M$, $\theta^p \colon I_p \to M$ *is an integral curve of V starting at p.*

(e) $\theta_{t*} V_p = V_{\theta_t(p)}, \quad \forall p \in M, \ \forall t \in I_p$.

Proof.

(a) Let $p \in \mathcal{D}_t$, hence $(t,p) \in \mathcal{D}$. Since $\mathcal{D}$ is open, there exist $\delta > 0$ and a neighborhood U of p s.t. $]t-\delta, t+\delta[\times U \subset \mathcal{D}$. In particular, $\{t\} \times U \subset \mathcal{D}$, so $U \subset \mathcal{D}_t$ and $\mathcal{D}_t$ is open.
If $p \in \mathcal{D}_t$, then $t \in I_p$ and $t + (-t) = 0 \in I_p$. Hence $\theta_t(p) \in \mathcal{D}_{-t}$ and

$$(\theta_{-t} \circ \theta_t)(p) = \theta\big(-t, \theta(t,p)\big) = \theta(-t+t, p) = p.$$

Similarly, $\theta_{-t}(\mathcal{D}_{-t}) \subset \mathcal{D}_t$ and $(\theta_t \circ \theta_{-t})(q) = q$ for all $q \in \mathcal{D}_{-t}$. Hence $\theta_t^{-1} = \theta_{-t}$. Furthermore, θ_t and θ_{-t} are smooth in every open subset of M where they are defined.

(b) Follows directly from the definition of a flow.

(c) Let $U \subset M$ be open and $f \in C^\infty(U)$. Then for every $p \in U$

$$Vf(p) = V_p f = \dot{\theta}^p_0 f = \frac{d}{dt} f\big(\theta(t,p)\big)_{|t=0}.$$

Since f and θ are smooth, also $p \mapsto Vf(p)$ is smooth. Thus $V \in \mathcal{T}(M)$.

(d) Let $p \in M$ and $s \in I_p$. We have to prove that

$$\dot{\theta}^p_s = V_{\theta(s,p)}.$$

Denote $q = \theta(s,p)$ and let $f \in C^\infty(q)$. Then

$$V_q f = \dot{\theta}^q_0 f = \frac{d}{dt} f\big(\theta(t,q)\big)_{|t=0} = \frac{d}{dt} f\big(\theta(t+s,p)\big)_{|t=0} = \dot{\theta}^p_s f.$$

(e) Let $p \in M$ and $t \in I_p$. We write $q = \theta^p_t$ and prove that

$$\theta_{t*} V_p = V_q.$$

Let $f \in C^\infty(q)$. Then

$$\begin{aligned}(\theta_{t*} V_p) f = V_p(f \circ \theta_t) &= \frac{d}{ds}(f \circ \theta_t \circ \theta^p_s)(s)_{|s=0} \\ &= \frac{d}{ds} f\big(\theta_t(\theta^p_s)\big)_{|s=0} = \frac{d}{ds} f\big(\theta(t+s,p)\big)_{|s=0} \\ &= \dot{\theta}^p_t f = V_q f.\end{aligned}$$

$\square$

Lemma 3.34. *Let $\theta\colon \mathcal{D} \to M$ be a flow, V its infinitesimal generator, and $p \in M$. If $V_p = 0$, then θ^p is the constant path $\theta^p_t \equiv p$. If $V_p \neq 0$, then $\theta^p\colon I_p \to M$ is an immersion.*

Proof. Denote $\gamma = \theta^p$. Let $t \in I_p$ and write $q = \gamma(t)$. Now $\gamma_{*t}\colon T_t\mathbb{R} \to T_qM$ is the zero map (i.e. $\gamma_{*t}v = 0\ \forall v \in T_t\mathbb{R}$) $\iff$ $\dot\theta^p_t = \gamma_{*t}(\frac{\partial}{\partial t}) = 0$. By Theorem 3.33 (a) and (e), we have $V_q = \theta^p_{t*}V_p$ and $V_p = \theta^q_{-t*}V_q$. Hence $\gamma'(t) = V_q = 0 \iff \gamma'(0) = V_p = 0$. In other words, if $\gamma'(t) = 0$ for some $t \in I_p$, then $\gamma'(t) = 0\ \forall t \in I_p$. Hence, if $V_p = 0$, then $\gamma\colon I_p \to M$ is a smooth mapping such that $\gamma_* \equiv 0$, and consequently γ is a constant path (I_p connected). On the other hand, if $V_p \neq 0$, then $\gamma_{*t} \neq 0\ \forall t$, so γ is an immersion. $\square$

Example 3.35. Let $\mathcal{D} = \mathbb{R} \times \mathbb{R}^2$ and $\theta\colon \mathcal{D} \to \mathbb{R}^2$,

$$\theta(t,(x,y)) = (x\cos t + y\sin t, -x\sin t + y\cos t).$$

Then θ is a flow since θ is clearly smooth and

(a) $\theta(0,(x,y)) = (x\cos 0 + y\sin 0, -x\sin 0 + y\cos 0) = (x,y)$,

(b)

$$\begin{aligned}
&\theta(t,\theta(s,(x,y)))\\
&= ((x\cos s + y\sin s)\cos t + (-x\sin s + y\cos s)\sin t,\\
&\qquad - (x\cos s + y\sin s)\sin t + (-x\sin s + y\cos s)\cos t)\\
&= (x(\cos s\cos t - \sin s\sin t) + y(\sin s\cos t + \cos s\sin t),\\
&\qquad - x(\cos s\sin t + \sin s\cos t) + y(\cos s\cos t - \sin s\sin t))\\
&= (x\cos(s+t) + y\sin(s+t), -x\sin(s+t) + y\cos(s+t))\\
&= \theta(s+t,(x,y)).
\end{aligned}$$

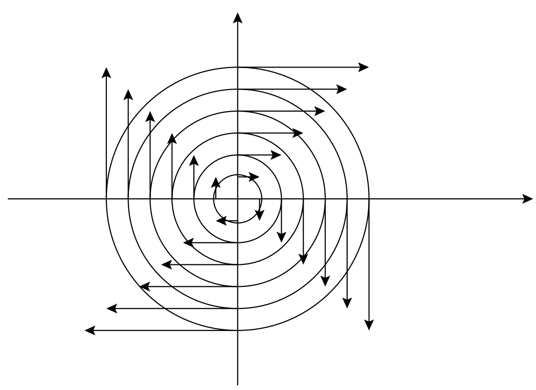
Its infinitesimal generator is

$$\begin{aligned}
V_{(x,y)} &= \frac{d}{dt}\theta(t,(x,y))_{|t=0} = (-x\sin 0 + y\cos 0, -x\cos 0 - y\sin 0)\\
&= (y,-x),
\end{aligned}$$

so using coordinate vector fields

$$V_{(x,y)} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

[Below are some values of V (vectors) and some integral curves.]



3.36 Flows of vector fields

We say that a flow $\theta\colon \mathcal{D} \to M$ is *maximal* if the flow domain $\mathcal{D}$ is the largest possible. In other words, if $\tilde{\theta}\colon \tilde{\mathcal{D}} \to M$ is a flow such that $\mathcal{D} \subset \tilde{\mathcal{D}}$ and $\theta = \tilde{\theta}|\mathcal{D}$, then $\tilde{\mathcal{D}} = \mathcal{D}$.

Theorem 3.37. *Let M be a smooth manifold and $V \in \mathcal{T}(M)$. Then there exists a unique maximal flow $\theta\colon \mathcal{D} \to M$ whose infinitesimal generator is V. Moreover, θ has the following properties:*

(a) For every $p \in M$, the path $\theta^p\colon I_p \to M$ is the unique maximal integral curve of V starting at p.

(b) If $s \in I_p$, then $I_{\theta(s,p)}$ is the interval $I_p - s = \{t - s\colon t \in I_p\}$.

The proof of Theorem 3.37 is based on the following theorem on the existence, uniqueness, and smoothness of solutions to systems of ordinary differential equations. Later we will prove part of it.

Theorem 3.38 (Existence, uniqueness, and smoothness)**.** *Let $U \subset \mathbb{R}^n$ be open and $V = (V^1, \ldots, V^n)\colon U \to \mathbb{R}^n$ smooth. Let $t_0 \in \mathbb{R}$ and $x \in U$. Consider the following initial value problem*

$$\begin{cases} (\gamma^i)'(t) &= V^i(\gamma(t)), \\ \gamma^i(t_0) &= x^i, \end{cases} \tag{3.39}$$

where $\gamma = (\gamma^1, \ldots, \gamma^n)\colon J \to U,\ t_0 \in J$, is a smooth path.

(a) Existence: *For every $(t_0, x_0) \in \mathbb{R} \times U$ there exist an open interval $J_0 \ni t_0$ and a neighborhood $U_0 \subset U$ of x_0 such that for every $x \in U_0$ there exists a C^∞-path $\gamma\colon J_0 \to U$ that is a solution to (3.39).*

(b) Uniqueness: *If $\gamma\colon J_0 \to U$ and $\tilde{\gamma}\colon \tilde{J}_0 \to U$ are solutions to (3.39), then $\gamma = \tilde{\gamma}$ on the interval $J_0 \cap \tilde{J}_0$.*

(c) Smoothness: *Let t_0, x_0, J_0 and U_0 be as in (a) and define the mapping $\theta\colon J_0 \times U_0 \to U$,*

$$\theta(t, x) = \gamma(t),$$

where $\gamma\colon J_0 \to U$ is the unique solution to (3.39) with initial value $\gamma(t_0) = x$. Then θ is smooth.

We split the proof of Theorem 3.37 into several parts.

Theorem 3.40. *Let $V \in \mathcal{T}(M)$. Then for every $p \in M$ there exists a unique maximal integral curve of V starting at p.*

Proof. By Theorem 3.38, for every $p \in M$ there exists an integral curve of V starting at p (cf. (3.30)). Let $\gamma\colon I \to M$ and $\tilde{\gamma}\colon I \to M$ be integral curves of V s.t. $\gamma(t_0) = \tilde{\gamma}(t_0)$ for some $t_0 \in I$. Denote

$$J = \{t \in I\colon \gamma(t) = \tilde{\gamma}(t)\}.$$

We claim that $J \neq \emptyset$ is both open and closed in I, and so $J = I$ (since I is connected).
Clearly $J \neq \emptyset$ since $t_0 \in J$.
Let $t_i \in J,\ t_i \to t \in I$. Since γ and $\tilde{\gamma}$ are continuous, we have

$$\tilde{\gamma}(t) = \lim_{i\to\infty} \tilde{\gamma}(t_i) = \lim_{i\to\infty} \gamma(t_i) = \gamma(t),$$

so $t \in J$. Thus $J \subset I$ is closed.
Let $t_1 \in J$. Now γ and $\tilde{\gamma}$ are solutions to the same system of differential equations with the same initial value $\gamma(t_1) = \tilde{\gamma}(t_1)$, therefore $\gamma \equiv \tilde{\gamma}$ on an interval $]t_1 - \varepsilon, t_1 + \varepsilon[$ by Theorem 3.38. Hence $]t_1 - \varepsilon, t_1 + \varepsilon[\subset J$, and therefore $J \subset I$ is open. We have proven that $J = I$.
Denote

$$\begin{aligned}
\Gamma &= \{\gamma_\alpha \mid \gamma_\alpha \colon J_\alpha \to M \text{ is an integral curve of } V \text{ starting at } p\}, \\
I_p &= \bigcup_{\gamma_\alpha \in \Gamma} J_\alpha
\end{aligned}$$

($\alpha \in \mathcal{A},\ \mathcal{A}$ some set of indices) and define

$$\gamma \colon I_p \to M,\ \gamma(t) = \gamma_\alpha(t),$$

for some $\gamma_\alpha \in \Gamma$ such that $t \in J_\alpha$. In other words, $\gamma | J_\alpha = \gamma_\alpha$. By the beginning of the proof γ is well-defined and thus it is the maximal integral curve of V starting at p. □

In what follows we denote by θ^p the maximal integral curve of V starting at p.

Let $V \in \mathcal{T}(M),\ p \in M$ and $\theta^p \colon I_p \to M$ the maximal integral curve of V starting at p. Define

$$\mathcal{D}(V) = \{(t,p) \in \mathbb{R} \times M \colon t \in I_p\}, \tag{3.41}$$

$$\mathcal{D}_t(V) = \{p \in M \colon (t,p) \in \mathcal{D}(V)\}, \tag{3.42}$$

$$\theta \colon \mathcal{D}(V) \to M,\ \theta(t,p) = \theta^p(t). \tag{3.43}$$

We also denote

$$\theta^p(t) = \theta_t(p) = \theta_t^p.$$

Lemma 3.44. *Let $\theta \colon \mathcal{D}(V) \to M$ be as above. Then*

(a) $\theta(0,p) = p \quad \forall p \in M$,

(b) $\theta\big(t, \theta(s,p)\big) = \theta(t+s, p) \quad \forall s \in I_p,\ t+s \in I_p$.

Proof. (a) is clear.
Fix $p \in M,\ s \in I_p$ and let $q = \theta_s^p$. If $\gamma \colon I_p - s \to M$ is the path

$$\gamma(t) = \theta^p(t+s),$$

then $\gamma(0) = q$ and γ is an integral curve of V starting at q (Lemma 3.31). By Theorem 3.38 (b) ("Uniqueness"), $\gamma = \theta^q$ in the set where both are defined. Since θ^q is maximal, it is defined at least on the interval $I_p - s$ (so $I_p - s \subset I_q$). Thus for every $t \in I_p - s$

$$\theta(t+s,p) = \gamma(t) = \theta^q(t) = \theta(t,q) = \theta\big(t, \theta(s,p)\big). \quad \square$$

Remark 3.45. Above we observed that $I_p - s \subset I_{\theta(s,p)}$ for all $s \in I_p$. Since $0 \in I_p$, it follows that $-s \in I_{\theta(s,p)}$, and therefore $\theta\big(-s, \theta(s,p)\big) = p$. The path $\gamma \colon I_{\theta(s,p)} + s \to M$,

$$\gamma(t) = \theta\big(t - s, \theta(s,p)\big),$$

is an integral curve of V starting at $\gamma(0) = \theta\big(-s, \theta(s,p)\big) = p$. We conclude that $\gamma = \theta^p$ in $I_{\theta(s,p)} + s$ since θ^p is maximal. Hence $I_{\theta(s,p)} + s \subset I_p$, or equivalently $I_{\theta(s,p)} \subset I_p - s$. We obtain $I_p - s = I_{\theta(s,p)}$ for every $s \in I_p$. (cf. Theorem 3.37 (b).)

Lemma 3.46. *Let $\theta\colon \mathcal{D}(V) \to M$ be as in (3.41)–(3.43). Then $\mathcal{D}(V) \subset \mathbb{R} \times M$ is open and θ is smooth.*

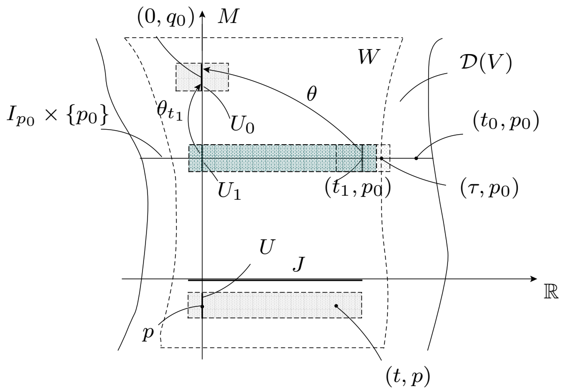
Proof. Let $W \subset \mathcal{D}(V)$ be the set of points $(t, p) \in \mathcal{D}(V)$ that have a neighborhood $J \times U \subset \mathcal{D}(V)$ where θ is defined and smooth and, moreover, that $U \subset M$ is a neighborhood of p and $J \subset \mathbb{R}$ is an open interval containing 0 and t. Clearly W is open in $\mathbb{R} \times M$ and $\theta|W$ is smooth. It remains to prove that $W = \mathcal{D}(V)$. Assume on the contrary that there exists $(t_0, p_0) \in \mathcal{D}(V) \setminus W$. We may suppose that $t_0 \geq 0$ (the case $t_0 < 0$ is similar). It follows from Theorem 3.38 (by using a chart at p_0) that θ is defined and smooth in some product neighborhood of $(0, p_0)$. Let

$$\tau = \sup\{t \in \mathbb{R}\colon (t, p_0) \in W\},$$

so $\tau > 0$. Since $\tau \leq t_0$ and I_{p_0} is an open interval containing 0 and t_0, we have $\tau \in I_{p_0}$. Let $q_0 = \theta^{p_0}(\tau)$. By Theorem 3.38 there exists a product neighborhood $]-\varepsilon, \varepsilon[\times U_0$ of $(0, q_0)$ where θ is defined and smooth. We apply Lemma 3.44 to prove that θ is smooth in some product neighborhood of (τ, p_0). Let $t_1 < \tau$ s.t. $t_1 + \varepsilon > \tau$ and $\theta^{p_0}(t_1) \in U_0$. Since $t_1 < \tau$, we have $(t_1, p_0) \in W$, so there exists a product neighborhood $]-\delta, t_1 + \delta[\times U_1$ of (t_1, p_0) where θ is defined and smooth. Since $\theta(t_1, p_0) \in U_0$, we may choose small enough U_1 so that $\theta(t_1, p) \in U_0\ \forall p \in U_1$. By Lemma 3.44

$$\theta_t(p) = \theta_{t-t_1} \circ \theta_{t_1}(p)$$

if $t_1 \in I_p$ and $t - t_1 \in I_{\theta(t_1,p)}$. It follows from the choice of t_1 that $\theta(t_1, p)$ is defined $\forall p \in U_1$ and is smooth with respect to p. Furthermore, $]-\varepsilon, \varepsilon[\subset I_{\theta(t_1,p)}\ \forall p \in U_1$ since $\theta(t_1, p) \in U_0\ \forall p \in U_1$ and θ is defined and smooth in $]-\varepsilon, \varepsilon[\times U_0$. It follows that $\theta_t(p) = \theta_{t-t_1} \circ \theta_{t_1}(p)$ is defined and smooth with respect to (t, p) if $p \in U_1$ and $|t - t_1| < \varepsilon$. Hence we may extend θ smoothly to the set $]-\delta, t_1 + \varepsilon[\times U_1$ that leads to a contradiction with the definition of τ since $t_1 + \varepsilon > \tau$. This proves that $W = \mathcal{D}(V)$. □



Let us underline the main steps of the proof:

1. Antithesis and the definition of τ.

2. $q_0 = \theta(\tau, p_0)$.

3. Theorem 3.38 $\Rightarrow$ $\exists$ a product neighborhood $]-\varepsilon, \varepsilon[\times U_0$ of $(0, q_0)$ where θ is smooth.

4. Choice of $t_1 \in]\tau - \varepsilon, \tau[$.

5. Behavior of θ in a neighborhood of $(0, q_0)$ is "transported" to a product neighborhood $]t_1 - \varepsilon, t_1 + \varepsilon[\times U_1$ of (t_1, p_0) by using the group laws.

6. Extended θ smoothly to a neighborhood of (τ, p_0) and obtained a contradiction.

Proof of Theorem 3.37. Let $\theta\colon \mathcal{D}(V) \to M$ be as in (3.41)–(3.43). Then:

1. Lemma 3.44 and 3.46 $\Rightarrow$ θ is a flow.
2. Theorem 3.40 and general properties of flows (Theorem 3.33) $\Rightarrow$ uniqueness.
3. The claim (a) follows directly from the construction.
4. The claim (b) is proven in Remark 3.45 after Lemma 3.44. $\square$

Definition 3.47. Let $V \in \mathcal{T}(M)$ and let $\theta\colon \mathcal{D}(V) \to M$ be the maximal flow whose infinitesimal generator is V. Then we say that θ is the *flow of* V.

We say that a vector field $V \in \mathcal{T}(M)$ is *complete* if it generates a global flow, i.e. $\mathcal{D}(V) = \mathbb{R}\times M$. Then every maximal integral curve is defined on whole $\mathbb{R}$.

Lemma 3.48 (Escape lemma)**.** *Let $V \in \mathcal{T}(M)$. If γ is an integral curve of V such that its maximal interval of definition, I, is not the whole $\mathbb{R}$, then $\gamma(I)$ can not be contained in any compact subset of M.*

Proof. Let $I =]a, b[$, $-\infty \le a < 0 < b \le +\infty$, and let θ be the flow of V. Then $\gamma = \theta^p$ (and $I = I_p$), where $p = \gamma(0)$. Assume on the contrary that $b < \infty$ and $\gamma(I) \subset K$ for some compact $K \subset M$. (The case $a > -\infty$ similarly.) Choose a sequence $t_i \to b$, $t_i \in I$. Since $\gamma(t_i) \in K$ and K is compact, there exist $q \in K$ and a subsequence, still denoted by (t_i), s.t. $\gamma(t_i) \to q$. Choose a product neighborhood $]-\varepsilon, \varepsilon[\times U$, $]-\varepsilon,\varepsilon[\subset I$ of $(0, q)$ where θ is defined. Fix a sufficiently large i such that $\gamma(t_i) = \theta(t_i, p) \in U$ and $t_i > b - \varepsilon$. Then $t \mapsto \theta\big(t - t_i, \theta(t_i, p)\big)$ is defined $\forall t \in]t_i - \varepsilon, t_i + \varepsilon[$ and it is an integral curve of V (starting at $\theta(t_i, p)$). Furthermore, $\theta\big(t - t_i, \theta(t_i, p)\big) = \theta(t, p) = \gamma(t)$ for $t_i - \varepsilon < t < b$. Define $\sigma\colon]a, t_i + \varepsilon[\to M$ by setting

$$\sigma(t) = \begin{cases} \gamma(t), & a < t < b, \\ \theta\big(t - t_i, \theta(t_i, p)\big), & t_i - \varepsilon < t < t_i + \varepsilon. \end{cases}$$

Then σ is an integral curve of V starting at p, hence γ is not maximal. This is contradiction and the lemma is proven. $\square$

The escape lemma implies the following.

Theorem 3.49. *If M is compact, every $V \in \mathcal{T}(M)$ is complete.*

3.50 Proof of the existence and uniqueness theorem

Let $U \subset \mathbb{R}^n$ be open. We say that a mapping $V\colon U \to \mathbb{R}^n$ is Lipschitz if there exists a constant $L > 0$ s.t.

$$|V(x) - V(y)| \le L|x - y| \quad \forall x, y \in U. \tag{3.51}$$

Theorem 3.52 (Existence)**.** *Let $U \subset \mathbb{R}^n$ be open and $V : U \to \mathbb{R}^n$ Lipschitz. Then for every $(t_0, x_0) \in \mathbb{R} \times U$ there exist a neighborhood $U_0 \subset U$ of x_0 and an open interval $J_0 \subset \mathbb{R}$, $t_0 \in J_0$ such that for every $x = (x^1, \ldots, x^n) \in U_0$ there exists a C^1-path $\gamma\colon J_0 \to U$ solving the initial value problem*

$$\begin{cases} (\gamma^i)'(t) &= V^i(\gamma(t)),\ \forall t \in J_0, \\ \gamma^i(t_0) &= x^i. \end{cases} \tag{3.53}$$

Proof. Suppose that $\gamma\colon J_0 \to U$ is a solution to (3.53). From the left-hand side of the equation we see that every γ^i, $i = 1, \ldots, n$, is differentiable, hence γ is continuous. Since both V^i and γ are continuous, also the right-hand side is continuous. Hence γ is continuously differentiable, so $\gamma \in C^1$. Integrating (3.53) with respect to t we obtain

$$\gamma^i(t) = x^i + \int_{t_0}^t V^i(\gamma(s))ds, \quad i = 1, \ldots, n. \tag{3.54}$$

Conversely, if $\gamma\colon J_0 \to U$ is a path that satisfies (3.54), then

$$(\gamma^i)'(t) = \frac{d}{dt}\int_{t_0}^t V^i(\gamma(s))ds = V^i(\gamma(t)) \quad \text{ja } \gamma^i(t_0) = x^i.$$

We define, for each path $\gamma\colon J_0 \to U$, a mapping $S_x\gamma\colon J_0 \to \mathbb{R}^n$,

$$S_x\gamma(t) = x + \int_{t_0}^t V(\gamma(s))ds \tag{3.55}$$

and we look for fixed points of S_x (i.e. paths γ s.t. $\gamma = S_x\gamma$) in some suitable metric space. Clearly $S_x\gamma$ is continuous (hence a path) and $S_x\gamma(t_0) = x$. For every $x_0 \in U$ choose $r > 0$ s.t. $\bar{B}(x_0, r) \subset U$. Denote $M = \max\{|V(x)|\colon x \in \bar{B}(x_0, r)\}$. Let $t_0 \in \mathbb{R}$, $\delta \le r/2$ and $J_0 =]t_0 - \varepsilon, t_0 + \varepsilon[$, where

$$\varepsilon < \min\{\frac{r}{2M}, \frac{1}{L}\},$$

and L is a Lipschitz constant of V in (3.51). For every $x \in U_0 := B(x_0, \delta)$ denote

$$\mathcal{M}_x = \{\gamma \mid \gamma\colon J_0 \to \bar{B}(x_0, r) \text{ path, } \gamma(t_0) = x\}$$

and define a metric in $\mathcal{M}_x$ by setting

$$d(\gamma, \tilde{\gamma}) = \sup\{|\gamma(t) - \tilde{\gamma}(t)|\colon t \in J_0\}.$$

If (γ_i) is a Cauchy sequence in $\mathcal{M}_x$, it is uniformly convergent (by the Cauchy criterion), so the limit $\gamma = \lim_i \gamma_i$ is a continuous mapping $J_0 \to \bar{B}(x_0, r)$, so γ is a path. Clearly $\gamma(t_0) = x$, so $\gamma \in \mathcal{M}_x$. Hence $(\mathcal{M}_x, d)$ is complete. Next we prove that the formula (3.55) defines a contraction $S_x\colon \mathcal{M}_x \to \mathcal{M}_x$. If $\gamma \in \mathcal{M}_x$ and $t \in J_0$, then

$$|S_x\gamma(t) - x_0| = |\int_{t_0}^t V(\gamma(s))ds + x - x_0| \le M|t - t_0| + |x - x_0| < M\varepsilon + \delta \le r,$$

so $S_x\gamma$ is a path $J_0 \to \bar{B}(x_0, r)$ and therefore $S_x\gamma \in \mathcal{M}_x$. Furthermore, S_x is a contraction since, for all $\gamma, \tilde{\gamma} \in \mathcal{M}_x$,

$$\begin{aligned} d(S_x\gamma, S_x\tilde{\gamma}) &= \sup_{t \in J_0}\Big|\int_{t_0}^t V(\gamma(s))ds - \int_{t_0}^t V(\tilde{\gamma}(s))ds\Big| \le \sup_{t \in J_0}\int_{t_0}^t \big|V(\gamma(s)) - V(\tilde{\gamma}(s))\big|\, ds \\ &\le \sup_{t \in J_0}\int_{t_0}^t L\underbrace{|\gamma(s) - \tilde{\gamma}(s)|}_{\le d(\gamma, \tilde{\gamma})} \le L\sup_{t \in J_0}|t - t_0| d(\gamma, \tilde{\gamma}) \le L\varepsilon d(\gamma, \tilde{\gamma}). \end{aligned}$$

Since $L\varepsilon < 1$, the Banach fixed point theorem 0.8 implies that S_x has a fixed point $\gamma \in \mathcal{M}_x$. Hence

$$\gamma(t) = S_x\gamma(t) = x + \int_{t_0}^t V(\gamma(s))ds,$$

so γ satisfies (3.54) and therefore also (3.53). □Note that the Banach fixed point theorem gives uniqueness among paths $J_0 \to \bar{B}(x_0, r)$.

