Number of Extensions of Non-Fregean Logics

Joanna Golińska Taneli Huuskonen

October 23, 2002

Abstract

We show that there are continuum many different extensions of SCI (the basic theory of non-Fregean propositional logic) that lie below WF (the Fregean extension) and are closed under substitution. Moreover, continuum many of them are independent from WB (the Boolean extension), continuum many lie above WB and are independent from WH (the Boolean extension with only two values for the equality relation), and only countably many lie between WH and WF.

1 Introduction

The expression "non-fregean logic" was introduced by Roman Suszko in the article *Non-Fregean Logic and Theories* [3]. Its cornerstone is the omission of the *Fregean axiom.* Recall that according to G. Frege, sentences are not only true or false, but they are also names of their truth values. Hence the Fregean axiom can be formulated as follows: all true statements (likewise, all false statements) have the same common referent, the truth (respectively, the falsehood). The Fregean axiom lies at the heart of classical logical calculi. In a model for a language based on classical logic there is no universe corresponding to the sentences of the language, but only a basis for an unambiguous division of the sentences into true and false ones. The Fregean axiom can be seen as the formal counterpart of a philosophical view on the meanings of sentences.

On the other hand, the philosophical foundation of the non-fregean logic can be summarized as follows: a description of the world is incomplete (in the non-technical sense of the word) if it consists solely of a description of objects, their properties and their relations to each other. A full and adequate description of reality should reflect also the fact that reality is a collection of possibilities, some of which are realized and which can be described with sentences. While maintaining the view that a logical sentence is always either true or false, we should also be able to take into account the fact that reality, which we want to describe, contains denotations for expressions having more than merely syntactic content. Therefore, we should acknowledge the fact that the denotations of names are objects, that the denotations of predicates are sets or relations and that the denotations of sentences are the *situations* described by them.

The above observations show that the basic philosophical assumption of non-Fregean logic is that the denotations of the sentences of a given language are different from their truth values; the universe of the denotations is commonly called the *universe of situations*. In order to be able to speak about the situations, we add to the language a new connective, known as the *identity connective*, which links a pair of sentences to truth when their denotations are the same in a given model, that is, when the sentences describe the same situation. According to Suszko, the identity connective is more basic than other non-truth-functional operators, for instance, the various modal operators. It is basic in the sense that it cannot be eliminated from the logics that have been used and studied without trivializing it into another name for the equivalence connective. In the general case, the identity connective is different from the equivalence connective: two sentences with the same truth value can have different denotations. In other words, the truth value of a sentence is distinct from the situation described by the sentence. Adding the identity connective to classical logic does not, however, conflict with two-valuedness. Non-Fregean logic is two-valued as well as extensional, and it is the weakest logic with that property, while classical logic is the strongest one. Moreover, two-valuedness implies that the universe of situations must have at least two elements. That is the only limitation that non-Fregean logic imposes on the size of the universe of situations. On the other hand, if we add the condition that the universe of situations has at most two elements, we obtain the classical logical calculus, and the identity connective becomes indistinguishable from the equivalence connective. In fact, the Fregean axiom claims exactly this: that the two connectives are the same. Therefore Suszko called the classical logical calculus *Fregean logic* and the calculus without this axiom *non-Fregean logic*.

Thus, in a sense non-Fregean logic is an extension of classical logic: the language for building formulae is expanded. However, from another and perhaps more relevant point of view, classical logic is a strengthening of non-Fregean logic. On the other hand, the latter provides such a general logical calculus that most known logics—classical first-order logic, classical sentential calculus, the many-valued logics of Łukasiewicz as well as some modal logics—can be formulated in its general framework.

One should not forget about the philosophical applications of non-Fregean logic either. Most importantly, this logic provides excellent tools for the precise formulation of an ontology, for the formalization of the correspondence theory of truth and such concepts as fact, necessity, possible world, state of affairs and event. However, the value of non-Fregean logic does not derive solely from its applications. Non-Fregean logic constitutes an autonomic logical calculus, which can be studied in its own right, irrespectively of any connections with reality.

