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NONLINEAR POTENTIAL THEORY AND QUASIREGULAR MAPPINGS ON RIEMANNIAN MANIFOLDS

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1. Introduction

In this paper we study solutions of a quasilinear elliptic equation

$$Tu = -\operatorname{div} \mathcal{A}_x(\nabla u) = 0 \tag{1.1}$$

on a Riemannian n-manifold M. Here $\langle \mathcal{A}_x(\nabla u), \nabla u\rangle \approx |\nabla u|^p$ and $1 < p \leq n$. The precise assumptions on $\mathcal{A}$ are given in section 2.

In the Euclidean n-space $\mathbf{R}^n$ solutions of (1.1) have been extensively studied recently by J. Heinonen, T. Kilpeläinen, P. Lindqvist, and O. Martio. They have developed a nonlinear potential theory where so called $\mathcal{A}$-superharmonic functions play a role similar to that of superharmonic functions in the classical potential theory, see [GLM], [HK], [HKM] and references there.

Our purpose is to extend this theory to cover Riemannian n-manifolds, too. Part of the problems in this theory are in a sense local and therefore they do not cause extra difficulties in the case of Riemannian manifolds. In this paper we are mainly interested in global problems, like finding counterparts for the Green function and for the classification theory of Riemann surfaces. In the latter one defines several classes of surfaces depending on the existence of harmonic functions of given kind on them, see [AS] and [SN].

The paper is organized as follows. Section 2 contains some properties of solutions of (1.1) known in the euclidean case. Here we also discuss how we can obtain these results on Riemannian manifolds. In section 3 we give a definition for a Green function and a proof of the existence theorem. The uniqueness of the Green function is also studied. In section 4 we apply some methods from section 3 to study solutions of (1.1) which have many singularities. Section 5 is devoted to the classification problem. We introduce some classes of manifolds and prove inclusions between them. Strictness of some inclusions is also discussed in this section, but a detailed discussion will appear in a forth-coming paper. In section 6 we study the Heisenberg group which can be an interesting example in the classification theory. In the final section we apply some potential theoretic methods in studying quasiregular mappings. We prove a generalization of the so called Comparison lemma which is an essential tool in the proof of the Picard type theorem for quasiregular mappings and in value distribution theory, see [Ri, Chapter IV and V]. We also extend the Picard type theorem for metrics in the image with only a regularity assumption.

Notation. Throughout this paper we let M (and N) be an n-dimensional, $n \geq 2$, noncompact, connected and orientable Riemannian manifold of class C^∞ equipped with a Riemannian metric $\langle\ ,\ \rangle$. For each point $x \in M$, the tangent space to M at x will be denoted by T_xM; and the tangent bundle, that is, the union of all tangent spaces of M, will be denoted by TM. The norm associated to the Riemannian metric will be denoted by $|\ \ |$ and the Riemannian volume form by dm.

Throughout the paper G will be an open subset of M and $D \subset\subset G$ means that $\bar{D}$, the closure of an open set D, is compact in G. The space of all functions $u \in locL^1(G)$ whose distributional gradient ∇u belongs to $L^p(G)$, $1 \leq p < \infty$, will be denoted by $L^1_p(G)$. We equip $L^1_p(G)$ with the seminorm $\|\nabla u\|_p$. Similarly, the Sobolev space $W^1_p(G)$ consists of all functions $u \in L^1_p(G)$ which belongs to $L^p(G)$, too. It is a Banach-space equipped with the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p .$$

The spaces $L^1_{p,0}(G)$ and $W^1_{p,0}$ are the closures of $C^\infty_0(G)$ in $L^1_p(G)$ and in $W^1_p(G)$, respectively. Recall that a vectorfield $X \in locL^1(G)$ is a distributional gradient of a function $u \in locL^1(G)$ if

$$\int_G u \operatorname{div} Y\, dm = -\int_G \langle X, Y\rangle\, dm$$

for all vectorfields $Y \in C^1_0(G)$.

2. $\mathcal{A}$-harmonic functions

Let $\mathcal{A} : TM \to TM$ be an operator satisfying the following assumptions for some numbers $1 < p \leq n$ and $0 < \alpha \leq \beta < \infty$:

(2.1) the mapping $\mathcal{A}_x = \mathcal{A}|T_xM : T_xM \to T_xM$ is continuous for a.e. $x \in M$, and the mapping $x \mapsto \mathcal{A}_x(X)$ is measurable for all measurable vectorfields X;

for a.e. $x \in M$ and for all $h \in T_xM$

$$\langle \mathcal{A}_x(h), h\rangle \geq \alpha\, |h|^p , \tag{2.2}$$

$$|\mathcal{A}_x(h)| \leq \beta\, |h|^{p-1} , \tag{2.3}$$

$$\langle \mathcal{A}_x(h_1) - \mathcal{A}_x(h_2), h_1 - h_2\rangle > 0, \tag{2.4}$$

whenever $h_1 \neq h_2$, and

$$\mathcal{A}_x(\lambda h) = |\lambda|^{p-2}\lambda \mathcal{A}_x(h) \tag{2.5}$$

for all $\lambda \in \mathbf{R}\setminus\{0\}$.

The class of all operators $\mathcal{A}$ which satisfy the conditions (2.1) - (2.5) with the constant p will be denoted by $\mathcal{A}_p(M)$. If $M = \mathbf{R}^n$, $\mathcal{A}$ is defined in $\mathbf{R}^n \times \mathbf{R}^n$ and we also write $\mathcal{A}(x,h)$ instead of $\mathcal{A}_x(h)$.

A function $u \in C(G) \cap locW_p^1(G)$ is said to be *$\mathcal{A}$-harmonic* in G if it is a weak solution of the equation (1.1), in other words, if u satisfies

$$\int_G \langle \mathcal{A}_x(\nabla u), \nabla\varphi\rangle \, dm = 0 \tag{2.6}$$

for all $\varphi \in C_0^\infty(G)$. If, moreover, u belongs to $L_p^1(G)$, it is equivalent to require (2.6) for all $\varphi \in L_{p,0}^1(G)$, see [Ri, VI.1.15].

The simplest operator satisfying the conditions (2.1) - (2.5) is the *p-Laplacian*

$$\mathcal{A}_x(h) = |h|^{p-2} h.$$

In this case continuous solutions are usually called *p-harmonic*.

We shall list below the most important properties of $\mathcal{A}$-harmonic functions. It follows directly from (2.6) that $\lambda u + \mu$ is $\mathcal{A}$-harmonic, if u is $\mathcal{A}$-harmonic and $\lambda, \mu \in \mathbf{R}$. Here the assumption (2.5) is also used. Furthermore, it is well known that solutions of (1.1) are locally Hölder continuous and that Harnack's inequality holds: If u is a nonnegative $\mathcal{A}$-harmonic function in G and if C is a connected compact subset of G, there is a constant $c = c(n, p, \beta/\alpha, C, G) > 1$ with

$$\sup_C u \leq c \inf_C u, \tag{2.7}$$

see [Se1] and [Tr]. A frequently used fact is that the class of $\mathcal{A}$-harmonic functions is closed under uniform convergence. As a consequence we obtain Harnack's principle: If u_i, $i = 1, 2, \ldots$, is an increasing sequence of $\mathcal{A}$-harmonic functions in a domain G, then $u = \lim_{i\to\infty} u_i$ is either $\mathcal{A}$-harmonic or identically $+\infty$ in G, [HK, 3.2 and 3.3].

It is worth noting that the Hölder continuity and Harnack's inequality have been proved for $M = \mathbf{R}^n$. However, these results can easily been obtained on Riemannian manifolds by using suitable chart mappings. Indeed, for every $x \in M$ we can choose a neighborhood U of x and a chart $\varphi : U \to B^n(0,r)$ which is 2-bilipschitz, see [LF, 2.2]. It turns out that $u \circ \varphi^{-1}$ is $\mathcal{A}_1$-harmonic in $B^n(0,r)$, if u is $\mathcal{A}$-harmonic in U. Here $\mathcal{A}_1$ is an operator, called the *pullback* of $\mathcal{A}$ by φ^{-1}, which satisfies the conditions (2.1) - (2.5) with the same constant p as $\mathcal{A}$ does and with constants α_1 and β_1 depending only on p, n and on the constants α and β of $\mathcal{A}$, see [MV, Section 3] and (2.9) below.

One of the most natural questions also in this nonlinear theory is the solvability of the Dirichlet boundary value problem. The following *Wiener criterion* has turned out to be very important in this problem. A closed set $C \subset \mathbf{R}^n$ is said to be *p-thin* at a point $x \in \mathbf{R}^n$ if

$$W(x,C) = \int_0^1 \left(\frac{\operatorname{cap}_p\big(\bar{B}^n(x,t) \cap C, B^n(x,2t)\big)}{\operatorname{cap}_p\big(\bar{B}^n(x,t), B^n(x,2t)\big)} \right)^{1/(p-1)} \frac{dt}{t} < \infty.$$

We recall the definition of the *p-capacity* of a *condenser*. Let F be a subset of G. The (outer) p-capacity of the condenser (F, G) is defined by

$$\operatorname{cap}_p(F,G) = \inf_{\substack{F \subset U \\ U \text{ open}}} {}_*\operatorname{cap}_p(U,G)$$

where, for any set $A \subset G$,

$${}_*\operatorname{cap}_p(A,G) = \sup_C \inf_u \int_G |\nabla u|^p \, dm.$$

In the latter the supremum is taken over all compact sets $C \subset A$ and u runs through all functions in $C_0^\infty(G)$ with $u \geq 1$ in C. See [Mz2] and [Re1] for a thorough discussion of variational capacities.

The connection between the Dirichlet boundary value problem and the thinness of a closed set is the following. Let G be a bounded domain in $\mathbf{R}^n$ and $v \in W_p^1(G)$. Then there exists a unique $\mathcal{A}$-harmonic function u in G with boundary values v, i.e. $u - v \in W_{p,0}^1(G)$. If, moreover, $v \in C(\bar{G})$ and the Wiener criterion $W(x, \mathbf{R}^n \setminus G) = \infty$ holds at $x \in \partial G$, then

$$\lim_{y \to x} u(y) = v(x), \tag{2.8}$$

see [Mz1]. On the other hand, [LM] shows that $W(x, \mathbf{R}^n \setminus G) = \infty$ if (2.8) is true for all $v \in W_p^1(G) \cap C(\bar{G})$ and if $n - 1 < p \leq n$.

To generalize the definition of the thinness of a set to Riemannian manifolds one can use again suitable charts. We say that a closed set $C \subset M$ is p-thin at $x \in M$ if there exist a neighborhood U of x and a 2-bilipschitz chart $\varphi : U \to B^n(0,r)$ such that $\varphi(U \cap C)$ is p-thin at $\varphi(x)$. The above mentioned result extends immediately to Riemannian manifolds. A domain $G \subset\subset M$ will be called *regular* if $W(x, M \setminus G) = \infty$ for every $x \in \partial G$. It follows from the existence of a triangulation that every open set can be exhausted by regular ones.

It is well known that the Laplace equation $\Delta u = 0$ in the plane is invariant under analytic functions. This connection between harmonic and analytic functions has a counterpart for equations (1.1). The so called *quasiregular* (qr) maps form a generalization of complex analytic functions in the plane to $\mathbf{R}^n$ and even more generally to Riemannian n-manifolds. See [MaR] for the definition of qr maps on Riemannian n-manifolds and [Ri] for properties of qr maps. Let $L \geq 1$. A quasiregular map $f : M \to N$ is of *L-bounded length distortion*, abbreviated L-BLD, if

$$|h|/L \leq |T_x f h| \leq L|h|$$

for almost every $x \in M$ and for all $h \in T_x M$.

Let then $f : M \to N$ be a qr map and $\mathcal{A}$ an operator in TN satisfying (2.1) - (2.5) with constants p, α and β. The pullback $f^{\#}\mathcal{A}$ of $\mathcal{A}$ is defined by

$$f^{\#}\mathcal{A}_x(h) = J_f(x) T_x f^{-1} \mathcal{A}_{f(x)}(T_x f^{-1*} h) \tag{2.9a}$$

whenever $J_f(x) > 0$. Otherwise, we set

$$f^{\#}\mathcal{A}_x(h) = |h|^{p-2} h. \tag{2.9b}$$

If f is L-BLD, then $f^{\#}\mathcal{A}$ satisfies (2.1) - (2.5) with constants p, $\alpha_1 = L^{2-n-p}\alpha$ and $\beta_1 = L^{n+p-2}\beta$. On the other hand, if $p = n$ and if f is a qr map, then we can choose $\alpha_1 = \alpha / K_O(f)$ and $\beta_1 = K_I(f)\beta$ for the constants of $f^{\#}\mathcal{A}$. Moreover, if u is $\mathcal{A}$-harmonic in N, then $u \circ f$ is $f^{\#}\mathcal{A}$-harmonic in M in the both cases. For these important results we refer to [Re2], [GLM] and [MV]. See [ET, p. 235] for the equivalence of the measurability conditions appearing in the above mentioned references.

We close this section by introducing *$\mathcal{A}$-superharmonic functions*. A lower semicontinuous function $u : G \to \mathbf{R} \cup \{\infty\}$ is $\mathcal{A}$-superharmonic if it satisfies the $\mathcal{A}$-comparison principle, i.e. if for each domain $D \subset\subset G$ and each $\mathcal{A}$-harmonic $h \in C(\bar{D})$, $h \leq u$ on ∂D implies $h \leq u$ in D. A function v is *$\mathcal{A}$-subharmonic* if $-v$ is $\mathcal{A}$-superharmonic. For basic properties of $\mathcal{A}$-superharmonic functions we refer to [HK]. Here we mention only the so called comparison principle: If u and $-v$ are $\mathcal{A}$-superharmonic in a domain $G \subset\subset M$ with

$$\limsup_{y \to x} v(y) \leq \liminf_{y \to x} u(y)$$

for all $x \in \partial G$ and if the left and the right hand side are not simultaneously ∞ or $-\infty$, then $v \leq u$ in G, see [HK, 3.7].

3. The Green function

The Green function has an important role in the theory of linear uniformly elliptic equations in divergence form

$$Lu = -\sum_{i,j=1}^{n} D_j(a^{ij} D_i u) = 0 \tag{3.1}$$

where the coefficients a^{ij} are supposed to be bounded measurable functions in a bounded domain $\Omega \subset \mathbf{R}^n$ such that the matrix (a^{ij}) is symmetric and uniformly positive definite in Ω. The Green function $g = g(\cdot, y)$, $y \in \Omega$, is a weak solution of the equation

$$Lg = \delta_y$$

with vanishing boundary values in the $W^1_{2,0}(\Omega)$- sense. Here and in the sequel δ_y will be the *Dirac measure* at y. The importance of the Green function can be seen in the following representation formula. For any bounded measure μ the solution of

$$Lu = \mu$$

vanishing on $\partial\Omega$ can be represented by

$$u(x) = \int g(x,y)\,d\mu(y), \tag{3.2}$$

see e.g. [LSW, Theorem 6.1] and [GW].

