Controlling the maximal singular integral by the singular integral

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Abstract

Let T be a smooth homogeneous Calderón-Zygmund singular integral operator in $\mathbb{R}^n$. The most basic form of control of the maximal singular integral $T^\star f$ by $T^\star f$ one may consider is the estimate of the $L^2(\mathbb{R}^n)$ norm of $T^\star f$ by a constant times the $L^2(\mathbb{R}^n)$ norm of Tf. It turns out that if T is an even higher order Riesz transform, then one has the stronger pointwise inequality $T^\star f(x) \leq C\, M(Tf)(x)$, where C is a constant and M is the Hardy-Littlewood maximal operator. In fact the L^2 estimate of $T^\star$ by T is equivalent, for even smooth homogeneous Calderón-Zygmund operators, to the pointwise inequality between $T^\star$ and $M(T)$. Our main result characterizes the L^2 and pointwise inequalities in terms of an algebraic condition expressed in terms of the kernel $\frac{\Omega(x)}{|x|^n}$ of T, where Ω is an even homogeneous function of degree 0, of class $C^\infty(S^{n-1})$ and with zero integral on the unit sphere S^{n-1}. Let $\Omega = \sum P_j$ the expansion of Ω in spherical harmonics P_j of degree j. Let A stand for the algebra generated by the identity and the smooth homogeneous Calderón-Zygmund operators. Then our characterizing condition states that T is of the form $R \circ U$, where U is an invertible operator in A and R is a higher order Riesz transform associated to a homogeneous harmonic polynomial P which divides each P_j in the ring of polynomials in n variables with real coefficients. An application of our methods gives a sufficient condition on the Beltrami coefficient that implies that the corresponding quasi-conformal mapping is bilipschitz.