Theorem 3.56 (Uniqueness)**.** *If $\gamma\colon J_0 \to U$ and $\tilde\gamma\colon J_0 \to U$ are solutions to the initial value problem (3.53) such that $\tilde\gamma(t_0) = \gamma(t_0)$), then $\gamma = \tilde\gamma$.*

Proof. Suppose that γ and $\tilde\gamma$ are solutions to (3.53) with initial values $\gamma(t_0) = x$ and $\tilde\gamma(t_0) = y$. Then

$$\begin{aligned}\frac{d}{dt}|\tilde\gamma(t)-\gamma(t)|^2 &= \frac{d}{dt}\left((\tilde\gamma(t)-\gamma(t))\cdot(\tilde\gamma(t)-\gamma(t))\right)\\ &= 2(\tilde\gamma(t)-\gamma(t))\cdot\frac{d}{dt}(\tilde\gamma(t)-\gamma(t))\\ &= 2(\tilde\gamma(t)-\gamma(t))\cdot(V(\tilde\gamma(t))-V(\gamma(t)))\\ &\le 2|\tilde\gamma(t)-\gamma(t)|\underbrace{|V(\tilde\gamma(t))-V(\gamma(t))|}_{\le L|\tilde\gamma(t)-\gamma(t)|}\\ &\le 2L|\tilde\gamma(t)-\gamma(t)|^2.\end{aligned}$$

We obtain

$$\begin{aligned}\frac{d}{dt}\left(e^{-2Lt}|\tilde\gamma(t)-\gamma(t)|^2\right) &= e^{-2Lt}\frac{d}{dt}|\tilde\gamma(t)-\gamma(t)|^2 - 2Le^{-2Lt}|\tilde\gamma(t)-\gamma(t)|^2\\ &\le e^{-2Lt}\left(2L|\tilde\gamma(t)-\gamma(t)|^2 - 2L|\tilde\gamma(t)-\gamma(t)|^2\right) = 0.\end{aligned}$$

Hence

$$e^{-2Lt}|\tilde\gamma(t)-\gamma(t)|^2 \le e^{-2Lt_0}|\tilde\gamma(t_0)-\gamma(t_0)|^2, \quad \forall t \ge t_0.$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}|\tilde\gamma(t)-\gamma(t)|^2 &\ge -2|\tilde\gamma(t)-\gamma(t)||V(\tilde\gamma(t))-V(\gamma(t))|\\ &\ge -2L|\tilde\gamma(t)-\gamma(t)|^2\end{aligned}$$

which implies that

$$\frac{d}{dt}\left(e^{2Lt}|\tilde\gamma(t)-\gamma(t)|^2\right) \ge 0,$$

so

$$e^{2Lt}|\tilde\gamma(t)-\gamma(t)|^2 \le e^{2Lt_0}|\tilde\gamma(t_0)-\gamma(t_0)|^2, \quad \forall t \le t_0.$$

Hence, for all $t \in J_0$, we have

$$|\tilde\gamma(t)-\gamma(t)| \le e^{L|t-t_0|}|\tilde\gamma(t_0)-\gamma(t_0)|. \tag{3.57}$$

It follows that $\tilde\gamma \equiv \gamma$ if $\tilde\gamma(t_0) = \gamma(t_0)$. □

Lemma 3.58 (Continuity)**.** *Let J_0 be an open interval, $t_0 \in J_0$, $U_0 \subset U$ open and $\theta\colon J_0 \times U_0 \to U$ any mapping s.t. for every $x \in U_0$, $\gamma\colon J_0 \to U$, $\gamma(t) = \theta(t,x)$, is the solution to (3.53) with initial value $\gamma(t_0) = x$. Then θ is continuous.*

Fix $(t,x) \in J_0 \times U_0$ and prove that θ is continuous at (t,x). Since continuity is a local property, we may assume that $\bar J_0 = [a,b] \subset \mathbb{R}$. It follows from the proof of (3.57) that

$$|\theta(t,\tilde x) - \theta(t,x)| \le e^{LT}|\tilde x - x|,$$

where $T = b - a$. Hence θ is Lipschitz with respect to x with a constant e^{LT}. Let $(\tilde{t}, \tilde{x}) \in J_0 \times U_0$. Since every solution to 3.53) satisfies the (integral) equation (3.54), we obtain

$$\theta(\tilde{t}, \tilde{x}) = \tilde{x} + \int_{t_0}^{\tilde{t}} V(\theta(s, \tilde{x}))ds$$

and similarly for the point (t, x). Hence

$$\begin{aligned} |\theta(\tilde{t}, \tilde{x}) - \theta(t, x)| &\le |\tilde{x} - x| + \Big| \int_{t_0}^{\tilde{t}} V(\theta(s, \tilde{x}))ds - \int_{t_0}^{t} V(\theta(s, x))ds \Big| \\ &\le |\tilde{x} - x| + \Big| \int_{t_0}^{\tilde{t}} |V(\theta(s, \tilde{x})) - V(\theta(s, x))| ds \Big| + \Big| \int_{\tilde{t}}^{t} |V(\theta(s, x))| ds \Big|. \end{aligned}$$

Since $s \mapsto \theta(s, x)$ is continuous, there exists $\delta_x > 0$ such that

$$M_x = \max\{\big|V(\theta(s, x))\big| \colon s \in [t - \delta_x, t + \delta_x]\} < \infty.$$

For $\tilde{t} \in [t - \delta_x, t + \delta_x]$, we get

$$\begin{aligned} |\theta(\tilde{t}, \tilde{x}) - \theta(t, x)| &\le |\tilde{x} - x| + \Big| L \int_{t_0}^{\tilde{t}} |\theta(s, \tilde{x}) - \theta(s, x)| ds \Big| + \Big| \int_{\tilde{t}}^{t} M_x ds \Big| \\ &\le |\tilde{x} - x| + LTe^{LT} |\tilde{x} - x| + M_x |\tilde{t} - t|. \end{aligned}$$

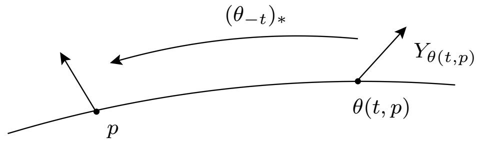
Thus θ is continuous at (t, x). □

Theorem 3.59 (Smoothness)**.** *Let $U \subset \mathbb{R}^n$ be open and $V \colon U \to \mathbb{R}^n$ Lipschitz. Suppose that $U_0 \subset U$ is open, $J_0 \subset \mathbb{R}$ is an open interval, $t_0 \in J_0$, and $\theta \colon J_0 \times U_0 \to U$ is any mapping s.t. for every $x \in U_0$, $\gamma \colon J_0 \to U$, $\gamma(t) = \theta(t, x)$, is a solution to (3.53). If $V \in C^k(U)$ for some $k \ge 1$, also $\theta \in C^k(J_0 \times U_0)$.*

Proof. Omitted. [See e.g. Lee [L2].]

3.60 Lie derivative of a vector field

Let $X \in \mathcal{T}(M)$, $Y \in \mathcal{T}(M)$, $p \in M$ and let θ be the flow of X. Then $\theta_{-t}(\theta_t^p) = p$ for $t \in I =]-\delta, \delta[$, where $\delta > 0$ is sufficiently small.



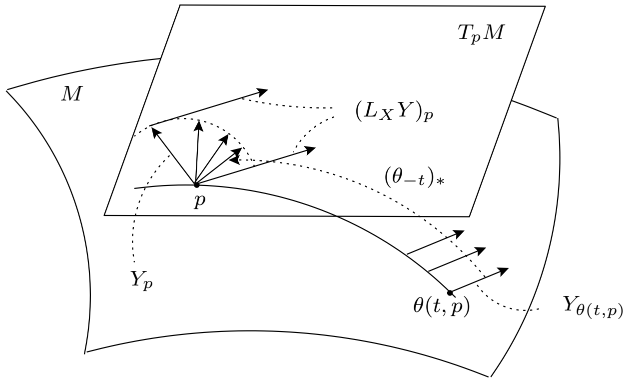
We can define a âŸžC^∞-path $I \to T_pM$,

$$t \mapsto (\theta_{-t})_* Y_{\theta(t,p)}.$$

The tangent (vector) to this path at $t = 0$,

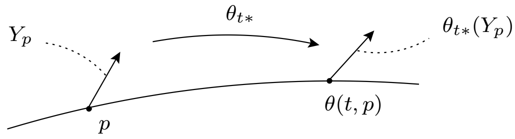
$$\lim_{t \to 0} \frac{(\theta_{-t})_* Y_{\theta(t,p)} - Y_p}{t} = \frac{d}{dt} \big((\theta_{-t})_* Y_{\theta(t,p)}\big)_{|t=0} \tag{3.61}$$

is called the *Lie derivative* of Y with respect to X and it is denoted by $\big(L_X Y\big)_p$.



Another way to write (3.61): For every t, with $|t|$ small enough, let $\theta_{t*}Y$ be the vector field that is defined by

$$(\theta_{t*}Y)_{\theta(t,p)} = \theta_{t*}(Y_p).$$



Then

$$\begin{aligned}\frac{(\theta_{-t})_*Y_{\theta(t,p)} - Y_p}{t} &= \frac{Y_p - (\theta_{-t})_*Y_{\theta(t,p)}}{-t}\\ &= \frac{Y_p - \big(\theta_{-t*}Y\big)_{\theta(-t,\theta(t,p))}}{-t}\\ &= \frac{Y_p - (\theta_{-t*}Y)_p}{-t}\\ &= \frac{Y_p - (\theta_{s*}Y)_p}{s}, \quad s=-t.\end{aligned}$$

Theorem 3.62. *Let $X, Y \in \mathcal{T}(M)$ and $p \in M$. Then*

$$(L_XY)_p = [X,Y]_p.$$

For the proof we need the following:

Lemma 3.63. *Let $h\colon]-\delta,\delta[\times U \to \mathbb{R}$ be a C^∞-function such that $h(0,q)=0$ for all $q \in U$, $U \subset M$ open. Then there exists a C^∞-function $g\colon]-\delta,\delta[\times U \to \mathbb{R}$ s.t. $h(t,q) = tg(t,q)$. In particular,*

$$g(0,q) = D_1h(0,q). \quad (D_1 = \frac{\partial}{\partial t})$$

Proof. Define $g(t,q) = \int_0^1 D_1h(ts,q)ds$. □

Proof of Theorem 3.62. Let θ be the flow of X and let $f \in C^\infty(p)$. Define

$$h(t,q) = f\big(\theta(t,q)\big) - f(q) = \big(f\circ\theta_t\big)(q) - f(q).$$

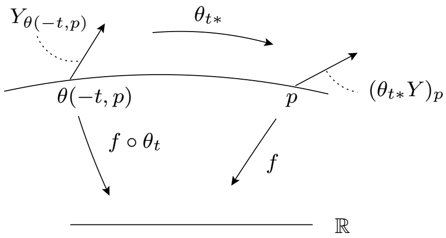
From Lemma 3.63 we get $g_t(q) = g(t,q)$ s.t.

$$(f\circ\theta_t)(q) = f(q) + h(t,q) = f(q) + tg_t(q) \tag{3.64}$$

and

$$\begin{aligned}
g_0(q) &= \frac{\partial h}{\partial t}(0,q) = \lim_{t\to 0}\frac{h(t,q) - \overbrace{h(0,q)}^{=0}}{t} = \lim_{t\to 0}\frac{f\big(\theta(t,q)\big) - f(q)}{t}\\
&= \lim_{t\to 0}\frac{(f\circ\theta^q)(t) - (f\circ\theta^q)(0)}{t} = \big(f\circ\theta^q\big)'(0)\\
&= X_q f.
\end{aligned}$$

Let us find the limit $\lim_{t\to 0}\frac{1}{t}\big(Y - (\theta_{t*}Y)\big)_p f$.



First we obtain

$$\big(\theta_{t*}Y\big)_p f = \Big(\theta_{t*}Y_{\theta(-t,p)}\Big)f = Y_{\theta(-t,p)}(f\circ\theta_t) \overset{(3.64)}{=} Y_{\theta(-t,p)}f + tY_{\theta(-t,p)}g_t.$$

Hence

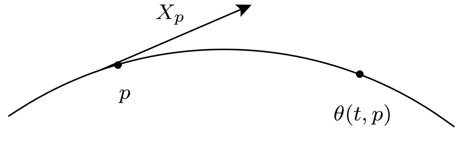
$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t}\big(Y - (\theta_{t*}Y)\big)_p f &= \lim_{t\to 0}\frac{1}{t}\Big(Y_p f - Y_{\theta(-t,p)}f - tY_{\theta(-t,p)}g_t\Big)\\
&= \lim_{t\to 0}\frac{1}{t}\Big(Y_p f - Y_{\theta(-t,p)}f\Big) - \lim_{t\to 0}Y_{\theta(-t,p)}g_t.
\end{aligned}$$

Now

$$\lim_{t\to 0}Y_{\theta(-t,p)}g_t = Y_p g_0 = Y_p(Xf)$$

and

$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t}\Big(Y_p f - Y_{\theta(-t,p)}f\Big) &= \lim_{-s\to 0}\frac{1}{-s}\Big(Y_p f - Y_{\theta(s,p)}f\Big)\\
&= \lim_{s\to 0}\frac{1}{s}\Big((Yf)(\theta(s,p)) - (Yf)(p)\Big)\\
&= X_p(Yf).
\end{aligned}$$



We get

$$\lim_{t\to 0}\frac{1}{t}\big(Y - (\theta_{t*}Y)\big)_p f = X_p(Yf) - Y_p(Xf) = [X,Y]_p f. \quad \square$$

4 Tensors and tensor fields

4.1 Tensors

Let $V_1, \dots, V_k$ and W be (real) vector spaces. A mapping $F\colon V_1 \times \cdots \times V_k \to W$ is called *multi linear* (more precisely *k-linear*) is it is linear with respect to every variable, i.e.

$$F(v_1, \dots, av_i + bv_i', \dots, v_k) = aF(v_1, \dots, v_i, \dots, v_k) + bF(v_1, \dots, v_i', \dots, v_k)$$

for all $i = 1, \dots, k$ and $a, b \in \mathbb{R}$.

Example 4.2.

1. If V is an inner product space, the inner product $\langle \cdot, \cdot \rangle\colon V \times V \to \mathbb{R}$ is 2-linear (or *bilinear*). Example. The usual dot product in $\mathbb{R}^n$. Use: We can define the norm of a vector or the angle between two vectors.

2. Cross product in $\mathbb{R}^3$ is a bilinear mapping $\cdot \times \cdot\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. Use: We can compute the area of a parallelogram or find a vector that is orthogonal to given vectors.

3. The determinant is an n-linear mapping $\det\colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$. Interpretation: If $v_1, \dots, v_n \in \mathbb{R}^n$, $v_i = (v_i^1, \dots, v_i^n)$, then

$$\det(v_1, \dots, v_n) = \det \begin{pmatrix} v_1^1 & \cdots & v_n^1 \\ \vdots & \ddots & \vdots \\ v_1^n & \cdots & v_n^n \end{pmatrix}.$$

Use: We may study linear independence of vectors $v_1, \dots, v_n$ and compute the volume of parallelepiped spanned by the vectors.

Let V be a finite dimensional (real) vector space. A linear map $\omega\colon V \to \mathbb{R}$ is called a *covector* on V and the vector space of all covectors is called the *dual* of V and it is denoted by V^*.

Let us denote

$$\langle \omega, v \rangle = \langle v, \omega \rangle = \omega(v) \in \mathbb{R}, \quad \omega \in V^*,\ v \in V.$$

Lemma 4.3. *Let V be an n-dimensional vector space and $(v_1, \dots, v_n)$ a basis of V. Then the covectors $\omega^1, \dots, \omega^n$, defined by*

$$\omega^j(v_i) = \delta_i^j,$$

form a basis of V^*. In particular, $\dim V^* = \dim V$.

Proof. (Exerc.)

[Note.: Above δ_i^j is the Kronecker delta, i.e. $\delta_i^j = 1$ if $i = j$, and $\delta_i^j = 0$ if $i \neq j$.]

Definition 4.4.

1. a *k-covariant tensor on V* is a k-linear map

$$V^k \to \mathbb{R}, \quad V^k = \underbrace{V \times \cdots \times V}_{k \text{ copies}}.$$

2. an *l-contravariant tensor on V* is an l-linear map

$$V^{*l} \to \mathbb{R}, \quad V^{*l} = \underbrace{V^* \times \cdots \times V^*}_{l \text{ copies}}.$$

3. a *k-covariant, l-contravariant tensor on V* (or a (k,l)*-tensor* for short) is a $(k+l)$-linear map

$$V^k \times V^{*l} \to \mathbb{R}.$$

We denote

$$\begin{aligned} T^k(V) &= \text{ the space of } k\text{-covariant tensors,} \\ T_l(V) &= \text{ the space of } l\text{-contravariant tensors,} \\ T_l^k(V) &= \text{ the space of } k\text{-covariant, } l\text{-contravariant tensors.} \end{aligned}$$

Remark 4.5. 1. $T^k(V)$, $T_l(V)$ and $T_l^k(V)$ are vector spaces in a natural way.

2. We make a convention that 0-covariant tensors and 0-contravariant tensors are real numbers, so $T^0(V) = T_0(V) = \mathbb{R}$.

Example 4.6. 1. Every linear mapping $\omega\colon V \to \mathbb{R}$ is a 1-covariant tensor. Hence $T^1(V) = V^*$. Similarly, $T_1(V) = V^{**} = V$.

2. If V is an inner product space, then any inner product on V is a 2-covariant tensor (bilinear real-valued mapping, i.e. a *bilinear form*).

3. The determinant is an n-covariant tensor on $\mathbb{R}^n$.

Definition 4.7. The *tensor product* of tensors $F \in T_l^k(V)$ and $G \in T_q^p(V)$ is the tensor $F \otimes G \in T_{l+q}^{k+p}(V)$,

$$F \otimes G(v_1, \dots, v_{k+p}, \omega^1, \dots, \omega^{l+q}) = F(v_1, \dots, v_k, \omega^1, \dots, \omega^l) G(v_{k+1}, \dots, v_{k+p}, \omega^{l+1}, \dots, \omega^{l+q}).$$

Lemma 4.8. *If* $(v_1, \dots, v_n)$ *is a basis for* V *and* $(\omega^1, \dots, \omega^n)$ *the corresponding dual basis for* V^* $(\omega^i(v_j) = \delta^i_j)$*, then tensors*

$$\omega^{i_1} \otimes \cdots \omega^{i_k} \otimes v_{j_1} \otimes \cdots \otimes v_{j_l}, \quad 1 \le j_p, i_q \le n,$$

form a basis for $T_l^k(V)$*. In particular,* $\dim T_l^k(V) = n^{k+l}$.

Proof. (Exerc.)

Remark 4.9. We noticed earlier that $T_1(V) = V^{**} = V$ (i.e. every vector $v \in V$ is a 1-contravariant tensor) and $T^1(V) = V^*$ (i.e. every covector is a 1-covariant tensor). Thus

$$\omega^{i_1} \otimes \cdots \omega^{i_k} \otimes v_{j_1} \otimes \cdots \otimes v_{j_l} \in T_l^k(V),$$

i.e. it is a (k,l)-tensor.

4.10 Cotangent bundle

Earlier we defined the *differential* of a function $f \in C^\infty(p)$ at p as the linear map $df_p\colon T_pM \to \mathbb{R}$,

$$df_p v = vf, \quad v \in T_pM.$$

Hence $df_p \in T_pM^*$ (= the dual of T_pM). We call T_pM^* the *cotangent space* of M at p. If (U,x), $x = (x^1, \dots, x^n)$, is a chart at p and $\big((\partial_1)_p, \dots, (\partial_n)_p\big)$ is the basis for T_pM formed by the coordinate vectors, then the differentials $dx^i_p,\ i=1,\dots,n$, at p of functions x^i form the dual basis for T_pM^*. Thus the differential of a function $f\in C^\infty(p)$ at p is

$$df_p=(\partial_i)_p f\,dx^i_p. \qquad \text{(Exerc.) [Note: Einstein summation}$$

We define the *cotangent bundle* TM^* over M as the disjoint union of all cotangent spaces

$$TM^*=\bigsqcup_{p\in M}T_pM^*.$$

It has a natural projection $\pi\colon TM^*\to M,\ T_pM^*\ni\omega\mapsto p\in M$. Sections of TM^* are called *covector fields* on M or *(differential) 1-forms* on M. In other words, they are mappings $\omega\colon M\to TM^*$ such tha $\pi\circ\omega=id$. The *differential* of a function $f\in C^\infty(M)$ is the covector field

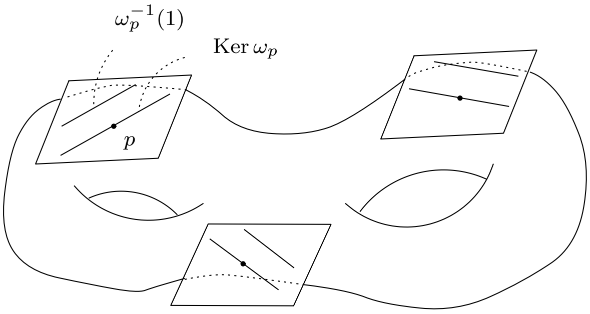
$$df\colon M\to TM^*,\quad df(p)=df_p\colon T_pM\to\mathbb{R}.$$

The geometric visualization of vector and covector fields. A vector field attach a vector to each point liittää jokaiseen pisteeseen vektorin whereas a covector field ω attach to each point p, where $\omega_p\neq 0$, a codimension 1 subspace of T_pM (a hyperplane)

$$\operatorname{Ker}\omega_p=\{v\in T_pM\colon \omega_p(v)=0\}$$

and an affine (codimension 1) hyperplane

$$\omega_p^{-1}(1)=\{v\in T_pM\colon \omega_p(v)=1\}.$$



Similarly to the case of the tangent bundle, we may prove that the cotangent bundle has a canonical smooth structure. The set of all smooth covector fields is denoted by $\mathcal{T}^1(M)$ (or $\mathcal{T}^1_0(M)$, $\mathcal{T}^*(M),\mathcal{T}^{0,1}(M)$).

If $(U,x),\ x=(x^1,\dots,x^n)$, is a chart and ω is a covector field on U, there exist functions $\omega_i\colon U\to\mathbb{R},\ i=1,\dots,n$, s.t.

$$\omega=\omega_i dx^i.$$

The functions ω_i are called the *component functions* of ω with respect to (U,x). As in the case of vector fields we have:

Lemma 4.11. *Let ω be a covector field on M. Then the following are equivalent:*

(a) $\omega\in\mathcal{T}^1(M)$.

(b) the component functions of ω are smooth with respect all charts.

(c) If $U\subset M$ is open and $V\in\mathcal{T}(U)$ is a smooth vector field in U, the function $p\mapsto\omega_p(V_p)$ is smooth.

Proof. Exerc. [Cf. Lemma 3.3] ☐

4.12 Tensor bundles over M

Let M be a smooth manifold.

Definition 4.13. We define tensor bundles over M as disjoint unions:

1. *k-covariant tensor bundle*
$$T^k M = \bigsqcup_{p\in M} T^k(T_pM),$$
2. *l-contravariant tensor bundle*
$$T_l M = \bigsqcup_{p\in M} T_l(T_pM), \quad \text{ja}$$
3. *(k,l)-tensor bundle*
$$T_l^k M = \bigsqcup_{p\in M} T_l^k(T_pM).$$

We identify:

$$\begin{aligned}
T^0M &= T_0M = M \times \mathbb{R},\\
T^1M &= TM^*,\\
T_1M &= TM,\\
T_0^kM &= T^kM,\\
T_l^0M &= T_lM.
\end{aligned}$$

All tensor bundles have natural C^∞ structures, so we can consider smooth sections. We say that a section $s\colon M \to T_l^kM$ is a *(k,l)-tensor field* (i.e. $\pi \circ s = id_M$, so $s(p) \in T_l^k(T_pM)$). Similarly a *smooth* (k,l)-tensor field is a smooth section $M \to T_l^kM$. Similarly, we define (smooth) *k-covariant tensor fields* and *l-contravariant tensor fields.* By our convention both 0-covariant and 0-contravariant tensors are real numbers, hence (smooth) 0-covariant tensor fields and (smooth) 0-contravariant tensor fields are (smooth) real-valued functions. We denote

$$\begin{aligned}
\mathcal{T}^k(M) &= \{\text{smooth sections of } T^kM\}\\
&= \{\text{smooth } k\text{-covariant tensor fields}\}\\
\mathcal{T}_l(M) &= \{\text{smooth sections of } T_lM\}\\
&= \{\text{smooth } l\text{-contravariant tensor fields}\}\\
\mathcal{T}_l^k(M) &= \{\text{smooth sections of } T_l^kM\}\\
&= \{\text{smooth } (k,l)\text{-tensor fields}\}.
\end{aligned}$$

If (U,x), $x = (x^1,\dots,x^n)$, is a chart and σ is a tensor field over U, we may write

$$\begin{aligned}
\sigma &= \sigma_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}, \quad \text{if } \sigma \text{ is a } k\text{-covariant tensor field},\\
\sigma &= \sigma^{j_1\cdots j_l} \partial_{j_1} \otimes \cdots \otimes \partial_{j_l}, \quad \text{if } \sigma \text{ is an } l\text{-contravariant tensor field, and}\\
\sigma &= \sigma^{j_1\cdots j_l}_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_l}, \quad \text{if } \sigma \text{ is a } (k,l)\text{-tensor field.}
\end{aligned}$$

The functions $\sigma_{i_1\cdots i_k}$, $\sigma^{j_1\cdots j_l}$ and $\sigma^{j_1\cdots j_l}_{i_1\cdots i_k}$ are called the *component functions* of σ with respect to the chart (U,x). We have again:

Lemma 4.14. *Let σ be a (k,l)-tensor field over M. Then the following are equivalent.*

(a) $\sigma \in \mathcal{T}^k_l(M)$.

(b) the component functions of σ are smooth with respect to every chart.

(c) If $U \subset M$ is open and $X_1, \ldots, X_k \in \mathcal{T}(U)$ are smooth vector fields on U and $\omega^1, \ldots, \omega^l \in \mathcal{T}^1(M)$ are smooth covector fields on U, then the function

$$p \mapsto \sigma\big(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l\big)_p \in \mathbb{R}$$

is smooth.

Proof. Exerc. [Cf. Lemma 3.3] $\square$

"Pullback". Let $f\colon M \to N$ be a smooth mapping and $k \in \{0,1,2,\ldots\}$. For every $p \in M$ we define a mapping (*pullback*)

$$f^*\colon T^k(T_{f(p)}N) \to T^k(T_pM)$$

by setting

$$f^*S(v_1, \ldots, v_k) = S(f_*v_1, \ldots, f_*v_k), \quad S \in T^k(T_{f(p)}N),\ v_1, \ldots, v_k \in T_pM.$$

Furthermore, we define the “pullback" operation for (smooth) k-covariant tehsor fields: Let $f\colon M \to N$ be smooth and let σ be a k-covariant tensor field on N. We define a k-covariant tensor field $f^*\sigma$ on M by setting

$$(f^*\sigma)_p = f^*(\sigma_{f(p)}), \quad p \in M.$$

In other words, if $p \in M$ and $v_1, \ldots, v_k \in T_pM$, then

$$(f^*\sigma)_p(v_1, \ldots, v_k) = \sigma_{f(p)}(f_*v_1, \ldots, f_*v_k).$$

Theorem 4.15. *Let M, N, P be smooth manifolds, $f\colon M \to N$ and $g\colon N \to P$ smooth mappings, $\sigma \in \mathcal{T}^k(N)$, $\tau \in \mathcal{T}^l(N)$ and $h \in C^\infty(N)$. Then*

(a) $f^*dh = d(h \circ f)$.

(b) $f^*(h\sigma) = (h \circ f) f^*\sigma$.

(c) $f^*(\sigma \otimes \tau) = f^*\sigma \otimes f^*\tau$.

(d) $f^*\sigma \in \mathcal{T}^k(M)$.

(e) $f^*\colon \mathcal{T}^k(N) \to \mathcal{T}^k(M)$ *is linear.*

(f) $(g \circ f)^* = f^* \circ g^*$.

(g) $(id_N)^*\sigma = \sigma$.

Proof. Let us prove some claims.

(b) Let $p \in M$ and $v_1, \ldots, v_k \in T_pM$. Then

$$\begin{aligned}
\big(f^*(h\sigma)\big)_p(v_1, \ldots, v_k) &= (h\sigma)_{f(p)}(f_*v_1, \ldots, f_*v_k) \\
&= h(f(p))\sigma_{f(p)}(f_*v_1, \ldots, f_*v_k) \\
&= (h \circ f)(p)\sigma_{f(p)}(f_*v_1, \ldots, f_*v_k) \\
&= \big((h \circ f) f^*\sigma\big)_p(v_1, \ldots, v_k).
\end{aligned}$$

(f) Let $\beta \in \mathcal{T}^k(P)$. Then $(g \circ f)^*\beta \in \mathcal{T}^k(M)$. Similarly,

$$(f^* \circ g^*)\beta = f^*(\underbrace{g^*\beta}_{\in \mathcal{T}^k(N)}) \in \mathcal{T}^k(M).$$

Let $p \in M$ and $v_1, \dots, v_k \in T_pM$. Then

$$\begin{aligned}
\big((g \circ f)^*\beta\big)_p(v_1, \dots, v_k) &= \beta_{g(f(p))}\big((g \circ f)_*v_1, \dots, (g \circ f)_*v_k\big) \\
&= \beta_{g(f(p))}\big(g_*(f_*v_1), \dots, g_*(f_*v_k)\big) \\
&= (g^*\beta)_{f(p)}(f_*v_1), \dots, f_*v_k) \\
&= \big(f^*(g^*\beta)\big)_p(v_1, \dots, v_k) \\
&= \big((f^* \circ g^*)\beta\big)_p(v_1, \dots, v_k).
\end{aligned}$$

We leave the others as exercise. □

Note: If f is not a diffeomorphism, we can not, in general, define a pullback operation for l-contravariant tensor fields or (k, l)-tensor fields.

4.16 Symmetric tensors and tensor fields

Let T be a k-covariant tensor on V. We say that T is *symmetric* if

$$T(v_1, \dots, \overset{i}{v_i}, \dots, \overset{j}{v_j}, \dots, v_k) = T(v_1, \dots, \overset{i}{v_j}, \dots, \overset{j}{v_i}, \dots, v_k)$$

for all $1 \le i < j \le k$. [Here the indices i and j over vectors indicate, of course, the positions of the vectors.] We denote

$$\Sigma^k(V) = \{T \in T^k(V) \colon T \text{ symmetric}\}.$$

Clearly $\Sigma^k(V)$ is a subspace of $T^k(V)$. We define a mapping, *symmetrization*, $\operatorname{Sym}\colon T^k(V) \to T^k(V)$,

$$\operatorname{Sym} T = \frac{1}{k!} \sum_{\sigma \in S_k} {}^{\sigma}T,$$

where S_k is the permutation group of $\{1, \dots, k\}$ and ${}^{\sigma}T$ is the k-covariant tensor

$${}^{\sigma}T(v_1, \dots, v_k) = T\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big).$$

Hence

$$\operatorname{Sym} T(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} T\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big).$$

The number of elements in S_k is $k!$, so $\operatorname{Sym} T$ is the "mean" of tensors ${}^{\sigma}T$ over all permutations $\sigma \in S_k$. Furthermore, we make a *convention* that ${}^{\tau}({}^{\sigma}T) = {}^{\tau\sigma}T$, where $\tau\sigma(i) = \tau(\sigma(i))$.

Lemma 4.17. *1.* Sym *is a linear map* $T^k(V) \to \Sigma^k(V)$.

2. $(\operatorname{Sym}) \circ (\operatorname{Sym}) = \operatorname{Sym}$.

3. $T \in \Sigma^k(V) \iff T = \operatorname{Sym} T$.

Proof. Clearly Sym is linear. Let $T \in T^k(V)$. If $\tau \in S_k$ is an arbitrary permutation, then $\operatorname{Sym} T\big(v_{\tau(1)}, \dots, v_{\tau(k)}\big) = {}^{\tau}(\operatorname{Sym} T)(v_1, \dots, v_k)$ and

$$\begin{aligned} {}^{\tau}(\operatorname{Sym} T) &= {}^{\tau}\Big(\frac{1}{k!}\sum_{\sigma\in S_k} {}^{\sigma}T\Big) = \frac{1}{k!}\sum_{\sigma\in S_k} {}^{\tau}({}^{\sigma}T) \\ &= \frac{1}{k!}\sum_{\sigma\in S_k} {}^{\tau\sigma}T = \frac{1}{k!}\sum_{\eta\in S_k} {}^{\eta}T \\ &= \operatorname{Sym} T, \end{aligned}$$

where $\eta = \tau\sigma$ runs through all the elements in S_k along with σ. Since $\tau \in S_k$ was arbitrary, $\operatorname{Sym} T$ is symmetric. On the other hand, if $T \in \Sigma^k(V)$, then

$$\begin{aligned} & T(v_1, \dots, v_k) = T\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big) \quad \forall \sigma \in S_k \\ \Rightarrow\ & k!T(v_1, \dots, v_k) = \sum_{\sigma\in S_k} T\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big) \\ \Rightarrow\ & T(v_1, \dots, v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} T\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big) = \operatorname{Sym} T(v_1, \dots, v_k). \end{aligned}$$

Hence

$$T \in \Sigma^k(V) \ \Rightarrow\ T = \operatorname{Sym} T.$$

In particular,

$$\operatorname{Sym}(\underbrace{\operatorname{Sym} T}_{\in \Sigma^k(V)}) = \operatorname{Sym} T.$$

On the other hand, if $T = \operatorname{Sym} T$, then $T \in \Sigma^k(V)$ since $\operatorname{Sym} T$ is symmetric. $\square$

If $S \in \Sigma^k(V)$ and $T \in \Sigma^l(V)$, their tensor product need not be symmetric, $S \otimes T \notin \Sigma^{k+l}(V)$ in general. We define the *symmetric product* of tensors $S \in \Sigma^k(V)$ and $T \in \Sigma^l(V)$ as $ST = \operatorname{Sym}(S \otimes T) \in \Sigma^{k+l}(V)$,

$$ST(v_1, \dots, v_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} S(v_{\sigma(1)}, \dots, v_{\sigma(k)})T\big(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}\big).$$

A symmetric k-covariant tensor field over M is a k-covariant tensor field whose value at every point $p \in M$ is a symmetric tensor. Similarly, we define symmetric l-contravariant tensors and tensor fields.