It is clear that the representation formula (3.2) has no counterpart for equations (1.1). However, in this section we are going to define a Green function for (1.1) and prove some properties of it. Singular solutions of equations like (1.1) have been studied by J. Serrin in his fundamental papers [Se1], [Se2] and in the case of the p-Laplacian by S. Kichenassamy and L. Veron ([KV], [K]) in $\mathbf{R}^n$ and by V. M. Kesel'man on Riemannian n-manifolds ([Ke]). In [Ke] the name Green function was used and results similar to those in Theorem 5.2 were stated without proofs.

We start by introducing the *$\mathcal{A}$-capacity* of a condenser. Suppose that $G \subset M$ is an open set and $C \subset G$ is compact. Let $G_i \subset\subset M$ be an increasing sequence of open subsets of G such that $C \subset G_i$ and $\cup_i G_i = G$. Let $\varphi \in C^\infty(G)$ be such that $\varphi = 1$ in a neighborhood of C and $\operatorname{spt}\varphi$, the support of φ, is a compact subset of G_1. Then there exists a unique $\mathcal{A}$-harmonic function u_i in $G_i \setminus C$ with $u_i - \varphi \in W^1_{p,0}(G_i \setminus C)$. We set $u_i = 1$ in C and $u_i = 0$ in $G \setminus G_i$. Then (u_i) is increasing and the limit $u = \lim_{i\to\infty} u_i$ is $\mathcal{A}$-harmonic in $G \setminus C$ by Harnack's principle. The function u is called the *$\mathcal{A}$-potential* of (C, G) and the number

$$\operatorname{cap}_{\mathcal{A}}(C,G) = \int_{G\setminus C} \langle \mathcal{A}_x(\nabla u), \nabla u\rangle \, dm \tag{3.3}$$

is said to be the $\mathcal{A}$-capacity of (C,G). If F is any subset of G, the $\mathcal{A}$-capacity of $E = (F,G)$ is defined by

$$\operatorname{cap}_{\mathcal{A}} E = \inf_{\substack{F\subset U \\ U \text{ open}}} {}_*\operatorname{cap}_{\mathcal{A}}(U,G) \tag{3.4}$$

where, for any set $A \subset G$,

$$_*\operatorname{cap}_{\mathcal{A}}(A,G) = \sup_{\substack{C \subset A \\ C \text{ compact}}} \operatorname{cap}_{\mathcal{A}}(C,G).$$

It follows from the proof of 3.6 that (3.3) and (3.4) give the same $\mathcal{A}$-capacity of (F,G) if F is a compact subset of G. This will be proved after Lemma 3.7. If $\mathcal{A}_x(h) = |h|^{p-2}\, h$, we obtain the usual p-capacity. It it easy to show that

$$\alpha \operatorname{cap}_p E \le \operatorname{cap}_{\mathcal{A}} E \le \frac{\beta^p}{\alpha^{p-1}} \operatorname{cap}_p E, \tag{3.5}$$

see [Mz1, p. 231].

The following two lemmas are trivial if (1.1) is the Euler equation of some variational integral

$$\int F_x(\nabla u)\, dm$$

where $F_x(h) \approx |h|^p$. The proofs for general $\mathcal{A}$ were presented to the author by T. Kilpeläinen in a private communication. The warmest thanks are due to him.

3.6. Lemma. *Let $E_i = (F_i, G)$, $i = 1, 2$, be condensers such that $F_1 \subset F_2$. Then*

$$\operatorname{cap}_{\mathcal{A}} E_1 \le \operatorname{cap}_{\mathcal{A}} E_2.$$

Proof. We may assume that F_1 and F_2 are compact subsets of G. Let u_i be the $\mathcal{A}$-potential of E_i, $i = 1, 2$. Suppose first that $G \setminus F_2$ is regular. Then u_2 is continuous in G, $u_2 = 1$ in F_2 and $\lim_{x\to z} u_2(x) = 0$ whenever $z \in \partial G$. For every $0 < \delta < 1$ the function

$$\varphi = \min\left(\frac{u_2 - \delta u_1}{1-\delta},\ 1\right)$$

is also continuous in G, $\varphi = 1$ in F_2 and $\lim_{x\to z} \varphi(x) = 0$ for every $z \in \partial G$. It follows from [Ma, 2.2] that $u_2 - \varphi \in L^1_{p,0}(G \setminus F_2)$ and thus

$$\begin{aligned}\operatorname{cap}_{\mathcal{A}} E_2 &= \int_{G\setminus F_2} \langle \mathcal{A}_x(\nabla u_2), \nabla\varphi\rangle\, dm \\ &= 1/(1-\delta) \int_D \langle \mathcal{A}_x(\nabla u_2), \nabla u_2 - \delta\nabla u_1\rangle\, dm\end{aligned}$$

where $D = \{x \in G : u_2(x) - \delta u_1(x) \le 1-\delta\}$. On the other hand, $u_1 - \varphi \in L^1_{p,0}(G \setminus F_1)$ and hence

$$\operatorname{cap}_{\mathcal{A}} E_1 = 1/(1-\delta) \int_D \langle \mathcal{A}_x(\nabla u_1), \nabla u_2 - \delta\nabla u_1\rangle\, dm.$$

These together with (2.4) and (2.5) imply

$$\begin{aligned}\delta^{p-1}\operatorname{cap}_{\mathcal{A}} E_1 &= 1/(1-\delta)\int_D \langle \mathcal{A}_x(\delta\nabla u_1), \nabla u_2 - \delta\nabla u_1\rangle\, dm \\ &\le 1/(1-\delta)\int_D \langle \mathcal{A}_x(\nabla u_2), \nabla u_2 - \delta\nabla u_1\rangle\, dm \\ &= \operatorname{cap}_{\mathcal{A}} E_2.\end{aligned}$$

Letting $\delta \to 1$ we obtain $\operatorname{cap}_{\mathcal{A}} E_1 \le \operatorname{cap}_{\mathcal{A}} E_2$. Recall that we supposed $G \setminus F_2$ to be regular.

Let then (C_j, G_j), $j = 1, 2, \ldots$, be a sequence of condensers such that $C_j \subset G_j$ is compact, $G_j \setminus C_j$ is regular, $G_j \setminus C_j \subset\subset G_{j+1} \setminus C_{j+1}$ and finally $\cup_j (G_j \setminus C_j) = G \setminus F_2$. We proved above that $\operatorname{cap}_{\mathcal{A}}(F_1, G_j) \le \operatorname{cap}_{\mathcal{A}}(C_j, G_j)$. Hence

$$\begin{aligned}\operatorname{cap}_{\mathcal{A}}(F_1, G) &= \lim_{j\to\infty} \operatorname{cap}_{\mathcal{A}}(F_1, G_j) \\ &\le \lim_{j\to\infty} \operatorname{cap}_{\mathcal{A}}(C_j, G_j) = \operatorname{cap}_{\mathcal{A}}(F_2, G)\end{aligned}$$

by [Mz1, Lemma 1] and [HK, 2.32]. □

3.7. Lemma. *Let $E_i = (F, G_i)$, $i = 1, 2$, be condensers such that $G_2 \subset G_1$. Then*

$$\operatorname{cap}_{\mathcal{A}} E_1 \le \operatorname{cap}_{\mathcal{A}} E_2.$$

Proof. The claim can be proved as 3.6 by replacing the function φ by

$$\psi = \max\left(0, \frac{u_2 - \delta u_1}{1-\delta}\right). \quad \square$$

We shall next prove that (3.3) and (3.4) give the same $\mathcal{A}$-capacity of (F, G) if $F \subset G$ is compact. To prove this, let $F \subset G$ be compact and write

$$\gamma = \inf_{\substack{F \subset U \\ U \text{ open}}} {}_*\operatorname{cap}_{\mathcal{A}}(U, G).$$

It follows immediately from the definition of γ that $\gamma \ge \operatorname{cap}_{\mathcal{A}}(F, G)$. It remains to show that $\gamma \le \operatorname{cap}_{\mathcal{A}}(F, G)$. Suppose that $U_i \subset\subset G$ is a decreasing sequence of open sets such that $F \subset U_i$ and $\cap_i \bar{U}_i = F$. Let C be a compact subset of U_i. In the proof of 3.6 we showed that $\operatorname{cap}_{\mathcal{A}}(\bar{U}_i, G) \ge \operatorname{cap}_{\mathcal{A}}(C, G)$. Hence

$$\operatorname{cap}_{\mathcal{A}}(\bar{U}_i, G) \ge {}_*\operatorname{cap}_{\mathcal{A}}(U_i, G) \ge \gamma.$$

If u_i is the $\mathcal{A}$-potential of $(\bar{U}_i, G)$, the sequence (u_i) is decreasing and the limit $u = \lim_{i\to\infty} u_i$ is the $\mathcal{A}$-potential of (F, G). As in the end of the proof of 3.6 we obtain

$$\lim_{i\to\infty} \operatorname{cap}_{\mathcal{A}}(\bar{U}_i, G) = \operatorname{cap}_{\mathcal{A}}(F, G).$$

Hence $\operatorname{cap}_{\mathcal{A}}(F, G) \geq \gamma$. We have proved that $\gamma = \operatorname{cap}_{\mathcal{A}}(F, G)$.

The next lemma will be used frequently in the sequel. We shall use the following notation. Let C be a compact set in $G \subset\subset M$ and let u be the $\mathcal{A}$-potential of $E = (C, G)$. Suppose, for simplicity, that u is continuous in G, $u = 1$ in C, and $\lim_{x\to y} u(x) = 0$ for all $y \in \partial G$. Let $E(a, b)$ be the condenser

$$(\{x \in G : u(x) \geq b\}, \{x \in G : u(x) > a\})$$

where $0 \leq a < b \leq 1$. Then the following lemma holds.

3.8. Lemma. *For all* $0 \leq a < b \leq 1$

$$\operatorname{cap}_{\mathcal{A}} E(a, b) = \frac{\operatorname{cap}_{\mathcal{A}} E}{(b-a)^{p-1}}.$$

Proof. Let $D = \{x \in G : a < u(x) < b\}$. The function v, defined by

$$v(x) = \begin{cases} 0 & \text{if } u(x) \leq a \\ \dfrac{u(x) - a}{b - a} & \text{if } x \in D \\ 1 & \text{if } u(x) \geq b, \end{cases}$$

is the $\mathcal{A}$-potential of $E(a, b)$. Using (2.5) we obtain

$$\begin{aligned} \operatorname{cap}_{\mathcal{A}} E &= \int_G \langle \mathcal{A}_x(\nabla u), \nabla v \rangle \, dm = \int_D \langle \mathcal{A}_x(\nabla u), \nabla v \rangle \, dm \\ &= (b-a)^{p-1} \int_D \langle \mathcal{A}_x(\nabla v), \nabla v \rangle \, dm = (b-a)^{p-1} \operatorname{cap}_{\mathcal{A}} E(a, b). \quad \square \end{aligned}$$

We are now ready to give a definition for a Green function. We shall first define it in a regular domain $G \subset\subset M$.

3.9. Definition. Let y be a point in a regular domain $G \subset\subset M$. A function $g = g(\cdot, y)$ is a *Green function* in G for the equation (1.1) if it satisfies the following conditions:

(3.10) $\quad g$ is $\mathcal{A}$-harmonic in $G \setminus \{y\}$,

(3.11) $\quad \lim_{x\to z} g(x) = 0$

for every $z \in \partial G$,

$$\lim_{x \to y} g(x) = \infty, \tag{3.12}$$

and

$$\begin{aligned}\operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x) \geq b\}, \{x \in G : g(x) > a\}\big) \\ = (b-a)^{1-p}\end{aligned} \tag{3.13}$$

for all $b > a \geq 0$.

Before studying the existence of a Green function we shall prove some properties following quite immediately from the definition.

3.14. Lemma. *Let g be a Green function and $b > a \geq 0$. Then*

$$\int_D \langle \mathcal{A}_x(\nabla g), \nabla g \rangle \, dm = b - a$$

where $D = \{x \in G : a < g(x) < b\}$.

Proof. It is clear that the function $(g-a)/(b-a)$ is the $\mathcal{A}$-potential of the condenser

$$E = \big(\{x \in G : g(x) \geq b\}, \{x \in G : g(x) > a\}\big).$$

Hence

$$\int_D \langle \mathcal{A}_x(\nabla g), \nabla g \rangle \, dm = (b-a)^p \operatorname{cap}_{\mathcal{A}} E = b - a. \quad \square$$

A simple consequence of 3.14 is that $g \notin locL_p^1(G)$. However, $g \in locL_q^1(G)$ for every $0 < q < n(p-1)/(n-1)$ by [Li, 1.4].

3.15. Lemma. *Let $g = g(\cdot, y)$ be a Green function. Then*

$$Tg = \delta_y$$

in the sense of distributions, i.e.

$$\int_G \langle \mathcal{A}_x(\nabla g), \nabla \varphi \rangle \, dm = \varphi(y)$$

for every $\varphi \in C_0^\infty(G)$.

Proof. It follows from [Se2, Theorem 3] that

$$Tg = \lambda \delta_y$$

for some $\lambda \in \mathbf{R}$. Choose $\varphi \in C_0^\infty(G)$ such that $\varphi = 1$ in the set $C = \{x \in G : g(x) \geq 1\}$. Then $g - \varphi \in L^1_{p,0}(G \setminus C)$ and hence

$$\lambda = \int_G \langle \mathcal{A}_x(\nabla g), \nabla \varphi \rangle \, dm = \int_{G \setminus C} \langle \mathcal{A}_x(\nabla g), \nabla g \rangle \, dm = 1$$

by 3.14. □

As a first step in studying the existence of a Green function we shall prove the following result.

