Dirichlet problem at infinity on Gromov hyperbolic metric measure spaces*

Ilkka Holopainen†
Department of Mathematics and Statistics
P.O. Box 68 (Gustaf Hällströmin katu 2b)
FIN-00014 University of Helsinki, Finland
ilkka.holopainen@helsinki.fi

Urs Lang
Departement Mathematik
ETH Zentrum
8092 Zürich, Switzerland
lang@math.ethz.ch

Aleksi Vähäkangas‡
Department of Mathematics and Statistics
P.O. Box 68 (Gustaf Hällströmin katu 2b)
FIN-00014 University of Helsinki, Finland
aleksi.vahakangas@helsinki.fi

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1 Introduction

The interplay between the existence of non-constant harmonic functions on a complete Riemannian manifold M and the global geometrical structure of M has inspired researchers in geometric analysis for more than thirty years. A crude idea is that harmonic functions are, in a sense, rare on manifolds satisfying various non-negative curvature assumptions, whereas they should exist in abundance on simply connected manifolds with negative sectional curvature $K \leq -a^2 < 0$.

In 1975 Yau [65] proved that, on a complete Riemannian manifold M of non-negative Ricci curvature, every positive harmonic function is constant.

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In other words, M has the so-called strong Liouville property. Since the fundamental works of Cheng and Yau [20], Greene and Wu [29], and Yau [65], [67] in the mid-70's, there has been a lot of research related to various Liouville-type problems not only for harmonic functions, but also in the non-linear setting of p-harmonic functions. There is a vast literature on such Liouville-type theorems for harmonic functions and therefore we just refer to the survey articles [31] by Grigor'yan and [53] by Li. Some of these results have their counterparts in the case of p-harmonic functions as well; see e.g. [37], [38], and [24].

On the other hand, regarding the negative curvature case, Greene and Wu [30] conjectured that a Cartan-Hadamard manifold M admits a non-constant bounded harmonic function if the sectional curvatures of M have an upper bound

$$K(x) \leq -C\, r^{-2}(x)$$

outside a compact set for some constant $C > 0$, where $r = d(\cdot, o)$ is the distance function to a fixed point $o \in M$. Recall that a Cartan-Hadamard manifold is a complete, connected and simply connected Riemannian n-manifold, $n \geq 2$, of non-positive sectional curvature. By the Cartan-Hadamard theorem, the exponential map $\exp_o\colon T_oM \to M$ is a diffeomorphism for every point $o \in M$. In particular, M is diffeomorphic to $\mathbb{R}^n$. It is well-known that M can be compactified by adding a *sphere at infinity*, denoted by $S(\infty)$, so that the resulting space $\bar{M} = M \cup S(\infty)$ will be homeomorphic to a closed Euclidean ball. The sphere at infinity is defined as the set of all equivalence classes of geodesic unit speed rays in M; two such rays γ_1 and γ_2 are equivalent if $\sup_{t\geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$. There is a natural topology, called the *cone topology,* on $\bar{M} = M \cup S(\infty)$ defined as follows. For any point $o \in M$ and any unit vector $v \in T_oM$, let

$$C_o(v, \alpha) = \{x \in \bar{M} \setminus \{o\} \colon \sphericalangle(v, \dot{\gamma}^x(0)) < \alpha\}$$

be the cone about v of angle $\alpha > 0$, where γ^x is either the unit speed geodesic from $o = \gamma^x(0)$ to $x \in M$ or the ray emanating from o that represents the class $x \in S(\infty)$, and $\sphericalangle(v, \dot{\gamma}^x(0))$ is the angle between the vectors v and $\dot{\gamma}^x(0)$ in T_oM. Then geodesic balls $B(q, r),\ q \in M,\ r > 0$, and truncated cones $C_o(v, \alpha) \setminus \bar{B}(o, s)$, with $v \in T_oM,\ \alpha > 0,\ s > 0$, form a basis for the cone topology. Furthermore, the cone topology is independent of the choice of $o \in M$ and, equipped with this topology, $\bar{M}$ is homeomorphic to the closed unit ball $\bar{B}^n \subset \mathbb{R}^n$ and $S(\infty)$ to the sphere $S^{n-1} = \partial B^n$; see [26].

A way to approach the conjecture of Greene and Wu is to study whether the so-called *Dirichlet problem at infinity* (or the *asymptotic Dirichlet problem*) is solvable on a Cartan-Hadamard manifold. That is to ask whether every continuous function on $S(\infty)$ has a (unique) harmonic extension to M. Of course, the answer, in general, is no since the simplest Cartan-Hadamard manifold $\mathbb{R}^n$ admits no positive harmonic functions other than constants.

The Dirichlet problem at infinity was solved by Choi [21] under the condition that the sectional curvature has a negative upper bound $K \le -a^2 < 0$ and any two points of the sphere at infinity can be separated by convex neighborhoods. Such appropriate convex sets were constructed by Anderson [7] for manifolds of pinched sectional curvature $-b^2 \le K \le -a^2 < 0$. At the same time, the Dirichlet problem at infinity was independently solved by Sullivan [63] under the same pinched curvature assumption by a quantitative study of how the negative curvature pushes random paths out to infinity. See also the earlier papers by Prat [56], [57], [58], and Kifer [47]. In [8], Anderson and Schoen presented a simple and direct solution to the Dirichlet problem again in the case of pinched negative curvature. Ballmann [9] solved the Dirichlet problem at infinity for non-positively curved, irreducible, rank one manifolds admitting a compact quotient. Major contributions to the Dirichlet problem were given by Ancona in a series of papers [3], [4], [5], and [6]. In [3] he was able to replace the lower curvature bound by a bounded geometry assumption that each ball up to a fixed radius is L-bi-Lipschitz equivalent to an open set in $\mathbb{R}^n$ for some fixed $L \ge 1$. He also considered a more general class of linear equations than merely the Laplace equation. On the other hand, in [6] he showed that the Dirichlet problem is not solvable, in general, if there are neither curvature lower bounds nor the bounded geometry assumption; see also [12]. Furthermore, in [4] Ancona studied the asymptotic Dirichlet problem on Gromov hyperbolic graphs and in [5] on Gromov hyperbolic Riemannian manifolds with bounded geometry and a positive lower bound $\lambda_1(M) > 0$ for Dirichlet eigenvalues. See [16] and [48] for conditions on Gromov-hyperbolic manifolds M that imply the positivity of $\lambda_1(M)$. Recently, Brin and Kifer [14] have studied Brownian motion, harmonic functions, and Martin boundary on Gromov hyperbolic Euclidean complexes.

The present paper owes much to Cheng's work [19], where the Dirichlet problem at infinity was solved under a pointwise pinching condition on the sectional curvature. To be precise, let M be a Cartan-Hadamard manifold with $\lambda_1(M) > 0$. Suppose that there is a point $o \in M$ and a constant $C \ge 1$ such that at every point $x \in M$ we have $|K(\sigma)| \le C|K(\sigma')|$, whenever σ and σ' are 2-planes in T_xM containing the tangent vector at x of the geodesic joining o and x. Then the Dirichlet problem at infinity has a unique solution for every continuous function on $S(\infty)$. Cheng's method is based on Caccioppoli-type estimates and hence can be adapted to a wide class of equations and even to various minimization problems. See also [22] for an extension of the asymptotic Dirichlet problem to manifolds that are quasi-isometric to a Cartan-Hadamard manifold.

In addition to the works quoted above there are several papers where progress has been made to weaken the curvature assumptions typically by allowing curvature decay (or growth) at a certain rate. We refer to [11], [43], [42], and [52] to mention but a few.

In the general case of the p-Laplacian, $1 < p < \infty$, Pansu [55] showed the existence of nonconstant bounded p-harmonic functions with finite p-energy on Cartan-Hadamard manifolds of pinched curvature $-b^2 \leq K \leq -a^2$ for $p > (n-1)b/a$. The Dirichlet problem at infinity for the p-Laplacian was solved in [39] on Cartan-Hadamard manifolds of pinched negative sectional curvature by modifying the direct approach of Anderson and Schoen [8]. Recall that a *p-harmonic function* in an open subset U of a Riemannian manifold M is a continuous (weak) solution u of the p-Laplace equation

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0.$$

More precisely, u belongs to the local Sobolev space $W^{1,p}_{\mathrm{loc}}(U)$ and

$$\int_U \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \, d\mu = 0 \tag{1.1}$$

for every $\varphi \in C_0^\infty(U)$. Note that (1.1) is the Euler-Lagrange equation for the (p-energy) variational integral

$$\int_U |\nabla u|^p \, d\mu. \tag{1.2}$$

For the nonlinear potential theory associated with the p-Laplacian and more general quasilinear elliptic operators, we refer to the book [36] by Heinonen, Kilpeläinen, and Martio.

In this paper we solve the Dirichlet problem at infinity for p-harmonic functions in a very general setting of Gromov hyperbolic metric measure spaces. It is worth pointing out that the metric spaces in question do not have, in general, a manifold structure not to mention a smooth structure. Therefore, p-harmonic functions can not be defined as solutions of an equation like (1.1) but rather as minimizers of a variational integral such as (1.2). In recent years there has been a growing interest in analysis on metric measure spaces, in particular, in studying quasiconformal mappings, Sobolev spaces, differentiability of Lipschitz functions, and calculus of variations. The notion of an upper gradient of a function has turned out to be very important in these studies. Our approach uses minimal upper gradients, and p-harmonic functions are defined as p-energy minimizers among functions with the same boundary functions in relatively compact subsets.

To formulate our main result, let X be a connected, locally compact Gromov hyperbolic metric measure space equipped with a Borel regular measure μ. We impose a (local) bounded geometry condition by assuming that the measure μ is locally doubling, the measures of balls of a fixed sufficiently small radius have a uniform positive lower bound, and that X supports a local Poincaré-type inequality. Furthermore, we assume that X has at most an exponential volume growth and that a global Sobolev-type inequality holds for compactly supported functions. The latter assumption replaces the condition $\lambda_1(M) > 0$ mentioned above in the context of Ancona's and Cheng's work. Our main theorem reads as follows.

Theorem 1.1. *Let $f\colon \partial_G X \to \mathbb{R}$ be a bounded continuous function. Then there exists a continuous function $u\colon X^* \to \mathbb{R}$ which is p-harmonic in X and equal to f in $\partial_G X$.*

We refer to subsequent sections for exact assumptions and the notation employed in the theorem as well as in the following explanation of its proof.