Example 4.18. The most important symmetric tensor field on M is so-called *Riemannian metric.* It is a smooth symmetric 2-covariant tensor field $g \in \mathcal{T}^2(M)$ that is positive definite at every point $p \in M$. Then the pair (M, g) is called a *Riemannian manifold.* Then for each $p \in M$

$$g_p \colon T_pM \times T_pM \to \mathbb{R}$$

is an inner product. Hence we can define the norm $|v| = g_p(v, v)^{1/2}$ of a vector $v \in T_pM$ and a length of a C^∞-path $\gamma \colon [a, b] \to M$

$$\ell(\gamma) = \int_a^b |\dot\gamma_t| dt.$$

The length of a piecewise C^∞-path is the sum of the lengths of the pieces. Suppose that M is connected and $p, q \in M$. Define

$$d(p, q) = \inf_{\gamma} \ell(\gamma),$$

where the infimum is taken over all piecewise C^∞-paths joining p and q. Then $d\colon M \times M \to \mathbb{R}$ satisfies the axioms of a metric and the topology determined by d is the same as the original topology of M. By using a smooth partition of unity one can prove that there are Riemannian metrics on every smooth manifold.

5 Differential forms

In this section we consider alternating covariant tensors and tensor fields. The sign of an alternating tensor changes if two vectors are switched. Differential forms are alternating k-covariant tensor fields. They are very important since, for example:

1. they can be integrated over manifolds and submanifolds independently of local representations;
2. they form a link between “analysis" and topology on a manifold (de Rham cohomology).

Furthermore, the classical differential operators like grad (gradient), div (divergence) and curl (curl) as well as Green’s, Gauss’s and Stokes’s theorems can be presented using differential forms.

5.1 Exterior algebra, alternating tensors

We say that a k-covariant tensor $T \in T^k(V)$ is *alternating* (antisymmetric or skew-symmetric) if

$$T(v_1,\ldots,\overset{i}{v_i},\ldots,\overset{j}{v_j},\ldots,v_k) = -T(v_1,\ldots,\overset{i}{v_j},\ldots,\overset{j}{v_i},\ldots,v_k)$$

for all $1 \le i < j \le k$. We denote

$$\Lambda^k(V) = \{T \in T^k(V)\colon T \text{ alternating}\}$$

and call the elements of $\Lambda^k(V)$ *k-covectors*. Clearly $\Lambda^k(V)$ is a vector subspace of $T^k(V)$. [Note: In some books the notation $\Lambda^k(V^*)$ is used.]

We denote $(\sigma(1),\sigma(2),\ldots,\sigma(k)) = \sigma(1,\ldots,k)$ if $\sigma \in S_k$. A permuation $\sigma \in S_k$ is called *transposition* if it interchanges two elements and leaves all the others fixed. In other words,

$$\sigma(1,\ldots,\overset{i}{i},\ldots,\overset{j}{j},\ldots,k) = (1,\ldots,\overset{i}{j},\ldots,\overset{j}{i},\ldots,k)$$

for some $1 \le i < j \le k$. A permutation σ is *even (odd)*, denoted by sgn $= +1$ (sgn $= -1$), if it can be expressed as a composition of even (odd) number of transpositions. Define a mapping, called the *alternating projection*, $\operatorname{Alt}\colon T^k(V) \to T^k(V)$,

$$\operatorname{Alt} T = \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)({}^\sigma T), \quad \text{ts.}$$

$$\operatorname{Alt} T(v_1,\ldots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)T\big(v_{\sigma(1)},\ldots,v_{\sigma(k)}\big).$$

Lemma 5.2. *1.* Alt *is a linear mapping* $T^k(V) \to \Lambda^k(V)$.

2. $(\operatorname{Alt})\circ(\operatorname{Alt}) = \operatorname{Alt}$.

3. $T \in \Lambda^k(V) \iff T = \operatorname{Alt} T$.

Proof. Linearity is clear. Fix indices $1 \le i < j \le k$. Let $\tau \in S_k$,

$$\tau(1, \dots, \overset{i}{i}, \dots, \overset{j}{j}, \dots, k) = (1, \dots, \overset{i}{j}, \dots, \overset{j}{i}, \dots, k),$$

so $\operatorname{sgn} \tau = -1$. Let $T \in T^k(V)$. Then

$$\operatorname{Alt} T(v_1, \dots, \overset{i}{v_j}, \dots, \overset{j}{v_i}, \dots, v_k) = {}^{\tau}(\operatorname{Alt} T)(v_1, \dots, \overset{i}{v_i}, \dots, \overset{j}{v_j}, \dots, v_k),$$

and so

$$\begin{aligned}
{}^{\tau}(\operatorname{Alt} T) &= {}^{\tau}\Big(\frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, {}^{\sigma}T\Big) \\
&= -{}^{\tau}\Big(\frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \tau)(\operatorname{sgn} \sigma)\, {}^{\sigma}T\Big) \\
&= -\frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \tau\sigma)\, {}^{\tau\cdot\sigma}T \\
&= -\frac{1}{k!} \sum_{\eta \in S_k} (\operatorname{sgn} \eta)\, {}^{\eta}T \\
&= -\operatorname{Alt} T,
\end{aligned}$$

where $\eta = \tau\sigma$ and $\operatorname{sgn} \eta = (\operatorname{sgn} \tau)(\operatorname{sgn} \sigma)$. Hence $\operatorname{Alt} T \in \Lambda^k(V)$ $\forall T \in T^k(V)$.
If $T \in \Lambda^k(V)$, then

$$T(v_1, \dots, v_k) = (\operatorname{sgn} \sigma) T\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big) \quad \forall \sigma \in S_k,$$

so

$$\begin{aligned}
\operatorname{Alt} T(v_1, \dots, v_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} \underbrace{(\operatorname{sgn} \sigma) T\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big)}_{= T(v_1, \dots, v_k)} \\
&= T(v_1, \dots, v_k).
\end{aligned}$$

Hence

$$T \in \Lambda^k(V) \;\Rightarrow\; \operatorname{Alt} T = T.$$

In particular, $(\operatorname{Alt}) \circ (\operatorname{Alt}) = \operatorname{Alt}$. Finally, if $\operatorname{Alt} T = T$, then $T \in \Lambda^k(V)$. □

Example 5.3. 1. $T \in T^0(V) \;\Rightarrow\; \operatorname{Alt} T = T$.

2. $T \in T^1(V) \;\Rightarrow\; \operatorname{Alt} T = T$.

3. $T \in T^2(V) \;\Rightarrow$

$$\operatorname{Alt} T(v, w) = \frac{1}{2}\big(T(v, w) - T(w, v)\big).$$

4. $T \in T^3(V) \;\Rightarrow$

$$\operatorname{Alt} T(x, y, z) = \frac{1}{6}\big(T(x, y, z) + T(y, z, x) + T(z, x, y) - T(y, x, z) - T(z, y, x) - T(x, z, y)\big).$$

Definition 5.4. If $\alpha \in T^k(V)$ and $\beta \in T^l(V)$, their *wedge product* (or *exterior product*) is $\alpha \wedge \beta \in \Lambda^{k+l}(V)$,

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta).$$

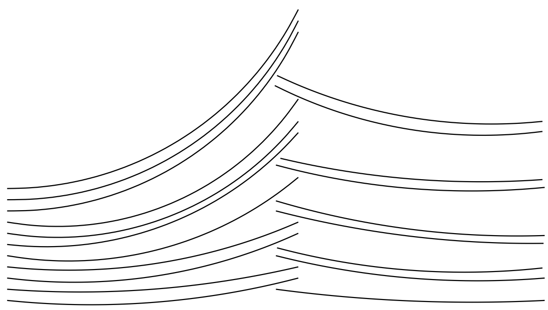
Thus

$$\alpha \wedge \beta(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big) \beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\big).$$

A permutation $\sigma \in S_{k+l}$, with

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k) \quad \text{ja} \quad \sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+l),$$

is called s (k,l) shuffle. Denote the set of all (k,l)-shuffles by $Sh(k,l)$.



Lemma 5.5. *If $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$, then*

$$\alpha \wedge \beta(v_1, \ldots, v_{k+l}) = \sum_{\sigma \in Sh(k,l)} (\operatorname{sgn} \sigma) \alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big) \beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\big).$$

Proof. Exerc. □

Theorem 5.6 (Properties of the wedge product)**.** *(a) Bilinearity:*

$$(a\alpha + b\beta) \wedge \eta = a(\alpha \wedge \eta) + b(\beta \wedge \eta) \qquad \forall \alpha, \beta \in T^k(V),\ \eta \in T^l(V),\ a, b \in \mathbb{R}.$$

(b)

$$\alpha \wedge \beta = (\operatorname{Alt} \alpha) \wedge \beta = \alpha \wedge (\operatorname{Alt} \beta) \quad \forall \alpha \in T^k(V),\ \beta \in T^l(V).$$

(c) Anticommutativity:

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \quad \forall \alpha \in T^k(V),\ \beta \in T^l(V).$$

(d) Associativity:

$$\alpha \wedge (\beta \wedge \eta) = (\alpha \wedge \beta) \wedge \eta = \frac{(k+l+p)!}{k!l!p!} \operatorname{Alt}(\alpha \otimes \beta \otimes \eta) \qquad \forall \alpha \in T^k(V),\ \beta \in T^l(V),\ \eta \in T^p(V).$$

(e) For all covector $\omega^1, \dots, \omega^k$ and for all vectors $v_1, \dots, v_k$

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \dots, v_k) = \det\bigl[\omega^j(v_i)\bigr].$$

Proof. (a): Bilinearity is clear.
(b): If $\tau \in S_k$, then

$$\begin{aligned}
\operatorname{Alt}{}^{\tau}T(v_1, \dots, v_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) T\bigl(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(k)}\bigr) \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \tau)(\operatorname{sgn}(\sigma\tau)) T\bigl(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(k)}\bigr) \\
&= (\operatorname{sgn} \tau) \operatorname{Alt} T(v_1, \dots, v_k).
\end{aligned}$$

Hence

$$\operatorname{Alt}({}^{\tau}\alpha) = (\operatorname{sgn} \tau) \operatorname{Alt} \alpha.$$

We obtain

$$\begin{aligned}
\operatorname{Alt}\bigl((\operatorname{Alt} \alpha) \otimes \beta\bigr) &= \operatorname{Alt}\Bigl(\frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)({}^{\sigma}\alpha \otimes \beta)\Bigr) \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \operatorname{Alt}({}^{\sigma}\alpha \otimes \beta) \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma') \operatorname{Alt}{}^{\sigma'}(\alpha \otimes \beta),
\end{aligned}$$

where $\sigma' \in S_{k+l}$ is a permutation s.t. $\sigma'(1, \dots, k, k+1, \dots, k+l) = (\sigma(1), \dots, \sigma(k), k+1, \dots, k+l)$, and thus $\operatorname{sgn} \sigma' = \operatorname{sgn} \sigma$ and ${}^{\sigma'}(\alpha \otimes \beta) = {}^{\sigma}\alpha \otimes \beta$. Since

$$\operatorname{Alt}{}^{\sigma'}(\alpha \otimes \beta) = (\operatorname{sgn} \sigma') \operatorname{Alt}(\alpha \otimes \beta),$$

we get

$$\begin{aligned}
\operatorname{Alt}\bigl((\operatorname{Alt} \alpha) \otimes \beta\bigr) &= \frac{1}{k!} \sum_{\sigma \in S_k} \underbrace{(\operatorname{sgn} \sigma')(\operatorname{sgn} \sigma')}_{=1} \operatorname{Alt}(\alpha \otimes \beta) \\
&= \operatorname{Alt}(\alpha \otimes \beta).
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha \wedge \beta &= \frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta) \\
&= \frac{(k+l)!}{k!l!} \operatorname{Alt}\bigl((\operatorname{Alt} \alpha) \otimes \beta\bigr) \\
&= (\operatorname{Alt} \alpha) \wedge \beta.
\end{aligned}$$

Similarly the other equation in (b).
(c): Let $\tau \in S_{k+l}$,

$$\tau(1, \dots, k+l) = (k+1, \dots, k+l, 1, \dots, k),$$

so $\operatorname{sgn} \tau = (-1)^{kl}$. Now

$$(\alpha \otimes \beta)(v_1, \dots, v_{k+l}) = (\beta \otimes \alpha)\bigl(v_{\tau(1)}, \dots, v_{\tau(k+l)}\bigr),$$

so $\alpha \otimes \beta = {}^{\tau}(\beta \otimes \alpha)$. Hence

$$\operatorname{Alt}(\alpha \otimes \beta) = (\operatorname{sgn} \tau) \operatorname{Alt}(\beta \otimes \alpha) = (-1)^{kl} \operatorname{Alt}(\beta \otimes \alpha)$$

that implies (c).
(d):

$$\begin{aligned}\alpha \wedge (\beta \wedge \eta) &= \frac{(k+l+p)!}{k!(l+p)!} \operatorname{Alt}\big(\alpha \otimes (\beta \wedge \eta)\big) \\ &= \frac{(k+l+p)!}{k!(l+p)!} \frac{(l+p)!}{l!p!} \operatorname{Alt}\big(\alpha \otimes \operatorname{Alt}(\beta \otimes \eta)\big) \\ &= \frac{(k+l+p)!}{k!l!p!} \operatorname{Alt}(\alpha \otimes \beta \otimes \eta)\end{aligned}$$

since $\operatorname{Alt}\big(\alpha \otimes \operatorname{Alt}(\beta \otimes \eta)\big) = \operatorname{Alt}\big(\alpha \otimes (\beta \otimes \eta)\big) = \operatorname{Alt}(\alpha \otimes \beta \otimes \eta)$. Computing similarly $(\alpha \wedge \beta) \wedge \eta$ we obtain (d).
(e): If $\alpha_i \in T^{d_i}(V)$, we obtain by repeating the property (d) that

$$\alpha_1 \wedge \cdots \wedge \alpha_k = \frac{(d_1 + \cdots + d_k)!}{d_1! \cdots d_k!} \operatorname{Alt}(\alpha_1 \otimes \cdots \otimes \alpha_k). \tag{5.7}$$

In particular, in case $d_i \equiv 1$ we get

$$\omega^1 \wedge \cdots \wedge \omega^k = k! \operatorname{Alt}(\omega^1 \otimes \cdots \otimes \omega^k), \tag{5.8}$$

so

$$\begin{aligned}\omega^1 \wedge \cdots \wedge \omega^k(v_1, \ldots, v_k) &= k! \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)(\omega^1 \otimes \cdots \otimes \omega^k)\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \omega^1\big(v_{\sigma(1)}\big) \cdots \omega^k\big(v_{\sigma(k)}\big) \\ &= \det\big[\omega^j(v_i)\big]. \quad \square\end{aligned}$$

Theorem 5.9. *Let V be an n-dimensional and let $\{\omega^1, \ldots, \omega^n\}$ be a basis of V^*. The the set*

$$\mathcal{E} = \{\omega^{i_1} \wedge \cdots \wedge \omega^{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n\}$$

of k-covectors forms a basis of $\Lambda^k(V)$. In particular,

$$\dim \Lambda^k(V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n$, then $\dim \Lambda^k(V) = 0$.

Proof. Let $\omega \in \Lambda^k(V)$. In particular, $\omega \in T^k(V)$, so

$$\omega = \omega_{i_1 \cdots i_k} \omega^{i_1} \otimes \cdots \otimes \omega^{i_k}$$

by Lemma 4.8. Since $\omega \in \Lambda^k(V)$, we have

$$\begin{aligned}\omega = \operatorname{Alt} \omega &= \omega_{i_1 \cdots i_k} \operatorname{Alt}\big(\omega^{i_1} \otimes \cdots \otimes \omega^{i_k}\big) \\ &\overset{(5.8)}{=} \big(\frac{\omega_{i_1 \cdots i_k}}{k!}\big) \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}.\end{aligned}$$

If above $i_j = i_l$ for some $j < l$, the corresponding term

$$\omega^{i_1} \wedge \cdots \wedge \omega^{i_j} \wedge \cdots \wedge \omega^{i_l} \wedge \cdots \wedge \omega^{i_k} = 0$$

because it is an alternating tensor. Hence we may assume that in every multi-index all numbers $i_1, \ldots, i_k$ are different. Furthermore, for every multi-index $i_1 \cdots i_k$ there exists a permutation $\sigma \in S_k$ such that $\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k)$, and so

$$\omega^{i_1} \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_k} = (\operatorname{sgn} \sigma) \omega^{\sigma(i_1)} \wedge \omega^{\sigma(i_2)} \wedge \cdots \wedge \omega^{\sigma(i_k)}.$$

Hence $\mathcal{E}$ spans $\Lambda^k(V)$. Suppose then that

$$\omega_{i_1 \cdots i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k} = 0.$$

Let $\{v_1, \ldots, v_n\}$ be the basis of V s.t. $\omega^j(v_i) = \delta^j_i$. Let $1 \leq j_1 < \cdots < j_k \leq n$, so

$$\omega_{i_1 \cdots i_k} \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}(v_{j_1}, \ldots, v_{j_k}) = 0. \tag{5.10}$$

If $i_l \notin \{j_1, \ldots, j_k\}$, it follows from (5.8) that

$$\omega^{i_1} \wedge \cdots \wedge \omega^{i_l} \wedge \cdots \wedge \omega^{i_k}(v_{j_1}, \ldots, v_{j_k}) = 0$$

since $\omega^j(v_i) = \delta^j_i$. Hence there remains only one term in the sum (5.10) and therefore also this term must vanish, i.e.

$$\omega_{j_1 \cdots j_k}(\omega^{j_1} \wedge \cdots \wedge \omega^{j_k}(v_{j_1}, \ldots, v_{j_k})) = 0.$$

This is possible only if $\omega_{j_1 \cdots j_k} = 0$ because

$$\omega^{j_1} \wedge \cdots \wedge \omega^{j_k}(v_{j_1}, \ldots, v_{j_k}) = \det[\delta^{j_l}_{j_r}] = \det I_k = 1.$$

Hence the covectors in $\mathcal{E}$ are linearly independent, and therefore $\mathcal{E}$ forms a basis of $\Lambda^k(V)$. The other cases are left as an exercise. □

Corollary 5.11. *If* $\dim V = n$, *then* $\dim \Lambda^n(V) = 1$. *If* $(\omega^1, \ldots, \omega^n)$ *is a basis of* V^*, *then* $\omega^1 \wedge \cdots \wedge \omega^n$ *spans* $\Lambda^n(V)$.

5.12 Differential forms on manifolds

Recall that $T^k M$ is the bundle of k-covariants tensors over M. We denote by

$$\Lambda^k M = \bigsqcup_{p \in M} \Lambda^k(T_p M)$$

the bundle of alternating k-covariants tensors [$\Lambda^k M$ is a smooth subbundle of $T^k M$].

A *differential k-form* ω is a section of $\Lambda^k M$:

$$M \to \Lambda^k M, \ p \mapsto \omega_p \in \Lambda^k(T_p M).$$

By our earlier convention a 0-covariant tensor is a real number, so a differential 0-form is a real-valued function.

The exterior product of a differential k-form α and a differential l-form β is defined pointwise by

$$(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p, \quad p \in M,$$

so $\alpha \wedge \beta$ is a differential $(k+l)$-form.

If (U,x), $x=(x^1,\dots,x^n)$, is a chart, then every differential k-form ω (in U) can be written (by Theorem 5.9) in a form

$$\omega = \sum_{i_1<\dots<i_k} \omega_{i_1\cdots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

Denote by $\mathcal{A}^k(M)$ the set of all smooth sections of $\Lambda^k M$. [Other frequently used notations are e.g. $\Omega^k(M)$, $\mathcal{E}^k(M)$ and $\bigwedge^k(M)$.] The pull-back of a differential form under a smooth mapping is a special case of a pull-back of a k-covariant tensor field: If $f\colon M \to N$ is smooth ω is a differential k-form on N, then $f^*\omega$ is the differential k-form on M defined by

$$(f^*\omega)_p(v_1,\dots,v_k) = \omega_{f(p)}(f_*v_1,\dots,f_*v_k).$$

Lemma 5.13. *Let $f\colon M \to N$ be smooth. Then:*

(a) $f^*\colon \mathcal{A}^k(N) \to \mathcal{A}^k(M)$ is linear.

(b) $f^*(\alpha\wedge\beta) = (f^*\alpha)\wedge(f^*\beta)$.

(c) If (U,y), $y=(y^1,\dots,y^n)$, is a chart on N, then

$$f^*\Big(\sum_{i_1<\dots<i_k} \omega_{i_1\cdots i_k} dy^{i_1}\wedge\dots\wedge dy^{i_k}\Big) = \sum_{i_1<\dots<i_k} (\omega_{i_1\cdots i_k}\circ f) d(y^{i_1}\circ f)\wedge\dots\wedge d(y^{i_k}\circ f).$$

Proof. Exerc. □

In the special case $k=n=\dim M=\dim N$ we obtain from (c) the following important (change of variables) formula.

Theorem 5.14. *Let M and N be smooth n-manifolds and $f\colon M\to N$ a smooth mapping. Let (U,x), $x=(x^1,\dots,x^n)$, be a chart on M and (V,y), $y=(y^1,\dots,y^n)$ a chart on N. If u is a real-valued function on V, then in $U\cap f^{-1}V$ we have:*

$$f^*(u dy^1\wedge\dots\wedge dy^n) = (u\circ f)(\det Df) dx^1\wedge\dots\wedge dx^n, \tag{5.15}$$

where $Df(p)$ is the matrix of f_{*p} with respect to bases $\{(\partial/\partial x^i)_p\}$ and $\{(\partial/\partial y^i)_{f(p)}\}$ (= the matrix of $(y\circ f\circ x^{-1})'(x(p))$ with respect to the standard basis of $\mathbb{R}^n$).

Proof. For every $p\in U$, $dx^1\wedge\dots\wedge dx^n$ spans $\Lambda^n(T_pM)$ (by Corollary 5.11), so it is enough to show that both sides of (5.15) are equal for $(\partial/\partial x^1,\dots,\partial/\partial x^n)$. Write $f^j = y^j\circ f$. By Lemma 5.13 (c),

$$f^*(u dy^1\wedge\dots\wedge dy^n) = (u\circ f) df^1\wedge\dots\wedge df^n,$$

and further by Theorem 5.6,

$$df^1\wedge\dots\wedge df^n(\partial/\partial x^1,\dots,\partial/\partial x^n) = \det\Big(df^j(\partial/\partial x^i)\Big) = \det\Big(\frac{\partial f^j}{\partial x^i}\Big) = \det Df.$$

Hence

$$\begin{aligned} &f^*(u dy^1\wedge\dots\wedge dy^n)(\partial/\partial x^1,\dots,\partial/\partial x^n) = (u\circ f)\det Df \\ =&(u\circ f)(\det Df)\underbrace{(dx^1\wedge\dots\wedge dx^n)(\partial/\partial x^1,\dots,\partial/\partial x^n)}_{=1}. \end{aligned}$$

□

Example 5.16. Let $\omega \in \mathcal{A}^2(\mathbb{R}^2),\ \omega = dx \wedge dy$. We write ω in polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$$

We get

$$\begin{aligned} \omega &= dx \wedge dy \\ &= d(r\cos\theta) \wedge d(r\sin\theta) \\ &= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr \\ &= r\, dr \wedge d\theta \end{aligned}$$

because $dr \wedge dr = d\theta \wedge d\theta = 0$ and $d\theta \wedge dr = -dr \wedge d\theta$ by skew-symmetricity.

5.17 Exterior derivative

Next we define a differential operator, the so-called *exterior derivative*, that attach to a smooth differential k-form $\alpha \in \mathcal{A}^k(M)$ a smooth $(k+1)$-form $d\alpha \in \mathcal{A}^{k+1}(M)$. The form $d\alpha$ is called the *exterior derivative* of α.

Theorem 5.18. *Let M be a smooth manifold. For every integer $k \geq 0$ there exists a unique ($\mathbb{R}$-)linear mapping $d = d_U^k \colon \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U),\ U \subset M$ open, satisfying the following:*

(i) d is $\wedge$-antiderivation: If $\alpha \in \mathcal{A}^k(U)$ and $\beta \in \mathcal{A}^l(U)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

(ii) If $k = 0$, then d is the differential

$$d\colon \underbrace{C^\infty(U)}_{=\mathcal{A}^0(U)} \to \mathcal{A}^1(U), \quad f \mapsto df.$$

(iii) $d^2 = d \circ d = 0$.

(iv) d commutes with the restriction: If $V \subset U \subset M$ are open and $\alpha \in \mathcal{A}^k(U)$, then $d(\alpha|V) = (d\alpha)|V$.

The condition (iv) means that d is a local operator.

Proof. Let us first prove the uniqueness:
Suppose that there exists an operator d that satisfies conditions (i)–(iv). Let $(U, x),\ x = (x^1, \ldots, x^n)$, be a chart. It follows from (iii) and (ii) that

$$d(dx^i) = 0,$$

for the differential dx^i of a coordinate function x^i since $x^i \in \mathcal{A}^0(U) = C^\infty(U)$. Let

$$\alpha = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Since $d(dx^i) = 0$, it follows from (i) that

$$d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = 0.$$

Hence by (i) and (ii)

$$d\alpha = \sum_{1 \le i_1 < \cdots < i_k \le n} (d\alpha_{i_1 \cdots i_k}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{5.19}$$

This means that d is uniquely determined in U by conditions (i)–(iii) and hence in the whole M by condition (iv).
To prove the existence we define d in every chart (U, x) by the formula (5.19). Clearly such d is $\mathbb{R}$-linear and satisfies (ii). To verify (i) we may assume, by linearity of d, that

$$\alpha = f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad \text{and} \quad \beta = g\, dx^{j_1} \wedge \cdots \wedge dx^{j_l}.$$

Then

$$\alpha \wedge \beta = fg\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l},$$

so by (5.19)

$$\begin{aligned}
&d(\alpha \wedge \beta) = d(fg) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\
&= (gdf + fdg) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\
&= \underbrace{df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}}_{=d\alpha} \wedge g\, dx^{j_1} \wedge \cdots \wedge dx^{j_l} + (-1)^k f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \underbrace{dg \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}}_{=d\beta} \\
&= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.
\end{aligned}$$

To check the condition (iii) it suffices to show that $d(d\alpha) = 0$ is

$$\alpha = f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

For every $f \in \mathcal{A}^0(U)$

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i,$$

so

$$\begin{aligned}
d(d\alpha) &= d\Big(\sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\Big) \\
&= \sum_{i=1}^n d\Big(\frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\Big) \\
&= \sum_{i=1}^n d\Big(\frac{\partial f}{\partial x^i}\Big) \wedge dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
&= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.
\end{aligned}$$

Hence

$$d(d\alpha) = 0,$$

since

$$\frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial^2 f}{\partial x^i \partial x^j} \quad \text{and} \quad dx^j \wedge dx^i = -dx^i \wedge dx^j.$$

We have proven that in every chart (U, x) the formula (5.19) defines $d = d_U$ that satisfies conditions (i)–(iii). It remains to prove that these "local" ds defines d in every open subset of M and that (iv) holds. It is enough to verify that the definition is independent of a chart. Suppose that $\tilde{d}$ is another operator defined by (5.19) in a chart (V, y), where $U \cap V \neq \emptyset$. Since also $\tilde{d}$ satisfies (i)–(iii), then $d = \tilde{d}$ in $U \cap V$ by the local uniqueness. □

Theorem 5.20. *Let $f\colon M \to N$ be smooth. Then for every $\alpha \in \mathcal{A}^k(N)$*

$$f^*(d\alpha) = d(f^*\alpha). \tag{5.21}$$

Proof. By locality and linearity it is enough to verify (5.21) in an arbitrary chart (V, y), $y = (y^1, \ldots, y^n)$, of N for a differential k-form

$$\alpha = u\, dy^{i_1} \wedge \cdots \wedge dy^{i_k},$$

where $u \in \mathcal{A}^0(V)$. Then

$$\begin{aligned} f^*\alpha &= (u \circ f) d(y^{i_1} \circ f) \wedge \cdots \wedge d(y^{i_k} \circ f), \\ d(f^*\alpha) &= d(u \circ f) \wedge d(y^{i_1} \circ f) \wedge \cdots \wedge d(y^{i_k} \circ f), \\ d\alpha &= du \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_k}, \\ f^*(d\alpha) &= f^*du \wedge f^*dy^{i_1} \wedge \cdots \wedge f^*dy^{i_k} \\ &= d(u \circ f) \wedge d(y^{i_1} \circ f) \wedge \cdots \wedge d(y^{i_k} \circ f). \quad \square \end{aligned}$$

Theorem 5.22. *If $\omega \in \mathcal{A}^1(M)$ and $X, Y \in \mathcal{T}(M)$, then*

$$d\omega(X, Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X, Y]\big). \tag{5.23}$$

Proof. Every smooth differential 1-form can be expressed locally as a sum of 1-forms $u\, dv$, where u and v are smooth real-valued functions. Thus it suffices to assume that

$$\omega = u\, dv.$$

Let X and Y be smooth vector fields. Then the left-hand side of (5.23) is

$$d(u\, dv)(X, Y) = du \wedge dv(X, Y) = du(X)dv(Y) - dv(X)du(Y) = (Xu)(Yv) - (Xv)(Yu),$$

and the right-hand side

$$\begin{aligned} &X\big(u\, dv(Y)\big) - Y\big(u\, dv(X)\big) - u\, dv\big([X, Y]\big) \\ =&X(uYv) - Y(uXv) - u[X, Y]v \\ =&\big((Xu)(Yv) + uX(Yv)\big) - \big((Yu)(Xv) + uY(Xv)\big) - u\big(X(Yv) - Y(Xv)\big) \\ &= (Xu)(Yv) - (Xv)(Yu). \quad \square \end{aligned}$$

Theorem 5.22 is a special case of the following that could be used to define the exterior derivative.

Theorem 5.24. *If $\omega \in \mathcal{A}^k(M)$ and $X_1, \ldots, X_{k+1} \in \mathcal{T}(M)$, then*

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} X_i\big(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}\big)$$
$$+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}\big),$$

where $\hat{X}$ indicates an omitted vector.

Proof. Omitted (see e.g. Lee [L2]).

Definition 5.25. We say that a differential form $\alpha \in \mathcal{A}^k(M)$ is

closed if $d\alpha = 0$ and

exact if $\alpha = d\beta$ for some $\beta \in \mathcal{A}^{k-1}(M)$.