3.16. Lemma. *Suppose that g' satisfies the conditions* (3.10) - (3.12). *Then $\lambda g'$ is a Green function for some $\lambda \in \mathbf{R}$.*

Proof. Fix $c > 0$ and let

$$d = \operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g'(x) \geq c\}, G\big)^{1/(1-p)}.$$

Then $g = dc^{-1}g'$ satisfies (3.10) - (3.12) and, moreover,

$$\text{(3.17)} \qquad \operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x) \geq d\}, G\big) = \operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g'(x) \geq c\}, G\big) = d^{1-p}.$$

Let then $b > a \geq 0$. Suppose first that $d \geq b$. Since g/d is the $\mathcal{A}$-potential of the condenser $\big(\{x \in G : g(x) \geq d\}, G\big)$, it follows from 3.8 and (3.17) that

$$\begin{aligned}
&\operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x) \geq b\}, \{x \in G : g(x) > a\}\big)\\
&\quad = \operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x)/d \geq b/d\}, \{x \in G : g(x)/d > a/d\}\big)\\
&\quad = \frac{\operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x) \geq d\}, G\big)}{(b/d - a/d)^{p-1}} = (b-a)^{1-p}.
\end{aligned}$$

Let then $d < b$. Using again 3.8 and (3.17) we obtain

$$\frac{\operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x) \geq b\}, G\big)}{(d/b)^{p-1}} = \operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x)/b \geq d/b\}, G\big) = d^{1-p}.$$

Hence

$$\operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x) \geq b\}, G\big) = b^{1-p}$$

and

$$\begin{aligned}
&\operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x) \geq b\}, \{x \in G : g(x) > a\}\big)\\
&\quad = \operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x)/b \geq 1\}, \{x \in G : g(x)/b > a/b\}\big)\\
&\quad = \frac{\operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g(x) \geq b\}, G\big)}{(1 - a/b)^{p-1}} = (b-a)^{1-p}.
\end{aligned}$$

We have proved that $dc^{-1}g'$ satisfies also the condition (3.13) and the claim follows. □

In the light of 3.16 it is easy to make the following observation. Suppose that $g \in C(G \setminus \{y\}) \cap loc W_p^1(G \setminus \{y\})$ is positive in $G \setminus \{y\}$ satisfying (3.11) and

$$Tg = \delta_y.$$

Then g is a Green function.

Unfortunately, we are not able to prove the uniqueness of a Green function for all operators $\mathcal{A}$. However, the following lemma is useful at least in the case $p = n$.

3.18. Lemma. *Suppose that there exist a constant c and a neighborhood U of y such that*

$$|g_1(x) - g_2(x)| \le c$$

for all $x \in U \setminus \{y\}$ whenever g_1 and g_2 are Green functions. Then $g_1 = g_2$, i.e. the Green function is unique.

Proof. Suppose that there exists a point $x_0 \in G \setminus \{y\}$ such that $g_2(x_0) > g_1(x_0)$. Let $\lambda < 1$ be such that $\lambda g_2(x_0) > g_1(x_0)$. Then y belongs to the boundary of the x_0-component of the open set $\{x \in G \setminus \{y\} : \lambda g_2(x) > g_1(x)\}$. Hence

$$\liminf_{x \to y} \frac{g_1(x)}{g_2(x)} \le \lambda < 1$$

which is a contradiction since

$$\lim_{x \to y} \frac{g_1(x)}{g_2(x)} = 1$$

by assumption. □

Next we shall show the existence of a Green function.

3.19. Theorem. *Let $G \subset\subset M$ be a regular domain and let $y \in G$. Then there exists a Green function $g = g(\cdot, y)$.*

Proof. Choose a neighborhood U of y and a 2-bilipschitz chart $\varphi : U \to B^n(0, R)$ with $\varphi(y) = 0$. Let (r_i) be a decreasing sequence such that $r_i \le R$ and $\lim_{i \to \infty} r_i = 0$. Write $D(r) = \varphi^{-1} B^n(0, r)$ when $0 < r \le R$. Let u_i be the $\mathcal{A}$-potential of $E_i = (\bar{D}(r_i), G)$. For $0 < r \le R$ write

$$m_i(r) = \min\{u_i(x) : x \in \partial D(r)\}$$

and

$$M_i(r) = \max\{u_i(x) : x \in \partial D(r)\}.$$

Then $m_i(r) = M_i(r) = 1$ if $r \le r_i$. Suppose that $r > r_i$. By the comparison principle, $u_i|\bar{D}(r) \ge m_i(r)$ and $u_i \le M_i(r)$ in $G \setminus \bar{D}(r)$. Since $M_i(r) - u_i$ is a nonnegative $\mathcal{A}$-harmonic function in $G\setminus \bar{D}(r)$, it follows from Harnack's inequality that $u_i(x) < M_i(r)$ for all $x \in G \setminus \bar{D}(r)$. Hence

$$\{x \in G : u_i(x) \ge M_i(r)\} \subset \bar{D}(r) \subset \{x \in G : u_i(x) \ge m_i(r)\}$$

and thus

$$\operatorname{cap}_{\mathcal{A}} E_i(0, m_i(r)) \ge \operatorname{cap}_{\mathcal{A}}(\bar{D}(r), G) \ge \operatorname{cap}_{\mathcal{A}} E_i(0, M_i(r)) \tag{3.20}$$

by 3.6. Here $E_i(0,b)$, $0 < b \le 1$, is the condenser $(\{x \in G : u_i(x) \ge b\}, G)$. Suppose now on that $r \le R/2$ and i is sufficiently large so that $r_i \le r/2$. By Harnack's inequality,

$$M_i(r) \le \lambda m_i(r) \tag{3.21}$$

where the constant λ is independent on r and i. Lemma 3.8, (3.20) and (3.21) then imply

$$\begin{aligned} M_i(r) \le \lambda m_i(r) &= \lambda \left(\frac{\operatorname{cap}_{\mathcal{A}} E_i}{\operatorname{cap}_{\mathcal{A}} E_i(0, m_i(r))} \right)^{1/(p-1)} \\ &\le \lambda \left(\frac{\operatorname{cap}_{\mathcal{A}} E_i}{\operatorname{cap}_{\mathcal{A}}(\bar{D}(r), G)} \right)^{1/(p-1)}, \end{aligned}$$

and similarly

$$m_i(r) \ge \lambda^{-1} \left(\frac{\operatorname{cap}_{\mathcal{A}} E_i}{\operatorname{cap}_{\mathcal{A}}(\bar{D}(r), G)} \right)^{1/(p-1)}.$$

Let

$$g_i = \operatorname{cap}_{\mathcal{A}} E_i^{1/(1-p)} u_i.$$

Then

$$\lambda^{-1} \operatorname{cap}_{\mathcal{A}}(\bar{D}(r), G)^{1/(1-p)} \le g_i(x) \le \lambda \operatorname{cap}_{\mathcal{A}}(\bar{D}(r), G)^{1/(1-p)}$$

for all $x \in \partial D(r)$, when r and i are as above. Hence (g_i) is locally uniformly bounded in $G\setminus\{y\}$ and it follows from the Hölder continuity estimate [Tr, Theorem 2.2] that (g_i) is equicontinuous in $G \setminus \{y\}$. Ascoli's theorem and a fairly standard diagonal process then give a subsequence, denoted again by (g_i), which converges uniformly on every compact subset of $G \setminus \{y\}$. The limit $g' = \lim_{i\to\infty} g_i$ is $\mathcal{A}$-harmonic in $G\setminus\{y\}$ and $\lim_{x\to y} g'(x) = \infty$. Moreover, $\lim_{x\to z} g'(x) = 0$ for every $z \in \partial G$ by a boundary estimate due to V.G. Maz'ya [Mz1, p. 236]. The existence of a Green function follows now from 3.16. In fact, g' satisfies also (3.13), see the proof of 3.25. □

3.22. Theorem. *If $p = n$, then the Green function is unique.*

Proof. Let φ and $D(r)$, $0 < r \le R$, be as in the proof of 3.19. We shall show that there exists a constant c such that $|g_1 - g_2| \le c$ in $D(R/2) \setminus \{y\}$, whenever g_1 and g_2 are Green functions. For $0 < r < R$ and $i = 1, 2$, let $M_i(r)$ and $m_i(r)$ be the maximum and, respectively, the minimum of g_i on $\partial D(r)$. As in the proof of 3.19 we get

$$m_i(r) \le \operatorname{cap}_{\mathcal{A}}\big(\bar{D}(r), G\big)^{1/(1-n)} \le M_i(r). \tag{3.23}$$

Moreover, either $M_i(r) = m_i(r)$ or

$$\begin{aligned}\operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g_i(x) \ge M_i(r)\}, \{x \in G : g_i(x) > m_i(r)\}\big)^{1/(1-n)}\\ = M_i(r) - m_i(r)\end{aligned}$$

by (3.13). Suppose that $M_i(r) > m_i(r)$ and let $\varepsilon > 0$ be so small that $M_i(r) > m_i(r) + 2\varepsilon$. The set $\{x \in G : g_i(x) \ge M_i(r) - \varepsilon\}$ contains a continuum $C_1(r)$ which joins $\partial D(r)$ and y. Similarly, there exists a continuum $C_0(r) \subset \{x \in G : g_i(x) \le m_i(r) + \varepsilon\}$ joining $\partial D(r)$ and $\partial D(R)$. Let $K_j(r) = \varphi C_j(r)$. Then

$$\begin{aligned}\operatorname{cap}_{\mathcal{A}}\big(\{x \in G : g_i(x) \ge M_i(r) - \varepsilon\}, \{x \in G : g_i(x) > m_i(r) - \varepsilon\}\big)\\ \ge 4^{1-n}\alpha \mathrm{M}_n(\Gamma_r)\end{aligned}$$

where $\mathrm{M}_n(\Gamma_r)$ is the n-modulus of the family of all closed paths which join $K_1(r)$ and $K_2(r)$ in $B^n(0,R)$. By [Vä, Theorem 10.12],

$$\mathrm{M}_n(\Gamma_r) \ge c_n \log \frac{3r/2}{r/2} = c_n \log 3$$

for all $r \le R/2$ where c_n is a positive constant depending only on n. Hence

$$M_i(r) - m_i(r) - 2\varepsilon \le 4(\alpha c_n \log 3)^{1/(1-n)}$$

and letting $\varepsilon \to 0$ we obtain

$$M_i(r) - m_i(r) \le c_1 \tag{3.24}$$

for all $r \le R/2$. Finally, (3.23) and (3.24) imply

$$\operatorname{cap}\big(\bar{D}(r), G\big)^{1/(1-n)} - c_1 \le g_i(x) \le \operatorname{cap}\big(\bar{D}(r), G\big)^{1/(1-n)} + c_1$$

for all $x \in \partial D(r)$, $r \le R/2$. Hence $|g_1 - g_2| \le 2c_1$ in $D(R/2) \setminus \{y\}$. The claim follows now from 3.18. □

S. Kichenassamy and L. Veron have proved the uniqueness of a Green function for (1.1) if $\mathcal{A}_x(h) = |h|^{p-2}h$, $1 < p \le n$, and $G \subset \mathbf{R}^n$, see [KV, Theorem 2.1]. Their methods are not available in the general case.

We shall define a Green function on M by using an exhaustion. This is one motivation for the next theorem.

3.25. Theorem. *Let G_1 and G_2 be regular domains with $y \in G_1 \subset G_2$. If $g_1(\cdot, y)$ is a Green function in G_1, then there exists a Green function $g_2(\cdot, y)$ in G_2 such that $g_1(\cdot, y) \le g_2(\cdot, y)$.*

Proof. Let $g_1(\cdot, y) = 0$ in $M \setminus G_1$ and write

$$C_i = \{x \in G_1 : g_1(x, y) \ge i\}$$

for $i = 1, 2, \ldots$. Let $g_i \in C(\bar{G}_2)$ be a sequence of functions such that g_i is $\mathcal{A}$-harmonic in $G_2 \setminus C_i$, $g_i = 0$ on ∂G_2 and

$$g_i | C_i = \operatorname{cap}_{\mathcal{A}}(C_i, G_2)^{1/(1-p)}.$$

Then $g_i \ge g_1(\cdot, y)$ on $\partial(G_1 \setminus C_i)$ since

$$\begin{aligned} g_i | \partial C_i &= \operatorname{cap}_{\mathcal{A}}(C_i, G_2)^{1/(1-p)} \ge \operatorname{cap}_{\mathcal{A}}(C_i, G_1)^{1/(1-p)} \\ &= i = g_1(\cdot, y) | \partial C_i. \end{aligned}$$

By the comparison principle, $g_i \ge g_1(\cdot, y)$ in $\bar{G}_1 \setminus C_i$ and hence also in $\bar{G}_2 \setminus C_i$. As in the proof of 3.19 we deduce that there exists a subsequence, denoted again by (g_i), converging locally uniformly in $\bar{G}_2 \setminus \{y\}$ to a function g which satisfies (3.10) - (3.12). We shall next prove that the condition (3.13) holds. Write

$$a_i = \operatorname{cap}_{\mathcal{A}}(C_i, G_2)^{1/(1-p)}$$

and let $c > 0$. For every $i \ge c$

$$\begin{aligned} \operatorname{cap}_{\mathcal{A}}\big(\{x \in G_2 : g_i(x) \ge c\}, G_2\big) &= \operatorname{cap}_{\mathcal{A}}\big(\{x \in G_2 : g_i(x)/a_i \ge c/a_i\}, G_2\big) \\ &= \frac{\operatorname{cap}_{\mathcal{A}}(C_i, G_2)}{(c/a_i)^{p-1}} = c^{1-p} \end{aligned}$$

since g_i/a_i is the $\mathcal{A}$-potential of (C_i, G_2) and $a_i \ge i \ge c$. Let $\varepsilon \in]0, c[$ and write $C = \{x \in G_2 : g(x) \ge c\}$. Since $g_i \to g$ uniformly in $g^{-1}(c)$, there exists $i_\varepsilon \ge c + \varepsilon$ such that

$$c - \varepsilon < g_{i_\varepsilon}(x) < c + \varepsilon$$

for all $x \in g^{-1}(c)$. Hence

$$\begin{aligned} (c-\varepsilon)^{1-p} &= \operatorname{cap}_{\mathcal{A}}\big(\{x \in G_2 : g_{i_\varepsilon}(x) \ge c - \varepsilon\}, G_2\big) \ge \operatorname{cap}_{\mathcal{A}}(C, G_2) \\ &= \operatorname{cap}_{\mathcal{A}}\big(\{x \in G_2 : g_{i_\varepsilon}(x) \ge c + \varepsilon\}, G_2\big) = (c+\varepsilon)^{1-p}. \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain

$$\operatorname{cap}_{\mathcal{A}}(C, G_2) = c^{1-p}$$

and thus g satisfies also (3.13). The theorem is thereby proved since g is a Green function in G_2 and $g \geq g_1(\cdot, y)$. □

We shall next define Green function on M. Let (G_i), $i = 1, 2, \ldots$, be an exhaustion of M by regular domains G_i such that $y \in G_1$, $\bar{G}_i \subset G_{i+1}$ and $M = \cup_i G_i$. Then there exists a Green function $g_i = g(\cdot, y)$ in G_i such that (g_i) is increasing. By Harnack's principle, $\lim_{i\to\infty} g_i$ is identically $+\infty$ or $\mathcal{A}$-harmonic in $M \setminus \{y\}$. In the latter case the limit $g = \lim_{i\to\infty} g_i$ is said to be a Green function on M. If $p = n$, the Green function on M is unique by 3.22.

