Behaviour of Lipschitz functions on negligible sets

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Abstract

In various connections it has been observed that the behaviour of (Lipschitz) functions $\mathbb{R}^n \to \mathbb{R}^m$ on Lebesgue null sets is considerably more regular in higher dimensions ($n \geq 2$) then in the one dimensional case ($n = 1$). Perhaps the clearest manifestation of this phenomenon is [4]: there is a Lebesgue null set $E \subset \mathbb{R}^2$ such that every Lipschitz function $\mathbb{R}^2 \to \mathbb{R}$ is differentiable at some point of E. A rather different flavour has the result of [1]: the singular part of the derivative of any BV function $\mathbb{R}^n \to \mathbb{R}^m$ (which is, by definition of BV, a vector-valued measure) is a rank one measure. In yet another direction, a closer inspection of arguments of, e.g., [2, 3, 5] showing that for some continuous $f : \mathbb{R}^n \to \mathbb{R}$, equations such as $\operatorname{div} u = f$ fail to have locally Lipschitz solutions reveals that the obstacles are caused by behaviour of f on a null set.

We study those features of this phenomenon that stem from the fact that (all or at least the relevant) null sets are much smaller than they seem. A Lebesgue null set $E \subset \mathbb{R}^n$ can be, by Fubini's theorem, covered by a finite union of (Borel) sets E_j where E_j is null on every line parallel to some vector e_j. A set of non-differentiability points of a Lipschitz function can be covered by a finite union of (Borel) set E_j where each E_j is null on every curve with derivative close to some vector e_j. This seemingly much stronger decomposition property could be responsible for the result of [4] quoted above. However, this is not the case: for subsets of $\mathbb{R}^2$ the two properties are equivalent. The key problem whether or not this is true also in $\mathbb{R}^n$, $n \geq 3$ remains open, but we can at least prove that every null set in $\mathbb{R}^n$ is contained in the union of a set having this property and of a purely $n-1$ unrectifiable set.

The talk will give a more precise discussion of the above decomposition property of a null set E and show that it is equivalent to the existence of a natural $n-1$ dimensional tangent field on E. This immediately leads to a better understanding of Alberti's rank one theorem and, with some additional work, to a complete description of sets of non-differentiability of Lipschitz maps $\mathbb{R}^n \to \mathbb{R}^k$ for $k \geq n$. In particular these sets do not depend on k as long as it stays $\geq n$, although [4] implies that they are different if $k = 1$.

References

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