Zeros of derivatives of real entire functions

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Abstract

An entire function is called real if it takes real values on the real axis. We are interested in the number of non-real zeros of the derivatives of a real entire function. This subject a long history, going back to the work of Fourier on the reality of the zeros of Bessel functions.

It was conjectured by Pólya and Wiman – and proved by Craven, Csordas and Smith – that if a real entire function f of order less than 2 has only finitely many non-real zeros, then $f^{(n)}$ has only real zeros for sufficiently large n. For the case that the order is greater than 2, Pólya and Wiman made the following conjectures:

Conjecture 1 (Wiman 1911). f'' has non-real zeros.

Conjecture 2 (Pólya 1943). The number of non-real zeros of $f^{(n)}$ tends to infinity with n.

Conjecture 1 was confirmed in 2003 by Bergweiler, Eremenko and Langley, building on earlier work by Levin and Ostrovskii and by Sheil-Small. Conjecture 2 was recently proved by Langley for functions of infinite order and by Bergweiler and Eremenko in the finite order case. Together with results of Ki and Kim we obtain the following alternative for a real entire function f: *either $f^{(n)}$ has no non-real zeros for all large n, or the number of non-real zeros of $f^{(n)}$ tends to infinity with n.*

The talk will first give further background to the above conjectures, explaining in particular how they relate to Fourier's work. Then we will describe some of the previous results that have been obtained, as well as the techniques that have been used. Finally we will sketch the main ideas in the proof of the second conjecture.