Suppose that $\operatorname{cap}_p(\bar{D}(r), M) > 0$ and thus also $\operatorname{cap}_{\mathcal{A}}(\bar{D}(r), M) > 0$, see 3.19 for the notation $D(r)$. Let (g_i) be as above. By (3.23),

$$\min_{\partial D(r)} g_i \leq \operatorname{cap}_{\mathcal{A}}\big(\bar{D}(r), G_i\big)^{1/(1-p)} \leq \operatorname{cap}_{\mathcal{A}}\big(\bar{D}(r), M\big)^{1/(1-p)}$$

and a Green function exists on M.

Let $c > 0$ be so large that the set $C = \{x \in M : g(x) \geq c\}$ is compact. We shall next show that

$$\operatorname{cap}_{\mathcal{A}}(C, M) = c^{1-p}. \tag{3.26}$$

Choose a function $\varphi \in C_0^\infty(M)$ such that $\varphi = c$ in a neighborhood U of C. Then

$$\begin{aligned}\int_{M\setminus C} \langle \mathcal{A}_x(\nabla g), \nabla g\rangle\, dm &= \int_{\operatorname{spt}\varphi\setminus U} \langle \mathcal{A}_x(\nabla g), \nabla \varphi\rangle\, dm \\ &= \lim_{i\to\infty} \int_{\operatorname{spt}\varphi\setminus U} \langle \mathcal{A}_x(\nabla g_i), \nabla \varphi\rangle\, dm = c\end{aligned}$$

by [HK, 2.32]. Since g/c is the $\mathcal{A}$-potential of (C, M), (3.26) follows. Similarly, one can show that the condition (3.13) and 3.15 hold for g, too.

We say that the *ideal boundary* of M is of *positive p-capacity* if there exists a compact set $C \subset M$ such that $\operatorname{cap}_p(C, M) > 0$. In this case we write $\operatorname{cap}_p \partial M > 0$.

We have already proved the following existence theorem.

3.27. Theorem. *There exists a Green function on M if and only if $\operatorname{cap}_p \partial M > 0$.*

We close this section by stating without proof the following invariance property of a Green function under BLD-homeomorphisms and quasiconformal mappings.

3.28. Theorem. *Let M and N be Riemannian n-manifolds and $\mathcal{A}$ an operator in TN satisfying (2.1) - (2.5) with a constant p. Suppose that $f : M \to N$ is a BLD-homeomorphism and that g is a Green function on N for $\operatorname{div} \mathcal{A}_x(\nabla u) = 0$. Then $g \circ f$ is a Green function on M for $\operatorname{div} f^{\#}\mathcal{A}_x(\nabla u) = 0$. In the case $p = n$ it suffices to assume that f is quasiconformal.*

The proof is based on the following observation: Let $E = (C, G)$ be a condenser on M such that $C \subset G$ is compact. Then $fC \subset fG$ is compact. Moreover, if u is the $\mathcal{A}$-potential of $fE = (fC, fG)$, then $u \circ f$ is the $\mathcal{A}$-potential of E and $\operatorname{cap}_{f^{\#}\mathcal{A}} E = \operatorname{cap}_{\mathcal{A}} E$.

4. Singular solutions

In this section we study solutions of (1.1) with several singularities. For example, we shall construct an $\mathcal{A}$-harmonic function in $\mathbf{R}^n \setminus \{a_1, b_1, a_2, b_2, \dots\}$ which has positive singularities at every a_i and negative singularities at every b_i. In this example $\{a_1, a_2, \dots\}$ and $\{b_1, b_2, \dots\}$ are assumed to be compact and $\mathcal{A} \in \mathcal{A}_n(\mathbf{R}^n)$. Recall the notation $\mathcal{A}_n(\mathbf{R}^n)$ from section 2.

We shall start with the following lemma. Here and in the sequel $c_1, c_2, \dots$ are positive constants depending only on n, p, α and β.

4.1. Lemma. *Let $C = \{a_1, a_2, \dots\}$ be a compact subset of a regular domain $D \subset\subset M$. Then there exists an $\mathcal{A}$-harmonic function u in $D \setminus C$ such that*

$$\lim_{x \to y} u(x) = \infty \tag{4.2}$$

for every $y \in C$, and

$$\lim_{x \to z} u(x) = 0 \tag{4.3}$$

for every $z \in \partial D$.

Proof. Let (γ_j) be a sequence of positive numbers such that $\sum_{j=1}^{\infty} \gamma_j = 1$. Let

$$g_j = \gamma_j^{1/(p-1)} g(\cdot, a_j)$$

where $g(\cdot, a_j)$ is a Green function in D with the pole at a_j. For $i, j = 1, 2, \dots$, write

$$C_{j,i} = \bigcup_{k=1}^{j} \{x \in D : g_k(x) \geq i\}.$$

Then $C_{j,i} \subset D$ is compact and

$$\begin{aligned} \operatorname{cap}_{\mathcal{A}}(C_{j,i}, D) &\leq c_1 \operatorname{cap}_p(C_{j,i}, D) \leq c_1 \sum_{k=1}^{j} \operatorname{cap}_p\big(\{x \in D : g_k(x) \geq i\}, D\big) \\ &\leq c_2 \sum_{k=1}^{j} \operatorname{cap}_{\mathcal{A}}\big(\{x \in D : g_k(x) \geq i\}, D\big) = c_2 i^{1-p} \sum_{k=1}^{j} \gamma_k \\ &\leq c_2 i^{1-p} \end{aligned}$$

by (3.5) and (3.13). Hence

$$i \le c_3 \operatorname{cap}_{\mathcal{A}}(C_{j,i}, D)^{1/(1-p)}. \tag{4.4}$$

Let $u_{j,i} \in C(\bar{D})$ be $\mathcal{A}$-harmonic in $D \setminus C_{j,i}$ such that $u_{j,i} = 0$ on ∂D and $u_{j,i} = i$ in $C_{j,i}$. By the comparison principle,

$$u_{j,i} \ge \max_{1 \le k \le j} g_k(x) \tag{4.5}$$

for every $x \in D \setminus C_{j,i}$. In particular, $u_{j,i+1} \ge u_{j,i}$ on $\partial(D \setminus C_{j,i})$ and thus the sequence $u_{j,i}$, $i = 1, 2, \dots$, is increasing for every j.

To estimate $u_{j,i}$ from above, let $B_1 = B(y, r)$, $r > 0$, be a ball such that $B(y, 2r) \subset D \setminus C_{j,i}$ for all j if i is sufficiently large. Fix such i and j. Let v be the $\mathcal{A}$-potential of $(C_{j,i}, D)$. We write m for the minimum of v in $\bar{B}_1$. By 3.6 and 3.8,

$$m = \left(\frac{\operatorname{cap}_{\mathcal{A}}(C_{j,i}, D)}{\operatorname{cap}_{\mathcal{A}}(\{x \in D : v(x) \ge m\}, D)} \right)^{1/(p-1)} \le \left(\frac{\operatorname{cap}_{\mathcal{A}}(C_{j,i}, D)}{\operatorname{cap}_{\mathcal{A}}(\bar{B}_1, D)} \right)^{1/(p-1)}$$

since $v \ge m$ in $\bar{B}_1$. This together with Harnack's inequality implies

$$v(x) \le c_4 \left(\frac{\operatorname{cap}_{\mathcal{A}}(C_{j,i}, D)}{\operatorname{cap}_{\mathcal{A}}(\bar{B}_1, D)} \right)^{1/(p-1)} \tag{4.6}$$

for every $x \in \bar{B}_1$. On the other hand, $u_{j,i} = iv$ and hence by (4.4) and (4.6),

$$u_{j,i} \le c_5 \operatorname{cap}_{\mathcal{A}}(\bar{B}_1, D)^{1/(1-p)}$$

in $\bar{B}_1$. It follows now from Harnack's principle that

$$u_j = \lim_{i \to \infty} u_{j,i} \tag{4.7}$$

is $\mathcal{A}$-harmonic in $D \setminus \{a_1, a_2, \dots, a_j\}$. Moreover, the sequence (u_j) is increasing since $u_{j+1,i} \ge u_{j,i}$. Hence

$$u = \lim_{j \to \infty} u_j \tag{4.8}$$

is $\mathcal{A}$-harmonic in $D \setminus C$. Now u is a desired function since (4.2) is clear by the construction and (4.3) follows from the boundary estimate [Mz1, p. 236]. □

4.9. Lemma. *Let $C = \{a_1, a_2, \dots, a_j\}$ be a subset of a regular domain D and let $\gamma_i > 0$, $i = 1, 2, \dots, j$. Then there exist an $\mathcal{A}$-harmonic function w in $D \setminus C$ and constants c and $r > 0$ such that*

$$\lim_{x \to z} w(x) = 0 \tag{4.10}$$

for every $z \in \partial D$,

$$\gamma_i^{1/(p-1)} g(x, a_i) \le w(x) \le \gamma_i^{1/(p-1)} g(x, a_i) + c \tag{4.11}$$

for every $x \in B(a_i, r)$, $i = 1, 2, \dots, j$, where $g(\cdot, a_i)$ is a Green function in D with the pole at a_i, and

$$Tw = \sum_{i=1}^{j} \gamma_i \delta_{a_i}. \tag{4.12}$$

Proof. We may assume that $\sum_{i=1}^{j} \gamma_i = 1$. We claim that u_j in (4.7) is a desired function. The condition (4.10) is clear and the left hand side of (4.11) follows from (4.5). Let $r > 0$ be so small that the balls $B(a_i, 2r) \subset D$, $i = 1, 2, \dots, j$ are disjoint and let

$$c = \max\{u_j(x) : x \in \partial B(a_i, r),\ i = 1, 2, \dots, j\}.$$

It follows from the construction of u_j that the right hand side of (4.11) holds if c and r are chosen as above. To prove (4.12), let $\varphi \in C_0^\infty(D)$. We write $\varphi = \psi + \eta$ where ψ, $\eta \in C_0^\infty(D)$ such that $\operatorname{spt} \eta \subset \{x \in D : u_j(x) > c\}$ and $\psi = 0$ in the set $\{x \in D : u_j(x) > 2c\}$. The set $\{x \in D : u_j(x) > c\}$ is a union of disjoint neighborhoods U_i of the points a_i, $i = 1, 2, \dots, j$. Then

$$\begin{aligned}
\int_D \langle \mathcal{A}_x(\nabla u_j), \nabla \varphi \rangle \, dm &= \int_D \langle \mathcal{A}_x(\nabla u_j), \nabla \psi \rangle \, dm + \int_D \langle \mathcal{A}_x(\nabla u_j), \nabla \eta \rangle \, dm \\
&= \sum_{i=1}^{j} \int_{U_i} \langle \mathcal{A}_x(\nabla u_j), \nabla \eta \rangle \, dm = \sum_{i=1}^{j} \gamma_i \eta(a_i) \\
&= \sum_{i=1}^{j} \gamma_i \varphi(a_i).
\end{aligned}$$

The equality

$$\int_{U_i} \langle \mathcal{A}_x(\nabla u_j), \nabla \eta \rangle \, dm = \gamma_i \eta(a_i)$$

follows from the fact that $\gamma_i^{1/(1-p)}(u_j - c)$ is a Green function in U_i. The lemma is thereby proved. □

From now on we assume that $\mathcal{A}$ is an operator in $\mathbf{R}^n \times \mathbf{R}^n$ which satisfies (2.1) - (2.5) with the constant $p = n$.

4.13. Theorem. *Let $C^+ = \{a_1, a_2, \dots\}$ and $C^- = \{b_1, b_2, \dots\}$ be disjoint compact subsets of $\mathbf{R}^n$. Suppose that $\mathcal{A} \in \mathcal{A}_n(\mathbf{R}^n)$. Then there exists an $\mathcal{A}$-harmonic function e in $\mathbf{R}^n \setminus (C^+ \cup C^-)$ such that*

$$\lim_{x \to y} e(x) = \infty \tag{4.14}$$

for every $y \in C^+$, and

$$\lim_{x \to z} e(x) = -\infty \tag{4.15}$$

for every $y \in C^- \cup \{\infty\}$.

Proof. Let $D \subset\subset \mathbf{R}^n$ be a regular domain such that $G = \mathbf{R}^n \setminus \bar{D}$ is also a domain and that $\partial D = \partial G$. Suppose, moreover, that $C^+ \subset D$ and $C^- \subset G$. Let u be $\mathcal{A}$-harmonic in $D \setminus C^+$ such that (4.2) and (4.3) hold. We may assume that

$$\operatorname{cap}_{\mathcal{A}}\big(\{x \in D : u(x) \ge b\}, \{x \in D : u(x) > a\}\big) = (b-a)^{1-n}$$

for all $b > a \ge 0$, see the proof of 3.16. As in 4.1 we can prove that there is an $\mathcal{A}$-harmonic function v in G such that $\lim_{x \to y} v(x) = -\infty$ for every $y \in C^- \cup \{\infty\}$ and $\lim_{x \to z} v(x) = 0$ for every $z \in \partial G$. Again we can assume that

$$\operatorname{cap}_{\mathcal{A}}\big(\{x \in G : v(x) \ge b\}, \{x \in G : v(x) > a\}\big) = (b-a)^{1-n}$$

for all $a < b \le 0$. For each $i = 1, 2, \dots$, write

$$C_i = \{x \in D : u(x) \ge i\},$$
$$G_i = \{x \in G : v(x) > \lambda_i\} \cup \bar{D}$$

and

$$E_i = (C_i, G_i)$$

where $\lambda_i \in \mathbf{R}$ will be specified later. Let $e_i \in C(\mathbf{R}^n)$ be $\mathcal{A}$-harmonic in $G_i \setminus C_i$ such that

$$e_i | C_i = \operatorname{cap}_{\mathcal{A}} E_i^{1/(1-n)}/2$$

and

$$e_i|\mathbf{R}^n \setminus G_i = -\operatorname{cap}_{\mathcal{A}} E_i^{1/(1-n)}/2.$$