The proof of the main theorem is carried out in several steps that we briefly describe next. Suppose first that the given bounded continuous function on $\partial_G X$ is Lipschitz. We extend it to a Lipschitz function f on X in such a way that the minimal p-weak upper gradient $|\nabla f|$ belongs to $L^Q(X)$ for some sufficiently large Q. The extension is performed by combining the usual McShane-Whitney lemma with a discrete convolution (Lemma 3.1 and (3.4)). The Gromov hyperbolicity of X implies that $\operatorname{Lip} f(x)$, the pointwise upper Lipschitz constant of f at $x \in X$, decays exponentially to zero as $|x-o| \to \infty$ (Lemma 3.2). This exponential decay together with the volume growth condition then imply the L^Q-integrability of $|\nabla f|$. Using an exhaustion of X by relatively compact domains and solving the Dirichlet problem with boundary values f in each of these domains, we obtain a sequence of p-harmonic functions converging, after passing to a subsequence, locally uniformly to a function that is p-harmonic on all of X. To show that the p-harmonic limit function has the right boundary values, we apply, by adapting Cheng's ideas, the crucial Lemmata 5.1 and 5.2. This solves the Dirichlet problem at infinity with Lipschitz-continuous boundary values. The general case follows by another limiting argument.

Our paper is organized as follows. Section 2 contains basics on Gromov hyperbolic metric spaces. In particular, we describe the Gromov boundary $\partial_G X$ and the closure X^*, as well as the topology of X^*. In Section 3 we perform the Lipschitz extensions that were mentioned above. In Section 4 we introduce upper gradients, Sobolev spaces, and p-harmonic functions on metric measure spaces. The most important section in the paper is Section 5, where we prove the key Lemmata 5.1 and 5.2. Results from previous sections are applied in Section 6 in the actual proof of Theorem 1.1. Section 7 is devoted to the uniqueness of the solution with given boundary values. It turns out that the sequential compactness of the Gromov closure is crucial for the uniqueness. Finally, in Section 8 we provide examples of Gromov hyperbolic Riemannian manifolds that fulfill the assumptions in Theorem 1.1.

Throughout the paper c and C are positive constants, and $c(a, b, \ldots)$ denotes a positive constant depending on $a, b, \ldots$. The actual value of c (and C) may vary, even within a line. For a metric space $X = (X, d)$, we usually denote the distance $d(x, y)$ between points $x, y \in X$ also by $|x - y|$. Furthermore, $B(x, r)$ stands for the open ball $\{y \in X : |x - y| < r\}$.

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2 Hyperbolic metric spaces

In this section we recall the basic notions related to Gromov hyperbolic metric spaces. Our notation and terminology is similar to that in [64]. For points $x, y, o \in X$ in a metric space X, the *Gromov product* of x and y with respect to the basepoint o is defined by

$$(x \,|\, y)_o = \tfrac{1}{2}\big(|x-o| + |y-o| - |x-y|\big).$$

Note that $0 \le (x \,|\, y)_o \le \min\{|x-o|, |y-o|\}$. Moreover, if $o' \in X$ is another basepoint, then

$$|(x \,|\, y)_o - (x \,|\, y)_{o'}| \le |o - o'| \tag{2.1}$$

for all $x, y \in X$. The metric space X is called *(Gromov) δ-hyperbolic*, with $\delta \ge 0$, if

$$(x \mid z)_o \ge \min\{(x \mid y)_o, (y \mid z)_o\} - \delta \tag{2.2}$$

for all $x, y, z, o \in X$. Equivalently, (2.2) can be written as

$$|x-z| + |y-o| \le \max\{|x-y| + |z-o|, |y-z| + |x-o|\} + 2\delta. \tag{2.3}$$

The space X is called *(Gromov) hyperbolic* if it is δ-hyperbolic for some $\delta \ge 0$.

We assume from now on that X is δ-hyperbolic. We fix a basepoint $o \in X$ and abbreviate $(x \,|\, y) = (x \,|\, y)_o$. A sequence $\bar{x} = (x_i)$ of points in X is called a *Gromov sequence*, or a *sequence converging at infinity*, if

$$\lim_{i,j\to\infty} (x_i \,|\, x_j) = \infty. \tag{2.4}$$

The condition (2.4) is independent of the choice of the basepoint o by (2.1). It is worth observing that

$$|x_i - o| = (x_i \,|\, x_i) \to \infty$$

for a Gromov sequence (x_i). We say that two Gromov sequences $\bar{x} = (x_i)$ and $\bar{y} = (y_i)$ are *equivalent*, and write $\bar{x} \sim \bar{y}$, if $(x_i \,|\, y_i) \to \infty$ as $i \to \infty$. This defines an equivalence relation on the set of all Gromov sequences; the relation is transitive due to (2.2). Note that $\bar{x}$ is equivalent to all of its subsequences. The *Gromov boundary* of X, also called the *boundary at infinity* of X, is the set of all equivalence classes

$$\partial_G X = \{[\bar{x}] \colon \bar{x} \text{ is a Gromov sequence in } X\}.$$

The set

$$X^* = X \cup \partial_G X$$

is called the *Gromov closure* of X.

In order to fix an appropriate topology on X^* we next define the Gromov product $(a \,|\, b)$ for all $a, b \in X^*$. Following [23, p. 18] (and [1, 4.4], [64, 5.7]) we set

$$(a \,|\, b) = \inf\{\liminf_{i,j\to\infty}(x_i \,|\, y_j) \colon \bar{x} \in a, \bar{y} \in b\}$$

for $a, b \in \partial_G X$, and

$$(a \,|\, y) = (y \,|\, a) = \inf\{\liminf_{i\to\infty}(x_i \,|\, y) \colon \bar{x} \in a\}$$

for $a \in \partial_G X$ and $y \in X$. It then follows that

$$(a \,|\, c) \geq \min\{(a \,|\, b), (b \,|\, c)\} - \delta \tag{2.5}$$

for all $a, b, c \in X^*$. Note that $(a \,|\, a) = \infty$ for $a \in \partial_G X$ and that $(a \,|\, b) < \infty$ for $a \neq b$. Let $\varepsilon > 0$ and define

$$\varrho_\varepsilon(a, b) = \begin{cases} \exp\bigl(-\varepsilon(a \,|\, b)\bigr), & \text{if } a \neq b, \\ 0, & \text{if } a = b. \end{cases}$$

By (2.5), we then have

$$\varrho_\varepsilon(a, c) \leq e^{\varepsilon\delta} \max\{\varrho_\varepsilon(a, b), \varrho_\varepsilon(b, c)\}$$

for all $a, b, c \in X^*$. For $a, b \in X^*$ we define

$$d_\varepsilon(a, b) = \inf \sum_{j=1}^{k} \varrho_\varepsilon(a_j, a_{j-1}), \tag{2.6}$$

where the infimum is taken over all finite sequences $a = a_0, \ldots, a_k = b$ in X^*. If $\varepsilon > 0$ is so small that $e^{\varepsilon\delta} \leq 2$, then d_ε is a metric satisfying

$$\frac{1}{4}\varrho_\varepsilon(a, b) \leq d_\varepsilon(a, b) \leq \varrho_\varepsilon(a, b)$$

for all $a, b \in X^*$, cf. [27]. The idea of defining d_ε on all of X^* rather than just on $\partial_G X$, as it is usually done, is taken from [64]. Note however that our definition of $\varrho_\varepsilon(a, a)$ for $a \in X$ differs from that in [64, 5.13], so that we get an honest metric d_ε, not just a so-called metametric. We say that the metric space (X^*, d_ε) is *obtained from* the δ-hyperbolic space X.

Let $\mathcal{T}_d$ denote the original topology of X induced by the metric d. Each metric d_ε induces a topology $\mathcal{T}_{d_\varepsilon}$ on X^*. Observe that, for fixed $a \in X$, the function $b \mapsto \varrho_\varepsilon(a, b)$ is not continuous at a in the original topology $\mathcal{T}_d$ since $\varrho_\varepsilon(a, b) \to \exp(-\varepsilon|a - o|)$ as $|a - b| \to 0$. Consequently, the open ball $B_{d_\varepsilon}(a,r)$, $a \in X$, with respect to d_ε is the singleton $\{a\}$ for all $r \le \frac{1}{2}\exp(-\varepsilon|a-o|)$. Hence $\mathcal{T}_{d_\varepsilon}|X$ is discrete, i.e. all subsets of X are open with respect to $\mathcal{T}_{d_\varepsilon}$. Since we want to maintain the original topology of X, we choose

$$\mathcal{T}^* = \{U \in \mathcal{T}_{d_\varepsilon} : U \cap X \in \mathcal{T}_d\}$$

for the topology of X^* for the rest of the paper; see [1, 4.7] and [64, 5.29]. It is important to observe that $\mathcal{T}_{d_\varepsilon}$ and hence $\mathcal{T}^*$ is independent of the choices of o and ε.

3 Lipschitz extensions

Suppose that (X^*, d_ε) is obtained from the δ-hyperbolic space X and that $f\colon \partial_G X \to \mathbb{R}$ is L-Lipschitz. In this section we extend f first to a Lipschitz function on X^* with respect to d_ε and then to a continuous function in X^* that will be Lipschitz on X with respect to the original metric d.

First we recall the following well-known McShane-Whitney extension theorem.

Lemma 3.1. *Let (Y,d) be a metric space, $A \subset Y$, and $f\colon A \to \mathbb{R}$ an L-Lipschitz function. Then the function $F\colon Y \to \mathbb{R}$,*

$$F(x) = \inf\{f(a) + Ld(a,x)\colon a \in A\}, \tag{3.1}$$

is L-Lipschitz in Y and $F|A = f$.

Applying (3.1) to the metric space (X^*, d_ε) and to the L-Lipschitz function $f\colon \partial_G X \to \mathbb{R}$ we obtain a function $F\colon X^* \to \mathbb{R}$ which is L-Lipschitz with respect to the metric d_ε. Note that F need not be continuous in the topology $\mathcal{T}^*$ that we use for X^*. However,

$$|F(x) - F(y)| \le L\exp\bigl(-\varepsilon(x\,|\,y)\bigr) \le L\exp\bigl(\varepsilon|x-y|\bigr)\exp\bigl(-\varepsilon|x-o|\bigr) \tag{3.2}$$

for all $x, y \in X$. In particular,

$$|F(x) - F(y)| \le L. \tag{3.3}$$

Next we define, by using a Lipschitz partition of unity, another extension of $f\colon \partial_G X \to \mathbb{R}$ which will be Lipschitz in the original metric d. For that purpose let us assume that X is *locally doubling*. That is, we assume that there are positive constants R_0 and $N \in \mathbb{N}$ such that each ball $B(x,r)$ of radius $r \le R_0$ can be covered by N sets of diameter $\le r/2$. Let $P \subset X$ be an $R_0/10$-net which means that $|x-y| \ge R_0/10$ for all points $x, y \in P$, $x \ne y$, and that $X = \bigcup_{x\in P} B(x, R_0/10)$. By the doubling assumption, every point $x \in X$ belongs to at most N balls $B(y, R_0/6)$, $y \in P$, since $B(x, R_0/6)$ can be covered by N sets of diameter $\leq R_0/12$ each of which contains at most one point of P. For every $x \in P$ let $\eta_x\colon X \to \mathbb{R}$ be the function

$$\eta_x(y) = \begin{cases} 1, & |y-x| < R_0/10, \\ \frac{5R_0 - 30|y-x|}{2R_0}, & R_0/10 \leq |y-x| < R_0/6, \\ 0, & |y-x| \geq R_0/6, \end{cases}$$

and define $\varphi_x\colon X \to \mathbb{R}$ by

$$\varphi_x = \frac{\eta_x}{\sum_{y \in P} \eta_y}.$$

Each φ_x is L'-Lipschitz, with $L' = 45N/R_0$, and the collection $\{\varphi_x\colon x \in P\}$ forms a partition of unity subordinate to $\{B(x, R_0/5)\colon x \in P\}$. Then we define $f\colon X \to \mathbb{R}$ by setting

$$f(x) = \sum_{y \in P} \varphi_y(x) F(y) \tag{3.4}$$

for all $x \in X$, where $F\colon X^* \to \mathbb{R}$ is the extension of the given L-Lipschitz function $f\colon \partial_G X \to \mathbb{R}$ defined by (3.1). Above we purposely use the same notation f for the function in (3.4) as for the given function on $\partial_G X$ since we will prove that then f defines a continuous function in X^*.