Since $d \circ d = 0$, every exact form is closed. The converse is not true in general. The answer to the question when every closed p-form on M is exact depends, in fact, on topological properties of M and not at all on the smooth structure of M. Denote

$$\begin{aligned} \mathcal{Z}^p(M) &= \operatorname{Ker}\,[d\colon \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)] \\ &= \{\text{closed } p\text{-forms on } M\}, \\ \mathcal{B}^p(M) &= \operatorname{Im}\,[d\colon \mathcal{A}^{p-1}(M) \to \mathcal{A}^p(M)] \\ &= \{\text{exact } p\text{-forms on } M\}. \end{aligned}$$

Let us make a convention that $\mathcal{A}^p(M)$ is a trivial vector space if $p < 0$ or $p > \dim M$. The vector space (quotient space)

$$H^p_{dR}(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}$$

is called the pth *de Rham cohomology group* of M. Its elements are the equivalence classes $[\omega]$ of closed p-forms. Closed forms ω and ω' are equivalent (i.e. belong to the same equivalence class) if $\omega - \omega'$ is exact. Now every closed p-form on M is exact if and only if $H^p_{dR}(M) = 0$. The following de Rham theorem gives the connection to the topology of M: For every smooth manifold M and non-negative integer p, the de Rham cohomology group $H^p_{dR}(M)$ is isomorphic with the singular cohomology group $H^p(M, \mathbb{R})$ (see e.g. the books by Lee [L2] or Bredon [Br]).

Definition 5.26 (Interior multiplication)**.** Let V be a finite dimensional vector space and $X \in V$. We say that the linear mapping $i_X\colon \Lambda^k(V) \to \Lambda^{k-1}(V)$,

$$i_X\omega(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1}),$$

is the *interior multiplication* (or *contraction*) with X. If $k = 0$, we make a convention that $i_X\omega = 0$. We also denote

$$X \lrcorner\, \omega = i_X\omega.$$

Similarly, if $X \in \mathcal{T}(M)$ and $\omega \in \mathcal{A}^k(M)$, we define $i_X\omega \in \mathcal{A}^{k-1}(M)$ pointwise by setting

$$(i_X\omega)_p = i_{X_p}\omega_p, \quad p \in M.$$

Theorem 5.27. *Let $X \in \mathcal{T}(M)$. Then:*

(i) $i_X\colon \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ *is* $C^\infty(M)$*-linear:*

$$i_X(f\alpha + g\beta) = f i_X\alpha + g i_X\beta, \quad f, g \in C^\infty(M),\ \alpha, \beta \in \mathcal{A}^k(M).$$

(ii) $i_X \circ i_X = 0$.

(iii) i_X *is a* $\wedge$*-antiderivation:*

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X\beta), \quad \alpha \in \mathcal{A}^k(M).$$

(iv) $i_{fX}\omega = f i_X\omega$ *if* $f \in C^\infty(M)$.

(v) $i_X df = Xf$ *if* $f \in C^\infty(M)$.

Proof. (iii): Let $\beta \in \mathcal{A}^l(M)$. Denote $X_{k+l} = X$, so

$$\begin{aligned}
&i_X(\alpha \wedge \beta)(X_1, X_2, \dots, X_{k+l-1}) = (\alpha \wedge \beta)(\underbrace{X_{k+l}}_{=X}, X_1, \dots, X_{k+l-1}) \\
&= (-1)^{k+l-1} \alpha \wedge \beta(X_1, \dots, X_{k+l}) \\
&= \frac{(-1)^{k+l-1}}{k!l!} \sum_{\tilde\sigma \in S_{k+l}} (\operatorname{sgn}\tilde\sigma) \alpha(X_{\tilde\sigma(1)}, \dots, X_{\tilde\sigma(k)}) \beta(X_{\tilde\sigma(k+1)}, \dots, X_{\tilde\sigma(k+l)}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(i_X\alpha) \wedge \beta(X_1, X_2, \dots, X_{k+l-1}) \\
&= \frac{1}{(k-1)!l!} \sum_{\sigma \in S_{k+l-1}} (\operatorname{sgn}\sigma) i_X\alpha(X_{\tilde\sigma(1)}, \dots, X_{\tilde\sigma(k-1)}) \beta(X_{\tilde\sigma(k)}, \dots, X_{\tilde\sigma(k+l-1)}) \\
&= \frac{k}{k!l!} \sum_{\sigma \in S_{k+l-1}} (\operatorname{sgn}\sigma) \alpha(X_{k+l}, X_{\tilde\sigma(1)}, \dots, X_{\tilde\sigma(k-1)}) \beta(X_{\tilde\sigma(k)}, \dots, X_{\tilde\sigma(k+l-1)})
\end{aligned}$$

and

$$\begin{aligned}
&\alpha \wedge i_X\beta(X_1, X_2, \dots, X_{k+l-1}) \\
&= \frac{1}{k!(l-1)!} \sum_{\sigma \in S_{k+l-1}} (\operatorname{sgn}\sigma) \alpha(X_{\tilde\sigma(1)}, \dots, X_{\tilde\sigma(k)}) i_X\beta(X_{\tilde\sigma(k+1)}, \dots, X_{\tilde\sigma(k+l-1)}) \\
&= \frac{l}{k!l!} \sum_{\sigma \in S_{k+l-1}} (\operatorname{sgn}\sigma) \alpha(X_{\tilde\sigma(1)}, \dots, X_{\tilde\sigma(k)}) \beta(X_{k+l}, X_{\tilde\sigma(k+1)}, \dots, X_{\tilde\sigma(k+l-1)}).
\end{aligned}$$

Denote $S^i_{k+l} = \{\tilde\sigma \in S_{k+l} \colon \tilde\sigma(i) = k+l\}$. Then

$$\begin{aligned}
&i_X(\alpha \wedge \beta)(X_1, X_2, \dots, X_{k+l-1}) \\
&= \frac{(-1)^{k+l-1}}{k!l!} \Big(\sum_{i=1}^{k} \sum_{\tilde\sigma \in S^i_{k+l}} (\operatorname{sgn}\tilde\sigma) \alpha(X_{\tilde\sigma(1)}, \dots, \overset{i}{X_{k+l}}, \dots, X_{\tilde\sigma(k)}) \beta(X_{\tilde\sigma(k+1)}, \dots, X_{\tilde\sigma(k+l)}) \\
&+ \sum_{j=1}^{l} \sum_{\tilde\sigma \in S^{k+j}_{k+l}} (\operatorname{sgn}\tilde\sigma) \alpha(X_{\tilde\sigma(1)}, \dots, X_{\tilde\sigma(k)}) \beta(X_{\tilde\sigma(k+1)}, \dots, \overset{j}{X_{k+l}}, \dots, X_{\tilde\sigma(k+l)}) \Big).
\end{aligned}$$

For every $\tilde\sigma \in S^i_{k+l}$ define $\sigma \in S_{k+l-1}$ by setting

$$\sigma(j) = \begin{cases} \tilde\sigma(j), & 1 \le j \le i-1 \\ \tilde\sigma(j+1), & i \le j \le j+k-1. \end{cases}$$

Conversely, for any given $\sigma \in S_{k+l-1}$ there is a unique $\tilde\sigma \in S^i_{k+l}$ s.t. the above holds. On the other hand, $\sigma \in S_{k+l-1}$ can be interpreted as the element of S_k that keeps $(k+l)$ fixed. The sign $\operatorname{sgn}\sigma$ is the same in both interpretations. If σ and $\tilde\sigma$ correspond to each other in the way above, then

$$\begin{aligned} \sigma(1,\dots,k+l) &= (\tilde\sigma(1),\dots,\tilde\sigma(i-1),\tilde\sigma(i+1),\dots,\tilde\sigma(k+l),k+l) \\ &= \sigma_i(\tilde\sigma(1),\dots,\tilde\sigma(i-1),\overset{k+l}{\tilde\sigma(i)},\tilde\sigma(i+1),\dots,\tilde\sigma(k+l)), \end{aligned}$$

where $\sigma_i \in S_{k+l}$,

$$\sigma_i(1,\dots,k+l) = (1,\dots,i-1,i+1,\dots,k+l,i), \quad \operatorname{sgn}\sigma_i = (-1)^{k+l-i}.$$

So $\operatorname{sgn}\tilde\sigma = (-1)^{k+l-i}\operatorname{sgn}\sigma$. We obtain

$$\begin{aligned} &\sum_{\tilde\sigma\in S^i_{k+l}} (\operatorname{sgn}\tilde\sigma)\alpha(X_{\tilde\sigma(1)},\dots,\overset{i}{X_{k+l}},\dots,X_{\tilde\sigma(k)})\beta(X_{\tilde\sigma(k+1)},\dots,X_{\tilde\sigma(k+l)}) \\ &\sum_{\sigma\in S_{k+l-1}} (-1)^{k+l-i}(\operatorname{sgn}\sigma)\alpha(X_{\sigma(1)},\dots,X_{\sigma(i-1)},\overset{i}{X_{k+l}},X_{\sigma(i)},\dots,X_{\sigma(k-1)})\beta(X_{\sigma(k)},\dots,X_{\sigma(k+l-1)}) \\ &= \underbrace{(-1)^{k+l-i}(-1)^{i-1}}_{=(-1)^{k+l-1}} \sum_{\sigma\in S_{k+l-1}} (\operatorname{sgn}\sigma)\alpha(X_{k+l},X_{\sigma(1)},\dots,X_{\sigma(k-1)})\beta(X_{\sigma(k)},\dots,X_{\sigma(k+l-1)}). \end{aligned}$$

We do the same for all $1 \le i \le k$ to obtain

$$\begin{aligned} &\sum_{i=1}^{k}\sum_{\tilde\sigma\in S^i_{k+l}} (\operatorname{sgn}\tilde\sigma)\alpha(X_{\tilde\sigma(1)},\dots,\overset{i}{X_{k+l}},\dots,X_{\tilde\sigma(k)})\beta(X_{\tilde\sigma(k+1)},\dots,X_{\tilde\sigma(k+l)}) \\ &k(-1)^{k+l-1}\sum_{\sigma\in S_{k+l-1}} (\operatorname{sgn}\sigma)\alpha(X_{k+l},X_{\sigma(1)},\dots,X_{\sigma(k-1)})\beta(X_{\sigma(k)},\dots,X_{\sigma(k+l-1)}). \end{aligned}$$

Similarly we conclude that

$$\begin{aligned} &\sum_{j=1}^{l}\sum_{\tilde\sigma\in S^{k+j}_{k+l}} (\operatorname{sgn}\tilde\sigma)\alpha(X_{\tilde\sigma(1)},\dots,X_{\tilde\sigma(k)})\beta(X_{\tilde\sigma(k+1)},\dots,\overset{j}{X_{k+l}},\dots,X_{\tilde\sigma(k+l)}) \\ &= l(-1)^{k+l-1}\sum_{\sigma\in S_{k+l-1}} (\operatorname{sgn}\sigma)\alpha(X_{\sigma(1)},\dots,X_{\sigma(k)})\beta(X_{k+l},X_{\sigma(k+1)},\dots,X_{\sigma(k+l-1)}). \end{aligned}$$

Hence (iii) holds.

The other cases are left as an exercise. □

Lie derivatives of tensor fields. Using the flow of a vector field X we may define Lie derivatives of smooth tensor fields with respect to X. We consider only k-covariant tensor fields.

Let $\tau \in \mathcal{T}^k(M)$ be a smooth k-covariant tensor field, $X \in \mathcal{T}(M)$ and let θ be the flow of X. If $p \in M$ and $|t|$ is small enough, θ_t is a diffeomorphism between some neighborhoods of p and $\theta(t,p)$. Hence we can define the *Lie derivative* of τ with respect to X pointwise as the limit

$$(L_X\tau)_p = \lim_{t\to0}\frac{(\theta_t^*\tau)_p - \tau_p}{t} = \frac{d}{dt}\,(\theta_t^*\tau)_{p|t=0}\,.$$

It turns out that, indeed, the limit above exists at every point p and the mapping $p \mapsto (L_X\tau)_p$ is a smooth k-covariant tensor field.

Theorem 5.28. *Suppose that $X, Y \in \mathcal{T}(M)$ are smooth vector fields, $f \in C^\infty(M)$, σ and τ are smooth covariant vector fields, and ω and η are smooth differential forms. Then*

(a) $L_X f = Xf$.

(b) $d(L_X\omega) = L_X(d\omega)$.

(c) $L_X(f\sigma) = (L_X f)\sigma + f L_X \sigma$.

(d) $L_X(\sigma \otimes \tau) = (L_X\sigma) \otimes \tau + \sigma \otimes (L_X\tau)$.

(e) $L_X(\omega \wedge \eta) = (L_X\omega) \wedge \eta + \omega \wedge (L_X\eta)$.

(f) $L_X(i_Y\omega) = i_{(L_XY)}\omega + i_Y(L_X\omega)$.

(g) If $\sigma \in \mathcal{T}^k(M)$ *and* $Y_1, \ldots, Y_k \in \mathcal{T}(M)$, *then*

$$\begin{aligned} L_X\big(\sigma(Y_1, \ldots, Y_k)\big) = (L_X\sigma)(Y_1, \ldots, Y_k) + \sigma(L_XY_1, Y_2, \ldots, Y_k) + \\ \cdots + \sigma(Y_1, \ldots, Y_{k-1}, L_XY_k). \end{aligned}$$

(h) $L_{fX}\omega = fL_X\omega + df \wedge i_X\omega$.

Proof. We will prove some claims.
(b): Since the exterior derivative is linear, we obtain by Theorem 5.20 that

$$\begin{aligned} d(L_X\omega) &= d\big(\lim_{t\to 0} \frac{1}{t}(\theta_t^*\omega - \omega)\big) \\ &= \lim_{t\to 0} \frac{1}{t}\left(d(\theta_t^*\omega) - d\omega\right) \\ &= \lim_{t\to 0} \frac{1}{t}\left(\theta_t^*(d\omega) - d\omega\right) \\ &= L_X(d\omega). \end{aligned}$$

As an example we prove (c):
Let $\sigma \in \mathcal{T}^k(M)$, $f \in C^\infty(M), p \in M$, and $v_1, \ldots, v_k \in T_pM$. Then

$$\begin{aligned} \left(L_X(f\sigma)\right)_p(v_1, \ldots, v_k) &= \lim_{t\to 0} \frac{1}{t}\left(\left(\theta_t^*(f\sigma)\right)_p - (f\sigma)_p\right)(v_1, \ldots, v_k) \\ &= \lim_{t\to 0} \frac{1}{t}\left((f\sigma)_{\theta(t,p)}(\theta^p_{t*}v_1, \ldots, \theta^p_{t*}v_k) - (f\sigma)_p(v_1, \ldots, v_k)\right) \\ &= \lim_{t\to 0} \frac{1}{t}\left((f\circ\theta^p)(t)\sigma_{\theta(t,p)}(\theta^p_{t*}v_1, \ldots, \theta^p_{t*}v_k) - (f\circ\theta^p)(0)\sigma_p(v_1, \ldots, v_k)\right) \\ &= \lim_{t\to 0} \frac{1}{t}\left(\varphi(t)\psi(t) - \varphi(0)\psi(0)\right) \\ &= \varphi(0)\psi'(0) + \varphi'(0)\psi(0), \end{aligned}$$

where

$$\begin{aligned} &\varphi(t) = (f\circ\theta^p)(t), \quad \varphi(0) = f(p), \quad \varphi'(0) = X_pf, \\ &\psi(t) = \left(\theta_t^*\sigma\right)_p(v_1, \ldots, v_k), \quad \psi(0) = \sigma_p(v_1, \ldots, v_k), \text{ and } \psi'(0) = \left(L_X\sigma\right)_p(v_1, \ldots, v_k). \end{aligned}$$

Other cases are left as an exercise. $\square$

Theorem 5.29 (Cartan's magic formula)**.** *Let $X \in \mathcal{T}(M)$ and let $\omega \in \mathcal{A}^k(M)$. Then*

$$L_X\omega = i_X d\omega + d i_X\omega. \tag{5.30}$$

Proof. We prove the claim by induction with respect to k. For $k=0$ Theorems 5.27 (v) and 5.28 (a) imply that

$$L_X f = Xf = i_X df.$$

[Note: By our convention $i_X f = 0$ for smooth functions f.]
Next we prove (5.30) for smooth 1-forms. By linearity and locality we may assume that

$$\omega = u\,dv,$$

where u and v are smooth functions. Now

$$L_X(u\,dv) = uL_X dv + (L_X u)dv = u\,d(L_X v) + (Xu)dv = ud(Xv) + (Xu)dv$$

and

$$\begin{aligned} i_X d(udv) + di_X(udv) &= i_X(du \wedge dv) + d\big((\overbrace{i_X u}^{=0}) \wedge dv + ui_X dv\big) \\ &= (i_X du) \wedge dv - du \wedge (i_X dv) + d(uXv) \\ &= (Xu)dv - (Xv)du + ud(Xv) + (Xv)du \\ &= ud(Xv) + (Xu)dv. \end{aligned}$$

Hence (5.30) holds for $k=1$. Suppose that (5.30) holds for all smooth l-forms, where $l<k$ and $k>1$. Let $\omega \in \mathcal{A}^k(M)$ and write it locally as

$$\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Denote

$$\alpha_{i_1\cdots i_k} = \omega_{i_1\cdots i_k} dx^{i_1}$$

and

$$\beta_{i_1\cdots i_k} = dx^{i_2} \wedge \cdots \wedge dx^{i_k}.$$

Now we see that ω can be written as a sum of terms $\alpha \wedge \beta$, where α is a smooth 1-form and β a smooth $(k-1)$-form. It is enough to verify the formula for such term. By the induction hypothesis and by Theorem 5.28 (e) the left-hand side of (5.30) is

$$\begin{aligned} L_X(\alpha \wedge \beta) &= (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta) \\ &= \big(i_X d\alpha + di_X\alpha\big) \wedge \beta + \alpha \wedge \big(i_X d\beta + di_X\beta\big). \end{aligned}$$

Furthermore, both d and i_X are $\wedge$-antiderivations, so the right-hand side of (5.30) is

$$\begin{aligned} i_X d(\alpha \wedge \beta) + di_X(\alpha \wedge \beta) &= i_X(d\alpha \wedge \beta - \alpha \wedge d\beta) + d\big((i_X\alpha) \wedge \beta - \alpha \wedge i_X\beta\big) \\ &= (i_X d\alpha) \wedge \beta + d\alpha \wedge i_X\beta - (i_X\alpha) \wedge d\beta + \alpha \wedge i_X d\beta \\ &\quad + (di_X\alpha) \wedge \beta + i_X\alpha \wedge d\beta - d\alpha \wedge i_X\beta + \alpha \wedge di_X\beta \\ &= \big(i_X d\alpha + di_X\alpha\big) \wedge \beta + \alpha \wedge \big(i_X d\beta + di_X\beta\big) \\ &= L_X(\alpha \wedge \beta). \end{aligned}$$

$\square$

6 Integration of differential forms

Let $U \subset M$ be open and let ω be a differential form on U. Define the *support* of ω as

$$\operatorname{supp} \omega = U \cap \overline{\{p \in U : \omega_p \neq 0\}}.$$

Suppose then that $U \subset \mathbb{R}^n$ is open and ω is a continuous compactly supported differential n-form in U, i.e. $\operatorname{supp} \subset U$ is compact. Then

$$\omega = u(x) dx^1 \wedge \cdots \wedge dx^n, \quad x = (x^1, \ldots, x^n) \in \mathbb{R}^n,$$

where $u \colon U \to \mathbb{R}$ is continuous and compactly supported ($u \in C_0(U)$). Define

$$\int_U \omega = \int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} u(x) dx,$$

where we have Riemann integral on the right-hand side. [In fact, it is enough to assume that u is Lebesgue integrable over $\mathbb{R}^n$ and there is Lebesgue integral on the right-hand side.] Let then $f \colon W \to U$ be a diffeomorphism, where $W \subset \mathbb{R}^n$ is open. Suppose that U and W are connected. By the "change of variables fromula" (5.15),

$$f^*\omega = (\det Df)(u \circ f) dx^1 \wedge \cdots \wedge dx^n.$$

Using change of variables in Riemann (Lebesgue) integral we obtain
(6.1)

$$\int_{\mathbb{R}^n} f^*\omega = \int_W f^*\omega = \int_W (u \circ f)(x) \det Df(x) dx = \operatorname{sgn} \det Df(x) \int_U u(y) dy = \operatorname{sgn} \det Df(x) \int_{\mathbb{R}^n} \omega.$$

Note that the sign of $\det Df$ can not change in U since f is a diffeomorphism ($\det Df \neq 0$) and U is connected.

Suppose then that M is an *oriented* differentiable n-manifold. Let $\{(U_\alpha, x_\alpha)\}$ be an orientation, that is, a smooth atlas such that for every α and β for which $U_\alpha \cap U_\beta \neq \emptyset$, the Jacobian determinant of $x_\beta \circ x_\alpha^{-1}$ is positive at every point $q \in x_\alpha(U_\alpha \cap U_\beta)$:

$$\det\bigl(x_\beta \circ x_\alpha^{-1}\bigr)'(q) > 0, \quad \forall q \in x_\alpha(U_\alpha \cap U_\beta).$$

Let (U, φ), $\varphi \colon U \to W \subset \mathbb{R}^n$, be a chart in the orientation of M. Suppose that ω is a continuous differential n-form whose support $\operatorname{supp} \omega \subset U$ is compact. Then $(\varphi^{-1})^*\omega$ is a continuous compactly supported differential n-form in W. We define

$$\int_M \omega = \int_{\mathbb{R}^n} (\varphi^{-1})^*\omega. \tag{6.2}$$

Let (V, ψ), $\psi \colon V \to \tilde{W}$, be another chart in the orientation of M s.t. $\operatorname{supp} \omega \subset V$. Then $\operatorname{supp} \omega \subset U \cap V$, so we may assume (to simplify notation) that $U = V$. Now $f = \psi \circ \varphi^{-1} \colon W \to \tilde{W}$ is a diffeomorphism whose Jacobian determinant is positive in W. Since $\varphi^{-1} = \psi^{-1} \circ f$, we have $(\varphi^{-1})^* = f^* \circ (\psi^{-1})^*$, and therefore

$$\int_{\mathbb{R}^n} (\varphi^{-1})^*\omega = \int_{\mathbb{R}^n} f^* \circ (\psi^{-1})^*\omega = \int_{\mathbb{R}^n} f^*\bigl((\psi^{-1})^*\omega\bigr) = \int_{\mathbb{R}^n} (\psi^{-1})^*\omega$$

by (6.1). We conclude that the definition (6.2) is independent of the choice of the chart (within the orientation).

Next we want to define the integral over M of an arbitrary compactly supported continuous differential n-form.

6.3 Smooth partition of unity

Let X be a topological space. We say that a collection $\mathcal{U} \subset \mathcal{P}(X)$ is *locally finite* if every point of X has a neighborhood that intersects at most finitely many members of $\mathcal{U}$.

Definition 6.4 (Partition of unity)**.** Let X be a topological space and $\mathcal{F} = \{U_\alpha : \alpha \in \mathcal{A}\}$ an open cover of X. A collection $\{\psi_i : i \in I\}$ of continuous functions $\psi_i : X \to \mathbb{R}$ is a *partition of unity subordinate to* $\mathcal{F}$ if

(a) $0 \le \psi_i(x) \le 1$ for all $i \in I$ and $x \in X$,

(b) $\forall i \in I\ \exists \alpha \in \mathcal{A}$ s.t. $\operatorname{supp} \psi_i \subset U_\alpha$,

(c) $\{\operatorname{supp} \psi_i\}_{i \in I}$ is locally finite,

(d) $\sum_{i\in I} \psi_i(x) = 1$ for every $x \in X$.

The index set I can be arbitrary, in particular, it may be uncountable. By the condition (c) every $y \in X$ has a neighborhood where the sum in (d) has only finitely many non vanishing terms. Hence there is no problem with the sum.

If M is a smooth manifold and each ψ_i is smooth, we call $\{\psi_i\}$ a smooth partition of unity.

Let $\mathcal{U}$ be an open cover of X. We say that an open cover $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$. A topological space X is *paracompact* if X is Hausdorff and every open cover of X has a locally finite refinement.

Lemma 6.5. *Every topological manifold has a countable, locally finite open cover $\{V_j\}_{j\in\mathbb{N}}$ by precompact sets V_j. Furthermore, the sets V_j can be chosen such that $i \in \{j-1, j, j+1\}$ if $V_j \cap V_i \neq \emptyset$.*

Proof. Let M be a topological manifold and let $\{B_j\}_{j\in\mathbb{N}}$ be a countable cover where every B_j is precompact (see Theorem 0.20). Next we prove that M has a countable cover $\{U_j\}_{j\in\mathbb{N}}$ s.t. for every $j \in \mathbb{N}$

(a) U_j is open and precompact,

(b) $\bar{U}_j \subset U_{j+1}$,

(c) $B_j \subset U_j$.

Denote $U_1 = B_1$. Suppose that there exist sets U_j, $j = 1, \dots, k$, satisfying (a)–(c). Since $\bar{U}_k$ is compact and $\{B_j\}$ is an open cover of M, there exists $m_k \in \mathbb{N}$ s.t.

$$\bar{U}_k \subset B_1 \cup B_2 \cup \cdots \cup B_{m_k}.$$

We set $U_{k+1} = B_1 \cup B_2 \cup \cdots \cup B_{m_k}$. Then (a) and (b) hold for the index $j = k+1$, too. By increasing m_k if needed we may assume that $m_k \ge k+1$, so $B_{k+1} \subset U_{k+1}$ and (c) holds for the index $j = k+1$. We have proven by induction that there exists a countable family $\{U_j\}_{j\in\mathbb{N}}$ satisfying (a)–(c). Furthermore, it follows from (c) that $\{U_j\}_{j\in\mathbb{N}}$ is an open cover of M since $\{B_j\}_{j\in\mathbb{N}}$ is a cover of M. Finally we form a countable, locally finite open cover $\{V_j\}_{j\in\mathbb{N}}$ by precompact sets by setting $V_1 = U_3$ and $V_j = U_{j+2} \setminus \bar{U}_j$ when $j \ge 2$. Then every $\bar{V}_j$ is compact since it is a closed subset of a compact set $\bar{U}_{j+2}$. If $p \in M$, let k be the smallest positive integer such that $p \in U_{k+2}$. Then $p \in V_k$ and V_k intersects only with V_{k-1}, V_k, and V_{k+1}. Hence $\{V_j\}_{j\in\mathbb{N}}$ is locally finite. $\square$

Theorem 6.6. *Let M be a smooth n-manifold. Every open cover of M has a countable, locally finite refinement $\{W_i\}_{i\in\mathbb{N}}$ such that*

(i) there exist diffeomorphisms $\varphi_i\colon W_i \to B^n(0,3) \subset \mathbb{R}^n$ and

(ii) the sets $U_i = \varphi_i^{-1}B^n(0,1)$ cover M.

In particular, M is paracompact.

Proof. Let $\mathcal{X}$ be an arbitrary open cover of M and let $\{V_j\}_{j\in\mathbb{N}}$ be a countable, locally finite open cover of M by precompact sets given by Lemma 6.5. For every $p \in M$ let

$$W'_p = \bigcap_{V_j \ni p} V_j.$$

Since $\mathcal{X}$ is an open cover of M, $p \in X_p$ for some $X_p \in \mathcal{X}$. Denote $W''_p = W'_p \cap X_p$, so each $W''_p \subset X$ for some $X \in \mathcal{X}$. Let (U, φ) be a chart at p such that $\varphi(p) = 0$. We may assume that $B^n(0,3) \subset \varphi(U \cap W''_p)$. Denote $W_p = \varphi^{-1}B^n(0,3)$ and $U_p = \varphi^{-1}B^n(0,1)$. The family $\{U_p\colon p \in \bar{V}_k\}$ is an open cover of $\bar{V}_k$ for every k. Since $\bar{V}_k$ is compact, it can be covered by finitely many such $U^1_k, \dots, U^{m_k}_k$. Let $(W^1_k, \varphi^1_k), \dots, (W^{m_k}_k, \varphi^{m_k}_k)$ be the corresponding charts. Then the family $\{W^i_k\colon k \in \mathbb{N}, i \in \{1, \dots, m_k\}\}$ is a countable open cover of M that is a refinement of $\mathcal{X}$ and satisfies the conditions (i) and (ii). Next we prove that $\{W^i_k\}_{k,i}$ is locally finite. It is enough to show that each W^i_k intersects at most finitely many $W^{i'}_{k'}$. Assume on the contary that there are indices k_0 and i_0 such that $W^{i_0}_{k_0} \cap W^i_k \neq \emptyset$ for infinitely many W^i_k. For every k there exist only m_k sets W^i_k, so there must be infinitely many k such that $W^{i_0}_{k_0} \cap W^i_k \neq \emptyset$. By the construction $W^{i_0}_{k_0} \subset V_{j_0}$ for some j_0 and each $W^i_k \subset V_j$ for some j, so there exists V_j that contain W^i_k for infinitely many k. On the other hand, $W^i_k \cap V_k \neq \emptyset$, so V_j intersects infinitely many V_k. This leads to a contradiction since each V_j intersects only V_{j-1}, V_j, and V_{j+1}. □

Theorem 6.7. *Let M be a smooth manifold and $\mathcal{U} = \{U_\alpha\colon \alpha \in \mathcal{A}\}$ an arbitrary open cover of M. Then there exists a smooth partition of unity $\{\psi_i\colon i \in \mathbb{N}\}$ subordinate to $\mathcal{U}$.*

Proof. Let $\mathcal{U} = \{U_\alpha\colon \alpha \in \mathcal{A}\}$ be an open cover of M and $\{W_i\}_{i\in\mathbb{N}}$ a locally finite refinement of $\mathcal{U}$ such that conditions (i) and (ii) in Theorem 6.6 hold. Let $f_i\colon M \to \mathbb{R}$ be a smooth function such that $0 \le f_i \le 1$, $f_i \equiv 1$ in U_i, and $\operatorname{supp} f_i \subset W_i$ (see below). Define functions $\psi_i\colon M \to \mathbb{R}$,

$$\psi_i = \frac{f_i}{\sum_j f_j}.$$

Since $\{W_i\}$ is locally finite, each point of M has a neighborhood where the sum $\sum_j f_j$ (in the denominator) has only finitely many non-vanishing term. Furthermore, $\sum_j f_j(x) \ge 1$ for every x since $\{U_i\}$ covers M. Hence $\psi_i \in C^\infty(M)$, $0 \le \psi_i \le 1$, $\operatorname{supp}\psi_i \subset W_i$, and $\sum_i \psi_i(x) = 1$ for every $x \in M$. Let us prove next the existence of such f_i. First we notice that functions $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

and $h\colon \mathbb{R} \to \mathbb{R}$,

$$h(t) = \frac{f(2-t)}{f(2-t) + f(t-1),}$$

are smooth (exerc.). Furthermore, $h(t) = 1\ \forall t \leq 1,\ h(t) = 0\ \forall t \geq 2$ and $0 \leq h(t) \leq 1\ \forall t \in \mathbb{R}$. Hence the function $H\colon \mathbb{R}^n \to \mathbb{R},\ H(x) = h(|x|)$, is smooth, $0 \leq H(x) \leq 1\ \forall x \in \mathbb{R}^n,\ H(x) = 1\ \forall x \in \bar{B}^n(0,1)$ and $\operatorname{supp} H \subset \bar{B}^n(0,2)$. Finally, we define functions $f_i\colon M \to \mathbb{R}$ by setting

$$f_i(p) = \begin{cases} H(\varphi_i(p)), & p \in W_i, \\ 0, & p \in M \setminus \bar{W}_i. \end{cases}$$

□

6.8 Integration of a differential n-form

Let M be oriented and let ω be an arbitrary continuous, compactly supported differential n-form on M. Let $\mathcal{U} = \{U_\alpha\}$ be an orientation of M and $\{f_i\colon i \in I\}$ a smooth partition of unity subordinate to $\mathcal{U}$.