Choose λ_i such that $m_i = \min\{e_i(x) : x \in \partial D\} = 0$. Let M_i be the maximum of e_i on ∂D. Then

$$\begin{aligned}&\operatorname{cap}_{\mathcal{A}}\big(\{x \in G_i : e_i(x) \geq M_i\}, \{x \in G_i : e_i(x) > 0\}\big)\\ &\quad\geq \inf_{F_1, F_2} \alpha \mathrm{M}_n\big(\Delta(F_1, F_2; G_1)\big) > 0\end{aligned}$$

where F_1 and F_2 are two continua which join ∂D and C^+, and ∂D and ∂G_1, respectively, and $\mathrm{M}_n\big(\Delta(F_1, F_2; G_1)\big)$ is the n-modulus of the family of all paths which join F_1 and F_2 in G_1. The function

$$\operatorname{cap}_{\mathcal{A}} E_i^{1/(n-1)}\big(e_i + \operatorname{cap}_{\mathcal{A}} E_i^{1/(1-n)}/2\big)$$

is the $\mathcal{A}$-potential of E_i. Applying 3.8 to this function yields

$$\begin{aligned}\operatorname{cap}_{\mathcal{A}} E_i^{1/(1-n)}/2 - M_i &= \operatorname{cap}_{\mathcal{A}}\big(C_i, \{x \in G_i : e_i(x) > M_i\}\big)^{1/(1-n)}\\ &\leq \operatorname{cap}_{\mathcal{A}}(C_i, D)^{1/(1-n)} = i.\end{aligned}$$

On the other hand,

$$M_i = \operatorname{cap}_{\mathcal{A}}\big(\{x \in G_i : e_i(x) \geq M_i\}, \{x \in G_i : e_i(x) > 0\}\big)^{1/(1-n)} \leq \kappa$$

where κ do not depend on i. Hence

$$\operatorname{cap}_{\mathcal{A}} E_i^{1/(1-n)}/2 \leq i + \kappa,$$

and similarly

$$\operatorname{cap}_{\mathcal{A}} E_i^{1/(1-n)}/2 \geq i.$$

By the comparison principle,

$$u \leq e_i \leq u + \kappa$$

in $D \setminus C_i$. To estimate e_i in $G_i \setminus D$, we shall first compare λ_i with $\operatorname{cap}_{\mathcal{A}} E_i^{1/(1-n)}$. As above we obtain

$$\begin{aligned}\operatorname{cap}_{\mathcal{A}} E_i^{1/(1-n)}/2 &= \operatorname{cap}_{\mathcal{A}}\big(\{x \in G_i : e_i(x) \geq 0\}, G_i\big)^{1/(1-n)}\\ &\leq \operatorname{cap}_{\mathcal{A}}(\bar{D}, G_i)^{1/(1-n)} = -\lambda_i,\end{aligned}$$

and similarly

$$M_i + \operatorname{cap}_{\mathcal{A}} E_i^{1/(1-n)}/2 = \operatorname{cap}_{\mathcal{A}}\big(\{x \in G_i : e_i(x) \geq M_i\}, G_i\big)^{1/(1-n)}$$
$$\geq \operatorname{cap}_{\mathcal{A}}(\bar{D}, G_i)^{1/(1-n)} = -\lambda_i.$$

Hence

$$\lambda_i \leq -\operatorname{cap}_{\mathcal{A}} E_i^{1/(1-n)}/2 \leq \lambda_i + \kappa.$$

Again by the comparison principle,

$$v \leq e_i \leq v + \kappa$$

in $G_i \setminus D$. As in the end of the proof of 3.19 we find a subsequence which converges locally uniformly to a function e which satisfies the conditions of the theorem. □

The next theorem will partly generalize [K, Theorem 1].

4.16. Theorem. *Let $x_i \in \mathbf{R}^n$ and let $\gamma_i \in \mathbf{R}$, $i = 1, 2, \ldots, m$, be such that $\sum_{i=1}^m \gamma_i = 0$. Suppose that $\mathcal{A} \in \mathcal{A}_n(\mathbf{R}^n)$. Then there exists an $\mathcal{A}$-harmonic function u in $\mathbf{R}^n \setminus \{x_1, \ldots, x_m\}$ such that $\lim_{|x| \to \infty} u(x) = 0$ and*

$$Tu = \sum_{i=1}^{m} \gamma_i \delta_{x_i}.$$

Proof. If $\gamma_i = 0$ for all i, then $u = 0$ is a desired function. Suppose that $\gamma_i \neq 0$ for some i. Let $x_0 \in \mathbf{R}^n \setminus \{x_1, \ldots, x_m\}$ and let f be a Möbius transformation such that $f(x_0) = \infty$ and $f(\infty) = x_{i_0}$ for some i_0 with $\gamma_{i_0} < 0$. Let $C^+ = \{f^{-1}(x_i) : \gamma_i > 0\}$ and $C^- = \{f^{-1}(x_i) : \gamma_i < 0,\ i \neq i_0\}$ and let D and $G = \mathbf{R}^n \setminus \bar{D}$ be regular domains such that $C^+ \subset D$, $C^- \subset G$ and $\partial D = \partial G$. By 4.9, there exist $f^{\#}\mathcal{A}$-harmonic functions w_1 and w_2 in $D \setminus C^+$ and in $G \setminus C^-$, respectively, which satisfy (4.10) - (4.12) with $a_i = f^{-1}(x_i)$. If $i = i_0$, (4.11) is supposed to be true outside some compact set. By assumption, $\sum_{i=1}^m \gamma_i^+ = \sum_{i=1}^m \gamma_i^-$ where $\gamma_i^+ = \max(\gamma_i, 0)$ and $\gamma_i^- = \min(\gamma_i, 0)$. As in the proof of 4.13 we can construct an $f^{\#}\mathcal{A}$-harmonic function v in $\mathbf{R}^n \setminus (C^+ \cup C^-)$ such that $v(x_0) = 0$ and that $v - w_i$ is bounded in $D \setminus C^+$ if $i = 1$ or in $G \setminus C^-$ if $i = 2$. The function $u = v \circ f^{-1}$ is then a desired function. □

5. Classification

The classification theory of Riemann surfaces is an interesting and important part of the classical function theory. In this theory one classifies surfaces according to the nonexistence of certain harmonic functions on them and one proves inclusions between these classes. The most interesting part of the theory is to study the strictness of these inclusions. For a thorough discussion of the classification theory we refer to [AS], [SN] and [Sa].

In this section we shall study the corresponding classification problem in the nonlinear case. We do not study the strictness of inclusions very deeply, but this will be discussed in details in a forth-coming paper. Recall the notation $\mathcal{A}_p(M)$ from section 2.

5.1. Definition. We say that M belongs to the class O_G^p if there is no Green function on M for any $\mathcal{A} \in \mathcal{A}_p(M)$.

5.2. Theorem. *The following conditions are equivalent:*

(5.3) $M \in O_G^p$,

(5.4) *the ideal boundary of M is of p-capacity zero,*

(5.5) *every positive $\mathcal{A}$-superharmonic function on M is constant for all $\mathcal{A} \in \mathcal{A}_p(M)$.*

Proof. The equivalence of (5.3) and (5.4) is already stated in 3.27. The condition (5.5) implies (5.3) since a Green function on M is positive and $\mathcal{A}$-superharmonic on M. It remains to prove that (5.4) implies (5.5). Suppose that the ideal boundary of M is of p-capacity zero and that u is a positive nonconstant $\mathcal{A}$-superharmonic function on M. Let $x_0 \in M$ be such that $u(x_0) < \infty$ and let $\varepsilon > 0$ be such that $u(x_0) - \varepsilon > 0$. Since u is lower semicontinuous, there exists a ball $B_0 = B(x_0, r) \subset\subset M$ such that $u(x) \geq u(x_0) - \varepsilon$ for all $x \in \bar{B}_0$. Let (G_i) be an exhaustion of M by regular domains $G_i \subset\subset M$. Suppose that $\bar{B}_0 \subset G_1$ and $G_i \subset G_{i+1}$. Then there exists a function $h_i \in C(M)$ such that h_i is $\mathcal{A}$-harmonic in $G_i \setminus \bar{B}_0$, $h_i | M \setminus G_i = 0$ and $h_i | \bar{B}_0 = u(x_0) - \varepsilon$. The function $\lim_{i\to\infty} h_i/(u(x_0) - \varepsilon)$ is the $\mathcal{A}$-potential of $(\bar{B}_0, M)$ and since the ideal boundary of M is of p-capacity zero, $\lim_{i\to\infty} h_i(x) = u(x_0) - \varepsilon$ for all $x \in M$. On the other hand, $u \geq h_i$ on the boundary of $G_i \setminus \bar{B}_0$. By the comparison principle, $u \geq h_i$ in $G_i \setminus \bar{B}_0$. Hence $u \geq u(x_0) - \varepsilon$ on M for all $\varepsilon > 0$ small enough. Letting $\varepsilon \to 0$ we obtain $u \geq u(x_0)$. Since u is nonconstant, there exists a point $x_1 \in M$ with $u(x_1) > u(x_0)$. By the same argument as above we obtain $u \geq u(x_1)$. This is a contradiction and thus u is constant. □

5.6. Definition. A Riemannian manifold M belongs to the class O_{HP}^p (O_{HB}^p) if every positive (resp. bounded) $\mathcal{A}$-harmonic function on M is constant for all $\mathcal{A} \in \mathcal{A}_p(M)$.

5.7. Theorem.

$$O_G^p \subset O_{HP}^p \subset O_{HB}^p.$$

Proof. Suppose that $M \in O_G^p$ and that u is a positive $\mathcal{A}$-harmonic and thus also $\mathcal{A}$-superharmonic function on M. By 5.2, u is constant which proves the first inclusion. Let then $M \in O^p_{HP}$ and let u be a bounded $\mathcal{A}$-harmonic function on M. Then $u+\lambda$ is positive and $\mathcal{A}$-harmonic on M for some constant $\lambda \in \mathbf{R}$. By assumption, $u+\lambda$ and hence also u is constant. □

5.8. Definition. We say that M belongs to the class O^p_{HD} (O^p_{HBD}) if every (bounded) $\mathcal{A}$-harmonic function $u \in L^1_p(M)$ is constant for all $\mathcal{A} \in \mathcal{A}_p(M)$.

The proof of the following theorem can also be found in [Ki, p. 273].

5.9. Theorem.

$$O^p_{HB} \subset O^p_{HD} = O^p_{HBD}.$$

Proof. Suppose that $M \in O^p_{HB}$ and that $u \in L^1_p(M)$ is $\mathcal{A}$-harmonic in M. For each $i = 1, 2, \dots$, we write

$$u_i = \max(-i, \min(i, u)).$$

Let (G_j) be an exhaustion of M by regular domains $G_j \subset G_{j+1} \subset\subset M$. Then there exist functions $v_{i,j} \in C(M) \cap L^1_p(M)$ such that $v_{i,j}$ is $\mathcal{A}$-harmonic in G_j and $v_{i,j} = u_i$ in $M \setminus G_j$. Moreover, $v_{i,j} - u_i \in L^1_{p,0}(M)$ and $-i \le v_{i,j} \le i$. As in the proof of 3.19 we find a subsequence, denoted again by $(v_{i,j})$, which converges locally uniformly to a function v_i. Then v_i is a bounded $\mathcal{A}$-harmonic function on M and thus it is constant. It follows that $u_i \in L^1_{p,0}(M)$ since $u_i - v_i \in L^1_{p,0}(M)$. Thus

$$\int_M \langle \mathcal{A}_x(\nabla u), \nabla u \rangle \, dm = \lim_{i\to\infty} \int_M \langle \mathcal{A}_x(\nabla u), \nabla u_i \rangle \, dm = 0$$

and $\nabla u = 0$ a.e. Hence u is constant and $M \in O^p_{HD}$. To prove the equality $O^p_{HD} = O^p_{HBD}$, let $M \in O^p_{HBD}$ and let $u \in L^1_p(M)$ be $\mathcal{A}$-harmonic on M. We claim that the function v_i, constructed as above, is constant. Since v_i is a bounded $\mathcal{A}$-harmonic function on M, it suffices to prove that $\|\nabla v_i\|_p < \infty$. Let $w_{i,j} \in C(M)$ be p-harmonic in G_j such that $w_{i,j} = v_{i,j}$ in $M \setminus G_j$. Then

$$\int_M \langle \mathcal{A}_x(\nabla v_{i,j}), \nabla v_{i,j} \rangle \, dm = \int_M \langle \mathcal{A}_x(\nabla v_{i,j}), \nabla w_{i,j} \rangle \, dm.$$

This together with Hölder's inequality, (2.2) and (2.3) imply

$$\int_M |\nabla v_{i,j}|^p \, dm \le (\beta/\alpha)^p \int_M |\nabla w_{i,j}|^p \, dm.$$

Since $\|\nabla w_{i,j}\|_p$ is decreasing in j, the sequence $\|\nabla v_{i,j}\|_p$, $j = 1, 2, \dots$, is uniformly bounded and hence $\|\nabla v_i\|_p < \infty$. It follows from the assumption $M \in O^p_{HBD}$ that v_i is constant. As above we conclude that u is constant and $M \in O^p_{HD}$. On the other hand, $O^p_{HD} \subset O^p_{HBD}$ and thus $O^p_{HBD} = O^p_{HD}$. □

In the rest of this section we shall study the strictness of inclusions in 5.7. We start with the following simple consequence of removability theorems in [Se1].

5.10. Lemma. *Suppose that $M \in O^p_{HP} \setminus O^p_G$. Let $M' = M \setminus \{y\}$, $y \in M$, equipped with the induced Riemannian structure. Then $M' \in O^p_{HB} \setminus O^p_{HP}$.*

Proof. Let $\mathcal{A} \in \mathcal{A}_p(M')$ be an operator in TM'. We extend it to TM by setting $\mathcal{A}_y(h) = |h|^{p-2}h$ for all $h \in T_yM$. By assumption, there exists a Green function $g(\cdot, y)$ on M. It is a positive $\mathcal{A}$-harmonic function on M'. Hence $M' \notin O^p_{HP}$. Let then u be a bounded $\mathcal{A}$-harmonic function on M'. By [Se1, Theorem 10], there exists an $\mathcal{A}$-harmonic function $u^\star$ on M with $u^\star|M' = u$. Since $u^\star$ is bounded and $M \in O^p_{HP}$ $(\subset O^p_{HB})$, $u^\star$ is constant. Hence $M \in O^p_{HB}$ and the lemma follows. □

In the case $1 < p < n$ the strictness of the inclusions $O^p_G \subset O^p_{HP} \subset O^p_{HB}$ is trivial.

5.11. Theorem. *If $1 < p < n$, then $\mathbf{R}^n \in O^p_{HP} \setminus O^p_G$ and $\mathbf{R}^n \setminus \{y\} \in O^p_{HB} \setminus O^p_{HP}$.*

Proof. Since $\operatorname{cap}_p(\bar{B}^n(0,r), \mathbf{R}^n) > 0$, (see e.g. [Mz1]), $\mathbf{R}^n \notin O^p_G$. Let then u be a positive nonconstant $\mathcal{A}$-harmonic function in $\mathbf{R}^n$. We may assume that $\inf u = 0$. It follows from Harnack's inequality [Tr] that $\sup u = 0$. This is a contradiction and thus $\mathbf{R}^n \in O^p_{HP}$. The second claim follows then from 5.10.