For the next lemma we recall that the *pointwise upper Lipschitz constant* of a function $g\colon X \to \mathbb{R}$ at $x \in X$ is defined by

$$\operatorname{Lip} g(x) = \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|g(x) - g(y)|}{r}.$$

Here $B(x, r)$ is a ball with respect to the original metric d.

Lemma 3.2. *The function f defined by* (3.4) *is a bounded Lipschitz function with respect to the original metric d, its pointwise upper Lipschitz constant satisfies an estimate*

$$\operatorname{Lip} f(x) \leq cL \exp\bigl(-\varepsilon|x - o|\bigr) \tag{3.5}$$

for all $x \in X$, where $c = 90N^2 R_0^{-1} \exp(2\varepsilon R_0)$, and

$$\lim_{x \to a} f(x) = f(a) \tag{3.6}$$

for all $a \in \partial_G X$.

Proof. Let $x, z \in X$ and denote by P' the set of all points $y \in P$ with $\varphi_y(x) + \varphi_y(z) > 0$. Then P' contains at most $2N$ points. To estimate

$|f(x)-f(z)|$ from above we first compute

$$\begin{aligned}|f(x)-f(z)| &= \Big|\sum_{y\in P}\varphi_y(x)F(y)-\sum_{y\in P}\varphi_y(z)F(y)\Big|\\ &= \Big|\sum_{y\in P}\varphi_y(x)\big(F(y)-F(x)\big)-\sum_{y\in P}\varphi_y(z)\big(F(y)-F(x)\big)\Big|\\ &= \Big|\sum_{y\in P'}\big(\varphi_y(x)-\varphi_y(z)\big)\big(F(y)-F(x)\big)\Big|\\ &\le \sum_{y\in P'}|\varphi_y(x)-\varphi_y(z)||F(y)-F(x)|.\end{aligned}$$

Combining this with (3.3) shows that $|f(x)-f(z)|\le 2L$ and hence f is bounded. Suppose that $z\in B(x,r)$, with $0<r\le R_0$. Then $|x-y|\le 2R_0$ for all $y\in P'$ and therefore

$$\begin{aligned}|F(x)-F(y)| &\le L\exp\big(\varepsilon|x-y|\big)\exp\big(-\varepsilon|x-o|\big)\\ &\le L\exp(2\varepsilon R_0)\exp\big(-\varepsilon|x-o|\big).\end{aligned}$$

Since each φ_y is L'-Lipschitz, with $L'=45N/R_0$, and $\operatorname{card}P'\le 2N$, we obtain an estimate

$$\begin{aligned}|f(x)-f(z)| &\le cL\exp\big(-\varepsilon|x-o|\big)|x-z|\\ &\le rcL\exp\big(-\varepsilon|x-o|\big),\end{aligned}$$

where $c=90N^2R_0^{-1}\exp(2\varepsilon R_0)$. This proves both the Lipschitz condition for points $x,z\in X$, with $|x-z|<R_0$, and the estimate (3.5). Suppose then that $|x-z|\ge R_0$. Since $|f(x)-f(z)|\le 2L$, we get

$$|f(x)-f(z)|\le\frac{2L}{R_0}|x-z|.$$

Hence f is Lipschitz with respect to the metric d. To prove (3.6), it suffices to show that

$$\lim_{X\ni x\to a}f(x)=f(a)$$

for all $a\in\partial_G X$. If $a\in\partial_G X$ and $x\in X$, we get

$$\begin{aligned}|f(a)-f(x)| &\le |f(a)-F(x)|+|F(x)-f(x)|\\ &\le Ld_\varepsilon(a,x)+\sum_{y\in P}\varphi_y(x)|F(x)-F(y)|\\ &\le Ld_\varepsilon(a,x)+L\exp(\varepsilon R_0)\exp\big(-\varepsilon|x-o|\big)\\ &\to 0\end{aligned}$$

as $d_\varepsilon(a,x)\to 0$. □

4 p-harmonic functions on a metric measure space

In this section $X = (X, d, \mu)$ is a connected, locally compact, and non-compact metric measure space with a metric d and a Borel regular measure μ. We assume that the measure μ is *locally doubling*, that is, there exist positive constants C_d and R_d such that

$$0 < \mu(B(x, 2r)) \le C_d\, \mu(B(x, r)) < \infty \tag{4.1}$$

for every ball $B(x, r) \subset X$, with $0 < r \le R_d$. It is well-known that the local doubling condition (4.1) implies that X is locally doubling as a metric space. To see this, given $B(x, 2r)$, choose a maximal collection of pairwise disjoint balls $B(x_i, r/2)$ with $x_i \in B(x, 2r)$, $i = 1, \dots, N$. Then the balls $B(x_i, r)$ cover $B(x, 2r)$. It follows that

$$\begin{aligned} N\mu(B(x, 5r/2)) &\le \sum_{i=1}^{N} \mu(B(x_i, 9r/2)) \le C_d^4 \sum_{i=1}^{N} \mu(B(x_i, r/2)) \\ &\le C_d^4 \mu(B(x, 5r/2)), \end{aligned}$$

provided $r \le (4/9)R_d =: \tilde{R}_d$. Hence, $B(x, 2r)$ can be covered by $\tilde{C}_d := [C_d^4]$ balls of radius r.

Let Γ be a family of paths in X and let $1 \le p < \infty$. The *p-modulus* of Γ is defined as

$$M_p(\Gamma) = \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all Borel functions $\rho\colon X \to [0, +\infty]$ satisfying

$$\int_\gamma \rho \, ds \ge 1$$

for every locally rectifiable path $\gamma \in \Gamma$. We say that a property of paths hold for *p-almost all paths* if the family of paths for which the property fails is of zero p-modulus. A Borel function $g\colon X \to [0, +\infty]$ is said to be an *upper gradient* of a function $u\colon X \to [-\infty, +\infty]$ if, for every rectifiable path $\gamma\colon [a, b] \to X$,

$$\big|u(\gamma(b)) - u(\gamma(a))\big| \le \int_\gamma g \, ds \tag{4.2}$$

whenever both $u(\gamma(a))$ and $u(\gamma(b))$ are finite, and $\int_\gamma g\, ds = +\infty$ otherwise. We say that g is a *p-weak upper gradient* of u if (4.2) holds for p-almost all paths $\gamma\colon [a, b] \to X$. If u has a p-weak upper gradient in $L^p(X)$, then it also has a *minimal p-weak upper gradient*, denoted by $|\nabla u|$, in the sense that $|\nabla u| \le g$ μ-a.e. for every p-weak upper gradient $g \in L^p(X)$ of u; see [33, 7.16].

We also assume that the space (X, d, μ) supports a *local weak $(1, p)$-Poincaré inequality* which in this paper means that there exist constants

$C_P > 0$, $R_P > 0$, and $\tau \geq 1$ such that for all balls $B = B(x,r) \subset X$, with $0 < r \leq R_P$,

$$\fint_B |u - u_B|\, d\mu \leq C_P r \left(\fint_{\tau B} g^p \, d\mu \right)^{1/p} \tag{4.3}$$

whenever u is an integrable function in $\tau B = B(x, \tau r)$ and g is a p-weak upper gradient of u. Here

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

Let $\tilde{N}^{1,p}(X)$ be the set of all functions $u \in L^p(X)$ that have a p-weak upper gradient $g \in L^p(X)$. We equip $\tilde{N}^{1,p}(X)$ with the seminorm

$$\|u\|_{\tilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients g of u. As usual, we identify functions $u, v \in \tilde{N}^{1,p}(X)$, and write $u \sim v$, if

$$\|u - v\|_{\tilde{N}^{1,p}(X)} = 0.$$

The *Sobolev space* $N^{1,p}(X)$ is then the space $\tilde{N}^{1,p}(X)/\sim$ with the (well-defined) norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{\tilde{N}^{1,p}(X)}.$$

We say that u belongs to the *local Sobolev space* $N^{1,p}_{\mathrm{loc}}(X)$ if $u \in N^{1,p}(U)$ for every measurable $U \Subset X$. Here $U \Subset X$ means that $\bar{U}$ is compact. Every function $u \in \tilde{N}^{1,p}(X)$ is *absolutely continuous on p-almost every path* in the sense that $u \circ \gamma$ is absolutely continuous on $[0, \ell(\gamma)]$ for p-almost every rectifiable arc length parameterized path $\gamma \colon [0, \ell(\gamma)] \to X$. Furthermore, for p-almost every path γ,

$$|(u \circ \gamma)'(t)| \leq |\nabla u|(\gamma(t))$$

for almost every $t \in [0, \ell(\gamma)]$. For this and other basic properties of the Sobolev spaces $N^{1,p}(X)$ we refer to [59]. In [18] Cheeger gives an alternative definition which leads to the same Banach space if $1 < p < \infty$; see [59]. Furthermore, Cheeger [18] proved the deep result that $N^{1,p}(X)$ is reflexive if $1 < p < \infty$.

The *(Sobolev) p-capacity* of a set $E \subset X$ is defined by

$$C_p(E) = \inf \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$, with $u|E \geq 1$. For a subset $\Omega \subset X$ let $N^{1,p}_0(\Omega)$ be the space of all elements in $N^{1,p}(X)$ whose representatives u satisfy

$$C_p\big(\{x \in X \setminus \Omega \colon u(x) \neq 0\}\big) = 0.$$

The space $N_0^{1,p}(\Omega)$ equipped with the norm

$$\|u\|_{N_0^{1,p}(\Omega)} = \|u\|_{N^{1,p}(X)}$$

is called the *Sobolev space with zero boundary values*. It is worth observing that $N^{1,p}(X) = N_0^{1,p}(X)$, and therefore $N_0^{1,p}(\Omega)$, for a subset $\Omega \subset X$, depends on the ambient space X. In the literature the Sobolev spaces as above are usually called *Newtonian spaces*.