Define

$$\int_M \omega = \sum_{i \in I} \int_M f_i \omega.$$

In the sum there are only finitely many (non-vanishing) terms since $\operatorname{supp}\omega$ is compact and $\{\operatorname{supp} f_i\}$ is locally finite. Let us prove that the definition above is independent of the choices of an atlas and a partition of unity if the chosen atlases define the *same* orientation. Let $\mathcal{V} = \{V_\beta\}$ be another atlas that determines the same orientation as $\{U_\alpha\}$ and let $\{g_j : j \in J\}$ be a smooth partition of unity subordinate to $\mathcal{V}$. Then

$$f_i = f_i \sum_{j \in J} g_j = \sum_j f_i g_j,$$

so

$$\int_M f_i \omega = \sum_j \int_M f_i g_j \omega$$

and, furthermore,

$$\begin{aligned} \int_M \omega = \sum_i \int_M f_i \omega &= \sum_i \sum_j \int_M f_i g_j \omega \\ &= \sum_j \sum_i \int_M f_i g_j \omega = \sum_j \int_M g_j \omega \end{aligned}$$

as it should be. It is worth noticing that

$$\int_M f_i g_j \omega$$

is related to the atlas $\{U_\alpha\}$ when appearing on the first line above and to the atlas $\{V_\beta\}$ when appearing on the second line. These integrals are the same since both atlases determine the same orientation.

The change of variables formula generalizes as follows: Let M^n and N^n be smooth oriented n-manifolds and $f\colon M \to N$ a sense preserving diffeomorphism. If ω is a continuous compactly supported differential n-form on N, then

$$\int_M f^* \omega = \int_N \omega.$$

All the above hold for arbitrary Lebesgue integrable (compactly supported) differential n-forms.

7 Stokes's theorem

In his section we will state and prove Stokes's theorem. For that purpose we need the notion of a manifold with boundary and some more information on orientation.

7.1 Orientation

The following characterization is often used as a definition of orientability.

Theorem 7.2. *A smooth n-manifold M is orientable if and only if there exists a smooth differential n-form (an orientation form) $\omega \in \mathcal{A}^n(M)$ that does not vanish at any point (i.e. for every $p \in M$ there exist vectors $v_1, \dots, v_n \in T_pM$ s.t. $\omega_p(v_1, \dots, v_n) \neq 0$).*

Proof. Assume first that M is orientable ans let $\{(U_\alpha, x_\alpha)\}$ bean orientation. For every α define in U_α a smooth n-form

$$\omega^\alpha = dx_\alpha^1 \wedge dx_\alpha^2 \wedge \cdots \wedge dx_\alpha^n.$$

In other words, $\omega^\alpha = x_\alpha^*(dx^1 \wedge \cdots \wedge dx^n)$. If (U_β, x_β) is another chart in the orientation s.t. $U_\alpha \cap U_\beta \neq \emptyset$, then by Theorem 5.14

$$\omega^\beta = \det D(x_\beta \circ x_\alpha^{-1})\omega^\alpha. \tag{7.3}$$

Furthermore, the function $p \mapsto \det D(x_\beta \circ x_\alpha^{-1})(p)$ is positive in $U_\alpha \cap U_\beta$ since $(U_\alpha, x_\alpha$ and (U_β, x_β) belong to the same orientation. Let $\{\psi_i\}_{i\in\mathbb{N}}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$. For every $i \in \mathbb{N}$ choose α_i s.t. $\operatorname{supp}\psi_i \subset U_{\alpha_i}$. Next we define

$$\omega = \sum_{i\in\mathbb{N}} \psi_i \omega^{\alpha_i}.$$

Clearly ω is a smooth n-form. To show that ω does not vanish at any point, ix an arbitrary $p \in M$, so $p \in U_{\alpha_j}$ for some j. By (7.3)

$$\omega_p^{\alpha_i} = a_i \omega_p^{\alpha_j},$$

where the coefficients a_i are non-negative and $a_i > 0$ if $p \in U_{\alpha_i}$. Let $\partial_1, \dots, \partial_n$ be the coordinate vector fields associated to a chart $(U_{\alpha_j}, x_{\alpha_j})$, so

$$\omega^{\alpha_j}(\partial_1, \dots, \partial_n) \equiv 1$$

in U_{α_j}. Since $\sum_i \psi_i(p) = 1$, there exists $k \in \mathbb{N}$ s.t. $\psi_k(p) > 0$. Then $p \in \operatorname{supp}\psi_k \subset U_{\alpha_k}$, so $a_k > 0$ and

$$\begin{aligned}
\omega_p(\partial_1, \dots, \partial_n) &= \sum_i \psi_i(p)\omega_p^{\alpha_i}(\partial_1, \dots, \partial_n) \\
&= \sum_i \psi_i(p) a_i \omega_p^{\alpha_j}(\partial_1, \dots, \partial_n) \\
&= \sum_i \psi_i(p) a_i \geq \psi_k(p) a_k > 0.
\end{aligned}$$

Suppose then that there exists $\omega \in \mathcal{A}^n(M)$ s.t. $\omega_p \neq 0\ \forall p \in M$. Let $\{(U_\alpha, x_\alpha)\}$ be an atlas. We may assume that every $U\alpha$ is connected (replace U_α by its components and re-index if necessary). Then

$$\omega|U_\alpha = \omega_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n,$$

where ω_α is a smooth function that does not vanish at any point of U_α and does not change its sign. By changing the sign of a coordinate function (e.g. x_α^1) if necessary, we may assume that $\omega_\alpha > 0$ in U_α for all α. Let then (U_α, x_α) and (U_β, x_β) be charts such that $U_\alpha \cap U_\beta = W \neq \emptyset$. Then

$$(x_\alpha^{-1})^*\omega = (\omega_\alpha \circ x_\alpha^{-1})dx^1 \wedge \cdots \wedge dx^n$$

in $x_\alpha W$ and

$$(x_\beta^{-1})^*\omega = (\omega_\beta \circ x_\beta^{-1})dx^1 \wedge \cdots \wedge dx^n$$

in $x_\beta W$. On the other hand,

$$(x_\alpha^{-1})^*\omega = \big(x_\beta^{-1} \circ (x_\beta \circ x_\alpha^{-1})\big)^*\omega = (x_\beta \circ x_\alpha^{-1})^*\big((x_\beta^{-1})^*\omega\big)$$

in $x_\alpha W$, so

$$\begin{aligned}(\omega_\alpha \circ x_\alpha^{-1})dx^1 \wedge \cdots \wedge dx^n &= (x_\beta \circ x_\alpha^{-1})^*\big((x_\beta^{-1})^*\omega\big)\\ &= (x_\beta \circ x_\alpha^{-1})^*\big((\omega_\beta \circ x_\beta^{-1})dx^1 \wedge \cdots \wedge dx^n\big)\\ &= (\omega_\beta \circ x_\beta^{-1}) \circ (x_\beta \circ x_\alpha^{-1}) \det(x_\beta \circ x_\alpha^{-1})' dx^1 \wedge \cdots \wedge dx^n\end{aligned}$$

by Theorem 5.14. It follows that

$$\det(x_\beta \circ x_\alpha^{-1})'(q) > 0$$

for every $q \in x_\alpha W$ since both $\omega_\alpha \circ x_\alpha^{-1}$ and $(\omega_\beta \circ x_\beta^{-1}) \circ (x_\beta \circ x_\alpha^{-1})$ are positive functions in $x_\alpha W$. $\square$

Let M be a smooth manifold and $S \subset M$ a smooth submanifold. We say that a mapping $V\colon S \to TM$ is a *vector field along* S if $V_p \in T_pM$ for all $p \in S$. If S is a submanifold of codimension 1 (i.e. a *hypersurface*), then a vector $v \in T_pM$, $p \in S$, is called *transversal* to S if v and T_pS span T_pM. Furthermore, a vector field V along S is transversal to S if V_p is transversal to S for all $p \in S$.

Theorem 7.4. *If M is orientable, $S \subset M$ a hypersurface, and V a smooth transversal vector field along S. Then S is orientable. If ω is an orientation form on M, then $(i_V\omega)|S$ is an orientation form on S.*

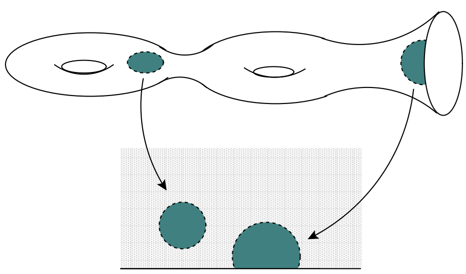
Proof. (Exerc.)

7.5 Smooth manifolds with boundary

Denote

$$\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \colon x^n > 0\} \quad \text{and} \quad \bar{\mathbb{H}}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \colon x^n \geq 0\}.$$

A topological space M is a topological n-manifold with boundary if M is Hausdorff, N_2, and every point of M has a neighborhood that is homeomorphic with some open subset of $\bar{\mathbb{H}}^n$ (w.r.t. relative topology).



A point $p \in M$ is called a boundary point of M if there exists a chart $(U, x),\ x = (x^1, \dots, x^n) \colon U \to \bar{\mathbb{H}}^n$, s.t. $x^n(p) = 0$. The set of boundary points is denoted by ∂M (this is not a boundary in topological sense).

To define the notion of a smooth manifold with boundary, recall that a mapping $f \colon A \to \mathbb{R}^m$, where $A \subset \mathbb{R}^n$ is arbitrary, is called smooth if every point $x \in A$ has a neighborhood V and a smooth mapping $F \colon V \to \mathbb{R}^m$ s.t. $F|V \cap A = f|V \cap A$.

Let $U \subset \bar{\mathbb{H}}^n$ be open and $f \colon U \to \mathbb{R}^m$ smooth. Suppose that $U \cap \partial\mathbb{H}^n \neq \emptyset$. We define partial derivatives of f at $p \in U \cap \partial\mathbb{H}^n$ as follows. Let V be a neighborhood of p (in $\mathbb{R}^n$) and $F \colon V \to \mathbb{R}^m$ a smooth mapping s.t. $F|V \cap U = f|V \cap U$. Then F has continuous partial derivatives of every order in V

$$\frac{\partial^{|\alpha|} F_i}{\partial^\alpha x}, \quad i = 1, \dots, m,$$

and

$$\frac{\partial^{|\alpha|} F_i}{\partial^\alpha x} = \frac{\partial^{|\alpha|} f_i}{\partial^\alpha x}$$

in $V \cap \operatorname{int} U$. By continuity, we may define partial derivatives of f at p independently of the choice of F by setting

$$\frac{\partial^{|\alpha|} f_i}{\partial^\alpha x}(p) = \frac{\partial^{|\alpha|} F_i}{\partial^\alpha x}(p).$$

A smooth n-manifold with boundary can now be defined as in the usual (without boundary) case. A pair (U, φ) is called a chart on an n-manifold M wih boundary if $U \subset M$ is open and $\varphi \colon U \to \varphi U \subset \bar{\mathbb{H}}^n$ is a homeomorphism (w.r.t. relative topology). Charts (U, φ) and (V, ψ) are C^∞-compatible if the mappings $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ and $\varphi \circ \psi^{-1} \colon \psi(U \cap V) \to \varphi(U \cap V)$ are smooth. A smooth n-manifold with boundary is a pair $(M, \mathcal{A})$, where M is a topological n-manifold with boundary and $\mathcal{A}$ is a maximal C^∞-atlas on M. A chart (U, φ) is an *interior chart* if $\varphi U \subset \mathbb{H}^n$, otherwise it is a *boundary chart* (so $\varphi U \cap \partial\mathbb{H}^n \neq \emptyset$ in that case). The smoothness of a mapping between two (smooth) manifolds with boundary is defined by using local representations. Tangent vectors and the tangent space T_pM at a boundary point $p \in \partial M$ as well as the tangent map f_{*p} are defined as in the usual case of a manifold without boundary. In particular, T_pM is an n-dimensional vector space also for points $p \in \partial M$.

It can be shown that ∂M is a topological $(n-1)$-manifold that has a canonical smooth structure so that the inclusion $i \colon \partial M \hookrightarrow M$ is a smooth embedding.

7.6 Stokes's theorem

For the Stokes's theorem we need a way to attach a suitable orientation to ∂M for an oriented M.

Let M be a smooth n-manifold with boundary ∂M. We consider ∂M as a smooth $(n-1)$-dimensional submanifol. We say that a vector $v \in T_pM$, where $p \in \partial M$, is *inward-pointing* if $v \notin T_p\partial M$ and there exists a smooth path $\gamma \colon [0, \varepsilon[\to M$ s.t. $\gamma(0) = p$ and $\dot\gamma_0 = v$. Similarly, a vector $v \in T_pM$ is *outward-pointing* if $-v$ is inward-pointing. A vector field V along ∂M is inward-pointing (outward-pointing) if V_p is inward-pointing (outward-pointing) for every $p \in \partial M$.

Lemma 7.7. *Let M be a smooth manifold with boundary, $p \in \partial M$, $(U, x),\ x = (x^1, \dots, x^n) \colon U \to \bar{\mathbb{H}}^n$, a chart at p and let $\partial_1, \dots, \partial_n$ be the corresponding coordinate vector fields in U. Then a vector $v = v^i(\partial_i)_p \in T_pM$ is inward-pointing if and only if $v^n > 0$.*

Proof. (Exerc.)

We can obtain the following existence result by applying partition of unity.

Lemma 7.8. *Let M be a smooth manifold with boundary. Then there exists a smooth outward-pointing vector field along ∂M.*

Proof. Let $\{(U_\alpha, x_\alpha)\}$ be a family of boundary charts of M such that $\partial M \subset \cup_\alpha U_\alpha$. Then

$$V^\alpha = -\frac{\partial}{\partial x_\alpha^n}|\partial M \cap U_\alpha$$

is a smooth outward-pointing vector field along $\partial M \cap U_\alpha$ for all α. Let $\{\psi_i\}$ be a smooth partition of unity subordinate to $\{U_\alpha \cap \partial M\}$ on ∂M. For every i choose α_i such that $\operatorname{supp} \psi_i \subset U_{\alpha_i} \cap \partial M$. Then

$$V = \sum_i \psi_i V^{\alpha_i}$$

is a smooth vector field along ∂M. It remains to prove that V is ouward-pointing. Let $p \in \partial M$. Since $\sum_i \psi_i(p) = 1$, there exists k s.t. $\psi_k(p) > 0$, so $p \in U_{\alpha_k}$. Let $\partial_1, \ldots, \partial_n$ be the coordinate vector fields associated to the chart $(U_{\alpha_k}, x_{\alpha_k})$ in U_{α_k}. Denote by $(\psi_i V^{\alpha_i})_p^n$ the nth component of a vector $(\psi_i V^{\alpha_i})_p$ with respect to the basis $\{(\partial_1)_p, \ldots, (\partial_n)_p\}$. Then each $(\psi_i V^{\alpha_i})_p^n$ is non-positive and, moreover, $(\psi_k V^{\alpha_k})_p^n = -\psi_k(p) < 0$. Then V_p^n, the nth component of V_p is negative:

$$V_p^n = \sum_i (\psi_i V^{\alpha_i})_p^n \leq -\psi_k(p) < 0.$$

□

Suppose that M is an oriented smooth n-manifold with boundary and let ω be an orientation n-form associated to the orientation of M (see Theorem 7.2). More precisely: Let $\{(U_\alpha, x_\alpha)\}$ be an orientation of M and ω an orientation n-form s.t. in every U_α

$$\omega|U_\alpha = \omega_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n,$$

where ω_α is a smooth *positive* function. Suppose, moreover, that V is a smooth otward-pointing vector field along ∂M. Then $(i_V\omega)|\partial M$ is an orientation $(n-1)$-form on ∂M. Finally, we equip ∂M with the orientation determined by $(i_V\omega)|\partial M$. Such orientation on ∂M is called the *induced* (or the *Stokes*) *orientation*. An arbitrary orientation n-form associated to the original orientation of M and any outward-pointing vector field along ∂M yield the same orientation on ∂M (Exerc.).

Example 7.9. Let $M = \bar{\mathbb{H}}^n$ and let

$$\omega = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

be the standard orientation n-form on $\mathbb{R}^n$ (and also on $\bar{\mathbb{H}}^n$). We identify $\partial\bar{\mathbb{H}}^n$ and $\mathbb{R}^{n-1}$ by identifying points $(x^1, \ldots, x^{n-1}, 0) \in \partial\bar{\mathbb{H}}^n$ and $(x^1, \ldots, x^{n-1}) \in \mathbb{R}^{n-1}$. The vector field

$$V = -\frac{\partial}{\partial x^n}$$

restricted to $\partial\bar{\mathbb{H}}^n$ is a smooth outward-pointing vector field along $\partial\bar{\mathbb{H}}^n$. Let us compute $i_V\omega$ by using the $\wedge$-antiderivation property (Theorem 5.27 (iii)):

$$\begin{aligned}
i_V\omega &= i_V(dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n) \\
&= (i_V dx^1) \wedge (dx^2 \wedge \cdots \wedge dx^n) - dx^1 \wedge i_V(dx^2 \wedge \cdots \wedge dx^n) \\
&= dx^1(V) dx^2 \wedge \cdots \wedge dx^n - dx^1 \wedge (i_V dx^2) \wedge (dx^3 \wedge \cdots \wedge dx^n) + dx^1 \wedge dx^2 \wedge i_V(dx^3 \wedge \cdots \wedge dx^n) \\
&\ \ \vdots \\
&= \sum_{i=1}^n (-1)^{i-1} \underbrace{dx^i(V)}_{=-\delta_n^i} dx^1 \wedge dx^2 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n \\
&= -(-1)^{n-1} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1} \\
&= (-1)^n dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1}.
\end{aligned}$$

We observe that $(i_V\omega)|\partial\bar{\mathbb{H}}^n$ is the standard orientation $(n-1)$-form on $\mathbb{R}^{n-1}$ (and hence the induced orientation of $\partial\bar{\mathbb{H}}^n$ is the same as the standard orientation of $\mathbb{R}^{n-1}$) if and only if n is even.

Theorem 7.10 (Stokes's theorem)**.** *Let M be a smooth oriented n-manifold with boundary and let ω be a smooth compactly supported $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega, \tag{7.11}$$

where ∂M has the induced orientation.

Proof. (a) Let us prove the claim first in the case $M = \bar{\mathbb{H}}^n$. Since $\operatorname{supp}\omega$ is compact, there exists $R > 0$ s.t. $\operatorname{supp}\omega \subset Q = [-R, R] \times \cdots \times [-R, R] \times [0, R]$. We may write ω in the form

$$\omega = \sum_{i=1}^n \omega_i dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n,$$

so

$$\begin{aligned}
d\omega &= \sum_{i=1}^n d\omega_i \wedge dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n \\
&= \sum_{i,j=1}^n \frac{\partial\omega_i}{\partial x^j} \underbrace{dx^j \wedge dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n}_{=\delta_i^j dx^j \wedge dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n} \\
&= \sum_{i=1}^n \frac{\partial\omega_i}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \frac{\partial\omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n.
\end{aligned}$$

Then

$$\begin{aligned}
\int_M d\omega &= \sum_{i=1}^n (-1)^{i-1} \int_Q \frac{\partial\omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \int_Q \frac{\partial\omega_i}{\partial x^i} dm(x) \\
&= \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial\omega_i}{\partial x^i} dx^1 \cdots dx^n.
\end{aligned}$$

Next we change, in each term, the order of integration so that we integrate first with respect to x^i. If $i \neq n$, we get

$$\int_{-R}^{R} \frac{\partial \omega_i}{\partial x^i}(x^1, \ldots, x^i, \ldots, x^n) dx^i = \omega_i(x^1, \ldots, R, \ldots, x^n) - \omega_i(x^1, \ldots, -R, \ldots, x^n) = 0$$

for all $(x^1, \ldots, \hat{x}^i, \ldots, x^n)$ since $\operatorname{supp} \omega_i \subset Q$. Hence

$$\sum_{i=1}^{n-1} (-1)^{i-1} \int_Q \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n = 0$$

and so

$$\begin{aligned} \int_M d\omega &= (-1)^{n-1} \int_Q \frac{\partial \omega_n}{\partial x^n} dx^1 \wedge \cdots \wedge dx^n \\ &= (-1)^{n-1} \int_{-R}^{R} \int_{-R}^{R} \cdots \int_0^R \frac{\partial \omega_n}{\partial x^n} dx^n dx^1 \cdots dx^{n-1}. \end{aligned}$$

On the other hand

$$\int_0^R \frac{\partial \omega_n}{\partial x^n}(x^1, \ldots, x^{n-1}, x^n) dx^n = -w_n(x^1, \ldots, x^{n-1}, 0),$$

so

$$\int_M d\omega = (-1)^n \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n(x^1, \ldots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}.$$

Next we integrate ω over $\partial \bar{\mathbb{H}}^n$ by using the standard orientation of $\mathbb{R}^{n-1}$ on $\partial \bar{\mathbb{H}}^n$ (i.e. we integrate ω over $\mathbb{R}^{n-1} \times \{0\}$). We obtain

$$\int_{\mathbb{R}^{n-1} \times \{0\}} \omega = \sum_{i=1}^{n} \int_{Q \cap \mathbb{R}^{n-1} \times \{0\}} \omega_i dx^1 \wedge \cdots \wedge \hat{dx}^i \wedge \cdots \wedge dx^n.$$

Now $dx^n | \mathbb{R}^{n-1} \times \{0\} = 0$, so only one non-zero term (when $i = n$) remains:

$$\int_{\mathbb{R}^{n-1} \times \{0\}} \omega = \int_{Q \cap \mathbb{R}^{n-1} \times \{0\}} \omega_n dx^1 \wedge \cdots \wedge dx^{n-1}.$$

By Example 7.9 the induced orientation of $\partial \bar{\mathbb{H}}^n$ $(= \partial M)$ is the same as the standard orientation of $\mathbb{R}^{n-1}$ if and only if n is even. Hence integrating ω with respect to the induced orientation yields

$$\int_{\partial M} \omega = (-1)^n \int_{Q \cap \partial M} \omega_n dx^1 \wedge \cdots \wedge dx^{n-1} = \int_M d\omega$$

as desired.

(b) Let then M be an arbitrary smooth oriented n-manifold with boundary. Suppose first that ω is a smooth compactly supported $(n-1)$-form s.t. $\operatorname{supp} \omega \subset U$ for some chart (U, φ) belonging to the orientation of M. Then

$$\int_M d\omega = \int_{\bar{\mathbb{H}}^n} (\varphi^{-1})^* d\omega = \int_{\bar{\mathbb{H}}^n} d((\varphi^{-1})^* \omega).$$

By (a)

$$\int_{\bar{\mathbb{H}}^n} d((\varphi^{-1})^* \omega) = \int_{\partial \bar{\mathbb{H}}^n} (\varphi^{-1})^* \omega,$$

when $\partial\bar{\mathbb{H}}^n$ has the induced orientation. Since φ is a sense preserving diffeomorphism, ((U,φ) belongs to the orientation of M) and φ_* maps outward-pointing vectors of $\partial M \cap U$ to outward-pointing vectors of $\partial\bar{\mathbb{H}}^n$, $\varphi|\partial M \cap U$ is a sense preserving diffeomorphism onto its image $\varphi U \cap \partial\bar{\mathbb{H}}^n$. Hence

$$\int_{\partial\bar{\mathbb{H}}^n} (\varphi^{-1})^*\omega = \int_{\partial M} \omega.$$

(c) Suppose finally that ω is an arbitrary smooth compactly supported $(n-1)$-form on M. Let $\{(U_\alpha, x_\alpha)\}$ be an orientation of M and let $\{\psi_i\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$. Applying what we proved above for forms $\psi_i\omega$ we get

$$\begin{aligned}
\int_{\partial M} \omega &= \sum_i \int_{\partial M} \psi_i\omega = \sum_i \int_M d(\psi_i\omega) \\
&= \sum_i \int_M (d\psi_i \wedge \omega + \psi_i d\omega) \\
&= \int_M d(\sum_i \psi_i) \wedge \omega + \int_M (\sum_i \psi_i) d\omega \\
&= \int_M d\omega
\end{aligned}$$

since $\sum_i \psi_i \equiv 1$, and therefore $d(\sum_i \psi_i) \equiv 0$. $\square$

Corollary 7.12. *Let M be a smooth, compact, oriented n-manifold without boundary (i.e. $\partial M = \emptyset$). Then the integral of every exact n-form over M vanishes:*

$$\int_M d\omega = 0 \quad \forall \omega \in \mathcal{A}^{n-1}(M).$$

Corollary 7.13. *Let M be a smooth, compact, oriented n-manifold with boundary. If $\omega \in \mathcal{A}^{n-1}(M)$ is closed, then the integral of ω over ∂M vanishes:*

$$\int_{\partial M} \omega = 0 \quad \textit{if } d\omega = 0.$$

The *divergence* of a smooth vector field X with respect to an orientation n-form ω is the function $\operatorname{Div}_\omega X \colon M \to \mathbb{R}$ s.t. $(\operatorname{Div}_\omega X)\omega = L_X\omega$.

Corollary 7.14 (The divergence formula, the Gauss formula)**.** *Let M be a smooth, oriented n-manifold with boundary and let ω be an orientation n-form on M (so-called volume form). Then for every compactly supported vector field $X \in \mathcal{T}(M)$*

$$\int_M (\operatorname{Div}_\omega X)\omega = \int_{\partial M} i_X\omega.$$

Corollary 7.15 (Integration by parts)**.** *Let M be a smooth, compact, oriented n-manifold with boundary, $X \in \mathcal{T}(M)$, $\alpha \in \mathcal{A}^k(M)$, and $\beta \in \mathcal{A}^{n-k}(M)$, $0 \le k \le n$. Then*

$$\int_M (L_X\alpha) \wedge \beta = \int_{\partial M} i_X(\alpha \wedge \beta) - \int_M \alpha \wedge (L_X\beta).$$

8 Whitney embedding and approximation

First recall: If M is a smooth manifold and $\{U_\alpha\}$ an open cover of M, there exists a smooth partitioen of unity $\{\psi_i\}_{i\in\mathbb{N}}$ subordinate to $\{U_\alpha\}$.

Proposition 8.1. *Let $\{U_\alpha\}$ and $\{\psi_i\}$ be as above. Let $g_\alpha\colon U_\alpha \to \mathbb{R}$ be arbitrary smooth functions. Fo each $i \in \mathbb{N}$ choose α_i such that $\operatorname{supp}\psi_i \subset U_{\alpha_i}$. Then the function $g\colon M \to \mathbb{R}$,*

$$g = \sum_{i\in\mathbb{N}} \psi_i g_{\alpha_i}$$

is smooth.

Proof. The claim follows from the local finiteness of $\{\operatorname{supp}\psi_i\}$. □

Lemma 8.2. *Let M be a smooth manifold and $K \subset M$ a compact set. Let $g\colon K \to \mathbb{R}$ be smooth. [Recall that this means that g extends locally at each point $p \in K$ to a smooth function in a neighborhood of p.] Then g extends to a smooth function $\bar{g}\colon M \to \mathbb{R}$.*

Proof. Cover K by open sets U_α (open in M) such that there exist $g_\alpha\colon U_\alpha \to \mathbb{R}$ with $g_\alpha|U_\alpha \cap K = g$. Add $M \setminus K$ and the zero function to get an open covering of M. Passing to a refinement we may assume that this covering is locally finite (see Theorem 6.6). Let $\{\psi_i\}$ be a smooth partition of unity subordinate to this covering. Then $\bar{g}\colon M \to \mathbb{R}$, $\bar{g} = \sum_i \psi_i g_{\alpha_i}$ will do. □

Theorem 8.3 (Whitney embedding theorem, compact case)**.** *If M^n is a compact n-dimensional smooth manifold, there exists a smooth embedding $g\colon M^n \to \mathbb{R}^{2n+1}$.*

Proof. Since M^n is compact, there exists a finite atlas $\{(U_1,\varphi_1),\dots,(U_k,\varphi_k)\}$. We may assume that there are open sets $V_i \subset U_i$ such that $\bar{V}_i \subset U_i$. Now there are smooth functions $\lambda_i\colon M \to \mathbb{R}$ such that $\lambda_i|\bar{V}_i = 1$ and $\operatorname{supp}\lambda_i \subset U_i$ (this follows e.g. from Lemma 8.2). Define ψ_i by setting

$$\psi_i(p) = \begin{cases} \lambda_i(p)\varphi_i(p), & p \in U_i, \\ 0, & p \notin U_i. \end{cases}$$

Then ψ_i is smooth. Next define $\theta\colon M \to (\mathbb{R}^n)^k \times \mathbb{R}^k$ by

$$\theta(p) = \big(\psi_1(p),\dots,\psi_k(p),\lambda_1(p),\dots,\lambda_k(p)\big).$$

Then

$$\theta_* = \psi_{1*} \times \cdots \times \psi_{k*} \times \lambda_{1*} \times \cdots \times \lambda_{k*}.$$

<u>Claim 1.</u> θ is an immersion.
Fix $p \in M$. Then $p \in U_j$ for some j. Since $\lambda_j = 1$ in a neighborhood of p, ψ_j coincides with φ_j in a neighborhood of p. Thus $\psi_{j*} = \varphi_{j*}$ near p. Since φ_j is a chart, φ_{j*} is injective, and therefore θ_* is injective.
<u>Claim 2.</u> θ is injective.
If $\theta(p) = \theta(q)$, then $\psi_i(p) = \psi_i(q)$ and $\lambda_i(p) = \lambda_i(q)$ for every i. Now $p \in V_j$ for some j, so $\lambda_j(p) = 1$. Hence $\psi_j(p) = \lambda_j(p)\varphi_j(p) = \varphi_j(p)$, and therefore

$$\varphi_j(p) = \underbrace{\lambda_j(p)}_{=1}\psi_j(p) = \underbrace{\lambda_j(q)}_{=1}\varphi_j(q) = \varphi_j(q).$$

This implies $p = q$ since φ_j is a chart.