On the other hand, [Ki, 1.8] shows that there are no domains G in $\mathbf{R}^n$ such that $G \in O^n_{HD} \setminus O^n_G$. Indeed, we have

5.12. Theorem. *If $G \subset \mathbf{R}^n$ is a domain such that $G \in O^n_{HD}$, then $G \in O^n_G$.*

The first thing which comes to mind in studying the strictness of the inclusion $O^p_G \subset O^p_{HP}$ is to use Harnack's inequality just as we did in the proof of 5.11. We shall next give Harnack's inequality in a form which is very useful in the above mentioned problem. The idea of the proof of 5.14 is essentially due to S. Granlund, [Gr].

Suppose that $D \subset\subset M$ is a domain and that $C \subset D$ is compact. For $n-1 < p \le n$ we write

$$\lambda_p(C, D) = \inf_{F_1, F_2} \mathrm{M}_p\big(\Delta(F_1, F_2; D)\big) \tag{5.13}$$

where F_1 and F_2 are continua which join C and $M \setminus D$ and $\mathrm{M}_p\big(\Delta(F_1, F_2; D)\big)$ is the p-modulus of the family of all paths which join F_1 and F_2 in D.

5.14. Theorem. *Let C and D be as above and let $\mathcal{A} \in \mathcal{A}_p(M)$, $n-1 < p \le n$. Then there exists a constant c_0 depending only on p and β/α such that*

$$\log \frac{M_C}{m_C} \le c_0 \left(\frac{\operatorname{cap}_p(\bar{D}, M)}{\lambda_p(C, D)} \right)^{1/p} \tag{5.15}$$

whenever u is a positive $\mathcal{A}$-harmonic function on M. Here $M_C = \max\{u(x) : x \in C\}$ and $m_C = \min\{u(x) : x \in C\}$.

Proof. We may assume that $M_C > m_C$. Let $\varepsilon > 0$ be so small that $M_C - \varepsilon > m_C + \varepsilon$. The sets $\{x \in M : u(x) \geq M_C - \varepsilon\}$ and $\{x \in M : u(x) \leq m_C + \varepsilon\}$ contain continua F_1 and F_2, respectively, which join C and $M \setminus D$. Write

$$w = \frac{\log u - \log(m_C + \varepsilon)}{\log(M_C - \varepsilon) - \log(m_C + \varepsilon)}.$$

Then

$$\int_D |\nabla w|^p \, dm \geq \mathrm{M}_p\big(\Delta(F_1, F_2; D)\big) \geq \lambda_p(C, D).$$

On the other hand,

$$\int_D |\nabla \log u|^p \, dm \leq c(p, \beta/\alpha) \operatorname{cap}_p(\bar{D}, M) \tag{5.16}$$

by [HK, 2.24]. Hence

$$\log \frac{M_C - \varepsilon}{m_C + \varepsilon} \leq c_0 \left(\frac{\operatorname{cap}_p(\bar{D}, M)}{\lambda_p(C, D)} \right)^{1/p}$$

and the theorem follows by letting $\varepsilon \to 0$. □

The inequality (5.16) also proves the inclusion $O_G^p \subset O_{HP}^p$ since the right hand side vanishes if $M \in O_G^p$.

Let us illustrate how we can use 5.14 in studying the strictness of the inclusion $O_G^p \subset O_{HP}^p$. Suppose that $M \notin O_G^p$ and we want to show that $M \in O_{HP}^p$. If u is a positive nonconstant $\mathcal{A}$-harmonic function on M we may assume that $\inf u = 0$. Then

$$\sup_C \log \frac{M_C}{m_C} = \infty$$

where the supremum is taken over all compact sets $C \subset M$. If we can find for every compact set $C \subset M$ a domain D such that the right hand side in (5.15) is uniformly bounded, we are done. Unfortunately, $\operatorname{cap}_p(\bar{D}, M) \to \infty$ as D gets larger and larger and thus $\lambda_p(C, D)$ has to grow at least as fast as $\operatorname{cap}_p(\bar{D}, M)$ does. Usually it is difficult or even impossible to obtain any good estimate for $\lambda_p(C, D)$. However, we think that the inequality (5.15) tells us something essential about the geometry of $M \in O_{HP}^p \setminus O_G^p$ near the infinity. We can, for example, give a very short proof for the strict inclusion $O_G^2 \subset O_{HP}^2$ for Riemann surfaces just by using 5.14, see e.g. [SN, p. 304] for the classical proof. Moreover, we can construct quite simple examples in any dimensions n to prove the strictness of the inclusion $O^n_G \subset O^n_{HP}$. These questions will be discussed in details in a forth-coming paper.

We close this section by asking whether we can replace the phrase "for all $\mathcal{A} \in \mathcal{A}_p(M)$" in 5.1, 5.6 and 5.8 by "for some $\mathcal{A} \in \mathcal{A}_p(M)$" and still get the same classes. It follows from 5.2 and from the proof of 5.9 that the answer is yes for the definitions 5.1 and 5.8. On the other hand, for the definitions of O^2_{HP} and O^2_{HB}, $n = 2$, the answer is no by [Ly]. The other cases remain open.

6. The Heisenberg group

In this section we shall give estimates for the modulus of certain path families on the Heisenberg group H_1.

The *Heisenberg group* H_1 is the Lie group consisting of all points $(x, y, z) \in \mathbf{R}^3$ with the operation

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z'+2x'y-2xy').$$

It is easy to see that

$$E_1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}$$

form a basis of left-invariant vector fields on H_1. Let $\langle\ ,\ \rangle$ be a left-invariant Riemannian metric on H_1 such that E_1, E_2 and E_3 are mutually orthonormal. The associated Riemannian volume form is

$$dm = dx \wedge dy \wedge dz. \tag{6.1}$$

We also define a singular Riemannian metric $\langle\ ,\ \rangle_0$ on H_1 such that E_1 and E_2 are orthonormal and that $|E_3|_0^2 = \langle E_3, E_3\rangle_0 = \infty$. We say that a vector h is *horizontal* if $|h|_0 < \infty$. A C^1-path $\gamma = (x, y, z)$ is said to be horizontal if $\dot\gamma(t)$ is a horizontal vector for all t. Note that $\dot\gamma = x'E_1 + y'E_2 + (z' - 2yx' + 2xy')E_3$ and thus γ is horizontal if and only if $z' = 2yx' - 2xy'$. Since E_1 and E_2 together with their commutator $[E_1, E_2] = -4E_3$ span TH_1, it is possible to join any two points q_1, $q_2 \in H_1$ by a horizontal path γ. The distance d_∞, defined by

$$d_\infty(q_1, q_2) = \inf_\gamma \int_a^b |\dot\gamma(t)|_0 \, dt,$$

is called the *Carnot-Carathéodory metric*. Here the infimum is taken over all horizontal paths $\gamma : [a, b] \to H_1$ such that $\gamma(a) = q_1$ and $\gamma(b) = q_2$.

In the first part of the section we shall prove using ideas of [ReK] that the ideal boundary of H_1 is of positive 3-capacity. This was first proved by P. Pansu,

[Pa], by means of isoperimetric inequalities. As in [ReK] we define a norm $N(q)$ of a point $q=(x,y,z)$ by

$$N(q)=\left((x^2+y^2)^2+z^2\right)^{1/4}.$$

The *horizontal gradient* of N is given by

$$\nabla_0 N=(E_1N)E_1+(E_2N)E_2$$

in $H_1\setminus\{0\}$. Then

$$|\nabla_0 N|_0^2=(E_1N)^2+(E_2N)^2=\frac{x^2+y^2}{N^2}.$$

If $x^2+y^2>0$, we define

$$V=\frac{N^2}{x^2+y^2}\nabla_0 N.$$

We write $x=\varrho\cos\vartheta, y=\varrho\sin\vartheta$ and use (ϱ,ϑ,z) as coordinates on H_1. Then

$$\begin{cases}\dfrac{\partial}{\partial x} &= \cos\vartheta\dfrac{\partial}{\partial\varrho}-\varrho\sin\vartheta\dfrac{\partial}{\partial\vartheta}\\ \dfrac{\partial}{\partial y} &= \sin\vartheta\dfrac{\partial}{\partial\varrho}+\varrho\cos\vartheta\dfrac{\partial}{\partial\vartheta}\end{cases}$$

and we have

$$V=\frac{1}{N}\left(\varrho\frac{\partial}{\partial\varrho}-z\frac{\partial}{\partial\vartheta}+2z\frac{\partial}{\partial z}\right).$$

Fix $a>0$. If q is not on the z-axis and if $N(q)=a$, we can write

$$q=(a\cos^{1/2}\alpha\cos\varphi, a\cos^{1/2}\alpha\sin\varphi, a^2\sin\alpha)$$

where $-\pi/2<\alpha<\pi/2$ and $0\le\varphi<2\pi$. Let $\gamma_q=(x,y,z)$ be an integral curve of V such that $\gamma_q(a)=q$. Then $N(\gamma_q(r))=r$ for every $r>0$ since $VN=1$. Writing

$$\begin{cases}x' &= -\varrho\vartheta'\sin\vartheta+\varrho'\cos\vartheta\\ y' &= \varrho\vartheta'\cos\vartheta+\varrho'\sin\vartheta\end{cases}$$

we get that

$$\begin{aligned}\dot\gamma &= x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+z'\frac{\partial}{\partial z}\\ &= \varrho'\frac{\partial}{\partial\varrho}+\varrho^2\vartheta'\frac{\partial}{\partial\vartheta}+z'\frac{\partial}{\partial z}.\end{aligned}$$

It is now easy to verify that $\gamma_q = (\varrho\cos\vartheta, \varrho\cos\vartheta, z)$ is given by

$$(6.2)\qquad \begin{cases} \varrho = \varrho(r) = r\cos^{1/2}\alpha \\ \vartheta = \vartheta(r) = \varphi - \tan\alpha\log(r/a) \\ z = z(r) = r^2\sin\alpha. \end{cases}$$

We can use the numbers r, α and φ as coordinates of any point $q = (x, y, z)$, $x^2 + y^2 > 0$. In these coordinates

$$|V|_0^2 = \frac{N^2}{x^2+y^2} = \cos^{-1}\alpha.$$

The Riemannian volume form (6.1) can now be written as

$$dm = r^3\, dr \wedge d\alpha \wedge d\varphi.$$

Let then $\tilde{\Gamma}_{a,b}$, $b > a$, be the family of all paths $\gamma_q|[a,b] \to H_1$ where γ_q is given by (6.2) and q runs over $\Sigma_a = \{q \in H_1 : N(q) = a\}$. All the paths in $\tilde{\Gamma}_{a,b}$ are horizontal, hence $|\dot{\gamma}_q(t)| = |\dot{\gamma}_q(t)|_0 = \cos^{-1/2}\alpha$. Moreover, they join Σ_a and $\Sigma_b = \{q \in H_1 : N(q) = b\}$. Let $\varrho \in F(\tilde{\Gamma}_{a,b})$, be admissible for $\tilde{\Gamma}_{a,b}$, see [Vä, Section 6] for notation. By Hölder's inequality,

$$\begin{aligned} 1 \le \left(\int_{\gamma_q} \varrho\, ds\right)^3 &= \left(\int_a^b \varrho(\gamma_q(r))\, |\dot{\gamma}_q(r)|\, dr\right)^3 \\ &\le 4\left(a^{-1/2} - b^{-1/2}\right)^2 \cos^{-3/2}\alpha \left(\int_a^b \varrho(\gamma_q(r))^3 r^3\, dr\right) \end{aligned}$$

for every $\gamma_q \in \tilde{\Gamma}_{a,b}$. Hence

$$\int_a^b \varrho(\gamma_q(r))^3 r^3\, dr \ge \frac{\cos^{3/2}\alpha}{4\left(a^{-1/2} - b^{-1/2}\right)^2}$$

and

$$\int_{H_1} \varrho^3\, dm \ge \frac{\pi\int_0^{\pi/2}\cos^{3/2}\alpha\, d\alpha}{\left(a^{-1/2} - b^{-1/2}\right)^2}.$$

It follows that

$$\mathrm{M}_3(\tilde{\Gamma}_{a,b}) \ge c_1\left(a^{-1/2} - b^{-1/2}\right)^{-2}$$

where $c_1 = \pi\int_0^{\pi/2}\cos^{3/2}\alpha\, d\alpha$. Letting $b \to \infty$ we obtain

$$\mathrm{M}_3(\tilde{\Gamma}_{a,\infty}) \ge c_1 a > 0.$$

Finally, if $\Gamma_{a,\infty}$ is the family of all paths which join Σ_a to infinity, then

$$\mathrm{M}_3(\Gamma_{a,\infty}) \geq \mathrm{M}_3(\tilde{\Gamma}_{a,\infty}) \geq c_1 a.$$

Let then u be a function defined by

$$u(q) = \begin{cases} 1 & \text{if } N(q) \leq a \\ \dfrac{N(q)^{-1/2} - b^{-1/2}}{a^{-1/2} - b^{-1/2}} & \text{if } a < N(q) < b \\ 0 & \text{if } N(q) \geq b. \end{cases}$$

Then $\varrho_0 = |\nabla u| \in F(\Gamma_{a,b})$ where $\Gamma_{a,b}$ is the family of all paths which join Σ_a and Σ_b. Since

$$\begin{aligned} |\nabla N|^2 &= (E_1 N)^2 + (E_2 N)^2 + (E_3 N)^2 \\ &= \frac{x^2 + y^2}{N^2} + \frac{z^2}{4N^6} \\ &= \cos\alpha + \frac{\sin^2\alpha}{4r^2}, \end{aligned}$$

we get that

$$\begin{aligned} \varrho_0(r, \alpha, \varphi) &= \frac{\left(\cos\alpha + \dfrac{\sin^2\alpha}{4r^2}\right)^{1/2}}{2r^{3/2}\left(a^{-1/2} - b^{-1/2}\right)} \\ &\leq \frac{\left(\cos\alpha + \dfrac{\sin^2\alpha}{4a_0^2}\right)^{1/2}}{2r^{3/2}\left(a^{-1/2} - b^{-1/2}\right)} \end{aligned}$$

if $b > r > a \geq a_0 > 0$. Hence

$$\mathrm{M}_3(\Gamma_{a,b}) \leq \frac{\pi \int_0^{\pi/2} \left(\cos\alpha + \dfrac{\sin^2\alpha}{4a_0^2}\right)^{3/2} d\alpha}{\left(a^{-1/2} - b^{-1/2}\right)^2}.$$

We have proved:

6.3. Theorem. *Let $b > a \geq a_0 > 0$ and let $\Gamma_{a,b}$ be the family of all paths which join Σ_a and Σ_b. Then*

$$c_1\left(a^{-1/2} - b^{-1/2}\right)^{-2} \leq \mathrm{M}_3(\Gamma_{a,b}) \leq c_2\left(a^{-1/2} - b^{-1/2}\right)^{-2}$$

where

$$c_1 = \pi \int_0^{\pi/2} \cos^{3/2}\alpha \, d\alpha$$

and

$$c_2 = \pi \int_0^{\pi/2} \left(\cos\alpha + \frac{\sin^2\alpha}{4a_0^2}\right)^{3/2} d\alpha.$$

In particular, $c_2 \to c_1$ if $a_0 \to \infty$.