The local doubling condition (4.1) and the local weak $(1,p)$-Poincaré inequality (4.3) imply, by [34, Theorem 5.1], that there are constants $c > 0$, $R_S > 0$, and $\lambda > 1$ such that a *local Sobolev-Poincaré inequality*

$$\left(\fint_B |u - u_B|^{\lambda p}\, d\mu\right)^{1/\lambda p} \le cr \left(\fint_{5\tau B} g^p\, d\mu\right)^{1/p} \tag{4.4}$$

holds for all balls $B = B(x,r) \subset X$, with $0 < r \le R_S$, whenever u is an integrable function in $5\tau B = B(x, 5\tau r)$ and g is a p-weak upper gradient of u. Furthermore, if $u \in N_0^{1,p}\big(B(x,r)\big)$, with $0 < r \le R_S$, then a *local Sobolev inequality*

$$\left(\fint_{B(x,r)} |u|^{\lambda p}\, d\mu\right)^{1/\lambda p} \le C_S r \left(\fint_{B(x,r)} g^p\, d\mu\right)^{1/p} \tag{4.5}$$

holds.

Let $1 < p < \infty$. Suppose that $\Omega \subset X$ is open and $\vartheta \in N^{1,p}(\Omega)$. A function $u \in N^{1,p}(\Omega)$ is called a *p-minimizer in Ω with boundary values ϑ* if $u - \vartheta \in N_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^p\, d\mu \le \int_\Omega |\nabla v|^p\, d\mu \tag{4.6}$$

for every $v \in N^{1,p}(\Omega)$, with $v - \vartheta \in N_0^{1,p}(\Omega)$. Recall that $|\nabla u|$ and $|\nabla v|$ are the minimal p-weak upper gradients of u and v in Ω, respectively. Let then $U \subset X$ be an open set. A function $u \in N_{\text{loc}}^{1,p}(U)$ is called a *p-minimizer in U* if (4.6) holds for every open set $\Omega \Subset U$ and for all functions $v \in N_{\text{loc}}^{1,p}(U)$, with $u - v \in N_0^{1,p}(\Omega)$. Furthermore, a function u is called *p-harmonic in U* if it is a continuous p-minimizer in U. It is proved in [51] and [54] that every p-minimizer in U can be redefined in a set of measure zero so that it becomes locally Hölder continuous in U.

For later purposes we record the following result on the existence and uniqueness of p-harmonic functions with prescribed boundary values in relatively compact open sets, cf. [18, 7.12, 7.14], [49, 3.2], [60, 5.6].

Lemma 4.1. *Suppose that $\Omega \Subset X$ is open and that $\vartheta \in N^{1,p}(\Omega)$ is bounded. Then there exists a unique p-harmonic function u in Ω, with $u - \vartheta \in N_0^{1,p}(\Omega)$.*

Since the assumptions in the references above slightly differ from ours, we sketch the proof.

Proof. Choose a sequence of functions $u_i \in N^{1,p}(\Omega)$ such that $u_i - \vartheta \in N_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u_i|^p \, d\mu \to I := \inf_v \int_\Omega |\nabla v|^p \, d\mu,$$

where the infimum is taken over all functions $v \in N^{1,p}(\Omega)$, with $v - \vartheta \in N_0^{1,p}(\Omega)$. Since ϑ is among such functions v, the sequence $(|\nabla u_i|)$ is bounded in $L^p(\Omega)$. By truncation, we may assume that

$$\inf_\Omega \vartheta \le u_i \le \sup_\Omega \vartheta$$

in Ω for all i. Thus

$$\sup_i \|u_i\|_{N^{1,p}(\Omega)} < \infty$$

since $\mu(\Omega) < \infty$ and ϑ is bounded. Consequently, we have

$$\sup_i \|u_i - \vartheta\|_{N_0^{1,p}(\Omega)} \le \sup_i \|u_i\|_{N^{1,p}(\Omega)} + \|\vartheta\|_{N^{1,p}(\Omega)} < \infty.$$

Hence there exist a subsequence (u_{i_j}) and a function $u \in N^{1,p}(\Omega)$ such that $u - \vartheta \in N_0^{1,p}(\Omega)$, $u_{i_j} \to u$ weakly in $L^p(\Omega)$, and

$$\int_\Omega |\nabla u|^p \, d\mu \le \liminf_{j\to\infty} \int_\Omega |\nabla u_{i_j}|^p \, d\mu \le I;$$

see [44, 3.1] and [49, 2.3 (vi)]. After redefining u in a set of measure zero, we have a p-harmonic function as desired.

To prove the uniqueness, suppose that u_1 and u_2 are p-harmonic in Ω, with $u_i - \vartheta \in N_0^{1,p}(\Omega)$, $i = 1, 2$. By [18, 7.14], $|\nabla(u_1 - u_2)| = 0$ μ-a.e. in Ω. The local Poincaré inequality (4.3) then implies that $u_1 - u_2$ is locally constant and hence constant in each component of Ω. Since $u_1 - u_2 \in N_0^{1,p}(\Omega)$, we have $u_1 = u_2$. □

Remark 4.2. Let (X, d, μ) be a metric measure space, where X is a Riemannian manifold, d is the Riemannian distance, and μ is the Riemannian measure. If u is a smooth real-valued function on X, then $|\nabla u|$, the norm of the gradient of u, is the minimal p-weak upper gradient of u for all $p \ge 1$. Thus the spaces $N^{1,p}(X)$, $N^{1,p}_{\mathrm{loc}}(X)$, and $N_0^{1,p}(\Omega)$, where $\Omega \Subset X$, coincide with the corresponding usual Sobolev spaces $W^{1,p}(X)$, $W^{1,p}_{\mathrm{loc}}(X)$, and $W_0^{1,p}(\Omega)$. Furthermore, p-harmonic functions in an open set $U \subset X$ defined as continuous p-minimizers in U are, equivalently, continuous (weak) solutions of the p-Laplace equation

$$-\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big) = 0,$$

that is,

$$\int_U \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \, d\mu = 0$$

for every $\varphi \in C_0^\infty(U)$.

5 Main estimates

We assume throughout this section that the assumptions introduced in Section 4 hold for (X, d, μ). More precisely, X is a connected, locally compact, and non-compact metric space equipped with a Borel regular measure μ. Furthermore, we assume that the measure μ satisfies the local doubling condition (4.1) and that the local weak $(1, p)$-Poincaré inequality (4.3) hold on X, with fixed $1 < p < \infty$. We will prove two lemmata that are crucial in our proof of the main result.

Lemma 5.1. *Suppose that a global (p, p)-Sobolev inequality*

$$\|u\|_p \le C \,\||\nabla u|\|_p \tag{5.1}$$

holds for all compactly supported functions $u \in N^{1,p}(X)$. Let $\Omega \Subset X$ be an open set and $f \in N^{1,p}(\Omega)$ a bounded continuous function. Then for every $q \ge p$ there exists a constant $c = c(p, q, C)$ such that

$$\|u - f\|_{L^q(\Omega)} \le c\,\||\nabla f|\|_{L^q(\Omega)}, \tag{5.2}$$

where $u \in N^{1,p}(\Omega)$ is the unique p-harmonic function in Ω with $u - f \in N_0^{1,p}(\Omega)$.

Proof. To prove (5.2) we combine ideas from [19] and [50]. Recall that u is absolutely continuous on p-almost every path in Ω and $u - f$ is absolutely continuous on p-almost every path in X. Consider first the set $\Omega^+ = \{x \in \Omega \colon u(x) > f(x)\}$ and define $w \colon \Omega^+ \to \mathbb{R}$ by

$$w = u - (\varepsilon(u - f))^{q-p+1},$$

where $q \ge p$ and $\varepsilon > 0$ is a constant such that

$$0 \le A(x) := (q - p + 1)\varepsilon^{q-p+1}\big(u(x) - f(x)\big)^{q-p} \le 1 \tag{5.3}$$

for all $x \in \Omega^+$. Let $\gamma \colon [0, \ell(\gamma)] \to \Omega^+$ be an arc length parameterized path on which u and $u - f$ are absolutely continuous. Then $w \circ \gamma \colon [0, \ell(\gamma)] \to \mathbb{R}$ is absolutely continuous, and therefore

$$(w \circ \gamma)'(t) = \big(1 - A(\gamma(t))\big)(u \circ \gamma)'(t) + A(\gamma(t))(f \circ \gamma)'(t)$$

for a.e. $t \in [0, \ell(\gamma)]$. Since $0 \le A \le 1$ in Ω^+, we obtain

$$|(w \circ \gamma)'(t)| \le ((1 - A)|\nabla u|)(\gamma(t)) + (A|\nabla f|)(\gamma(t))$$

for a.e. $t \in [0, \ell(\gamma)]$. Since this estimate holds for p-almost every path in Ω^+, we get an upper bound

$$|\nabla w| \le (1 - A)|\nabla u| + A|\nabla f|$$

μ-a.e. in Ω^+ for the minimal p-weak upper gradient of w. The convexity of $s \mapsto s^p$ and (5.3) then imply that

$$|\nabla w|^p \le (1-A)|\nabla u|^p + A|\nabla f|^p$$

μ-a.e. in Ω^+. Since the functions u and w have the same boundary values in Ω^+ and u is a p-minimizer in Ω^+, we obtain

$$\begin{aligned}\int_{\Omega^+} |\nabla u|^p \, d\mu &\le \int_{\Omega^+} |\nabla w|^p \, d\mu \\ &\le \int_{\Omega^+} |\nabla u|^p \, d\mu - \int_{\Omega^+} A|\nabla u|^p \, d\mu + \int_{\Omega^+} A|\nabla f|^p \, d\mu\end{aligned}$$

which implies

$$\int_{\Omega^+} (u-f)^{q-p}|\nabla u|^p \, d\mu \le \int_{\Omega^+} (u-f)^{q-p}|\nabla f|^p \, d\mu. \tag{5.4}$$

The minimal p-weak upper gradients of u, f, and $u-f$ satisfy estimates $|\nabla(u-f)| \le |\nabla u| + |\nabla f|$ and

$$|\nabla(u-f)|^p \le 2^{p-1}\big(|\nabla u|^p + |\nabla f|^p\big)$$

μ-a.e. in Ω^+. Combining the latter with (5.4) yields

$$\int_{\Omega^+} (u-f)^{q-p}|\nabla(u-f)|^p \, d\mu \le 2^p \int_{\Omega^+} (u-f)^{q-p}|\nabla f|^p \, d\mu. \tag{5.5}$$

Redefining $u-f = 0$ in $X \setminus \Omega^+$ and using (5.1), (5.5), and Hölder's inequality we get

$$\begin{aligned}\int_{\Omega^+} (u-f)^q \, d\mu &= \int_{\Omega^+} |(u-f)^{q/p}|^p \, d\mu \\ &\le c \int_{\Omega^+} |\nabla(u-f)^{q/p}|^p \, d\mu \\ &\le c \int_{\Omega^+} (u-f)^{q-p}|\nabla(u-f)|^p \, d\mu \\ &\le c \int_{\Omega^+} (u-f)^{q-p}|\nabla f|^p \, d\mu \\ &\le c \left(\int_{\Omega^+} (u-f)^q \, d\mu\right)^{(q-p)/q} \left(\int_{\Omega^+} |\nabla f|^q \, d\mu\right)^{p/q}.\end{aligned}$$

Hence we obtain

$$\int_{\Omega^+} (u-f)^q \, d\mu \le c \int_{\Omega^+} |\nabla f|^q \, d\mu.$$

Treating similarly the set $\{x \in \Omega \colon u(x) < f(x)\}$ proves the claim. □

In the next lemma, C_d and R_d are the constants in the local doubling condition (4.1) and $C_S > 0$, $R_S > 0$, and $\lambda > 1$ are the constants in the local Sobolev inequality (4.5).