Since M^n is compact, θ is a homeomorphism onto its image, so θ is an embedding of M^n into $\mathbb{R}^N$ for some (large) N. We regard this embedding as an inclusion $i\colon M^n \hookrightarrow \mathbb{R}^N$.
Claim 3. We can cut N down to $2n+1$.
Suppose that we can find a vector $w \in \mathbb{R}^n$ such that w is not tangent to M^n at any point and that there are no points $x, y \in M^n$ with $x-y$ parallel to w. Then the orthogonal projection of M^n onto the $(N-1)$-dimensional subspace $w^\perp$ is still injective and maps no nonzero tangent vector of M^n to zero. Hence we may embedd M^n into $\mathbb{R}^{N-1}$. Thus it suffices to show that such vector w exists if $N > 2n+1$. The argument is of "general position"-type. Suppose $N > 2n+1$ and consider the map $\sigma\colon TM^n \setminus \{\text{zero section}\} \to \mathbb{R}P^{N-1}$ taking a tangent vector $v \neq 0$ to a vector in $\mathbb{R}^N$ via the inclusion and then to the equivalence class of $v/|v|$ in $\mathbb{R}P^{N-1}$, $\sigma(v = [i_*v/|i_*c|]$. Also consider the map $\tau\colon M^n \times M^n \setminus \Delta_{M^n} \to \mathbb{R}P^{N-1}$ taking a pair $(x, y), x \neq y$, to the equivalence class of $(x-y)/|x-y|$, $\tau(x, y) = [(x-y)/|x-y|]$. Both σ and τ are smooth. The dimensions of the domains of σ and τ are $2n$ which is less than the dimension $N-1$ of the target manifold. By Sard's theorem, the images of both maps are of first category and hence the union of the images is also of first category which implies that there must be such a vector w. □

Remark 8.4. Sard's theorem: If $\phi\colon M^n \to \mathbb{R}^N$ is smooth, then the set of critical values has zero N-dimensional measure.
Critical value: If $\phi\colon M^m \to N^n$ is smooth, then $p \in M^m$ is called a *critical point* of ϕ if $\phi_*\colon T_pM^m \to T_{f(p)}N^n$ has rank $< n$, i.e. $\dim \phi_* T_pM^m < n$. The image $\phi(p) \in N^n$ of a critical point is called a *critical value*. All other points in N^n are called *regular* values (even if they do not belong to ϕM^m).
First category: A subset of a topological space is of first category if it is a countable union of nowhere dense subsets.

Theorem 8.5 (Whitney embedding theorem, general case)**.** *A smooth manifold M^n can be embedded as a submanifold and a closed subset of* $\mathbb{R}^{2n+1}$.

Proof. Cover M^n by open subsets with compact closures and take a smooth partition of unity $\{\lambda_i\}_{i\in\mathbb{N}}$ subordinate to a locally finite and countable refinement of that cover. Let $h(x) = \sum_k k\lambda_k(x)$. This is a smooth proper map $M^n \to [1, \infty) \subset \mathbb{R}$. Let $U_i = h^{-1}(i - 1/4, i + 5/4)$, $C_i = h^{-1}[i - 1(3, i + 4/3]$. Then U_i is open, C_i is compact, and $\bar{U}_i \subset \operatorname{int} C_i$. Furthermore, all C_{odd} are disjoint and, similarly, all C_{even} are disjoint. Now, for all i, the proof of Theorem 8.3 shows that there exists a smooth map $g_i\colon M^n \to \mathbb{R}^{2n+1}$ that is an embedding on $\bar{U}_i$ and is 0 outside C_i. Composing this with a diffeomorphism $\mathbb{R}^{2n+1} \to$ an open ball in $\mathbb{R}^{2n+1}$,we may assume that g_iM^n is bounded. Let $f_o = \sum g_{\text{odd}}$, $f_e = \sum g_{\text{even}}$ and $f = (f_o, f_e, h)\colon M^n \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$. Now $fM^n \subset K \times \mathbb{R}$ for some compact $K \subset \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}$ since f_oM^n and f_eM^n are bounded. Then f is proper since h is proper. If $f(x) = f(y)$, then $h(x) = h(y)$, and therefore x and y are in the same U_i. If this i is odd, then f_o is an embedding on U_i, and so $x = y$. Similarly, if i is even, we have $x = y$. Hence f is an embedding to a closed subset (by properness). Repeating a similar dimension reduction argument as in the proof of Theorem 8.3 we find a projection P of $\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$ to a $(2n+1)$-dimensional subspace H such that P is an immersion on fM^n. Moreover, P can be chosen such that the original h-axis is not in $\operatorname{Ker} P$. That is, if $\pi\colon \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R} \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}$ is the projection, then $\operatorname{Ker} \pi \cap \operatorname{Ker} P = \{0\}$. This implies that $\pi \times P$ is an inclusion, hence proper. Thus for a compact $C \subset H$, $K \times \mathbb{R} \cap P^{-1}C = (\pi \times P)^{-1}(K \times C)$ is compact, hence P is proper on $fM^n \subset K \times \mathbb{R}$. Therefore, $P \circ f$ is an embedding of M^n as a closed subset of $\mathbb{R}^{2n+1}$. □

8.6 Tubular neighborhoods

Definition 8.7 (Normal bundle)**.** Let M^n be a smooth submanifold of $\mathbb{R}^k$, $k > n$. The *normal bundle* of M^n in $\mathbb{R}^k$ is

$$NM = \{(x, v) \in M^n \times \mathbb{R}^k \colon v \perp T_x M\},$$

where the orthogonality $\perp$ is with respect to the standard inner product of $\mathbb{R}^k$.

Then NM is a vector bundle of rank $(k-n)$ over M. Define $\theta \colon NM \to \mathbb{R}^k$, $\theta(x, v) = x + v$, and $N(M, \varepsilon) = \{(x, v) \in NM \colon |v| < \varepsilon\}$.

Theorem 8.8 (Tubular neighborhood theorem)**.** *Let M^n be a comapct smooth submanifold of $\mathbb{R}^k$. Then there exists $\varepsilon > 0$ such that $\theta \colon N(M, \varepsilon) \to \mathbb{R}^k$ is a diffeomorphism onto the neighborhood $\{y \in \mathbb{R}^k \colon \operatorname{dist}(M, y) < \varepsilon\}$ of M^n in $\mathbb{R}^k$.*

Proof. We have canonical splitting

$$T_x\mathbb{R}^k = T_x M \oplus N_x M,$$

where $N_x M$ is the normal space to $T_x M$ at x in$\mathbb{R}^k$. For fixed x, $v \mapsto \theta(x, v) = x + v$, is just a translation, so θ_* is the standard inclusion (identity) on $N_x M \to \mathbb{R}^k$. Also $\theta_* \colon T_x M \to T_x\mathbb{R}^k$ is just the differential of the inclusion $M \hookrightarrow \mathbb{R}^k$, so this part of θ_* is the standard inclusion of $T_x M$ in $T_x\mathbb{R}^k = \mathbb{R}^k$. Thus

$$\theta_* \colon \mathbb{R}^k = T_x\mathbb{R}^k = T_x M \oplus N_x M \to \mathbb{R}^k$$

is the identity. Therefore, θ_* is an isomorphism at $(x, 0)$ for every $x \in M$, so θ is a diffeomorphism on some neighborhood of $(x, 0)$. Consequently, θ_* is an isomorphism at (x, v) for any x and for $|v|$ small. By compactness, there exists $\delta > 0$ such that θ_* is an isomorphism at all points of $N(M, \delta)$. Thus $\theta \colon N(M, \delta) \to \mathbb{R}^k$ ia a local diffeomorphism onto its image.
<u>Claim:</u> θ is injective on $N(M, \varepsilon)$ for some $0 < \varepsilon \leq \delta$.
Suppose that θ is not injective on $N(M, \varepsilon)$ for any $\varepsilon > 0$. Then there are sequences $(x_i, v_i) \neq (y_i, w_i)$ in NM such that $|v_i| \to 0$, $|w_i| \to 0$ and $\theta(x_i, v_i) = \theta(y_i, w_i)$. Since M is compact and metrizable, there exist subsequences (after reindexing) such that $x_i \to x$ and $y_i \to y$. Then $\theta(x_i, v_i) \to \theta(x, 0) = x$ and $\theta(y_i, w_i) \to \theta(y, 0) = y$, so that $x = y$. But then, for i large, both (x_i, v_i) and (y_i, w_i) are close to $(x, 0)$. Since θ is injective locally near $(x, 0)$, this gives a contradiction. Hence θ is injective on $N(M, \varepsilon)$ for some $\varepsilon > 0$. To finish, we claim that $\theta N(M, \varepsilon) = \{y \in \mathbb{R}^k \colon \operatorname{dist}(y, M) < \varepsilon\}$. The inclusion $\subset$ is clear. Suppose $\operatorname{dist}(y, M) < \varepsilon$ and choose $x \in M$ such that $|x - y| = \operatorname{dist}(y, M)$. Then $y - x$ is a normal to M at x of length $|y - x| < \varepsilon$, so $y \in \theta N(M, \varepsilon)$. $\square$

Note that the map $r = \pi \circ \theta^{-1} \colon \theta N(M, \varepsilon) \to M$ is a smooth retraction of the tubular neighborhood onto M ($r|M = \mathrm{id}$). Also r is homotopic to id_M via smooth homotopy, so r is a smooth "deformation retraction".

Theorem 8.9. *Let M^n be a smooth manifold and $A \subset M^n$ closed. Let $f \colon M^n \to \mathbb{R}^k$ be continuous on M^n and smooth on A (w.r.t. induced structure). Given $\varepsilon > 0$, there exists a smooth map $g \colon M^n \to \mathbb{R}^k$ such that $g(a) = f(a)$ for all $x \in A$ and $|f(x) - g(x)| < \varepsilon$ for every $x \in M$. Moreover, $f \simeq g$ rel A via an ε-small homotopy.*

Proof. For every $x \in M^n$, let $V_x \subset M^n$ be an open neighborhood of x and $h_x \colon V_x \to \mathbb{R}^k$ such that

(i) if $x \in A$, h_x is a smooth local extension of $f|A \cap V_x$,

(ii) if $x \notin A$, $V_x \cap A = \emptyset$ and $h_x(y) = h_x(x)$ for all $y \in V_x$,

(iii) if $y \in V_x$, then $|f(y) - f(x)| < \varepsilon/2$, $|h_x(y) - f(x)| < \varepsilon/2$ and $d(x, y) < \varepsilon/2$, where d is a metric on M.

Let $\{U_i\}$ be a locally finite countable refinement of $\{V_x\}$ and $\{\lambda_i\}$ a smooth partition of unity subordinate to $\{U_i\}$. For each $i \in \mathbb{N}$ choose x_i such that $U_i \subset V_{x_i}$. Note that $\lambda_i = 0$ on A if $x_i \notin A$. Define $g(y) = \sum_i \lambda_i(y) h_{x_i}(y)$.
Claim: g has the desired properties. We note that g is smooth by Propostion 8.1. Suppose $y \in A$. Then $g(y) = \sum_i \lambda_i(y) h_{x_i}(y) = \sum_i \lambda_i(y) f(y) = f(y)$. If $x \in M \setminus A$, taking the sums over those i for which $y \in U_i$ gives:

$$\begin{aligned}
|g(y) - f(y)| &= \left| \sum_i \lambda_i(y) h_{x_i}(y) - f(y) \right| \\
&= \left| \sum_i \lambda_i(y)(h_{x_i}(y) - f(x_i)) + \sum_i \lambda_i(y)(f(x_i) - f(y)) \right| \\
&\leq \sum_i \lambda_i(y) \left(|h_{x_i}(y) - f(x_i)| + |f(x_i) - f(y)|\right) \\
&< \sum_i \lambda_i(y)(\varepsilon/2 + \varepsilon/2) \leq \varepsilon.
\end{aligned}$$

Finally, the standard homotopy $H(x, t) = tf(x) + (1 - t)g(x)$ gives the desired ε-small homotopy rel A. $\square$

Theorem 8.10 (Smooth approximation theorem)**.** *Suppose that M^m and N^n are smooth manifolds, where N^n is compact with a metric d. Let $A \subset M^m$ be closed. Let $f \colon M^m \to N^n$ be a continuous map such that $f|A$ is smooth. Then or every $\varepsilon > 0$ there exists a map $h \colon M^m \to N^n$ such that*

(i) h is smooth,

(ii) $\mathrm{dist}\,(h(x), f(x)) < \varepsilon$ *for every $y \in M^m$,*

(iii) $h|A = f|A$,

(iv) $h \simeq f$ rel A by an ε-small homotopy.

Proof. Embed N^n into some R^k. By continuity of the inverse map of the embedding and compactness of N^n, hence by uniform continuity of the inverse, there exists $\delta > 0$ such that $|p - q| < \delta$ implies $\mathrm{dist}(p, q) < \varepsilon$. Hence we may use on N^n the induced metric from $\mathbb{R}^k$. Take a $\delta/2$-tubular neighborhood U of N^n in $\mathbb{R}^k$ (taking smaller δ if necessary) and let $r \colon U \to N^n$ be the associated normal retraction. Approximate f by a smooth map $g \colon M^M \to \mathbb{R}^k$ within $\delta/2$ (by Theorem 8.9). Then $gM^m \subset U$. Now for the map $h = r \circ g$ we have

(a) h is smooth,

(b) $|h(x) - f(x)| \leq |r(g(x)) - g(x)| + |g(x) - f(x))| < \delta/2 + \delta/2 = \delta$,

(c) $h|A = r \circ g|A = r \circ f|A = f|A$,

(d) $h \simeq f$ rel A by $H(x, t) = r\big(tg(x) + (1 - t)f(x)\big)$.

$\square$

Corollary 8.11. *Suppose that M^m and N^n are smooth manifolds with N^n compact. Then every continuous $f\colon M^m \to N^n$ is homotopic to a smooth map. If f and g are smooth and $f \simeq g$, then $f \simeq g$ by a smooth homotopy $H\colon I \times M^m \to N^n$.*

Proof. The first part follows directly from Theorem 8.10. Suppose then that $F\colon I \times M \to N$ is a homotopy between smooth maps f and g. Extend F to $\mathbb{R} \times M$ by making it constant with respect to $t \in \mathbb{R}$ on both ends $(-\infty, 0]$ and $[1, \infty)$. Then F is smooth on the subspace $\{0,1\} \times M$, and therefore Theorem 8.10 implies the existence of a smooth map $G\colon \mathbb{R} \times M \to N$ which coincides with F on $\{0,1\} \times M$. Finally, take $H = G|I \times M$. ☐

We can get rid of the assumption N^n being compact in Theorem 8.10 and Corollary 8.11 by considering an "$\varepsilon(x)$-tubular" neighborhood of N^n in $\mathbb{R}^k$ instead of a fixed ε-tubular neighborhood. Thus we obtain the following results.

Theorem 8.12. *Let M and N be smooth manifolds and let $f\colon M \to N$ be continuous. Then f is homotopic to a smooth mapping $\tilde{f}\colon M \to M$. If f is smooth on a closed set $A \subset M$, then there exists smooth $\tilde{f}\colon M \to N$ such that $f \simeq \tilde{f}$ rel A.*

Theorem 8.13. *If $f, g\colon M \to N$ are smooth homotopic maps, then they are smoothly homotopic. If $f \simeq g$ rel A for some closed $A \subset M$, then f and g are smoothly homotopic rel A.*

8.14 Some consequences

Theorem 8.15. *Let M^m be a smooth m-manifold and $n > m$. Then every continuous map $f\colon M^m \to \mathbb{S}^n$ is homotopic to a constant map.*

Proof. Let $g\colon M^m \to \mathbb{S}^n$ be smooth and $g \simeq f$. By Sard's theorem, there exists $p \in \mathbb{S}^n \setminus gM^m$. Then $\mathbb{S}^n \setminus \{p\}$ is homotopic to $\mathbb{R}^n$, so it is contractible, i.e. $\mathrm{id}_{\mathbb{S}^n \setminus \{0\}} \simeq c$, a constant map. Composing g with such a contraction implies $f \simeq c$. ☐

Theorem 8.16. *The sphere $\mathbb{S}^n$ is not a retract of the ball closed unit ball $\bar{B}^{n+1}$.*

Proof. Supppose that $f\colon \bar{B}^{n+1} \to \mathbb{S}^n$ is a retraction. Then $f_1 := \frac{1}{2} f(2\cdot)\colon \bar{B}^{n+1}(0,1/2) \to \mathbb{S}^n(1/2)$ is a retraction. Let $f_2\colon \mathbb{R}^{n+1} \setminus B^{n+1}(0,1/2) \to \mathbb{S}^n$ be the radial projection. Then $h\colon \mathbb{R}^{n+1} \to \mathbb{S}^n$,

$$h(x) = \begin{cases} f_2(x), & x \in \mathbb{R}^{n+1} \setminus B^{n+1}(0,1/2), \\ (f_2 \circ f_1)(x), & x \in \bar{B}(0,1/2), \end{cases}$$

is a retraction $\bar{B}^{n+1} \to \mathbb{S}^n$ that is smooth in a neighborhood of $\mathbb{S}^n$. Thus we can smooth out h without changing it near $\mathbb{S}^n$. Thus we may assume that (the original) f is smooth and that it is the radial projection near $\mathbb{S}^n$. Let $x \in \mathbb{S}^n$ be a regular value of f. Then $f^{-1}(z)$ is a compact 1-manifold with boundary and its boundary is the single point $f^{-1}(z) \cap \mathbb{S}^n = \{z\}$. But any compact 1-manifold with boundary is homeomorphic to a disjoint union of circles and closed unit intervals, and hence has an even number of boundary points, which is a contradiction. ☐

Corollary 8.17 (Brower's fixed point theorem)**.** *Every continuous map $f\colon \bar{B}^n \to \bar{B}^n$ has a fixed point.*

Proof. Suppose there is a continuous map $f\colon \bar{B}^n \to \bar{B}^n$ without fixed points. Define $r\colon \bar{B}^n \to \mathbb{S}^{n-1}$ by letting $r(x) \in \mathbb{S}^{n-1}$ be the intersection point of $\mathbb{S}^{n-1}$ and the ray from $f(x)$ to x. This is a continuous map, hence a retraction of $\bar{B}^n$ onto $\mathbb{S}^{n-1}$, which is a contradiction. Continuity of r is

intuitively clear but unpleasant to prove.
Another proof: Define a continuous map $g\colon \bar{B}^n(0,2) \to \bar{B}^n(0,2)$,

$$g(x) = \begin{cases} (2-|x|)f(x/|x|), & 1 \le |x| \le 2, \\ f(x), & |x| \le 1. \end{cases}$$

Clearly g has no fixed points since, for $|x| \le 1$, $g(x) = f(x)$ and thus such x cannot be a fixed point. Furthermore, if $|x| > 1$, then $|g(x)| < 1$, hence such x cannot be a fixed point either. Then define $r\colon \bar{B}^n(0,2) \to \mathbb{S}^{n-1}(0,2)$,

$$r(x) = 2(x - g(x))/|x - g(x)|.$$

Now r is continuous and if $|x| = 2$, then $g(x) = 0$ and therefore $r(x) = 2x/|x| = x$. So r is a retraction $\bar{B}^n(0,2) \to \mathbb{S}^{n-1}(0,2)$ which is a contradiction. □

9 A brief introduction to the de Rham cohomology

In this section we consider the de Rham cohomology briefly by introducing some central notions and results.

Let us recall the following definitions:

$$\begin{aligned} \mathcal{Z}^p(M) &= \operatorname{Ker}[d\colon \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)] \\ &= \{\text{closed } p\text{-forms on } M\}, \\ \mathcal{B}^p(M) &= \operatorname{Im}[d\colon \mathcal{A}^{p-1}(M) \to \mathcal{A}^p(M)] \\ &= \{\text{exact } p\text{-forms on } M\}. \end{aligned}$$

The vector space (quotient space)

$$H^p_{dR}(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}$$

is called the pth *de Rham cohomology group* of M. Its elements are the equivalence classes $[\omega]$ of closed p-forms ω. (A closed p-form $\omega' \in [\omega]$ if $\omega' - \omega$ is exact.)

By our convention $\mathcal{A}^p(M) = 0$ (= trivial vector space) if $p < 0$ or $p > \dim M$, hence $H^p_{dR}(M) = 0$ for these values of p.

Theorem 9.1. *Let M be a connected smooth manifold. Then $H^0_{dR}(M) = \mathcal{Z}^0(M) = \{f\colon M \to \mathbb{R} \mid f \text{ constant}\}$. In particular,* $\dim H^0_{dR}(M) = 1$.

Proof. Since $\mathcal{B}^0(M) = 0$, we have $H^0_{dR}(M) = \mathcal{Z}^0(M) = \{f\colon M \to \mathbb{R} \mid df = 0\}$. Furthermore, since M is connected, $\{f\colon M \to \mathbb{R} \mid df = 0\} = \{f\colon M \to \mathbb{R} \mid f \text{ constant}\}$. □

Let $f^*\colon \mathcal{A}^p(N) \to \mathcal{A}^p(M)$ be the pull-back of a smooth mapping $f\colon M \to N$. Since f^* and d commute (Thm. 5.20), we get

$$f^*\mathcal{Z}^p(N) \subset \mathcal{Z}^p(M) \quad \text{and} \quad f^*\mathcal{B}^p(N) \subset \mathcal{B}^p(M).$$

Hence we may define the linear map (the *induced cohomology map*)

$$f^*\colon H^p_{dR}(N) \to H^p_{dR}(M)$$

by setting

$$f^*[\omega] = [f^*\omega], \quad [\omega] \in H^p_{dR}(N).$$

The definition is independent of the choice of the representative (Exerc.). If, furthermore, $g\colon N \to P$ is smooth, then $(g\circ f)^* = f^*\circ g^*\colon H^p_{dR}(P) \to H^p_{dR}(M)$. In particular, the induced cohomology map of the identity map $id\colon M \to M$ is the identity map $(id)^*\colon H^p_{dR}(M) \to H^p_{dR}(M)$. Hence diffeomorphic manifolds have isomorphic de Rham cohomology groups.

It turns out that, in fact, the de Rham cohomology groups are topological invariants: If M and N are homeomorphic smooth manifolds, their de Rham cohomology groups are isomorphic. We will (partly) prove this next.

Homotopy invariance. Let X and Y be topological spaces and $f_0, f_1\colon X \to Y$ continuous. We say that f_0 and f_1 are *homotopic* (denoted by $f_0 \simeq f_1$) if there exists a continuous mapping (a homotopy from f_0 to f_1) $H\colon X \times I \to Y,\ I = [0,1]$, s.t. $\forall x \in X$

$$\begin{aligned} H(x,0) &= f_0(x),\\ H(x,1) &= f_1(x). \end{aligned}$$

If, in addition, $H(x,t) = f_0(x) = f_1(x)\ \forall t \in I$ and $\forall x \in A \subset X$, we say that f_0 and f_1 are homotopic with respect to A, denoted by $f_0 \simeq f_1 \operatorname{rel} A$.

A continuous mapping $f\colon X \to Y$ is called a *homotopy equivalence* (and X and Y *homotopy equivalent*) if there exists a continuous mapping $g\colon Y \to X$ s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

If $f, g\colon M \to N$ are smooth mappings, then a collection of linear maps $h\,(= h^p)\colon \mathcal{A}^p(N) \to \mathcal{A}^{p-1}(M)$ s.t.

$$\underbrace{d(h\omega) + h(d\omega)}_{=d(h^p\omega)+h^{p+1}(d\omega)} = g^*\omega - f^*\omega \quad \forall \omega \in \mathcal{A}^p(N) \tag{9.2}$$

is called a *homotopy operator* between f^* and g^*.

Lemma 9.3. *Let M be a smooth manifold, $I = [0,1]$ and $i_t\colon M \to M \times I,\ t \in [0,1]$, an embedding*

$$i_t(x) = (x,t).$$

Then there exists a homotopy operator between i_0^* and i_1^*.

Proof. We define, for $\omega \in \mathcal{A}^p(M \times I)$,

$$h\omega = \int_0^1 \left(i_{\partial/\partial t}\omega\right)\, dt \in \mathcal{A}^{p-1}(M),$$

where t is the (standard) coordinate in I and $\partial/\partial t$ the corresponding coordinate tangent vector. In other words,

$$\begin{aligned} (h\omega)_q(v_1,\ldots,v_{p-1}) &= \int_0^1 \left(i_{\partial/\partial t}\omega\right)_{(q,t)}(v_1,\ldots,v_{p-1})\, dt\\ &= \int_0^1 \omega_{(q,t)}(\partial/\partial t, v_1,\ldots,v_{p-1})\, dt. \end{aligned}$$

Let $x = (x^1,\ldots,x^n)$ be a chart at q. It is enough to prove that (9.2) holds for forms

(i) $\omega = f(x,t)dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}$ and

(ii) $\omega = f(x,t)dx^{i_1} \wedge \cdots \wedge dx^{i_p}$.

Suppose first that

$$\omega = f(x,t)dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}.$$

Then

$$\begin{aligned} d(h\omega) &= d\left(\left(\int_0^1 f(x,t)\,dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}\right) \\ &= \frac{\partial}{\partial x^j}\left(\int_0^1 f(x,t)\,dt\right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \\ &= \left(\int_0^1 \frac{\partial f}{\partial x^j}(x,t)\,dt\right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}. \end{aligned}$$

On the other hand,

$$d\omega = \frac{\partial f}{\partial x^j}\, dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}$$

since $dt \wedge dt = 0$. Hence

$$\begin{aligned} h(d\omega) &= h\left(\frac{\partial f}{\partial x^j}\, dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}\right) \\ &= \int_0^1 \frac{\partial f}{\partial x^j}(x,t) i_{\partial/\partial t}\left(dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}\right)\, dt \\ &= -\left(\int_0^1 \frac{\partial f}{\partial x^j}(x,t)\,dt\right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \\ &= -d(h\omega), \end{aligned}$$

and so

$$d(h\omega) + h(d\omega) = 0.$$

Since $t \circ i_0 \equiv 0$ and $t \circ i_1 \equiv 1$, we have $i_0^* dt = i_1^* dt = 0$, and therefore $i_0^*\omega = i_1^*\omega = 0$, so the formula (9.2) holds. Suppose then that

$$\omega = f(x,t)dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Now $i_{\partial/\partial t}\omega = 0$, so $d(h\omega) = 0$. Furthermore,

$$\begin{aligned} h(d\omega) &= h\left(\frac{\partial f}{\partial t} dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} + \alpha\right) \\ &= \left(\int_0^1 \frac{\partial f}{\partial t}(x,t)\,dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &= (f(x,1) - f(x,0))\, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &= i_1^*\omega - i_0^*\omega, \end{aligned}$$

where α is the sum of terms that do not contain dt. □

Theorem 9.4. *Let $f, g\colon M \to N$ be homotopic smooth mappings. Then the induced cohomology mappings $f^*, g^*\colon H^p_{dR}(N) \to H^p_{dR}(M)$ are equal for all p.*

Proof. For the proof, we need *smooth* homotopy $H\colon M \times I \to N$ from f to g (see Theorem 8.13). Then $H \circ i_0 = f$ and $H \circ i_1 = g$, where i_t is as in Lemma 9.3. Let

$$\tilde{h} = h \circ H^*\colon \mathcal{A}^p(N) \to \mathcal{A}^{p-1}(M),$$

where h is the homotopy operator constructed in Lemma 9.3. If $\omega \in \mathcal{A}^p(N)$, then

$$\begin{aligned}\tilde{h}(d\omega) + d(\tilde{h}\omega) &= h(H^* d\omega) + d(hH^*\omega) \\ &= hd(H^*\omega) + d(hH^*\omega) \\ &= i_1^* H^*\omega - i_0^* H^*\omega \\ &= (H \circ i_1)^*\omega - (H \circ i_0)^*\omega \\ &= g^*\omega - f^*\omega.\end{aligned}$$

Thus we have for the equivalence class $[\omega] \in H^p_{dR}(N)$ of a closed form $\omega \in \mathcal{A}^p(N)$

$$g^*[\omega] - f^*[\omega] = [g^*\omega - f^*\omega] = \left[\tilde{h}(d\omega) + d(\tilde{h}\omega)\right] = \left[d(\tilde{h}\omega)\right] = 0$$

since $d\omega = 0$ and $[\alpha] = 0$ for exact forms α. □

Theorem 9.5. *If M and N are homotopy equivalent smooth manifolds, then $H^p_{dR}(M)$ and $H^p_{dR}(N)$ are isomorphic for all p. The isomorphism is induced by any smooth homotopy equivalence $f \colon M \to N$.*

Proof. Let $f \colon M \to N$ and $g \colon N \to M$ be continuous s.t. $f \circ g \simeq id_N$ and $g \circ f \simeq id_M$. Then there exist *smooth* maps $\tilde{f} \colon M \to N$, $\tilde{f} \simeq f$, and $\tilde{g} \colon N \to M$, $\tilde{g} \simeq g$ (see Theorem 8.12 and 8.13). Then $\tilde{f} \circ \tilde{g} \simeq id_N$ and $\tilde{g} \circ \tilde{f} \simeq id_M$. By Theorem 9.4

$$\tilde{f}^* \circ \tilde{g}^* = (\tilde{g} \circ \tilde{f})^* = (id_M)^* = id_{H^p_{dR}(M)}.$$

Similarly, $\tilde{g}^* \circ \tilde{f}^*$ is the identity mapping on $H^p_{dR}(N)$, so $f^* \colon H^p_{dR}(N) \to H^p_{dR}(M)$ is an isomorphism. □

Since every homeomorphism is, in particular, a homotopy equivalence, we obtain as a corollary the invariance of de Rham cohomology groups under homeomorphisms.

Corollary 9.6. *If smooth manifolds M and N are homeomorphic, then their de Rham cohomology groups are isomorphic.*

Theorem 9.7 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^n$ be a star-shaped open set. Then $H^p_{dR}(U) = 0$ for all $p \geq 1$.*

Proof. Let U be star-shaped with respect to $y \in U$. Then id_U is homotopic with the constant mapping $c_y \colon U \to \{y\}$ $(H(x,t) = y + t(x-y))$. Hence $H^p_{dR}(U)$ is isomorphic with $H^p_{dR}(\{y\})$. Furthermore, $H^p_{dR}(\{y\}) = 0$ for $p \geq 1$. □

10 Cochain complexes and their cohomology

In this section we introduce and develop basic notions and theory on (general) cochain complexes and their cohomology.

A sequence of vector spaces and linear maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* if $\operatorname{Im} f = \operatorname{Ker} g$. So, $A \xrightarrow{f} B \to 0$ is exact if and only if f is surjective. On the other hand, $0 \to B \xrightarrow{g} C$ is exact if and only if g is injective. A sequence $A^* = \{A^i, d^i\}$,

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \to \cdots$$

of vector spaces A^i and linear maps d^i is called a *cochain complex* if $d^{i+1} \circ d^i = 0$. It is *exact* if $\operatorname{Ker} d^i = \operatorname{Im} d^{i-1}$ for all i. An exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{10.1}$$

is called *short exact*. So, f is injective, g is surjective and $\operatorname{Im} f = \operatorname{Ker} g$.