As a consequence we get the following results. Corollary 6.4 was first proved by P. Pansu. The proof of 6.5 can be found for example in [Ri, III.2.12].

6.4. Corollary. *The ideal boundary of H_1 is of positive 3-capacity.*

6.5. Corollary. *Every quasiregular mapping $f : \mathbf{R}^3 \to H_1$ is constant.*

Corollary 6.4 says that $H_1 \notin O_G^3$. It would be interesting to know whether $H_1 \in O_{HP}^3$. The rest of this section is devoted to this question although we can not give any answer to that.

Let $\gamma = \gamma_{a,\vartheta} = (x, y, z)$ be a path

$$\begin{cases} x(t) = 2a\sin(\frac{t}{2a})\cos(\vartheta - \frac{t}{2a}) \\ y(t) = 2a\sin(\frac{t}{2a})\sin(\vartheta - \frac{t}{2a}) \\ z(t) = 2at - 2a^2\sin(\frac{t}{a}) - 2\pi a^2, \end{cases}$$

where $0 \le \vartheta < 2\pi$, $0 < a < \infty$ and $0 < t < 2\pi a$. Then $\gamma_{a,\vartheta}$ is a horizontal path joining the points $(0, 0, 2\pi a^2)$ and $(0, 0, -2\pi a^2)$. Moreover, $|\dot\gamma| = 1$. If a and t are fixed, $\vartheta \mapsto \gamma_{a,\vartheta}(t)$ is a circle on a plane $z = z(a,t)$ with a center on z-axis. Let a be fixed. It follows from the formula

$$z'(t) = \frac{x(t)^2 + y(t)^2}{a} \tag{6.6}$$

that $t \mapsto z(a,t)$ is strictly increasing and therefore the paths $\gamma_{a,\vartheta}$, $0 \le \vartheta < 2\pi$, do not intersect. Suppose that $a_2 > a_1 > 0$. Write $\gamma_i = \gamma_{a_i,\vartheta_i} = (x_i, y_i, z_i)$, $i = 1, 2$, and let $\tilde\gamma_i = (x_i, y_i, 0)$ be the projection of γ_i to the (x,y)-plane. Then $\tilde\gamma_i$ is a circle with radius a_i. Since γ_i is horizontal and $|\dot\gamma_i| = 1$, both the length of $\gamma_i|[s,r]$ and the euclidean length of $\tilde\gamma_i|[s,r]$ are equal to $r - s$ for all $0 < s < r < 2\pi a_i$. Suppose that $\tilde\gamma_1(t_1) = \tilde\gamma_2(t_2)$ for some $t_i \in]0, 2\pi a_i[$. Then $t_1 > t_2$ if $t_2 \le \pi a_2$, and $2\pi a_2 - t_2 < 2\pi a_1 - t_1$ if $t_2 \ge \pi a_2$. It follows now from (6.6) that $z_1(t_1) \ne z_2(t_2)$ in both cases. We have proved that through every point (x,y,z), $x^2 + y^2 > 0$, goes exactly one path $\gamma_{a,\vartheta}$. Thus we can use (ϑ, a, t) as coordinates of $q = \gamma_{a,\vartheta}(t)$. We write

$$J(a,t) = \frac{\partial(x,y,z)}{\partial(a,\vartheta,t)}.$$

After an elementary calculation we get

$$J(a,t) = 4a^2\big((\pi - \frac{t}{a})\sin\frac{t}{a} + 4\sin^2\frac{t}{2a}\big).$$

Let $0 \le \alpha < \beta < \infty$ and let $\Gamma_{\alpha,\beta}$ be the family of all $\gamma_{a,\vartheta}$, $\alpha \le a \le \beta$, $0 \le \vartheta < 2\pi$. Let $\varrho \in F(\Gamma_{\alpha,\beta})$. Then for every $\gamma = \gamma_{a,\vartheta} \in \Gamma_{\alpha,\beta}$

$$\begin{aligned} 1 &\le \left(\int_\gamma \varrho\, ds\right)^3 = \left(\int_0^{2\pi a} \varrho(\gamma(t))\, |\dot\gamma(t)|\, J(a,t)^{1/3} J(a,t)^{-1/3}\, dt\right)^3 \\ &\le \left(\int_0^{2\pi a} \varrho(\gamma(t))^3 J(a,t)\, dt\right)\left(\int_0^{2\pi a} J(a,t)^{-1/2}\, dt\right)^2. \end{aligned}$$

Hence

$$\int_0^{2\pi a} \varrho(\gamma(t))^3 J(a,t)\, dt \ge \left(\int_0^{2\pi a} J(a,t)^{-1/2}\, dt\right)^{-2}$$

for every $\gamma \in \Gamma_{\alpha,\beta}$. If $\alpha \in [0, \pi/2]$, it follows from an estimate $\sin\alpha \ge \alpha/2$ that

$$(\pi - \alpha)\sin\alpha + 4\sin^2(\alpha/2) \ge 1 - (1 - \alpha/2)^2.$$

Hence

$$\int_0^{2\pi a} J(a,t)^{-1/2}\, dt = 2\int_0^{\pi a} J(a,t)^{-1/2}\, dt = c_3 < \infty$$

where c_3 does not depend on a. Thus we get

$$\begin{aligned} \int_{H_1} \varrho^3\, dm &\ge 2\pi \int_\alpha^\beta da \int_0^{2\pi a} \varrho(\gamma(t))^3 J(a,t)\, dt \\ &\ge 2\pi \int_\alpha^\beta \left(\int_0^{2\pi a} J(a,t)^{-1/2}\, dt\right)^{-2} da \\ &\ge c_4(\beta - \alpha) \end{aligned}$$

where c_4 is a positive constant. We have proved

$$\mathrm{M}_3(\Gamma_{\alpha,\beta}) \ge c_4(\beta - \alpha). \tag{6.7}$$

Note that the paths of $\Gamma_{\alpha,\beta}$ lie in a set $D = \{q \in H_1 : N(q) < \sqrt{2\pi}\beta\}$ and thus

$$\mathrm{M}_3\big(\Delta(F_1, F_2; D)\big) \ge c_4(\beta - \alpha)$$

where F_1 and F_2 are the line segments joining the points $(0,0,2\pi\alpha^2)$ and $(0,0,2\pi\beta^2)$ and the points $(0,0,-2\pi\alpha^2)$ and $(0,0,-2\pi\beta^2)$, respectively. In order to use 5.14 we should know corresponding estimates for all continua F_1 and F_2 which join $\{q \in H_1 : N(q) \le \sqrt{2\pi}\alpha\}$ and $H_1 \setminus D$. Since we do not have such estimates at the moment, we can not give any answer to the question whether $H_1 \in O^3_{HP}$.

7. Comparison lemma and the Picard type theorem

In this section we shall first prove a generalization of the Comparison lemma [Ri, IV.1.1]. This lemma is an essential tool in the proof of the Picard type theorem for quasiregular mappings and in the value distribution theory, see [Ri, Chapter IV and V]. The Picard type theorem says that a nonconstant K-quasiregular mapping of $\mathbf{R}^n$ into $\mathbf{R}^n$ can omit at most q_0 distinct points where q_0 depends only on n and K. In this section we shall also give a partial answer to a question, posed by M. Gromov, whether a similar result holds for quasiregular mappings of $\mathbf{R}^n$, $n \geq 3$, into $\mathbf{S}^n \setminus \{a_1, \dots, a_q\}$ if $\mathbf{S}^n \setminus \{a_1, \dots, a_q\}$ has an arbitrary Riemannian metric.

Let D be a domain in M with a C^∞ boundary Σ_0 which is homeomorphic to an $(n-1)$-sphere. Suppose that there exists a function $g \in C(\bar{D})$ which satisfies the following conditions: g is n-harmonic in D, $g = 0$ on Σ_0, for every $m > 0$ there exists a compact set $C \subset \bar{D}$ such that $g \geq m$ in $D \setminus C$, and

$$\operatorname{cap}_n\big(\{x \in D : g(x) \geq b\}, \{x \in D : g(x) > a\}\big) = (b-a)^{1-n}$$

for all $b > a \geq 0$. We say that g is a Green function in D for the n-Laplacian with the pole at infinity. We shall restrict ourselves to the case where $g \in C^\infty(D)$ and $|\nabla g| > 0$. Let $\langle\langle\ ,\ \rangle\rangle$ be a Riemannian metric in D defined by

$$\langle\langle\ ,\ \rangle\rangle = |\nabla g|^2 \langle\ ,\ \rangle.$$

We get the following new formulae for the gradient of a function, for the Riemannian volume form, etc. associated to the metric $\langle\langle\ ,\ \rangle\rangle$:

$$\text{(7.1)} \qquad \begin{aligned} &\hat{\nabla}\varphi = |\nabla g|^{-2}\nabla\varphi, \quad d\hat{m} = |\nabla g|^n\, dm, \quad \|h\| = |\nabla g|\,|h|, \\ &d\hat{s} = |\nabla g|\, ds, \quad d\hat{\mathcal{H}}^{n-1} = |\nabla g|^{n-1}\, d\mathcal{H}^{n-1}. \end{aligned}$$

Now g is (2-)harmonic in D with respect to the metric $\langle\langle\ ,\ \rangle\rangle$ since

$$\begin{aligned} \int_D \langle\langle \hat{\nabla} g, \hat{\nabla}\varphi\rangle\rangle\, d\hat{m} &= \int_D |\nabla g|^{2-4+n} \langle \nabla g, \nabla\varphi\rangle\, dm \\ &= \int_D \langle |\nabla g|^{n-2}\nabla g, \nabla\varphi\rangle\, dm = 0 \end{aligned}$$

for every $\varphi \in C_0^\infty(D)$. Since $g \in C^\infty$, $\hat{\operatorname{div}}(\hat{\nabla} g) = 0$. By the definition of the divergence,

$$\mathcal{L}_{\hat{\nabla} g}(d\hat{m}) = (\hat{\operatorname{div}}\hat{\nabla} g)\, d\hat{m} = 0$$

where $\mathcal{L}_{\hat{\nabla} g}(d\hat{m})$ is the Lie derivative of $d\hat{m}$ with respect to $\hat{\nabla} g$. This means that the volume is invariant under the flow of $\hat{\nabla} g$, see e.g. [AMR, 6.5.18].

Let $\Sigma_1 = g^{-1}(1)$. In the sequel $\gamma_y : [0,\infty[\to \bar{D}$, $y \in \Sigma_1$, will be a path such that $\gamma_y|]0,\infty[$ is an integral curve of $\hat{\nabla} g$ and $\gamma_y(1) = y$. Let $A \subset \Sigma_1$ be a Borel set. For every $t > 0$ we write

$$A_t = \{\gamma_y(t) : y \in A\},$$

and

$$A^{s,t} = \bigcup_{s \le r \le t} A_r.$$

Since the volume is invariant under the flow of $\hat{\nabla} g$,

$$\hat{m}(A^{s,s+\varepsilon}) = \hat{m}(A^{t,t+\varepsilon})$$

whenever t, s, $\varepsilon > 0$. On the other hand, $\|\hat{\nabla} g\| = 1$ and the co-area formula [Fe, 3.2.12, 3.2.46] implies

$$\begin{aligned} \hat{\mathcal{H}}^{n-1}(A_s) &= \lim_{\varepsilon\to 0} 1/\varepsilon \int_s^{s+\varepsilon} \Big(\int_{A_r} d\hat{\mathcal{H}}^{n-1}\Big)\, dr \\ &= \lim_{\varepsilon\to 0} \hat{m}(A^{s,s+\varepsilon})/\varepsilon = \lim_{\varepsilon\to 0} \hat{m}(A^{t,t+\varepsilon})/\varepsilon \\ &= \hat{\mathcal{H}}^{n-1}(A_t). \end{aligned} \tag{7.2}$$

We define a measure μ on Borel subsets of Σ_1 by

$$\mu(E) = \hat{\mathcal{H}}^{n-1}(E). \tag{7.3}$$

Note that

$$\mu(\Sigma_1) = \int_{g^{-1}(1)} |\nabla g|^{n-1}\, d\mathcal{H}^{n-1} = 1$$

by 3.14 and the co-area formula.

The next lemma is a counterpart of 3.8 for the modulus of a path family.