Lemma 5.2. *Let $1 < p < \infty$ and $Q \geq p$. Let $f \in N^{1,p}_{\mathrm{loc}}(X)$ be a bounded continuous function such that its minimal p-weak upper gradient $|\nabla f|$ is bounded. Suppose that $\Omega \Subset X$ and that u is a bounded p-harmonic function in Ω, with $u - f \in N^{1,p}_0(\Omega)$, and $u - f = 0$ in $X \setminus \Omega$. Then there exists a constant $d \in (0,1)$ such that for every $x \in X$ and $0 < R \leq \min\{R_d, R_S/2\}$ we have*

$$\sup_{B(x,R)} |u-f|^Q \leq C \left(\fint_{B(x,2R)} |u-f|^Q \, d\mu \right)^d , \tag{5.6}$$

where C depends on $p, Q, C_d, R_d, \lambda, C_S, R_S, \sup_X |u-f|$, and $\sup_X |\nabla f|$, but is independent of $x \in X$.

Proof. We start with a similar technique as in the proof of Lemma 5.1. Consider again the set $\Omega^+ = \{x \in X : u(x) > f(x)\}$ and define $w \colon \Omega^+ \to \mathbb{R}$ by

$$w = u - \eta^p \left(\varepsilon(u-f)\right)^{q-p+1},$$

where $q \geq Q$, $\varepsilon > 0$, and η is a Lipschitz function that will be chosen later. Denote

$$A = (q-p+1)\varepsilon^{q-p+1}\eta^p(u-f)^{q-p}$$

and choose $\varepsilon > 0$ so small that $0 \leq A \leq 1$ in Ω^+. As in the proof of Lemma 5.1 we conclude that

$$|\nabla w| \leq (1-A)|\nabla u| + A\left(|\nabla f| + \frac{p(u-f)}{(q-p+1)\eta}|\nabla \eta| \right)$$

and, furthermore,

$$|\nabla w|^p \leq (1-A)|\nabla u|^p + 2^{p-1}A\left(|\nabla f|^p + \left(\frac{p(u-f)}{(q-p+1)\eta}\right)^p |\nabla \eta|^p \right)$$

μ-a.e. in Ω^+. Since $u - f \in N^{1,p}_0(\Omega)$ and u is p-harmonic in Ω, we have

$$\begin{aligned}
\int_{\Omega^+} |\nabla u|^p \, d\mu &\leq \int_{\Omega^+} |\nabla w|^p \, d\mu \\
&\leq \int_{\Omega^+} |\nabla u|^p \, d\mu - \int_{\Omega^+} A|\nabla u|^p \, d\mu + 2^{p-1}\int_{\Omega^+} A|\nabla f|^p \, d\mu \\
&\quad + 2^{p-1}\left(\frac{p}{q-p+1}\right)^p \int_{\Omega^+} \frac{A(u-f)^p}{\eta^p}|\nabla \eta|^p \, d\mu,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_{\Omega^+} \eta^p (u-f)^{q-p}|\nabla u|^p \, d\mu \\
&\quad \leq 2^{p-1}\int_{\Omega^+} \eta^p (u-f)^{q-p}|\nabla f|^p \, d\mu + \frac{2^{p-1}p^p}{(q-p+1)^p}\int_{\Omega^+} (u-f)^q |\nabla \eta|^p \, d\mu.
\end{aligned}$$

Combining this with the estimate

$$|\nabla(u-f)^{q/p}|^p \le 2^{p-1}\frac{q^p}{p^p}(u-f)^{q-p}(|\nabla u|^p + |\nabla f|^p),$$

we conclude that

$$\begin{aligned}\int_{\Omega^+} &\eta^p |\nabla(u-f)^{q/p}|^p \, d\mu \\ &\le 2^{p-1}q^p(1+2^{p-1})p^{-p}\int_{\Omega^+}\eta^p(u-f)^{q-p}|\nabla f|^p \, d\mu \\ &\quad + \frac{2^{2p-2}q^p}{(q-p+1)^p}\int_{\Omega^+}(u-f)^q|\nabla\eta|^p \, d\mu.\end{aligned}$$

Arguing similarly for the set $\{x \in X : u(x) < f(x)\}$ and using the fact that

$$\frac{q}{q-p+1} \le p,$$

which follows from $q \ge p$, we obtain a Caccioppoli-type estimate

$$\begin{aligned}\int_X &\eta^p \big|\nabla|u-f|^{q/p}\big|^p \, d\mu \\ &\le cq^p\int_X \eta^p|u-f|^{q-p}|\nabla f|^p \, d\mu + c\int_X |u-f|^q|\nabla\eta|^p \, d\mu,\end{aligned} \tag{5.7}$$

where $c = c(p)$.

Fix $x \in X$ and $0 < R \le \min\{R_d, R_S/2\}$. For each $i = 0, 1, \dots$, we write $t_i = R + \lambda^{-i}R$, $B_i = B(x, t_i)$, and $q_i = Q\lambda^i$. Let $\eta_i : X \to \mathbb{R}$ be the Lipschitz function

$$\eta_i(y) = \begin{cases} 1, & |y-x| < t_{i+1}, \\ \frac{t_i - |y-x|}{t_i - t_{i+1}}, & t_{i+1} \le |y-x| < t_i, \\ 0, & |y-x| \ge t_i. \end{cases}$$

Then $\{y \in X : \eta_i(y) > 0\} = B_i \subset B(x, R_S)$ and the minimal p-weak upper gradient of η_i satisfies

$$|\nabla\eta_i| \le \frac{1}{t_i - t_{i+1}} = \frac{\lambda^{i+1}}{(\lambda-1)R}. \tag{5.8}$$

Applying the local Sobolev inequality (4.5) to $\eta_i|u-f|^{q_i/p} \in N_0^{1,p}(B_i)$ and using the local doubling condition (4.1) and the Caccioppoli-type estimate (5.7) we obtain

$$\begin{aligned}
&\Big(\fint_{B_{i+1}} |u-f|^{q_i\lambda}\,d\mu\Big)^{1/\lambda} \\
&\qquad \le c\Big(\fint_{B_i} \big|\eta_i|u-f|^{q_i/p}\big|^{p\lambda}\,d\mu\Big)^{1/\lambda} \\
&\qquad \le c t_i^p \fint_{B_i} \big|\nabla(\eta_i|u-f|^{q_i/p})\big|^p\,d\mu \\
&\qquad \le cR^p\Big(\fint_{B_i} \eta_i^p\big|\nabla|u-f|^{q_i/p}\big|^p\,d\mu + \fint_{B_i} |u-f|^{q_i}|\nabla\eta_i|^p\,d\mu\Big) \\
&\qquad \le cR^p\Big(q_i^p\fint_{B_i} \eta_i^p|u-f|^{q_i-p}|\nabla f|^p\,d\mu + \fint_{B_i} |u-f|^{q_i}|\nabla\eta_i|^p\,d\mu\Big).
\end{aligned}$$

Since $0 \le \eta_i \le 1$, we get by Hölder's inequality

$$\begin{aligned}
\fint_{B_i} \eta_i^p|u-f|^{q_i-p}|\nabla f|^p\,d\mu &\le \left(\fint_{B_i} |u-f|^{q_i}\,d\mu\right)^{1-p/q_i} \left(\fint_{B_i} |\nabla f|^{q_i}\,d\mu\right)^{p/q_i} \\
&\le M^p\left(\fint_{B_i} |u-f|^{q_i}\,d\mu\right)^{1-p/q_i},
\end{aligned}$$

where $M = \sup_X|\nabla f|$. Without loss of generality we may assume that

$$\sup_X|u-f| \le 1. \tag{5.9}$$

Inserting $q_i = Q\lambda^i$ and taking into account (5.8) and (5.9) we obtain

$$\begin{aligned}
&\Big(\fint_{B_{i+1}} |u-f|^{q_i\lambda}\,d\mu\Big)^{1/\lambda} \qquad (5.10)\\
&\le cR^p\left(M^pQ^p\lambda^{ip}\left(\fint_{B_i} |u-f|^{q_i}\,d\mu\right)^{1-p/q_i} + \frac{\lambda^{(i+1)p}}{(\lambda-1)^pR^p}\fint_{B_i} |u-f|^{q_i}\,d\mu\right) \\
&\le c(\lambda^p)^i\left(\fint_{B_i} |u-f|^{q_i}\,d\mu\right)^{1-p/q_i},
\end{aligned}$$

where $c = c(p,\lambda,Q,C_d,C_S,R_d,R_S,\sup|\nabla f|,\sup|u-f|)$. If we write

$$I_i = \left(\fint_{B_i} |u-f|^{Q\lambda^i}\,d\mu\right)^{1/\lambda^i},$$

then (5.10) reads as

$$I_{i+1} \le c^{1/\lambda^i}(\lambda^p)^{i/\lambda^i} I_i^{(Q\lambda^i-p)/Q\lambda^i}.$$

Applying this recursively we get

$$I_i \le c^{a_i}(\lambda^p)^{b_i} I_0^{c_i},$$

where $a_0 = b_0 = 0,\ c_0 = 1$, and

$$\begin{aligned} a_{i+1} &= \left(\frac{Q\lambda^i - p}{Q\lambda^i}\right) a_i + \frac{1}{\lambda^i}, \\ b_{i+1} &= \left(\frac{Q\lambda^i - p}{Q\lambda^i}\right) b_i + \frac{i}{\lambda^i}, \\ c_{i+1} &= \left(\frac{Q\lambda^i - p}{Q\lambda^i}\right) c_i. \end{aligned}$$

Clearly, $a_i \leq \lambda/(\lambda-1),\ b_i \leq b := \sum_{j=0}^{\infty} \frac{j}{\lambda^j} < \infty$, and

$$d := \lim_{i\to\infty} c_i = \prod_{i=0}^{\infty} \frac{Q\lambda^i - p}{Q\lambda^i} \in (0,1).$$

Now we can finally conclude that

$$\sup_{B(x,R)} |u-f|^Q \leq \limsup_{i\to\infty} I_i \leq c^{\lambda/(\lambda-1)} \lambda^{pb} \left(⨍_{B(x,2R)} |u-f|^Q \, d\mu \right)^d .$$

□

6 Solving the Dirichlet problem at infinity

In this section we combine the assumptions on X from previous sections. Thus X is a connected, locally compact, and non-compact δ-hyperbolic metric space equipped with a non-trivial Borel regular measure μ supported on all of X. Furthermore, we assume that the local doubling condition (4.1), the local weak $(1,p)$-Poincaré inequality (4.3), and the global (p,p)-Sobolev inequality (5.1) hold on X, the last two with a fixed $1 < p < \infty$.