The *cokernel* of a linear map $f\colon A \to B$ is $\operatorname{Cok}(f) = B/\operatorname{Im} f$. For a short exact sequence (10.1), g induces an isomorphism

$$g\colon \operatorname{Cok}(f) \xrightarrow{\cong} C.$$

Every long exact sequence induces short exact sequences

$$0 \to \operatorname{Im} d^{i-1} \to A^i \to \operatorname{Im} d^i \to 0$$

that can be used to determine A^i. Furthermore, the isomorphisms

$$A^{i-1}/\operatorname{Im} d^{i-2} \cong A^{i-1}/\operatorname{Ker} d^{i-1} \xrightarrow{d^{i-1}} \operatorname{Im} d^{i-1}$$

are used in concrete calculations.

Recall that a direct sum of vector spaces A and B is the vector space

$$\begin{aligned} A \oplus B &= \{(a,b)\colon a \in A, b \in B\}, \\ \lambda(a,b) &= (\lambda a, \lambda b),\ \lambda \in \mathbb{R}, \\ (a_1,b_1) + (a_2,b_2) &= (a_1+a_2, b_1+b_2). \end{aligned}$$

If $\{a_i\}$ and $\{b_j\}$ are bases of A and B, respectively, then $\{(a_i,0),(0,b_j)\}$ is a basis of $A \oplus B$.

Lemma 10.2. *Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be an exact sequence. Suppose that A and C are finite-dimensional. Then also B is finite-dimensional and* $\dim B \le \dim A + \dim C$.

Proof. Since C is finite-dimensional, also $\operatorname{Im}\beta$ is finite-dimensional. Hence there exist $v_1,\dots,v_k \in B$ such that $\operatorname{Im}\beta = \operatorname{span}\{\beta(v_i)\}$. Let $v \in B$ be arbitrary. Then $\beta(v) = \sum_{i=1}^k c_i\beta(v_i)$, and therefore $v' := v - \sum_{i=1}^k c_i v_i \in \operatorname{Ker}\beta = \operatorname{Im}\alpha$. Since A is finite-dimensional, also $\operatorname{Im}\alpha$ is finite-dimensional. Let $\{v_{k+1},\dots,v_m\}$ be a basis of $\operatorname{Im}\alpha$. Then

$$v = \sum_{i=1}^k c_i v_i + v' = \sum_{i=1}^k c_i v_i + \sum_{i=k+1}^m c_i v_i = \sum_{i=1}^m c_i v_i,$$

and therefore $\dim B \le \dim A + \dim C$ is finite. □

By modifying slightly the proof above we obtain:

Lemma 10.3. *Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of vector spaces. If A and C are finite-dimensional, then B is also finite-dimensional and $B \cong A \oplus C$.*

Definition 10.4. For a cochain complex A^*,

$$\cdots \to A^{p-1} \xrightarrow{d^{p-1}} A^p \xrightarrow{d^p} A^{p+1} \xrightarrow{d^{p+1}} A^{p+2} \to \cdots$$

we define the pth *cohomology vector space*

$$H^p(A^*) = \operatorname{Ker} d^p/\operatorname{Im} d^{p-1}.$$

Elements of $\operatorname{Ker} d^p$ are called *p-cocycles* (and are said to be closed) and elements of $\operatorname{Im} d^{p-1}$ are called *p-coboundaries* (or exact). Elements of $H^p(A^*)$ are called *cohomology classes*.

A *cochain map* $f\colon A^* \to B^*$ between cochain complexes consists of a family $f^p\colon A^p \to B^p$ of linear maps such that $d_B^p \circ f^p = f^{p+1} \circ d_A^p$:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{p-1} \arrow[r, "d_A^{p-1}"] \arrow[d, "f^{p-1}"'] & A^p \arrow[r, "d_A^p"] \arrow[d, "f^p"'] & A^{p+1} \arrow[r] \arrow[d, "f^{p+1}"] & \cdots \\
\cdots \arrow[r] & B^{p-1} \arrow[r, "d_B^{p-1}"] & B^p \arrow[r, "d_B^p"] & B^{p+1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 10.5. *A cochain map $f\colon A^* \to B^*$ induces a linear map $f^*\colon H^p(A^*) \to H^p(B^*)$ for all p.*

Proof. Let $a \in A^p$ be a p-cocycle ($d_A^p a = 0$) and $[a] = a + \operatorname{Im} d_A^{p-1}$ the corresponding cohomology class in $H^p(A^*)$. Define $f^*[a] = \big[f^p(a)\big]$. Then

(i)
$$d_B^p f^p(a) = f^{p+1} \underbrace{d_A^p(a)}_{=0} = 0,$$
so $f^p(a)$ is a p-cocycle and $\big[f^p(a)\big]$ is defined;

(ii) $\big[f^p(a)\big]$ is independent of the choice of a representative of $[a]$: If $[a_1] = [a_2]$, then $a_1 - a_2 \in \operatorname{Im} d_A^{p-1}$ and
$$f^p(\underbrace{a_1 - a_2}_{=d_A^{p-1}(x)}) = f^p d_A^{p-1}(x) = d_B^{p-1} f^{p-1}(x),$$
so $f^p(a_1) - f^p(a_2) \in \operatorname{Im} d_B^{p-1}$ and thus $f^p(a_1)$ and $f^p(a_2)$ define the same cohomology class;

(iii) clearly f^* is linear.

□

A short exact sequence of cochain complexes

$$0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$$

consists of cochain maps f and g such that

$$0 \to A^p \xrightarrow{f^p} B^p \xrightarrow{g^p} C^p \to 0$$

is exact for every p.

Lemma 10.6. *For a short exact sequence of cochain complexes, the sequence*

$$H^p(A^*) \xrightarrow{f^*} H^p(B^*) \xrightarrow{g^*} H^p(C^*)$$

is exact, i.e. $\operatorname{Im} f^* = \operatorname{Ker} g^*$.

Proof. Since $g^p \circ f^p = 0$, we have

$$g^* \circ f^*([a]) = g^*\big([f^p(a)]\big) = \big[\underbrace{g^p\big(f^p(a)\big)}_{=0}\big] = 0$$

for every cohomology class $[a] \in H^p(A^*)$. Hence $\operatorname{Im} f^* \subset \operatorname{Ker} g^*$.
Conversely, let $[b] \in \operatorname{Ker} g^* \subset H^p(B^*)$. Then $[g^p(b)] = g^*[b] = 0$, so $g^p(b) - 0 \in \operatorname{Im} d_C^{p-1}$. Hence

$g^p(b) = d_C^{p-1}(c)$. Since g^{p-1} is surjective, there exists $b_1 \in B^{p-1}$ such that $g^{p-1}(b_1) = c$. It follows that $g^p(d_B^{p-1}(b_1)) = d_C^{p-1}(g^{p-1}(b_1)) = d_C^{p-1}(c) = g^p(b)$, and therefore $g^p(b - d_B^{p-1}(b_1)) = 0$, i.e. $b - d_B^{p-1}(b_1) \in \operatorname{Ker} g^p = \operatorname{Im} f^p$. Hence there exists $a \in A^p$ such that $f^p(a) = b - d_B^{p-1}(b_1)$. We claim that a is a p-cocycle ($d_A^p a = 0$). Since f^{p+1} is injective it suffices to show that $f^{p+1}(d_A^p a) = 0$. But

$$f^{p+1}(d_A^p a) = d_B^p(f^p(a)) = d_B^p(b - d_B^{p-1}(b_1)) = d_B^p(b) = 0$$

since b is a p-cocycle (note that $[b] \in H^p(B^*)$). So we have found a cohomology class $[a] \in H^p(A^*)$ such that

$$f^*[a] = [f^p(a)] = [b - \underbrace{d_B^{p-1}(b_1)}_{\in \operatorname{Im} d_B^{p-1}}] = [b] = 0.$$

□

Remark 10.7. Th exact sequence in Lemma 10.6 need not be extendible to a short exact sequence. Even though $g^p \colon B^p \to C^p$ is surjective, the preimage $(g^p)^{-1}(c)$ of a p-cocycle $c \in C^p$ need not contain a p-cocycle.

However, on cohomology level this works:

Definition 10.8. For a short exact sequence of cochain complexes

$$0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0,$$

we define the linear map $\partial^* \colon H^p(C^*) \to H^{p+1}(A^*)$ by

$$\partial^*([c]) = \left[(f^{p+1})^{-1}(d_B^p((g^p)^{-1}(c)))\right].$$

The map ∂^* is called the *connecting homomorphism*.

To prove that ∂^* is well-defined we have to note several things. The definition requires that for every $b \in (g^p)^{-1}(c)$ we have $d_B^p(b) \in \operatorname{Im} f^{p+1}$ and then the uniquely determined $a \in A^{p+1}$ with $f^{p+1}(a) = d_B^p(b)$ is a $(p+1)$-cocycle. Finally, $[a] \in H^{p+1}(A^*)$ should be independent of the choice of $b \in (g^p)^{-1}(c)$.

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & A^{p-1} \arrow[r, "f^{p-1}"] \arrow[d, "d_A^{p-1}"'] & B^{p-1} \arrow[r, "g^{p-1}"] \arrow[d, "d_B^{p-1}"'] & C^{p-1} \arrow[r] \arrow[d, "d_C^{p-1}"'] & 0 \\
0 \arrow[r] & A^p \arrow[r, "f^p"] \arrow[d, "d_A^p"'] & B^p \arrow[r, "g^p"] \arrow[d, "d_B^p"'] & C^p \arrow[r] \arrow[d, "d_C^p"] \arrow[dll, squiggly, "\text{,,}\partial^*\text{``}"] & 0 \\
0 \arrow[r] & A^{p+1} \arrow[r, "f^{p+1}"'] \arrow[d] & B^{p+1} \arrow[r, "g^{p+1}"'] \arrow[d] & C^{p+1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

We claim:

(i) If $g^p(b) = c$ and $d_C^p(c) = 0$, then $d_B^p(b) \in \operatorname{Im} f^{p+1}$.

(ii) If $f^{p+1}(a) = d_B^p(b)$, then $d_A^{p+1}(a) = 0$.

(iii) If $g^p(b_1) = g^p(b_2) = c$ and $f^{p+1}(a_i) = d_B^p(b_i)$, then $[a_1] = [a_2] \in H^{p+1}(A^*)$.

Proof. (i) follows since $g^{p+1}d_B^p(b) = d_C^p(c) = 0$ and $\operatorname{Ker} g^{p+1} = \operatorname{Im} f^{p+1}$.
(ii): Since f^{p+2} is injective and $f^{p+2}d_A^{p+1}(a) = d_B^{p+1}f^{p+1}(a) = d_B^{p+1}d_B^p(b) = 0$, we have $d_A^{p+1}(a) = 0$.
(iii) follows since $b_1 - b_2 \in \operatorname{Ker} g^p = \operatorname{Im} f^p$, so $b_1 - b_2 = f^p(a)$ and hence $d_B^p(b_1) - d_B^p(b_2) = d_B^p f^p(a) = f^{p+1}d_A^p(a)$ and furthermore

$$\underbrace{(f^{p+1})^{-1}d_B^p(b_1)}_{=a_1} - \underbrace{(f^{p+1})^{-1}d_B^p(b_2)}_{=a_2} = d_A^p(a) \in \operatorname{Im} d_A^p.$$

Now a_i is a $(p+1)$-cocycle ($d_A^{p+1}(a_i) = 0$) since

$$f^{p+2}d_A^{p+1}(a_i) = d_B^{p+1}f^{p+1}(a_i) = d_B^{p+1}d_B^p(a_i) = 0$$

and f^{p+2} is injective. Thus $[a_1] = [a_2] \in H^{p+1}(A^*)$. ☐

Lemma 10.9. *The sequence*

$$H^p(B^*) \xrightarrow{g^*} H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*)$$

is exact.

Proof. Since

$$\partial^* g^*([b]) = \partial^*[g^p(b)] = \left[(f^{p+1})^{-1}(\underbrace{d_B^p(b)}_{=0})\right] = [0] = 0,$$

$\operatorname{Im} g^* \subset \operatorname{Ker} \partial^*$.
Conversely, let $\partial^*[c] = 0$. Choose $b \in B^p$ such that $g^p(b) = c$. Since $d_C^p(c) = 0$, $d_B^p(b) \in \operatorname{Im} f^{p+1}$ by (i). So, $d_B^p(b) = f^{p+1}(\tilde{a})$ and $d_A^{p+1}(\tilde{a}) = 0$ by (ii). Since $\tilde{a} \in \operatorname{Ker} d_A^{p+1} = \operatorname{Im} d_A^p$, $\tilde{a} = d_A^p(a)$ for some $a \in A^p$. Now

$$d_B^p(b - f^p(a)) = d_B^p(b) - d_B^p f^p(a) = d_B^p(b) - f^{p+1}\underbrace{d_A^p(a)}_{=\tilde{a}} = 0$$

and

$$g^p(b - f^p(a)) = g^p(b) - \underbrace{g^p f^p(a)}_{=0} = g^p(b) = c.$$

Hence $g^*[b - f^p(a)] = [c]$, so $\operatorname{Ker} \partial^* \subset \operatorname{Im} g^*$. ☐

Lemma 10.10. *The sequence*

$$H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*) \xrightarrow{f^*} H^{p+1}(B^*)$$

is exact.

Proof. Since

$$f^*\partial^*[c] = f^*\partial^*[\underbrace{g^p(b)}_{=c}] = f^*\left[(f^{p+1})^{-1}(d_B^p(b))\right] = [d_B^p(b)] = 0,$$

$\operatorname{Im} \partial^* \subset \operatorname{Ker} f^*$.
Conversely, suppose that $[a] \in \operatorname{Ker} f^*$ ($\subset H^{p+1}(A^*)$), that is $d_A^{p+1}(a) = 0$ and $[f^{p+1}(a)] = f^*[a] = 0$. Then $f^{p+1}(a)$ is exact, so $f^{p+1}(a) = d_B^p(b)$. Then $d_C^p g^p(b) = g^{p+1}d_B(b) = g^{p+1}f^{p+1}(a) = 0$, and so

$$\partial^*[g^p(b)] = \left[(f^{p+1})^{-1}(d_B^p(b))\right] = \left[(f^{p+1})^{-1}(f^{p+1}(a))\right] = [a].$$

Hence $\operatorname{Ker} f^* \subset \operatorname{Im} \partial^*$. ☐

We collect these to:

Theorem 10.11 (Long exact cohomology sequence, ”Zig-zag lemma“)**.** *Let*

$$0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$$

be a short exact sequence of cochain complexes. Then the sequence

$$\cdots \to H^p(A^*) \xrightarrow{f^*} H^p(B^*) \xrightarrow{g^*} H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*) \xrightarrow{f^*} H^{p+1}(B^*) \to \cdots$$

is exact.

Definition 10.12. Cochain maps $f, g\colon A^* \to B^*$ are ssaid to be *cochain homotopic* if there are linear maps $s\colon A^p \to B^{p-1}$ such that

$$d_B^{p-1} s + s d_A^p = f - g\colon A^p \to B^p$$

for every p.

Lemma 10.13. *For two cochain homotopic maps $f, g\colon A^* \to B^*$, we have*

$$f^* = g^*\colon H^p(A^*) \to H^p(B^*).$$

Proof. (cf. Theorem 9.4)
If $[a] \in H^p(A^*)$, then

$$(f^* - g^*)[a] = [f^p(a) - g^p(a)] = [d_B^{p-1} s(a) - s\, \underbrace{d_A^p(a)}_{=0}] = [\underbrace{d_B^{p-1} s(a)}_{\in \operatorname{Im} d_B^{p-1}}] = 0 \ \in H^p(B^*).$$

□

Lemma 10.14. *If A^* and B^* are cochain ncomplexes, then*

$$H^p(A^* \oplus B^*) = H^p(A^*) \oplus H^p(B^*).$$

Proof. Clearly

$$\operatorname{Ker}\left(d^p_{A\oplus B}\right) = \operatorname{Ker} d_A^p \oplus \operatorname{Ker} d_B^p$$

and

$$\operatorname{Im}\left(d^{p-1}_{A\oplus B}\right) = \operatorname{Im} d_A^{p-1} \oplus \operatorname{Im} d_B^{p-1}.$$

□

Theorem 10.15 (The 5-Lemma)**.** *Consider the following diagram of vector spaces and linear maps (or Abelian groups and homomorphisms):*

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow[\beta_1]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 & \xrightarrow[\beta_3]{} & B_4 & \xrightarrow[\beta_4]{} & B_5,
\end{array}$$

where horizontal sequences are exact and f_1, f_2, f_4 and f_5 are isomorphisms. Then also f_4 is an isomorphism.

Proof. (Exercise) □

Remark 10.16. The assumptions can be weakened as:

$$f_2, f_4 \text{ surjective and } f_5 \text{ injective} \ \Rightarrow \ f_3 \text{ surjective},$$
$$f_2, f_4 \text{ injective and } f_1 \text{ surjective} \ \Rightarrow \ f_3 \text{ injective}.$$

11 De Rham Theorem

In this section we focus on de Rham cohomology and its connection to Čech and singular cohomologies.

We start with the fundamental Meyer-Vietoris theorem. Suppose that U and V are open subsets of M such that $U \cup V = M$. We have the following inclusions

$$\begin{tikzcd} & U \arrow[dr, hook, "k"] & \\ U \cap V \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & M, \\ & V \arrow[ur, hook, "\ell"'] & \end{tikzcd}$$

pullbacks

$$\begin{tikzcd} & \mathcal{A}^p(U) \arrow[dr, hook, "i^*"] & \\ \mathcal{A}^p(M) \arrow[ur, hook, "k^*"] \arrow[dr, hook, "\ell^*"'] & & \mathcal{A}^p(U \cap V), \\ & \mathcal{A}^p(V) \arrow[ur, hook, "j^*"'] & \end{tikzcd}$$

and induced cohomology maps

$$\begin{tikzcd} & H^p_{dR}(U) \arrow[dr, hook, "i^*"] & \\ H^p_{dR}(M) \arrow[ur, hook, "k^*"] \arrow[dr, hook, "\ell^*"'] & & H^p_{dR}(U \cap V). \\ & H^p_{dR}(V) \arrow[ur, hook, "j^*"'] & \end{tikzcd}$$

Let us consider the following sequence

$$0 \to \mathcal{A}^p(M) \xrightarrow{k^* \oplus \ell^*} \mathcal{A}^p(U) \oplus \mathcal{A}^p(V) \xrightarrow{i^* - j^*} \mathcal{A}^p(U \cap V) \to 0, \tag{11.1}$$

where

$$(k^* \oplus \ell^*)\omega = (k^*\omega, \ell^*\omega),\ \omega \in \mathcal{A}^p(M),$$
$$(i^* - j^*)(\alpha, \beta) = i^*\alpha - j^*\beta,\ \alpha \in \mathcal{A}^p(U), \beta \in \mathcal{A}^p(V).$$

Since pullbacks commute with the exterior derivative d, we have

$$H^p_{dR}(M) \xrightarrow{k^* \oplus \ell^*} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{i^* - j^*} H^p_{dR}(U \cap V).$$

<u>Suppose</u> that

$$0 \to \mathcal{A}^*(M) \xrightarrow{k^* \oplus \ell^*} \mathcal{A}^*(U) \oplus \mathcal{A}^*(V) \xrightarrow{i^* - j^*} \mathcal{A}^*(U \cap V) \to 0 \tag{11.2}$$

is a short exact sequence (of cochain complexes), i.e. the sequence (11.1) is exact for every p. Then the "Zig-zag lemma", Theorem 10.11, implies

Theorem 11.3 (The Meyer-Vietoris sequence). *If $M = U \cup V$ for some open sets U and V, then the sequence*

$$\cdots \to H^p_{dR}(M) \xrightarrow{k^*\oplus\ell^*} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{i^*-j^*} H^p_{dR}(U\cap V) \xrightarrow{\partial^*} H^{p+1}_{dR}(M)$$
$$\xrightarrow{k^*\oplus\ell^*} H^{p+1}_{dR}(U) \oplus H^{p+1}_{dR}(V) \to \cdots$$

is exact.

This is a fundamental calculation technique that can be used to determine $H^*_{dR}(M)$ as a "function" of $H^*_{dR}(U), H^*_{dR}(V)$ and $H^*_{dR}(U\cap V)$. To prove Theorem 11.3 we need the following:

Lemma 11.4. *Suppose that U and V are open subsets of M such that $M = U \cup V$. Then the sequence*

$$0 \to \mathcal{A}^*(M) \xrightarrow{k^*\oplus\ell^*} \mathcal{A}^*(U) \oplus \mathcal{A}^*(V) \xrightarrow{i^*-j^*} \mathcal{A}^*(U\cap V) \to 0$$

is exact, where k, ℓ, i and j are inclusions as above.

Proof. Claim 1. $k^* \oplus \ell^*$ is injective:
Suppose $\sigma \in \mathcal{A}^p(M)$ satisfies $(k^*\oplus\ell^*)\sigma = (\sigma|U, \sigma|V) = (0,0)$. Since $M = U\cup V$, we have $\sigma = 0$.
Claim 2. $\mathrm{Im}(k^*\oplus\ell^*) = \mathrm{Ker}(i^*-j^*)$:
For every $\sigma \in \mathcal{A}^p(M)$,

$$(i^*-j^*)\circ(k^*\oplus\ell^*)\sigma = (i^*-j^*)(\sigma|U,\sigma|V) = i^*(\sigma|U) - j^*(\sigma|V) = \sigma|(U\cap V) - \sigma|(U\cap V) = 0,$$

so $\mathrm{Im}(k^*\oplus\ell^*) \subset \mathrm{Ker}(i^*-j^*)$.
Conversely, suppose

$$\underbrace{(\alpha,\beta)}_{\in\mathcal{A}^p(U)\oplus\mathcal{A}^p(V)} \in \mathrm{Ker}(i^*-j^*).$$

Thus

$$(i^*-j^*)(\alpha,\beta) = i^*\alpha - j^*\beta = \alpha|(U\cap V) - \beta|(U\cap V) = 0,$$

so $\alpha|(U\cap V) = \beta|(U\cap V)$. Hence we can define $\sigma \in \mathcal{A}^p(M)$ by

$$\sigma = \begin{cases} \alpha, & \text{in } U, \\ \beta, & \text{in } V. \end{cases}$$

Then $(k^*\oplus\ell^*)\sigma = (k^*\sigma, \ell^*\sigma) = (\alpha,\beta)$, hence $\mathrm{Ker}(i^*-j^*) \subset \mathrm{Im}(k^*\oplus\ell^*)$.
The only "nontrivial" part is the following.
Claim 3. i^*-j^* is surjective:
If $U\cap V = \emptyset$, M is disconnected and the claim is trivial. Let $\omega \in \mathcal{A}^p(U\cap V)$ be arbitrary. We need to find $\alpha \in \mathcal{A}^p(U)$ and $\beta \in \mathcal{A}^p(V)$ such that

$$\omega = (i^*-j^*)(\alpha,\beta) = i^*\alpha - j^*\beta = \alpha|(U\cap V) - \beta|(U\cap V).$$

Let $\{\varphi,\psi\}$ be a smooth partition of unity subordinate to $\{U,V\}$. Define

$$\alpha = \begin{cases} \psi\omega, & \text{in } U\cap V, \\ 0, & \text{in } U\setminus \operatorname{supp}\psi. \end{cases}$$

Since $\psi\omega = 0$ on $(U\cap V)\setminus\operatorname{supp}\psi$, so $\alpha \in \mathcal{A}^p(U)$. Similarly,

$$\beta = \begin{cases} -\varphi\omega, & \text{in } U\cap V, \\ 0, & \text{in } V\setminus \operatorname{supp}\varphi. \end{cases}$$

Then $\alpha|(U\cap V) - \beta|(U\cap V) = \psi\omega + \varphi\omega = (\psi+\varphi)\omega = \omega$. □

11.5 Some calculations and applications

Proposition 11.6 (Disjoint union). *Let M_j, $j \in \mathbb{N}$, be disjoint smooth n-manifolds and $M = \bigsqcup_{j\in\mathbb{N}} M_j$. Then, for every p, the inclusion maps $i_j\colon M_j \hookrightarrow M$ induce an isomorphism*

$$H^p_{dR}(M) \to \prod_{j\in\mathbb{N}} H^p_{dR}(M_j).$$

Proof. Pullbacks $i_j^*\colon \mathcal{A}^p(M) \to \mathcal{A}^p(M_j)$ induce an isomorphism $\mathcal{A}^p(M) \to \prod_{j\in\mathbb{N}} \mathcal{A}^p(M_j)$, $\omega \mapsto (\omega|M_1, \omega|M_2, \ldots)$. This map is injective since $(\omega|M_1, \omega|M_2, \ldots) = (0, 0, \ldots)$ if and only if $\omega = 0$. It is surjective since arbitrary p-forms on M_j define a p-form on M. $\square$

Theorem 11.7. *If M is simply connected, then $H^1_{dR}(M) = 0$.*

Remark 11.8. A smooth 1-form $\omega \in \mathcal{A}^1(M)$ s exact if and only if $\int_\gamma \omega = 0$ for every closed (piecewise) smooth $\gamma\colon [0,1] \to M$:
Indeed, if $\omega \in \mathcal{A}^1(M)$ is exact, $\omega = df$ for some $f \in C^\infty(M)$. So,

$$\int_\gamma \omega = \int_\gamma df = f(\gamma(1)) - f(\gamma(0)) = 0.$$

For the converse direction, let $\omega \in \mathcal{A}^1(M)$ and fix $x_0 \in M$. For every $x \in M$, let $\alpha\colon [0,1] \to M$ be an arbitrary smooth path form $\alpha(0) = x_0$ to $\alpha(1) = x$ and define $f(x) = \int_\alpha \omega$. We claim that f is well-defined and $df = \omega$. Let $\tilde\alpha\colon [0,1] \to M$ be another smooth path from $\tilde\alpha(0) = x_0$ to $\tilde\alpha(1) = x$. Then α followed by $-\tilde\alpha$, $-\tilde\alpha(s) = \tilde\alpha(1-s)$ is closed and piecewise smooth, hence

$$0 = \int_{\alpha(-\tilde\alpha} \omega = \int_\alpha \omega + \int_{-\tilde\alpha)} \omega = \int_\alpha \omega - \int_{-\tilde\alpha} \omega,$$

so f is well-defined. Finally, $df = \omega$ follows from "real analysis".

Proof of Theorem 11.7. (Idea)
Let $\omega \in \mathcal{A}^1(M)$ be closed. We need to show that ω is exact. Let $\gamma\colon [0,1] \to M$ be closed and piecewise smooth. Since M is simply connected, γ bounds a surface $\Sigma = H([0,1] \times [0,1])$ with piecewise smooth boundary $\partial\Sigma = \gamma$. Above H is a smooth homotopy between γ and the constant (path) $\gamma(0)$. (We have also slightly abused the notation.) By Stokes's theorem,

$$\int_\gamma \omega = \int_\Sigma d\omega = 0$$

since ω is closed. By Remark 11.8, ω is exact. $\square$

Theorem 11.9. *For $n \geq 1$*

$$H^p_{dR}(\mathbb{S}^n) = \begin{cases} \mathbb{R}, & p = 0 \text{ or } p = n, \\ 0, & 0 < p < n. \end{cases}$$

Proof. Since $\mathbb{S}^n$ is connected, $H^0_{dR}(\mathbb{S}^n) = 0$ by Theorem 9.1.
Let $p \geq 1$. We prove the claim by induction on the dimension n. If $\omega \in \mathcal{A}^1(\mathbb{S}^1)$ is an orientation (volume) form, $c_0 = \int_{\mathbb{S}^1} \omega \neq 0$. Hence ω is not exact and $[\omega] \in H^1_{dR}(\mathbb{S}^1)$ (note that every $\omega \in \mathcal{A}^1(\mathbb{S}^1)$ is closed). Let $\eta \in \mathcal{A}^1(\mathbb{S}^1)$ be arbitrary and $c := \frac{1}{c_0} \int_{\mathbb{S}^1} \eta$. Then $\int_{\mathbb{S}^1} (\eta - c\omega) = 0$, and so $\eta - c\omega$ is exact and $[\eta] = c[\omega]$. Hence $\dim H^1_{dR}(\mathbb{S}^1)$.
Let $n \geq 2$ and assume that the claim holds for $\mathbb{S}^{n-1}$. Since $\mathbb{S}^n$ is simply connected, $H^1_{dR}(\mathbb{S}^n) = 0$.

For $p > 1$, we apply Meyer-Vietoris theorem. Let $x_0, y_0 \in \mathbb{S}^n$ be the north and the south pole, $U = \mathbb{S}^n \setminus \{y_0\}$ and $V = \mathbb{S}^n \setminus \{y_0\}$. Then U and V are diffeomorphic with $\mathbb{R}^n$, so $H^q_{dR}(U) = H^q_{dR}(V) = 0$ for $q \geq 1$. By the Meyer-Vietoris theorem

$$\underbrace{H^{p-1}_{dR}(U) \oplus H^{p-1}_{dR}(V)}_{=0} \to H^{p-1}_{dR}(U \cap V) \to H^p_{dR}(\mathbb{S}^n) \to \underbrace{H^p_{dR}(U) \oplus H^p_{dR}(V)}_{=0}$$

is exact, hence $H^{p-1}_{dR}(U \cap V) \cong H^p_{dR}(\mathbb{S}^n)$. On the other hand, $U \cap V$ is diffeomorphic with $\mathbb{R}^n \setminus \{0\}$, hence homotopic with $\mathbb{S}^{n-1}$, so $H^p_{dR}(\mathbb{S}^n) \cong H^{p-1}_{dR}(\mathbb{S}^{n-1})$ and the claim holds for $\mathbb{S}^n$. $\square$

Corollary 11.10. *Let $n \geq 2,\ x \in \mathbb{R}^n$ and $M = \mathbb{R}^n \setminus \{x\}$. Then*

$$H^p_{dR}(M) = \begin{cases} \mathbb{R}, & p = 0 \text{ or } p = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, a closed $(n-1)$-form η is exact if and only if $\int_S \eta = 0$ for all $(n-1)$-spheres surrounding x, i.e. $S = \partial B$ for some open ball $B \ni x$.