7.4. Lemma. *Let $A \subset \Sigma_1$ be a Borel set, $b > a \ge 0$, and let $\Gamma_A^{a,b}$ be the family of all paths $\gamma = \gamma_y|[a,b]$, $y \in A$. Then*

$$\mathrm{M}_n(\Gamma_A^{a,b}) = \frac{\mu(A)}{(b-a)^{n-1}}.$$

Proof. The conformal change of the metric does not change $\mathrm{M}_n(\Gamma_A^{a,b})$, thus we can calculate $\mathrm{M}_n(\Gamma_A^{a,b})$ with respect to $\langle\langle\ ,\ \rangle\rangle$. Let $\varrho \in F(\Gamma_A^{a,b})$. Then Hölder's inequality implies

$$\begin{aligned} 1 &\le \left(\int_\gamma \varrho\, d\hat{s}\right)^n = \left(\int_a^b \varrho(\gamma(t))\|\dot\gamma(t)\|\, dt\right)^n \\ &\le (b-a)^{n-1}\int_a^b \varrho(\gamma(t))^n\, dt \end{aligned} \tag{7.5}$$

for every $\gamma \in \Gamma_A^{a,b}$. Note that $\|\dot\gamma(t)\| = 1$. Every point $x \in \bar{D}$ can be written uniquely in the form $x = (y,t) \in \Sigma_1 \times [0,\infty[$ such that $x = \gamma_y(t)$. Fubini's theorem together with (7.2) and (7.5) then imply

$$\begin{aligned} \int_D \varrho^n\, d\hat{m} &\ge \int_a^b \Big(\int_{g^{-1}(t)} \varrho^n\, d\hat{\mathcal{H}}^{n-1}\Big)\, dt \ge \int_a^b \Big(\int_{A_t} \varrho(x)^n\, d\hat{\mathcal{H}}^{n-1}(x)\Big)\, dt \\ &= \int_a^b \Big(\int_A \varrho(y,t)^n\, d\mu(y)\Big)\, dt = \int_A \Big(\int_a^b \varrho(y,t)^n\, dt\Big)\, d\mu(y) \\ &\ge \frac{\mu(A)}{(b-a)^{n-1}}. \end{aligned}$$

Hence

$$\mathrm{M}_n(\Gamma_A^{a,b}) \ge \frac{\mu(A)}{(b-a)^{n-1}}.$$

To complete the proof, let

$$\varrho = \frac{\chi(A \times [a,b])}{b-a}.$$

Then $\varrho \in F(\Gamma_A^{a,b})$ and

$$\int_D \varrho^n\, d\hat{m} = \frac{\mu(A)}{(b-a)^{n-1}}. \qquad \square$$

Let then $f : \mathbf{R}^n \to M$ be a nonconstant quasiregular mapping. For $z \in M$ and for a Borel set E such that $\bar{E}$ is a compact subset of $\mathbf{R}^n$ we set

$$n(E,z) = \sum_{x \in f^{-1}(z)\cap E} i(x,f)$$

where $i(x,f)$ is the local index of f at x, see [Ri, I.4]. Now $n(E,z)$ is finite and $z \mapsto n(E,z)$ is a Borel function. The average of $n(E,z)$ over $g^{-1}(s)$, $s > 0$, is

$$\begin{aligned} \nu(E,s) &= \int_{g^{-1}(s)} n(E,z)\, d\hat{\mathcal{H}}^{n-1}(z) \\ &= \int_{\Sigma_1} n(E,(y,s))\, d\mu(y). \end{aligned} \tag{7.6}$$

We write $n(r,z)$ for $n(\bar{B}^n(r),z)$ and call it the *counting function* of f. Similarly, we abbreviate $\nu(r,s) = \nu(\bar{B}^n(r),s)$.

7.7. Theorem. *Let* $\theta > 1$, $x \in \mathbf{R}^n$, *and* r, b, $a > 0$. *Then*

$$\nu\big(\bar{B}^n(x,\theta r),b\big) \geq \nu\big(\bar{B}^n(x,r),a\big) - \frac{K_I\omega_{n-1}\,|b-a|^{n-1}}{(\log\theta)^{n-1}}.$$

Proof. We may assume that $x = 0$ and $b > a$. For every $m = 1, 2, \ldots$ write

$$\begin{aligned} E_m &= \{y \in \Sigma_1 : n\big(\theta r,(y,b)\big) = n\big(r,(y,a)\big) - m\}, \\ E &= \bigcup_m E_m. \end{aligned}$$

Then

$$\begin{aligned} \int_{\Sigma_1} n\big(\theta r,(y,b)\big)\,d\mu(y) &= \int_{\Sigma_1\setminus E} n\big(\theta r,(y,b)\big)\,d\mu(y) \\ &\quad + \sum_m \int_{E_m} n\big(\theta r,(y,b)\big)\,d\mu(y) \\ &\geq \int_{\Sigma_1\setminus E} n\big(r,(y,a)\big)\,d\mu(y) + \sum_m \int_{E_m} \Big(n\big(r,(y,a)\big) - m\Big)\,d\mu(y) \\ &= \int_{\Sigma_1} n\big(r,(y,a)\big)\,d\mu(y) - \sum_m m\mu(E_m). \end{aligned}$$

Let $\Gamma_m = \Gamma_{E_m}^{a,b}$ be as in 7.4 and let $y \in E_m$. Then there exists a sequence $\beta_1, \ldots, \beta_k$, $k = n\big(r,(y,a)\big)$ of maximal $f|B^n(\theta r)$-lifts of γ_y starting at points in $f^{-1}(y,a) \cap \bar{B}^n(r)$ such that

$$\operatorname{card}\{j : \beta_j(t) = x\} \leq i(x,f)$$

for all x and t (which make sense), see [Ri, II.3]. Since $y \in E_m$, at least m of $\beta_1, \ldots, \beta_k$ end in $S^{n-1}(\theta r)$. Let Γ_m^* be the family of all these lifts when y runs over E_m. By Väisälä's inequality [Ri, II.9.1] and [MaR],

$$\mathrm{M}_n(\Gamma_m) \leq K_I \mathrm{M}_n(\Gamma_m^*)/m.$$

Since the path families Γ_m are separate, so are the families Γ_m^* and thus

$$\sum_m \mathrm{M}_n(\Gamma_m^*) = \mathrm{M}_n\Big(\bigcup_m \Gamma_m^*\Big) \leq \frac{\omega_{n-1}}{(\log\theta)^{n-1}}.$$

On the other hand,

$$\mathrm{M}_n(\Gamma_m) = \frac{\mu(E_m)}{(b-a)^{n-1}}$$

by 7.4. The theorem now follows since

$$\sum_m m\mu(E_m) = (b-a)^{n-1}\sum_m m\mathrm{M}_n(\Gamma_m)$$
$$\leq \frac{K_I\omega_{n-1}(b-a)^{n-1}}{(\log\theta)^{n-1}}. \quad \square$$

In the rest of this section we let $M = \mathbf{S}^n \setminus \{a_1, \ldots, a_q\}$ where $a_1, \ldots, a_q$, $q \geq 2$, are distinct points in $\mathbf{S}^n$, $n \geq 3$. Let

$$D_i = \{x \in \mathbf{S}^n : 0 < \sigma(x, a_i) < \sigma_0\}$$

where σ is the spherical metric on $\mathbf{S}^n$ and

$$\sigma_0 = \min_{1 \leq j < k \leq q} \sigma(a_j, a_k)/4.$$

7.8. Definition. We say that a Riemannian metric $\langle\, ,\, \rangle$ on M is *admissible* if for every $1 \leq i \leq q$ there exists a Green function $g_i \in C^\infty(D_i)$ for the n-Laplacian with the pole at a_i such that $|\nabla g_i| > 0$ in D_i.

We are now ready to prove a generalization of the Picard type theorem [Ri, IV.2.1].

7.9. Theorem. *For each $n \geq 3$ and $K \geq 1$ there exists a positive integer $q_0 = q_0(n, K)$ depending only on n and K such that every K-quasiregular mapping $f : \mathbf{R}^n \to M$ is constant whenever M is equipped with an admissible Riemannian metric and $q \geq q_0$.*

Proof. Let $\langle\, ,\, \rangle$ be an admissible Riemannian metric on M. We write $\Sigma_i = g_i^{-1}(1)$ where g_i is as in 7.8. Let μ_i be the measure on Borel subsets of Σ_i given by (7.3). Suppose that $f : \mathbf{R}^n \to M$ is a nonconstant K-quasiregular mapping. The average of $n(E, z)$ over $g_i^{-1}(s)$, $s > 0$, is denoted by $\nu_i(E, s)$, see (7.6). We abbreviate $\nu_i(E) = \nu_i(E, 1)$. Then

$$\nu_i(E) = \int_M n(E, z)\, d\mu_i(z)$$

since the support of μ_i is Σ_i. Moreover, there exists a constant $c_1 > 0$ such that

$$\mu_i(B(x, r)) \leq c_1 r^{n-1} = h(r)$$

for all i and for all balls $B(x, r) \subset M$. Since

$$\int_0^1 \frac{h(r)^{1/pn}}{r}\, dr < \infty$$

for all $p > 0$, we can use [MaR, 4.8]. Applying [MaR, 4.8] twice with the constant $\theta = 2$ yields

$$\nu_i(\bar{B}^n(x,4r)) \geq \nu_j(\bar{B}^n(x,r)) - d \tag{7.10}$$

for all $x \in \mathbf{R}^n$, $r > 0$ and for all $1 \leq i,\ j \leq q$ where d is a positive constant which is independent on x, r, i, and j, see also [MaR, Remark 5.12.6]. Note that the constant $c > 1$ in [MaR, 4.8] is unnecessary since we can prove [MaR, 4.8] using the sharper form of the Comparison lemma than that in the original proof. Let $c_2 = \omega_{n-1}(\log 2)^{1-n}$. We shall next show that $\nu_1(r) \to \infty$ as $r \to \infty$. Suppose that $\lim_{r\to\infty} \nu_1(r) = \lambda < \infty$. As in the proof of [MaR, 5.10] we can show that $\lim_{r\to\infty} n(r,z) \leq \lambda$ for all $z \in \Sigma_1$. It follows from 7.7 that the same is true for all $z \in D_1$. Let $B_i = \{x \in \mathbf{S}^n : 0 < \sigma(x,a_i) \leq \sigma_0/2\}$, $i = 1,\ 2$. By [Ri, III.2.12], f can omit at most a set of n-capacity zero. Hence $f^{-1}B_i \neq \emptyset$. Let C_i be a component of $f^{-1}B_i$. Since a_1 and a_2 are omitted, C_i tends to infinity. Let Γ_1' be the family of all paths $\gamma : [0,1] \to \mathbf{R}^n$ such that $\gamma(0) \in C_1$ and $\gamma(1) \in C_2$. For each $\gamma \in \Gamma_1'$ there exists $t_\gamma > 0$ such that $f(\gamma(t_\gamma)) \in \Sigma_1$ and $f(\gamma(t)) \in D_1$ for all $t \in [0, t_\gamma[$. Let Γ_2' be the family of all paths $\gamma|[0,t_\gamma[$, $\gamma \in \Gamma_1'$. Then

$$\mathrm{M}_n(\Gamma_1') \leq \mathrm{M}_n(\Gamma_2') \leq \lambda K_O \mathrm{M}_n(f\Gamma_2') < \infty,$$

by [Ri, II.(2.6)]. This is a contradiction since $\mathrm{M}_n(\Gamma_1') = \infty$. Thus we can choose $r > 0$ such that

$$\nu_1(r) > \max(8K_I c_2,\ 2d) \tag{7.11}$$

and then $b > 1$ such that

$$\nu_1(r) = 4K_I c_2 (b-1)^{n-1}. \tag{7.12}$$

The Comparison lemma 7.7 together with (7.10), (7.11) and (7.12) imply

$$\begin{aligned} \nu_i(8r,b) &\geq \nu_i(4r) - K_I c_2 (b-1)^{n-1} \\ &\geq \nu_1(r) - d - K_I c_2 (b-1)^{n-1} \\ &> \nu_1(r)/2 - K_I c_2 (b-1)^{n-1} \\ &= K_I c_2 (b-1)^{n-1} > 0. \end{aligned} \tag{7.13}$$

Thus there exists for each i a component H_i of $f^{-1}\{x \in M : g_i(x) \geq b\}$ which meets $\bar{B}^n(8r)$ and tends to infinity. Each H_i contains a compact set $F_i \subset \bar{B}^n(16r) \setminus B^n(8r)$ connecting $\mathbf{S}^n(8r)$ and $\mathbf{S}^n(16r)$. Let Γ_i be the family of all paths in $B^n(16r) \setminus \bar{B}^n(8r)$ which join F_i and $F_i^\star = \cup_{j\neq i} F_j$. We claim that

$$(\mathrm{M}_n(\Gamma_i) - K_O K_I c_2/n)\nu_1(r) \leq 8K_O K_I c_2 \nu_1(128r) \tag{7.14}$$

for all i. To prove this, we define an admissible function $\varrho_i \in F(\Gamma_i)$ by

$$\varrho_i(z) = \begin{cases} |\nabla g_i(z)|/(b-1) & \text{if } 1 \le g_i(z) \le b \\ 0 & \text{elsewhere.} \end{cases}$$

Then by [Ri, II.(2.6)] and the co-area formula,

$$\begin{aligned} \mathrm{M}_n(\Gamma_i) &\le K_O \int_M \varrho_i(z)^n n(16r,z)\,dm \\ &= K_O(b-1)^{-n}\int_1^b \Big(\int_{g_i^{-1}(t)} n(16r,z)\,|\nabla g_i(z)|^{n-1}\,d\mathcal{H}^{n-1}\Big)\,dt \\ &= K_O(b-1)^{-n}\int_1^b \nu_i(16r,t)\,dt. \end{aligned}$$

It follows now from 7.7 that

$$\begin{aligned} \mathrm{M}_n(\Gamma_i) &\le K_O(b-1)^{-n}\int_1^b \big(\nu_i(32r) + K_I c_2(t-1)^{n-1}\big)\,dt \\ &= K_O(b-1)^{1-n}\nu_i(32r) + K_O K_I c_2/n. \end{aligned} \tag{7.15}$$

On the other hand, $\nu_i(32r) < 2\nu_1(128r)$ by (7.10) and (7.11). This together with (7.12) and (7.15) imply (7.14). There exists a positive integer c_3 depending only on n such that the ball $\bar{B}^n(128r)$ can be covered by balls $\bar{B}^n(x_k, r/2)$, $k = 1, \dots, c_3$, such that $x_k \in \bar{B}^n(128r)$ for all k. Since $E \mapsto \nu_1(E)$ is a measure on Borel sets,

$$\nu_1(128r) \le \sum_{k=1}^{c_3} \nu_1\big(\bar{B}^n(x_k, r/2)\big).$$

Hence

$$\nu_1\big(\bar{B}^n(x_k, r/2)\big) \ge \nu_1(128r)/c_3 \tag{7.16}$$

for some $x_k = z_1$. By [Ri, IV.2.16], there exists a constant $c_4 > 0$ depending only on n such that

$$\mathrm{M}_n(\Gamma_i) \ge c_4 q^{1/(n-1)} \tag{7.17}$$

for some i. Suppose that

$$q \ge \big(K_O K_I c_2(8c_3 + 1/n)/c_4\big)^{n-1}. \tag{7.18}$$

It follows from (7.14), (7.16) and (7.17) that

$$\nu_1\big(\bar{B}^n(z_1, r/2)\big) \ge \nu_1(r).$$

We can repeat by replacing the ball $\bar{B}^n(r)$ by $\bar{B}^n(z_1, r/2)$ and continue similarly. We obtain a sequence $0 = z_0, z_1, \dots$ of points with $z_m \in \bar{B}^n(z_{m-1}, 2^{8-m}r)$ such that $\nu_m = \nu_1\big(\bar{B}^n(z_m, 2^{-m}r)\big) \ge \nu_1(r) > 0$. But the balls $\bar{B}^n(z_m, 2^{-m}r)$ converges to a point which implies $\nu_m \to 0$. This is a contradiction and the theorem is proved. □

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