We employ two additional assumptions on measures of balls. The first one is a global volume growth condition

$$\mu(B(o,R)) \leq Ce^{\beta R} \tag{6.1}$$

for all $R > 0$, where $\beta > 0$ and $C > 0$ are constants and $o \in X$ is fixed. The second new assumption is a uniform positive lower bound for measures of balls with fixed small radius. More precisely, we assume that there exist constants $C_v > 0$ and $0 < R_v \leq \min\{2R_d, R_S\}$ such that

$$\mu(B(x,R_v)) \geq C_v \tag{6.2}$$

for all $x \in X$.

Remark 6.1. Recall that a metric space is called *proper* if all closed balls and hence all bounded closed sets are compact. It is proved in [15] that a proper metric measure space (X, d, μ) that supports a global (p, p)-Sobolev inequality (5.1) and has infinite diameter necessarily has at least an exponential volume growth. That is, there exists a constant $c > 0$ such that

$$\mu\big(B(o, r)\big) \geq ce^{pr/C}$$

for all sufficient large r. Here C is the constant in (5.1) and $o \in X$ is fixed.

The main result of this section and the whole paper is Theorem 1.1. For the reader's convenience, we recall it from the introduction.

Theorem 6.2. *Let $f \colon \partial_G X \to \mathbb{R}$ be a bounded continuous function. Then there exists a continuous function $u \colon X^* \to \mathbb{R}$ which is p-harmonic in X and equal to f in $\partial_G X$.*

Suppose that the metric space (X^*, d_ε) is obtained from the δ-hyperbolic space X, cf. Section 2. We will solve the Dirichlet problem at infinity first for a Lipschitz function $f \colon \partial_G X \to \mathbb{R}$. As pointed out at the beginning of Section 4, the local doubling property of μ implies that X is locally doubling as a metric space. Hence we may apply (3.4) and Lemma 3.2 to define an extension of f and to estimate the pointwise upper Lipschitz constant $\operatorname{Lip} f$ of the extended function f. By [46, Lemma 4.1.2], $\operatorname{Lip} f$ is a Borel function. Furthermore, $\operatorname{Lip} f$ is an upper gradient of f by [18, Prop. 1.11]. Combining the estimate (3.5) for $\operatorname{Lip} f$ with the volume growth condition (6.1) then immediately gives the following lemma.

Lemma 6.3. *Let $f \colon \partial_G X \to \mathbb{R}$ be a Lipschitz function and let $f \colon X \to \mathbb{R}$ be defined by* (3.4)*. Then $f \in L^\infty(X)$ and $Lip f \in L^Q(X)$ for all $Q > \beta/\varepsilon$. In particular, for every $1 \leq p < \infty$, the minimal p-weak upper gradient $|\nabla f|$ of f belongs to $L^Q(X)$ for all $Q > \beta/\varepsilon$.*

Lemma 6.4. *Let $f \colon \partial_G X \to \mathbb{R}$ be a Lipschitz function. Then there exists a continuous function $u \colon X^* \to \mathbb{R}$ such that $u|\partial_G X = f$ and $u|X$ is p-harmonic in X.*

Proof. Let $F \colon X^* \to \mathbb{R}$ and $f \colon X \to \mathbb{R}$ be defined by (3.1) and (3.4), respectively. Since X is connected and locally compact, it is σ-compact; see e.g. [62]. Hence X is second countable. By using the local weak $(1, p)$-Poincaré inequality (4.3) and the assumption that every open set in X has positive measure we see that X is locally connected. Hence by the (global) connectivity, second countability, and local compactness of X, there exists an exhaustion of X by domains $\Omega_j \Subset X$. For each $j \in \mathbb{N}$, let u_j be the unique p-harmonic function in Ω_j such that $u_j - f \in N_0^{1,p}(\Omega_j)$, cf. Lemma 4.1. We set $u_j = f$ in $X \setminus \Omega_j$. The boundedness of f implies that the sequence (u_j) is uniformly bounded. Applying the Hölder continuity estimate [51, 5.2] (and [54]) we see that (u_j) is equicontinuous. Using the Ascoli-Arzela theorem, we obtain a subsequence, still denoted by (u_j), that converges locally uniformly to a function u which is p-harmonic in X by [61]. By Lemma 5.1 and Lemma 6.3,

$$\|u_j - f\|_{L^Q(X)} = \|u_j - f\|_{L^Q(\Omega_j)} \le c\,\||\nabla f|\|_{L^Q(\Omega_j)} \le c\,\||\nabla f|\|_{L^Q(X)} < \infty$$

for all $Q > \max\{p, \beta/\varepsilon\}$. Hence $u - f \in L^Q(X)$ by Fatou's lemma. Inserting the lower bound (6.2) into the estimate (5.6) and using Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned}\sup_{B(x,r)} |u-f|^Q &= \lim_{j\to\infty} \sup_{B(x,r)} |u_j - f|^Q \\ &\le \lim_{j\to\infty} C\left(\int_{B(x,R_v)} |u_j - f|^Q \, d\mu\right)^d \\ &= C\left(\int_{B(x,R_v)} |u - f|^Q \, d\mu\right)^d\end{aligned}$$

for all $x \in X$ and $0 < r \le R_v/2$. Since $u - f \in L^Q(X)$, we conclude that

$$\sup_{B(x,r)} |u-f|^Q \to 0$$

as $X \ni x \to a \in \partial_G X$. On the other hand, f is continuous in X^*, and therefore

$$|u(x) - f(a)| \le |u(x) - f(x)| + |f(x) - f(a)| \to 0$$

as $X \ni x \to a \in \partial_G X$. Setting $u|\partial_G X = f$ we obtain the result. □

Suppose then that $f\colon \partial_G X \to \mathbb{R}$ is a bounded continuous function. We will solve the Dirichlet problem with boundary values f by approximating f pointwise by Lipschitz functions and then applying the previous lemma to these approximating functions. An appropriate approximation is provided by the following lemma; see [35, Exercise 6.10], [2, Lemma 1.3.1].

Lemma 6.5. *Let (Y, d) be a metric space and let $f\colon Y \to [a, b]$ be a bounded continuous function. Then, for each $i \in \mathbb{N}$, the functions*

$$f_i(x) = \inf\{f(y) + i\, d(x,y) \colon y \in Y\}, \quad x \in Y, \tag{6.3}$$

and

$$g_i(x) = \sup\{f(y) - i\, d(x,y) \colon y \in Y\}, \quad x \in Y, \tag{6.4}$$

are i-Lipschitz. Furthermore, they satisfy

$$a \le f_i(x) \le f_{i+1}(x) \le f(x) \le g_{i+1}(x) \le g_i(x) \le b \tag{6.5}$$

and

$$\lim_{i\to\infty} f_i(x) = \lim_{i\to\infty} g_i(x) = f(x) \tag{6.6}$$

for all $x \in Y$*.*

Proof of Theorem 1.1. We apply Lemma 6.5 to a bounded continuous function $f\colon \partial_G X \to \mathbb{R}$. Let $f_i\colon \partial_G X \to \mathbb{R}$ and $g_i\colon \partial_G X \to \mathbb{R}$ be i-Lipschitz functions given by (6.3) and (6.4), respectively. Furthermore, let $F_i\colon X^* \to \mathbb{R}$ and $G_i\colon X^* \to \mathbb{R}$ be their i-Lipschitz extensions to the metric space (X^*, d_ε) given by (3.1). Finally, define f_i and g_i in X by setting

$$\begin{aligned} f_i(x) &= \sum_{y\in P} \varphi_y(x) F_i(y), \\ g_i(x) &= \sum_{y\in P} \varphi_y(x) G_i(y) \end{aligned}$$

for $x \in X$ as in (3.4). It is then clear that

$$f_i(x) \le f_{i+1}(x) \le g_{i+1}(x) \le g_i(x)$$

for all $i \in \mathbb{N}$ and $x \in X$. For each $i \in \mathbb{N}$, let u_i and v_i be continuous functions in X^* that are p-harmonic in X, with boundary values $u_i = f_i$ and $v_i = g_i$ in $\partial_G X$. We suppose that u_i and v_i are constructed as in the proof of Lemma 6.4. Then

$$u_i(x) \le u_{i+1}(x) \le v_{i+1}(x) \le v_i(x) \tag{6.7}$$

for all $x \in X^*$ and $i \in \mathbb{N}$. As in the proof of Lemma 6.4, we conclude that the increasing sequence (u_i) converges locally uniformly in X and the limit function, denoted by u, is p-harmonic in X. Furthermore,

$$u_i(x) \le u(x) \le v_i(x)$$

for all $x \in X^*$ and $i \in \mathbb{N}$. Since u_i and v_i are continuous in X^* and

$$\lim_{i\to\infty} u_i(a) = \lim_{i\to\infty} v_i(a) = f(a)$$

for $a \in \partial_G X$, we have

$$\lim_{\substack{x \to a \\ x\in X}} u(x) = f(a)$$

for all $a \in \partial_G X$. Setting $u = f$ in $\partial_G X$ then proves Theorem 1.1. □

7 Uniqueness of solutions

In this section we study the uniqueness (and non-uniqueness) of solutions to the Dirichlet problem at infinity. It is easy to see that, in general, p-harmonic functions in X with given boundary values in $\partial_G X$ need not be unique.

As an example of non-uniqueness, consider $X = \mathbb{H}^n \setminus \{x_0\}$, where $\mathbb{H}^n$ is the hyperbolic n-space and $x_0 \in \mathbb{H}^n$. Then X, equipped with the induced Riemannian structure of $\mathbb{H}^n$, is Gromov hyperbolic and satisfies all the assumptions that we employed in Theorem 1.1. On the other hand, $\partial_G X = \partial_G \mathbb{H}^n$ since the removal of x_0 does not add any new points to the Gromov boundary. It follows that there are many p-harmonic functions in X with zero boundary values on $\partial_G X$ showing the non-uniqueness.