Proof. Let $S = \partial B$ for some open ball $B \ni x$. Then $\mathbb{R}^n \setminus \{x\}$ is homotopic with S hence with $\mathbb{S}^{n-1}$. Thus the first claim follows from Theorem 11.9 and 9.5.
Let η be a closed $(n-1)$-form on M. Then η is exact on M if and only if $i^*\eta$ is exact on S, where $i\colon S \hookrightarrow M$ is he inclusion (note that i is a homotopy equivalence $i \circ \mathrm{id}_M \cong \mathrm{id}_M,\ \mathrm{id}_M \circ i \cong \mathrm{id}_{\mathbb{S}^n}$). If η (hence $(i^*\eta)$ is exact, then $\int_S \eta = \int_S i^*\eta = 0$ by Corollary 7.12. The converse direction follows from the following lemma. $\square$

Lemma 11.11. *An n-form $\omega \in \mathcal{A}^n(\mathbb{S}^n)$ is exact if and only if $\int_{\mathbb{S}^n} \omega = 0$.*

Proof. The implication $\boxed{\Rightarrow}$ follows from Corollary 7.12.
$\boxed{\Leftarrow}$ Let $\omega \in \mathcal{A}^n(\mathbb{S}^n)$. We prove the claim by induction on n. The case $n = 1$ follows basicly from Remark 11.8.
Suppose the claim holds for $\mathbb{S}^{n-1}$. Let $\mathbb{S}^n = N \cup S$, where $N = \{x \in \mathbb{S}^n \colon x_{n+1} \geq 0\}$ and $S = \{x \in \mathbb{S}^n \colon x_{n+1} \leq 0\}$. Let $O_N = \{x \in \mathbb{S}^n \colon x_{n+1} > -\varepsilon\}$ and $O_S = \{x \in \mathbb{S}^n \colon x_{n+1} < \varepsilon\}$ for $0 < \varepsilon < 1/2$. The common boundary $\mathbb{S}^{n-1} = \partial N = \partial S$ has oppposite Stokes orientations with respect to N and S. By Poincaré's lemma 9.7 (applied with diffeomorphisms $\varphi\colon O_S \to B^n(0,1),\ \psi\colon O_N \to B^n(0,1)$), there exist $\alpha_N \in \mathcal{A}^{n-1}(O_N)$ and $\alpha_S \in \mathcal{A}^{n-1}(O_S)$ such that $d\alpha_N = \omega$ on O_N and $d\alpha_S = \omega$ on O_S (note that $d\varphi^* = \varphi^* d,\ d\psi^* = \psi^* d$). By the assumption and Stokes's theorem,

$$\begin{aligned} 0 &= \int_{\mathbb{S}^n} \omega = \int_N \omega + \int_S \omega = \int_N d\alpha_N + \int_S d\alpha_S \\ &= \int_{\partial N} i^*\alpha_N + \int_{\partial S} i^*\alpha_S = \int_{\mathbb{S}^{n-1}} i^*\alpha_N - \int_{\mathbb{S}^{n-1}} i^*\alpha_S \\ &= \int_{\mathbb{S}^{n-1}} i^*(\alpha_N - \alpha_S), \end{aligned}$$

where $i\colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$ is the inclusion. By induction, $i^*(\alpha_N - \alpha_S)$ is exact. Let $O = O_N \cap O_S$ and $r\colon O \to \mathbb{S}^{n-1}$ be the retraction along meridians. Now $i \circ r \cong \mathrm{id}_O$. Since $d(\alpha_N - \alpha_S) = \omega - \omega = 0$, $\alpha_N - \alpha_S - r^*i^*(\alpha_N - \alpha_S)$ is exact (Theorem 9.4). But $i^*(\alpha_N - \alpha_S) \in \mathcal{A}^{n-1}(\mathbb{S}^{n-1})$ is exact (by the induction hypothesis), so $r^*i^*(\alpha_N - \alpha_S) \in \mathcal{A}^{n-1}(O)$ is exact. Hence $\alpha_N - \alpha_S \in \mathcal{A}^{n-1}(O)$ is exact. So, there exists $\beta \in \mathcal{A}^{n-1}(O)$ such that $d\beta = \alpha_N - \alpha_S$ on O. Finally, extend β by using a bump function to $\gamma \in \mathcal{A}^{n-1}(\mathbb{S}^n)$ such that $\gamma = \beta$ on O and $\gamma = 0$ on $\mathbb{S}^n \setminus V$, where V is open such that $\bar{O} \subset V$. Then

$$\lambda = \begin{cases} \alpha_N, & \text{on } N, \\ \alpha_S + d\gamma, & \text{on } S \end{cases}$$

is smooth and $d\lambda = \omega$ on $\mathbb{S}^n$ □

Corollary 11.12. *If $n \neq m$, then $\mathbb{R}^m$ and $\mathbb{R}^m$ are not heomeomorphic.*

Proof. Suppose that $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic. Then $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^m \setminus \{0\}$ are homeomorphic. Hence $H^p_{dR}(\mathbb{R}^n \setminus \{0\}) \cong H^p_{dR}(\mathbb{R}^m \setminus \{0\})$ for all p, but this implies $n = m$. □

Recall Theorem 8.16. There is no continuous map $f\colon \bar{B}^n \to \mathbb{S}^{n-1}$ with $f|\mathbb{S}^{n-1} = \mathrm{id}_{\mathbb{S}^{n-1}}$. We give another proof for this. We may assume that $n \neq 2$ since $\bar{B}^1 = [-1,1]$ is connected but $\mathbb{S}^0 = \{-1,1\}$ is disconnected. The map $r\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, $r(x) = x/|x|$, is homotopic to $\mathrm{id}_{\mathbb{R}^n\setminus\{0\}}$ by $H(t,x) = t\,\mathrm{id}_{\mathbb{R}^n\setminus\{0\}}(x) + (1-t)r(x)$. If $f\colon \bar{B}^n \to \mathbb{S}^{n-1}$ were a continuous map with $f|\mathbb{S}^{n-1} = \mathrm{id}_{\mathbb{S}^{n-1}}$, then $f(tr(x))$, $0 \le t \le 1$, would be a homotopy between a constant map and r. This would imply that $\mathbb{R}^n \setminus \{0\}$ is contractible, hence $H^{n-1}_{dR}(\mathbb{R}^n \setminus \{0\}) = 0$ which is a contradiction. □

Theorem 11.13 (Hairy ball theorem)**.** *There exists a continuous nowhere vanishing vector field V on $\mathbb{S}^n$ ($V_x \in T_x\mathbb{S}^n\ \forall x \in \mathbb{S}^n$) if and only if n is odd.*

Proof. Suppose n is odd, $n = 2m-1$, $m \ge 1$. Define $V\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$V(x_1, \dots, x_{2m}) = (-x_2, x_1, -x_4, x_3, \dots, -x_{2m}, x_{2m-1}).$$

Clearly, V is continuous, $|V(x)| = 1\ \forall x \in \mathbb{S}^n$ and $V(x) \cdot x = 0$. Hence V is a continuous nowhere vanishing vector field on $\mathbb{S}^n$

Conversely, suppose that such V exists on $\mathbb{S}^n$. Extend it to a map $Y\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ by setting

$$Y(x) = V(x/|x|),\ x \in \mathbb{R}^{n+1} \setminus \{0\}.$$

Then $Y(x) \neq 0$ and $Y(x) \cdot x = 0$ on $\mathbb{R}^{n+1} \setminus \{0\}$. The map

$$F(x,t) = \cos(t\pi)x + \sin(t\pi)Y(x)$$

defines a homotopy from $f_0 = \mathrm{id}_{\mathbb{R}^{n+1}\setminus\{0\}}$ to the antipodal map $f_1 = -\mathrm{id}_{\mathbb{R}^{n+1}\setminus\{0\}}$. Hence f_1^* is the *identity* on $H^n_{dR}(\mathbb{R}^{n+1} \setminus \{0\})$ which is 1-dimensional. On the other hand, $f_1^*\colon H^n_{dR}(\mathbb{R}^{n+1} \setminus \{0\}) \to H^n_{dR}(\mathbb{R}^{n+1} \setminus \{0\})$ operates by multiplication by $(-1)^{n+1}$ [Exerc.]. Hence n is odd. □

11.14 Čech cohomology (sketch)

Definition 11.15. Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ an open cover of M. We say that $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ is a *good cover* if for every finite set $\{\alpha_1, \dots, \alpha_k\} \subset \mathcal{A}$ of indices the intersection $U_{\alpha_1} \cap \dots \cap U_{\alpha_k}$ is either empty or diffeomorphic to $\mathbb{R}^n$ (hence contractible).

Lemma 11.16. *Every smooth manifold M has good covers.*

Proof. Equip M with a Riemannian metric. Then for every $x \in M$ there exists $r_x > 0$ such that (metric, geodesic) open balls $B(x,r)$ are convex for all $0 < r \le r_x$. Then $\{B(x,r_x)\}_{x\in M}$ is a good cover. □

Definition 11.17. A smooth manifold M has *finite topology* if there exists a finite good cover $\{U_1, \dots, U_N\}$.

Examples 11.18.

1. A compact manifold M has finite topology.
2. If $C \subset M$ is compact and U_0 is an open neighborhood of C, there exists an open neighborhood U of C such that $U \subset U_0$ and U has finite topology.

Theorem 11.19. *If M^n has finite topology, then all $H^p_{dR}(M)$ are finite dimensional.*

Proof. The claim follows from the Meyer-Vietoris theorem and Lemma 10.2 by induction on the number of sets in a good cover.
Let $\mathcal{U} = \{U_1, \dots, U_N\}$ be a good cover of M^n. If $N = 1$, then $M = U_1$ is doffeomorphic to $\mathbb{R}^n$, so

$$H^p_{dR}(M) = \begin{cases} \mathbb{R}, & p = 0, \\ 0, & p > 0. \end{cases}$$

Suppose the claim holds for $N-1$. Let $U = U_2 \cup \cdots \cup U_N$. Then $\{U_2, \dots, U_N\}$ is a good cover of U, hence $H^p_{dR}(U)$ are finite dimensional by the induction hypothesis. The manifold $U \cap U_1$ has a good cover $\{U_1 \cap U_2, \dots, U_1 \cap U_N\}$, so the cohomology groups of $U \cap U_1$ are finite dimensional. Now $M = U \cup U_1$ and we have the exact Meyer-Vietoris sequence

$$H^{p-1}_{dR}(U \cap U_1) \xrightarrow{\partial^*} H^p_{dR}(M) \to H^p_{dR}(U) \oplus H^p_{dR}(U_1)$$

and the claim follows from Lemma 10.2. $\square$

In fact, the dimensions of $H^p_{dR}(M)$ are determined by intersection properties of sets U_α, i.e. by the list of multi-indices for which the intersections are non-empty. The collection of such multi-indices is called the *nerve* of $\mathcal{U}$. This suggests that any cohomology theory whose input is the nerve of $\mathcal{U}$ and cohomology groups (vector spaces) as output will be isomorphic to de Rham cohomology.

Let $\mathcal{U} = \{U_1, \dots, U_d\}$ be a good cover of M. Denote by $N^k(\mathcal{U})$ the set of all multi-indices

$$I = (i_0, \dots, i_k), \; 1 \le i_0, \dots, i_k \le d,$$

such that

$$U_I := U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset.$$

Hence such U_I is diffeomorphic to $\mathbb{R}^n$. For example, $I = (i, i, \dots, i) \in N^k(\mathcal{U})$ since $U_I = U_i$. The disjoint union

$$N(\mathcal{U}) = \bigsqcup_{k \ge 0} N^k(\mathcal{U})$$

is called the *nerve* of $\mathcal{U}$ and $N^k(\mathcal{U})$ is the *k-skeleton* of $N(\mathcal{U})$. If $I = (i, i, \dots, i) \in N^k(\mathcal{U})$, then

$$I_j := (i_0, \dots, \widehat{i_j}, \dots, i_k) \in N^{k-1}(\mathcal{U}).$$

We associate a cochain complex $C^*/\mathcal{U})$ to $N(\mathcal{U})$ by defining the vector space

$$C^k(\mathcal{U}, \mathbb{R}) = \{u \colon N^k(\mathcal{U}) \to \mathbb{R}.$$

The differential (or coboundary operator) is the linear map $d \colon C^k(\mathcal{U}, \mathbb{R}) \to C^{k+1}(\mathcal{U}, \mathbb{R})$ defined by

$$du(I) = \sum_{j=0}^{k+1} (-1)^j I_j,$$

that is,

$$du(i_0,\dots,i_{k+1}) = \sum_{j=0}^{k+1}(-1)^j u(i_0,\dots,\widehat{i_j},\dots,i_{k+1}.$$

Then $d\circ d\ \big(: C^k(\mathcal{U},\mathbb{R}) \to C^{k+2}(\mathcal{U},\mathbb{R})\big) = 0$. Indeed, if $u \in C^k(\mathcal{U},\mathbb{R})$, then

$$d(du)(I) = \sum_{i=0}^{k+2}(-1)^i du(I_i) = \sum_{i=0}^{k+2}(-1)^i\left(\sum_{j<i}(-1)^j u(I_{i,j}) + \sum_{j>i}(-1)^{j-1}u(I_{i,j})\right) = 0$$

since each $u(I_{i,j}), i \neq j$, occurs twice, but with opposite signs. Note that the sign $(-1)^{j-1}$ for $j > i$ occurs since the (original) ith index i_i is missing and therefore i_j is the $(j-1)$st index.

The cochain complex

$$0 \to C^0(\mathcal{U},\mathbb{R}) \xrightarrow{d} C^1(\mathcal{U};\mathbb{R}) \xrightarrow{d} \cdots \xrightarrow{d} C^k(\mathcal{U},\mathbb{R}) \xrightarrow{d} \cdots$$

is called the *Čech cochain complex* of the cover $\mathcal{U}$ and the *Čech cohomology* vector spaces of the cover $\mathcal{U}$ are

$$H^k(\mathcal{U},\mathbb{R}) = \frac{\mathrm{Ker}\,(d\colon C^k(\mathcal{U},\mathbb{R}) \to C^{k+1}(\mathcal{U},\mathbb{R}))}{\mathrm{Im}\,(d\colon C^{k-1}(\mathcal{U},\mathbb{R}) \to C^k(\mathcal{U},\mathbb{R}))}.$$

Theorem 11.20. *Suppose that M has a finite good cover $\mathcal{U}$. Then for all $k \geq 0$*

$$H^k(\mathcal{U};\mathbb{R}) \cong H^k_{dR}(M).$$

Very rough idea of the proof: Both de Rham and Čech cohomology theories have Meyer-Vietoris sequences and satisfy the Poincaré lemma. Then the proof goes via induction on the number of sets in the good cover together with the 5-Lemma.

We can reduce the proof to the following general:

Proposition 11.21. *Let M^n be a smooth manifold. Suppose $P(U)$ is a statement about open subsets $U \subset M$ satisfying;*

(1) $P(U)$ is true if U is diffeomorphic to a convex subset of $\mathbb{R}^n$;

(2) $P(U),\ P(V),\ P(U\cap V) \Rightarrow\ P(U\cup V)$;

(3) $\{U_\alpha\}$ disjoint and $P(U_\alpha)\ \forall\alpha\ \Rightarrow\ P\big(\cup_\alpha U_\alpha\big)$.

Then $P(M)$ holds.

Proof of Propositio 11.21. Assume first that M^n is diffeomorphic to an open subset of $\mathbb{R}^n$. So, we may think of M^n being an open subset of $\mathbb{R}^n$. By (1), (2) and induction it follows that $P(U)$ holds if U is a union of finite number of convex open subsets of $\mathbb{R}^n$ because

$$(U_1\cup U_2\cup\cdots\cup U_n)\cap U_{n+1} = (U_1\cap U_{n+1})\cup\cdots(U_n\cap U_{n+1})$$

and a nonempty intersection of two convex sets is convex.
Let then $\{V_i\}$ be a collection of open sets such that $\bar{V_i}$ is compact. Take a (smooth) partition of unity $\{f_i\}$ subordinate to $\{V_i\}$ and define $f = \sum_j jf_j$. Let $A_j = f^{-1}[j,j+1]$. Then A_j is compact since f is proper. Cover A_j by finite union U_j of convex open subsets contained in $f^{-1}(j-\frac{1}{2}, j+\frac{3}{2})$. Then $A_j \subset U_j \subset f^{-1}(j-\frac{1}{2}, j+\frac{3}{2})$. So U_{even} are disjoint and similarly U_{odd} are disjoint. Since U_j

is a finite union of convex open sets, $P(U_j)$ holds. By (3), $P(U)$ and $P(V)$ hold, where $U = \cup_j U_{2j}$ and $V = \cup_j U_{2j+1}$. Now

$$U \cap V = \bigsqcup_j (U_{2j} \cap U_{2j+1})$$

is a disjoint union of sets $U_{2i} \cap U_{2j+1}$ that are either empty or finite unions of convex open sets. Hence $P(U \cap V)$ holds. Now it follows from (2) that $P(M) = P(U \cup V)$ holds. Hence $P(U)$ is true for all open U that are diffeomorphic to an open subset of $\mathbb{R}^n$. Replacing above "convex open" by "open" and repeating the argument completes the proof. ☐

The crucial step in the proof of Theorem 11.20 is to verify the property (2) in Proposition 11.21 $[P(U),\ P(V),\ P(U \cap V) \Rightarrow P(U \cup V)]$. Here we apply Meyer-Vietoris sequences and the 5-Lemma to exact sequences

$$\begin{tikzcd}
H^{p-1}_{dR}(U) \oplus H^{p-1}_{dR}(V) \arrow[r] \arrow[d, "\cong"'] & H^{p-1}_{dR}(U \cap V) \arrow[r] \arrow[d, "\cong"'] & H^p_{dR}(U \cup V) \arrow[r] \arrow[d] & {} \\
H^{p-1}(\{U\}, \mathbb{R}) \oplus H^{p-1}(\{V\}, \mathbb{R}) \arrow[r] & H^{p-1}(\{U \cap V\}, \mathbb{R}) \arrow[r] & H^p(\{U \cup V\}, \mathbb{R}) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^p_{dR}(U) \oplus H^p_{dR}(V) \arrow[r] \arrow[d, "\cong"'] & H^p_{dR}(U \cap V) \arrow[d, "\cong"'] \\
{} \arrow[r] & H^p(\{U\}, \mathbb{R}) \oplus H^p(\{V\}, \mathbb{R}) \arrow[r] & H^p(\{U \cap V\}, \mathbb{R}).
\end{tikzcd}$$

The 5-Lemma implies that also $H^p_{dR}(U \cup V) \to H^p(\{U \cup V\}, \mathbb{R})$ is an isomorphism.

11.22 Singular (co-)homology

Let $e_0, e_1, \dots$ be the standard basis of $\mathbb{R}^\infty$. The *standard p-simplex*, $p \geq 0$, is

$$\Delta_p = \left\{ \sum_{i=0}^{p} \lambda_i e_i \colon \sum_{i=0}^{p} \lambda_i = 1, 0 \leq \lambda_i \leq 1 \right\},$$

i.e. the convex hull of $\{e_0, \dots, e_p\}$. λ_i's are called *barycentric coordinates.*

Let X be a topological space. A *singular p-simplex* in X is a continuous map $\phi \colon \Delta_p \to X$. "Singular" refers to the lack of regularity. The *singular p-chain group* of X, $C_p(X)$, is the free Abelian group generated by all singular p-simplices in X. An element of $C_p(X)$, called a *singular p-chain*, is a finite formal linear combination of singular p-simplices with integer coefficients

$$\sum_\phi n_\phi \phi,$$

with $n_\phi \in \mathbb{Z}$ equal to zero for all but a finite number of ϕ.

Remark 11.23. Let us recall the notion of a free Abelian group generated by a set.
Let A be an arbitrary set and $F(A) = \{f \colon A \to \mathbb{Z} \mid f(a) \neq 0 \text{ for only finitely many } a \in A\}$. Define in $F(A)$, the sum $f + g$ by $(f+g)(a) = f(a) + g(a)$. Then $\big(F(A), +\big)$ is an Abelian group. For any $a \in A$, define $f_a = \chi_{\{a\}} \in F(A)$. Then $\{f_a \colon a \in A\}$ is a basis for $F(A)$ as a free Abelian group. By identifying a with f_a, we may interpret A as the basis of $F(A)$.

For $p+1$ points $v_0, \dots, v_p \in \mathbb{R}^N$ (not necessarily in general position) we define an *affine singular p-simplex* $[v_0, \dots, v_p] \colon \Delta_p \to \mathbb{R}^N$ as the map

$$\sum_{i=0}^{p} \lambda_i e_i \mapsto \sum_{i=0}^{p} \lambda_i v_i.$$

For each $i \in \{0, \dots, p\}$, let $F_i^p \colon \Delta_{p-1} \to \Delta_p$ be the affine singular $(p-1)$-simplex $F_i^p = [e_0, \dots, \widehat{e_i}, \dots, e_p]$. More precisely, F_i^p is the restriction to Δ_{p-1} of the affine map such that

$$\begin{aligned} e_0 &\mapsto e_0 \\ &\vdots \\ e_{i-1} &\mapsto e_{i-1} \\ e_i &\mapsto e_{i+1} \\ &\vdots \\ e_{p-1} &\mapsto e_p. \end{aligned}$$

F_i^p is called the *ith face map*. Note that F_i^p maps Δ_{p-1} homeomorphically onto the *boundary face* of Δ_p opposite the *vertex* e_i. Note also that, for $i > j$,

$$F_i^p \circ F_j^{p-1} = F_j^p \circ F_{i-1}^{p-1} \colon \Delta_{p-2} \to \Delta_p. \tag{11.24}$$

Let then $\phi \colon \Delta_p \to X$ be a singular p-simplex. We define a singular $(p-1)$-chain $\partial\phi$, called the *boundary* of ϕ by

$$\partial\phi = \sum_{i=0}^{p} (-1)\text{Ãő}\phi \circ F_i^p.$$

This definition extends uniquely to a homomorphism $\partial \colon C_p(X) \to C_{p-1}(X)$, called the *boundary operator*

$$\partial\Big(\sum_{\phi} n_\phi \phi\Big) = \sum_{\phi} n_\phi \partial\phi.$$

Proposition 11.25. *The composition $\partial \circ \partial \colon C_p(X) \to C_{p-2}(X)$ is zero.*

Proof. We have

$$\begin{aligned} \partial(\partial\phi) &= \sum_{j=0}^{p-1} \sum_{i=0}^{p} (-1)^{i+j} \phi \circ F_i^p \circ F_j^{p-1} \\ &= \sum_{0 \le j < i \le p} (-1)^{i+j} \phi \circ F_i^p \circ F_j^{p-1} + \sum_{0 \le i \le j \le p-1} (-1)^{i+j} \phi \circ F_i^p \circ F_j^{p-1}. \end{aligned}$$

Writing $j' = i,\ i' - 1 = j$ in the second sum and using (11.24) it bocomes

$$\sum_{0 \le j' < i' \le p} (-1)^{i'+j'-1} \phi \circ F_{j'}^p \circ F_{i'-1}^{p-1} = \sum_{0 \le j' < i' \le p} (-1)^{i'+j'-1} \phi \circ F_{i'}^p \circ F_{j'}^{p-1}$$

so the sums cancel term by term. $\square$

A singular p-chain σ is called a *p-cycle* if $\partial\sigma = 0$. The set of all p-cycles, denoted by $Z_p(X)$, is a subgroup of $C_p(X)$ as the kernel of the homomorphism ∂. Similarly, the image $\partial C_{p+1}(X)$ is the subgroup $B_p(X)$ of all *p-boundaries*. The quotient group

$$H_p(X) = Z_p(X)/B_p(X)$$

is the *pth (singular) homology group* of X. This is zero if every p-cycle is the boundary of a $(p+1)$-chain. Intuitively, this means that there are no p-dimensional "holes" in X.

If $f\colon X \to Y$ is continuous, let $f_\sharp\colon C_p(X) \to C_p(Y)$ be the homomorphism defined by

$$f_\sharp \sigma = f \circ \sigma$$

for all p-simplex σ. Since $f_\sharp$ commutes with ∂, $f_\sharp(\partial\sigma) = \partial f_\sharp \sigma$, $f_\sharp$ maps $Z_p(X)$ into $Z_p(Y)$ and $B_p(X)$ into $B_p(Y)$. Hence it passes to quotients and defines a homomorphism

$$f_\sharp\colon H_p(X) \to H_p(Y).$$

We have the Meyer-Vietoris sequence: Let X be a topological space and $U, V \subset X$ open sets such that $X = U \cup V$. For each p there exists a homomorphism $\partial_*\colon H_p(X) \to H_{p-1}(U \cap V$ such that the sequence

$$\cdots \xrightarrow{\partial_*} H_p(U \cap V) \xrightarrow{i_* \oplus j_*} H_p(U) \oplus H_p(V) \xrightarrow{k_* - \ell_*} H_p(X) \xrightarrow{\partial_*} H_{p-1}(U \cap V) \to \cdots$$

is exact.

We define the *singular pth cohomology group* $H^p(X;\mathbb{R})$ with coefficients in $\mathbb{R}$ as a real vector space that is isomorphic to the space $\operatorname{Hom}\left(H_p(X), \mathbb{R}\right)$ of group homomorphisms $H_p(X) \to \mathbb{R}$. Any continuous map $f\colon X \to Y$ induces a linear map $f^*\colon H^p(Y;\mathbb{R}) \to H^p(X;\mathbb{R})$,

$$(f^*\gamma)[c] = \gamma(f_\sharp[c])$$

for every $\gamma \in H^p(Y;\mathbb{R}) \cong \operatorname{Hom}\left(H_p(Y), \mathbb{R}\right)$ and $[c] \in H_p(X)$. Singular cohomology groups satisfy the Meyer-Vietoris theorem.

11.26 Smooth singular homology

The connection between singular and de Rham cohomologies is established by integrationg differential p-forms over singular p-chains. Given a singular p-simplex $\sigma\colon \Delta_p \to M$ and a differential p-form ω, we would like to integrate the pullback σ^* over Δ_p.
However, we face a problem. Pullback is defined for smooth maps (at least C^1) only. [Regularity could be weakened further but not for arbitrary continuous maps.]
To circumvent this problem we need to define a smoothing operator $s\colon C_p(M) \to C_p^\infty(M)$ (smooth p-chain group) such that it commutes with the boundary operator $s \circ \partial = \partial \circ s$ and $s \circ i = \mathrm{id}_{C_p^\infty(M)}$ $[i\colon C_p^\infty(M) \hookrightarrow C_p(M)]$ and we need a homotopy operator that shows that $i \circ s$ induces the identity map on $H_p(M)$.

Definition 11.27. The pth smooth singular homology group of M is the quotient

$$H_p^\infty(M) = \frac{\operatorname{Ker}\left[\partial\colon C_p^\infty(M) \to C_{p-1}^\infty(M)\right]}{\operatorname{Im}\left[\partial\colon C_{p+1}^\infty(M) \to C_p^\infty(M)\right]}.$$

Theorem 11.28. *For a smooth manifold M, the map $i_*\colon H_p^*(M) \to H_p(M)$ induced by the inclusion $i\colon C_p^\infty(M) \hookrightarrow C_p(M)$ is an isomorphism.*

The proof is technical (based on Whitney type approximation) construction of homotopy from each continuous simplex to a smooth one that respects the restriction to each boundary face of Δ_p.

11.29 de Rham homomorphism, the de Rham's theorem

Let $\sigma\colon \Delta_p \to M$ be a smooth p-simplex and ω a closed p-form on M. Define

$$\int_\sigma \omega := \int_{\Delta_p} \sigma^*\omega.$$

This definition makes sense since Δ_p has the induced orientation from $\mathbb{R}^{p+1}$. If $c = \sum_i c_i\sigma_i$ is a smooth p-chain, then

$$\int_c \omega := \sum_i c_i \int_{\sigma_i} \omega.$$

Stokes's theorem (we can neglect "corners"):
If c is a smooth p-chain and ω a smooth $(p-1)$-form, then

$$\int_{\partial c} \omega = \int_c d\omega.$$

We define the de Rham homomorphism $\Psi^*\colon H^p_{dR}(M) \to H^p(M;\mathbb{R}) \cong \operatorname{Hom}\big(H_p(M), \mathbb{R}\big)$ by

$$\Psi^*[\omega][c] = \int_{\tilde{c}} \omega, \quad \forall\ [\omega] \in H^p_{dR}(M),\ [c] \in H_p(M) \cong H^\infty_p(M),$$

where $\tilde{c}$ is any smooth p-cycle in $[c] \in H_p(M)$. It is well-defined. Indeed, if $c_1, c_2 \in [c]$ are smooth, then $c_1 - c_2 = \partial b$ for some smooth $(p+1)$-chain, so

$$\int_{c_1} \omega - \int_{c_2} \omega = \int_{\partial b} \omega = \int_b d\omega = 0$$

since $\omega \in [\omega]$ is closed. If $\omega = d\eta$ is exact, then

$$\int_{\tilde{c}} \omega = \int_{\tilde{c}} d\eta = \int_{\partial \tilde{c}} \eta = 0$$

since $\partial \tilde{c} = 0$. So, Ψ^* is a well-defined homomorphism.

Theorem 11.30 (The de Rham's theorem)**.** *The homomorphism*

$$\Psi^*\colon H^p_{dR}(M) \to H^p(M;\mathbb{R})$$

is an isomorphism for all p and for every smooth manifold M.

Proof. The idea of the proof is, of course, the same as in the proof of Theorem 11.20. We have Meyer-Vietoris sequences, Poincaré lemma and the 5-Lemma for both cohomologies.
Thus, let $P(U)$ be the property:

$$\Psi^*\colon H^p_{dR}(U) \to H^p(U;\mathbb{R})$$

is an isomorphism for an open set $U \subset M$.
Then we have:
(1): $P(U)$ holds if U is diffeomorphic to an open convex subset of $\mathbb{R}^n$ (Poincaré lemma).

(3): $\{U_\alpha\}$ disjoint and $P(U_\alpha)$ holds $\forall\ \alpha \Rightarrow P(\cup_\alpha U_\alpha)$ holds.
(2): Suppose $P(U)$, $P(V)$ and $P(U\cap)$ hold. Then

$$\begin{tikzcd}
H^{p-1}_{dR}(U) \oplus H^{p-1}_{dR}(V) \arrow[r] \arrow[d, "\Psi^*\oplus\Psi^*", "\cong"'] & H^{p-1}_{dR}(U\cap V) \arrow[r] \arrow[d, "\Psi^*", "\cong"'] & H^p_{dR}(U\cup V) \arrow[r] \arrow[d, "\text{homom.}", "\Psi^*"'] & {} \\
H^{p-1}(U;\mathbb{R}) \oplus H^{p-1}(V;\mathbb{R}) \arrow[r] & H^{p-1}(U\cap V;\mathbb{R}) \arrow[r] & H^p(U\cup V;\mathbb{R}) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^p_{dR}(U) \oplus H^p_{dR}(V) \arrow[r] \arrow[d, "\Psi^*\oplus\Psi^*", "\cong"'] & H^p_{dR}(U\cap V) \arrow[d, "\Psi^*", "\cong"'] \\
{} \arrow[r] & H^p(U;\mathbb{R}) \oplus H^p(V;\mathbb{R}) \arrow[r] & H^p(U\cap V;\mathbb{R}).
\end{tikzcd}$$

The 5-Lemma implies that also $H^p_{dR}(U\cup V) \to H^p(U\cup V;\mathbb{R})$ is an isomorphism, so $P(U\cup V)$ holds. Proposition 11.21 implies that $\Psi^*\colon H^p_{dR} \to H^p(M;\mathbb{R})$ is an isomorphism. $\square$

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