As another example, let $X = (0,1) \times \mathbb{R}$ be equipped with the induced standard metric measure space structure of $\mathbb{R}^2$. Then X is hyperbolic with $\partial_G X$ consisting of two points which we denote by $-\infty$ and $+\infty$. Moreover, X satisfies all the assumptions of Theorem 1.1. We point out that also the (p,p)-Sobolev inequality holds for compactly supported Sobolev functions in X. Theorem 1.1 then gives a p-harmonic function in X with prescribed limits, say a and b, at $\pm\infty$. However, we can find arbitrarily many p-harmonic functions in X with the same limits at $\pm\infty$ by giving (continuous) boundary values on $\{1\} \times \mathbb{R}$ and $\{0\} \times \mathbb{R}$ with the limits a and b at $\pm\infty$ and solving the Dirichlet problem in the usual way.

In the examples above the non-uniqueness is possible because X^* is not sequentially compact.

Theorem 7.1. *Suppose that a δ-hyperbolic space X satisfies the assumptions needed to define the asymptotic Dirichlet problem for p-harmonic functions. Suppose also that X^* is sequentially compact and $\partial_G X \neq \emptyset$. Then the solutions to the asymptotic Dirichlet problem are always unique (whenever they exist). That is, if $f : \partial_G X \to \mathbb{R}$ is continuous and $u, v : X^* \to \mathbb{R}$ are continuous functions such that $u|X$ and $v|X$ are p-harmonic and $u|\partial_G X = f = v|\partial_G X$, then $u = v$.*

Proof. First we observe that X is proper. Indeed, if (x_i) is a sequence in a closed ball $\bar{B}(x,r) \subset X$, there exists a subsequence converging to a point $x \in X^*$ since X^* is sequentially compact. Then necessarily $x \in X$ since $|x_i - o| \le r$ for all i. Thus $\bar{B}(x,r)$ is sequentially compact and hence compact because the topology of X is metrizable.

If $u \neq v$ in X, we may assume that $u(x_0) > v(x_0)$ for some $x_0 \in X$. Let K be the set

$$K = \{x \in X^* : u(x) > v(x) + \tfrac{1}{2}\big(u(x_0) - v(x_0)\big)\}.$$

Then K is a non-empty open subset of X since u and v are continuous, $u|\partial_G X = v|\partial_G X$, and $x_0 \in K$. We claim that $\bar{K}$ is compact. Since $\bar{K}$ is closed and X is proper, it suffices to prove that K is bounded. Suppose on the contrary that K is unbounded. Then there exists a sequence (y_i) in K such that $|y_i - o| \to \infty$. Since X^* is sequentially compact, (y_i) has a subsequence that converges to a point $y \in X^*$. Then $y \in \partial_G X$ since $|y_i - o| \to \infty$. The continuity of u and v then implies that $u(y) \geq v(y) + \frac{1}{2}\big(u(x_0) - v(x_0)\big)$ which is impossible because $u|\partial_G X = v|\partial_G X$. Thus $\bar{K}$ is compact. On the other hand, $u = v + \frac{1}{2}\big(u(x_0) - v(x_0)\big)$ in ∂K. Hence $u = v + \frac{1}{2}\big(u(x_0) - v(x_0)\big)$ in K by the uniqueness of p-harmonic functions in $K \Subset X$ with the same boundary values. We get a contradiction since $x_0 \in K$. Thus $u = v$ in X^*. □

Next we study conditions on a Gromov hyperbolic space X that imply the compactness of X^*. Let Y be a metric space and $c \geq 0$. We say that a sequence of points $y_0, y_1, \ldots, y_k$ in Y is a *c-roughly geodesic sequence* from y_0 to y_k if

$$|i - j| - c \leq |y_i - y_j| \leq |i - j| + c$$

for all $i, j \in \{0, 1, \ldots, k\}$. Furthermore, Y is called *c-roughly starlike with respect to* $y_0 \in Y$ if there exists a c-roughly geodesic sequence from y_0 to any point $y \in Y$. Note that if Y is c-roughly starlike with respect to $y_0 \in Y$, then it is c'-roughly starlike with respect to any point $y_0' \in Y$, with $c' = c + |y_0 - y_0'|$.

We will apply the following lemma which was pointed out to us by J. Väisälä.

Lemma 7.2. *Let X be a proper metric space which is also an embedded subspace of a sequentially compact topological space Y such that $Z = Y \setminus X$ is compact. Then Y is compact.*

Proof. Let $\mathcal{P}$ be an open cover of Y. Since Z is compact, $\mathcal{P}$ has a finite subcollection $\mathcal{Q}$ that covers Z. Let $U = \bigcup_{V \in \mathcal{Q}} V$ and $F = Y \setminus U = X \setminus U$. If F is bounded, it is compact since X is proper. Hence there is another finite subcollection of $\mathcal{P}$ that covers F, and so Y can be covered by a finite subcollection of $\mathcal{P}$ and we are done. If F is unbounded, there exists a sequence (x_i) in F such that $|x_i - o| \to \infty$ for any basepoint o. Since Y is sequentially compact, some subsequence (x_{i_j}) converges to a point $x \in Y$. Then $x \in Z$ and therefore $x \in V$ for some $V \in \mathcal{Q}$. Hence $x_{i_j} \in V \subset U$ for sufficiently large j which is a contradiction. □

The following lemma is a generalization of [28, p. 123].

Lemma 7.3. *Suppose that X is hyperbolic, proper, and c-roughly starlike with respect to $o \in X$. Then X^* and $\partial_G X$ are compact as well as sequentially compact.*

Proof. We first prove that any sequence (x_n) in X has a subsequence that converges in X^*. Since X is proper, we may assume that $|x_n - o| \to \infty$. For each $n \in \mathbb{N}$ let $y_{n,0} = o, y_{n,1}, \dots, y_{n,k_n} = x_n$ be a c-roughly geodesic sequence from o to x_n. Thus

$$|i - j| - c \le |y_{n,i} - y_{n,j}| \le |i - j| + c \tag{7.1}$$

for all $n \in \mathbb{N}$ and all $i, j \in \{0, 1, \dots, k_n\}$. By using the right-hand side of (7.1), the properness of X, and a standard diagonal argument we find a subsequence, still denoted by (x_n), and points $y_i \in X$ such that $y_{n,i} \to y_i$ for all i as $n \to \infty$. Furthermore, it follows from (7.1) that

$$|i - j| - c \le |y_i - y_j| \le |i - j| + c$$

for all $i, j \ge 0$. We claim that (x_n) is a Gromov sequence. Fix $N \in \mathbb{N}$ and let $n_0 \in \mathbb{N}$ be so large that

$$|y_{n,N} - y_N| \le 1$$

and

$$|x_n - o| \ge N + c$$

for all $n \ge n_0$. Then

$$\begin{aligned}
|x_i - x_j| &= |y_{i,k_i} - y_{j,k_j}| \\
&\le |y_{i,k_i} - y_{i,N}| + |y_{i,N} - y_N| + |y_N - y_{j,N}| + |y_{j,N} - y_{j,k_j}| \\
&\le (k_i - N) + (k_j - N) + 2 + 2c \\
&\le |x_i - o| + |x_j - o| - 2(N - 2c - 1)
\end{aligned}$$

whenever $i, j \ge n_0$. Hence

$$(x_i \,|\, x_j) = \frac{1}{2}\big(|x_i - o| + |x_j - o| - |x_i - x_j|\big) \ge N - 2c - 1$$

whenever $i, j \ge n_0$. Here $N \in \mathbb{N}$ is arbitrary so that $(x_i \,|\, x_j) \to \infty$ as $i, j \to \infty$. Hence (x_n) is a Gromov sequence. Let $a \in \partial_G X$ be the equivalence class of (x_n). Then $x_n \to a$ in the topology of X^*. We conclude that any sequence of points in X has a subsequence that converges in X^*.

Let then (x_n) be a sequence of points in X^*. We claim that (x_n) has a convergent subsequence. Note that (x_n) has a subsequence of points in X or a subsequence of points in $\partial_G X$. The former case was already covered above and hence we can assume that (x_n) is a sequence in $\partial_G X$. For each n let $y_n \in X$ be such that $d_\varepsilon(x_n, y_n) < 1/n$. By the above, there exist a subsequence (y_{n_k}) and $y \in \partial_G X$ such that $y_{n_k} \to y$. Then for the corresponding subsequence (x_{n_k}) we have

$$d_\varepsilon(x_{n_k}, y) \le d_\varepsilon(x_{n_k}, y_{n_k}) + d_\varepsilon(y_{n_k}, y) \to 0$$

and hence $x_{n_k} \to y$. We conclude that X^* is sequentially compact. This also shows that $\partial_G X$ is sequentially compact since (x_n) was an arbitrary sequence in $\partial_G X$. Hence $\partial_G X$ is compact because the topology of $\partial_G X$ is metrizable. Finally, Lemma 7.2 implies that X^* is compact. $\square$

Corollary 7.4. *Let X be a δ-hyperbolic metric measure space as in Section 6. Suppose, furthermore, that X is proper, c-roughly starlike with respect to $o \in X$, and that $\partial_G X \neq \emptyset$. Then, for every continuous function $f\colon \partial_G X \to \mathbb{R}$, there exists a unique continuous function $u\colon X^* \to \mathbb{R}$ that is p-harmonic in X and $u|\partial_G X = f$.*

We close this section with an example of a proper hyperbolic space X whose Gromov closure X^* is not sequentially compact. Thus this example shows that the properness of X alone is not sufficient for the sequential compactness of X^*.

Example 7.5. Let $\mathbb{H}^n$ be the n-dimensional hyperbolic space of constant sectional curvature -1. We consider $\mathbb{H}^n$ as the unit ball $B^n(0,1) \subset \mathbb{R}^n$ equipped with the hyperbolic metric. Furthermore, we consider $\mathbb{H}^n$ as an isometrically embedded subset of $\mathbb{H}^{n+1}$. Balls in $\mathbb{H}^n$ centered at 0 are denoted by $B^n_{\mathbb{H}}(r)$. We start with the 2-dimensional hyperbolic closed disc $A_2 := \bar{B}^2_{\mathbb{H}}(2)$ of radius 2 which we glue first into the 3-dimensional closed annulus $A_3 := \bar{B}^3_{\mathbb{H}}(3) \setminus B^3_{\mathbb{H}}(2)$ along the circle $\partial B^2_{\mathbb{H}}(2)$. Next we glue $A_2 \cup A_3$ into the 4-dimensional closed annulus $A_4 := \bar{B}^4_{\mathbb{H}}(4) \setminus B^4_{\mathbb{H}}(3)$ along the 2-sphere $\partial B^3_{\mathbb{H}}(3)$ and we repeat the process. Thus, for each $n \geq 3$, the union

$$X_n := \bigcup_{j=2}^{n} A_n$$

is glued into the $(n+1)$-dimensional annulus $A_{n+1} = \bar{B}^{n+1}_{\mathbb{H}}(n+1) \setminus B^{n+1}_{\mathbb{H}}(n)$ along the $(n-1)$-sphere $\partial B^n_{\mathbb{H}}(n)$. The space X is then the union

$$X := \bigcup_{j=2}^{\infty} A_n$$

so that each X_n is isometrically embedded into X. It is then clear that X is proper and Gromov hyperbolic. Furthermore, $\mathbb{H}^n \setminus B^n_{\mathbb{H}}(n-1)$ is isometrically embedded into X for all $n \in \mathbb{N}$ and therefore $\partial_G X$ is non-compact and X^* is not sequentially compact.

8 Examples

In this final section we give examples of δ-hyperbolic metric measure spaces that satisfy the conditions employed in Section 6. We restrict to complete Riemannian manifolds. Throughout the section, all manifolds are assumed to be connected and of dimension $n \geq 2$.

We begin with an example which, in a special case, shows that the global volume growth condition (6.1) holds on every complete Riemannian manifold of bounded (local) geometry. The notion of bounded geometry appears in the literature in somewhat varying meanings. Sometimes it refers to a lower bound for the Ricci curvature and to a positive lower bound for the injectivity radius. In this paper, we say that a complete n-dimensional Riemannian manifold M has *bounded geometry* if there exist constants $r_0 > 0$ and $L \geq 1$ such that each ball $B(x, r_0) \subset M$ is L-bi-Lipschitz equivalent to an open subset of $\mathbb{R}^n$. It is then clear that a local doubling condition (4.1), a local weak $(1,p)$-Poincaré inequality (4.3), and a lower bound (6.2) hold on M. Thus M satisfies all the local assumptions employed in Section 6.

Example 8.1. Let (X, d) be a length space, i.e. a metric space where the distance between any two points is the infimum of the lengths of all paths joining the two points. Let μ be a Borel regular measure on X that satisfies the local doubling condition (4.1) with constants C_d and R_d. Suppose that there exist positive constants c and r_0 such that

$$\mu(B(x, r_0)) \leq c \tag{8.1}$$

for all $x \in X$. We show that there exist positive constants β and C such that

$$\mu(B(x, R)) \leq Ce^{\beta R} \tag{8.2}$$

for all $x \in X$ and $R > 0$. To this end, we first recall (from the beginning of Section 4) that the local doubling condition (4.1) implies the existence of constants $\tilde{C}_d \in \mathbb{N}$ and $\tilde{R}_d > 0$ such that every ball $B(x, 2r)$ in X with $0 < r \leq \tilde{R}_d$ can be covered by $\tilde{C}_d$ balls of radius r. We claim that for all $k \in \mathbb{N}$, $x \in X$ and $0 < r \leq \tilde{R}_d$, the ball $B(x, (k+1)r)$ can be covered by $\tilde{C}_d^k$ balls of radius r. This holds for $k = 1$. Now suppose that the claim holds for some $k \in \mathbb{N}$, and consider a ball $B(x, (k+2)r)$. Since X is a length space, and since we are using open balls, for every $y \in B(x, (k+2)r)$ there is a point $y' \in B(x, (k+1)r)$ with $d(y, y') \leq r$. It follows that $B(x, (k+2)r)$ can be covered by $\tilde{C}_d^k$ balls of radius $2r$. Each of them can be covered by $\tilde{C}_d$ balls of radius r. Hence, the claim holds for $k+1$ and thus for every $k \in \mathbb{N}$. Finally, using (8.1), we obtain (8.2).

If M is a complete Riemannian manifold with bounded geometry, then, in addition to a local doubling condition (4.1), also an upper bound (8.1) holds on M. Thus M satisfies the volume growth condition (8.2), in particular (6.1).

Next we turn our attention to conditions that imply the Gromov hyperbolicity and the (p,p)-Sobolev inequality (5.1).

If M is a Cartan-Hadamard manifold whose sectional curvatures are bounded from above by a constant $-a^2 < 0$, then it is Gromov hyperbolic; see [32, 1.5] and [23, Théorème 5.1]. It is also well-known that a global $(1,1)$-Sobolev inequality holds on M. This follows, for instance, by applying Green's formula and the estimate (see e.g. [30, Theorem A])

$$\Delta r(x) \geq (n-1)a \coth(ar(x)) \geq (n-1)a \tag{8.3}$$

for the Laplacian of the distance function $r(x) = d(x, o)$, with fixed $o \in M$. Indeed, for a non-negative function $u \in C_0^\infty(M)$, we have

$$(n-1)a \int_M u\, d\mu \le \int_M u\Delta r\, d\mu = -\int_M \langle \nabla u, \nabla r\rangle\, d\mu \le \int_M |\nabla u|\, d\mu$$

by (8.3) and Green's formula (cf. [66, Proposition 3]). Thus we get a $(1,1)$-Sobolev inequality by approximation and a (p,p)-Sobolev inequality for every $p \ge 1$ by Hölder's inequality. Indeed, given u, the $(1,1)$-Sobolev inequality for $v := |u|^p$ yields

$$\|v\|_1 \le C\||\nabla v|\|_1 \le Cp \int_M v^{1-1/p}|\nabla u|\, d\mu \le Cp\|v\|_1^{1-1/p}\||\nabla u|\|_p,$$

thus $\|u\|_p \le Cp\||\nabla u|\|_p$.

The following is a generalization of Ancona's result [3, Theorem 9] to p-harmonic functions. It also extends [39, 2.1] since it replaces the lower curvature bound $K_M \ge -b^2$ in [39] by the bounded geometry assumption on M.

Theorem 8.2. *Let M be a Cartan-Hadamard manifold of bounded geometry whose sectional curvatures satisfy $K_M \le -a^2 < 0$. Given a continuous function h on $S(\infty)$, there exists a unique function $u \in C(\bar{M})$ which is p-harmonic in M and satisfies $u = h$ on $S(\infty)$.*

Proof. By the discussion above, M satisfies all the assumptions employed in Section 6. It is well-known that $\bar{M} = M \cup S(\infty)$ is canonically homeomorphic to the Gromov closure $M^* = M \cup \partial_G M$ (cf. [13, Sect. III.H.3]). Thus the existence of a function u follows from Theorem 1.1. Furthermore, the uniqueness of u follows from Theorem 7.1 since $\bar{M}$ is homeomorphic to a compact Euclidean ball. □

Let X and Y be metric spaces. A map $f\colon X \to Y$ is called a *quasi-isometry* if there exist constants $\lambda \ge 1$ and $\alpha \ge 0$ such that

$$\lambda^{-1}|x-y| - \alpha \le |f(x) - f(y)| \le \lambda|x-y| + \alpha$$

for all $x, y \in X$ and such that for each $y \in Y$ there exists an $x \in X$ with $|f(x) - y| \le \alpha$. If such a map exists, then X and Y are said to be *quasi-isometric*; this defines an equivalence relation on the class of metric spaces. In the literature quasi-isometries are also called rough quasi-isometries (e.g. in [10], [16]), roughly surjective quasi-isometries ([64]) or simply rough isometries ([45]). It is well-known that Gromov hyperbolicity is preserved by quasi-isometries between geodesic metric spaces. This holds for larger classes of spaces, e.g. for length spaces ([64, 3.18]).

Suppose that M and N are complete Riemannian manifolds of bounded geometry that are quasi-isometric. Then, by [25, Théorème 7.1], a global

Sobolev inequality (5.1) holds on M if and only if such an inequality holds on N; see also [45]. Thus we obtain the following corollary which generalizes a result of Cheng [19, 2.3].

Corollary 8.3. *Let N be a Cartan-Hadamard manifold of bounded geometry whose sectional curvatures satisfy $K_N \le -a^2 < 0$. Suppose that M is a complete Riemannian manifold of bounded geometry that is quasi-isometric to N. Then M is Gromov hyperbolic and for every continuous function h on $\partial_G M$ there exists a unique function $u \in C(M^*)$ which is p-harmonic in M and satisfies $u = h$ on $\partial_G M$.*

Next we recall Cao's result [16] on Cheeger's isoperimetric constant of Gromov hyperbolic manifolds or graphs. The Cheeger constant of a complete Riemannian n-manifold M is defined by

$$h(M) = \inf_{\Omega} \frac{\operatorname{Area}(\partial\Omega)}{\operatorname{Vol}(\Omega)},$$

where the infimum is taken over all non-empty open sets $\Omega \Subset M$ with smooth boundary. Similarly, the Cheeger constant of a connected graph $\Gamma = (V, E)$ is defined by

$$h(\Gamma) = \inf_{A} \frac{|\partial A|}{|A|},$$

where A ranges over all non-empty finite subsets of the vertex set V, $\partial A = \{v \in V : d(v, A) = 1\}$, d is the natural combinatorial distance on V taking values in $\{0\} \cup \mathbb{N}$, and $|A|$ denotes the cardinality of A. Following [16, Definition 1.2] we say that a complete Riemannian manifold (or a connected graph) X has a *quasi-pole* in a compact subset $\Omega \subset X$ if there exists a constant $C > 0$ such that each point of X lies in a C-neighborhood of the image of a geodesic ray emanating from Ω. Here, by a geodesic ray we mean an isometric embedding $\gamma\colon [0,\infty) \to X$ (or $\gamma\colon \{0\} \cup \mathbb{N} \to X$ in the case of a graph X, where X is identified with its vertex set). We point out that if X has a quasi-pole in a compact set Ω, then X is c-roughly starlike (cf. Section 7), for some c, with respect to any $o \in \Omega$.

Theorem 8.4. *[16, Main Theorem 1.1] Let X be a complete Riemannian manifold (or a connected graph) that admits a quasi-pole and has bounded geometry. Suppose that X is Gromov hyperbolic and the diameters of the components of $\partial_G X$ have a positive lower bound (with respect to a fixed metric d_ε). Then X has positive Cheeger constant.*

Since $h(M) > 0$ if and only if a global $(1,1)$-Sobolev inequality holds on M (see e.g. [17]), we obtain the following corollary.

Corollary 8.5. *Let M be a complete Riemannian manifold that admits a quasi-pole and has bounded geometry. Suppose that M is Gromov hyperbolic* *and the diameters of the components of $\partial_G M$ have a positive lower bound (with respect to a fixed metric d_ε). Given a continuous function h on $\partial_G M$, there exists a unique function $u \in C(M^*)$ which is p-harmonic in M and satisfies $u = h$ on $\partial_G M$.*

We finish the paper by two remarks on the possible generalizations of the methods that were used in the paper.

Remark 8.6. 1. Since the proofs of the key Lemmata 5.1 and 5.2 are based on Caccioppoli-type estimates (5.4) and (5.7), respectively, the method can be applied to p-harmonic functions on Gromov hyperbolic graphs of bounded degree. However, we will not study the asymptotic Dirichlet problem for p-harmonic functions on Gromov hyperbolic graphs in this paper. We refer to [40], [41], and [61] for the definition and properties of p-harmonic functions on graphs.

2. The proofs of Lemma 5.1 and 5.2 can be modified so that they apply to the study of the asymptotic Dirichlet problem for so-called $\mathcal{A}$-harmonic functions on Gromov hyperbolic Riemannian manifolds. This will be studied in detail in a forth-coming paper by the third author.

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