The sentential calculus based on the principles of non-Fregean logic is called the *Sentential Calculus with Identity* (SCI). In the present article we will only treat the sentential form of non-Fregean logic and its axiomatic extensions.

We thank Prof. Mieczysław Omyła for suggesting the subject to the first author and for his helpful remarks on an early version of this article, and Prof. Marcin Mostowski for his comments.

2 Basic definitions and classical results

In this section we present the basic definitions, axioms and lemmas that we need later on. They can be found in [5] and [2], among others. The latter is a more comprehensive source but only available in Polish, unfortunately.

Definition 2.1 The *SCI-language* is the language L built from the propositional variables p_0, p_1, $p_2 \ldots$ with the connectives $\neg$, $\rightarrow$ and $\equiv$. Its elements are called *SCI-formulas.*

We will regard the connectives $\wedge$, $\vee$ and $\leftrightarrow$ as the usual shorthand notations expressed in terms of $\neg$ and $\rightarrow$. For reasons that will be soon explained, this choice slightly affects the theory (see Remark 2.6 below). However, generalizing our results to another set of connectives is perfectly straightforward. Restricting our attention to the special case makes several definitions and lemmas considerably shorter.

The connective $\equiv$ is a new one, and it is used to express the condition that two formulas have the same denotation. So, its interpretation will be an equivalence relation satisfying some rather mild conditions in the general case.

Having defined the syntax, we now look at the semantics.

Definition 2.2 An *SCI-algebra* is a structure with signature

$$\tau_{SCI} = \langle \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv} \rangle,$$

where $\tilde{\neg}$ is a unary function symbol, and $\tilde{\rightarrow}$ and $\tilde{\equiv}$ are binary function symbols.

We will sometimes regard the SCI-language as an SCI-algebra, interpreting each operation in the obvious way.

Definition 2.3 An *SCI-model* is a pair $\langle \mathfrak{A}, D \rangle$, where

$$\mathfrak{A} = \langle A, \tilde{\neg}^{\mathfrak{A}}, \tilde{\rightarrow}^{\mathfrak{A}}, \tilde{\equiv}^{\mathfrak{A}} \rangle$$

is an SCI-algebra and D is a subset of A such that the following conditions are satisfied for all $a, b \in A$ (we omit the superscript $\mathfrak{A}$ for easier reading):

1. $\tilde{\neg} a \in D$ iff $a \notin D$,
2. $a \mathbin{\tilde{\rightarrow}} b \in D$ iff $a \notin D$ or $b \in D$,
3. $a \mathbin{\tilde{\equiv}} b \in D$ iff $a = b$.

Intuitively, the elements of an SCI-algebra are situations (denotations of sentences), and the function symbols correspond to the formation of new formulas with connectives. For instance, if a formula p denotes the situation s, then the denotation of $\neg p$ is $\tilde{\neg} s$. Moreover, the set D consists of the facts, that is, the denotations of the formulas that are true in a given model and interpretation.

Definition 2.4 Let $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ be an SCI-model. An $\mathfrak{M}$-*valuation* is a homomorphism $h : L \rightarrow \mathfrak{A}$. An SCI-formula α is *satisfied by* h *in* $\mathfrak{M}$ (written as $\mathfrak{M} \models_h \alpha$ for short) if $h(\alpha) \in D$, and a set A of SCI-formulas is satisfied by h in $\mathfrak{M}$ (written as $\mathfrak{M} \models_h A$) if every $\alpha \in A$ is.

The intuitive idea behind an $\mathfrak{M}$-valuation is simple: one chooses the values of the propositional variables freely, and they determine the values of all other formulas in the usual recursive way. The use of an algebra homomorphism is just a compact way of expressing this. Note that the values are elements of the SCI-algebra, that is, situations. The truth values are uniquely determined by the set D of facts.

Definition 2.5 Let $\mathfrak{M}$ be an SCI-model and α an SCI-formula. Then α is *satisfiable in* $\mathfrak{M}$ if it is satisfied by some $\mathfrak{M}$-valuation and *true in* $\mathfrak{M}$ (written as $\mathfrak{M} \models \alpha$) if it is satisfied by all $\mathfrak{M}$-valuations.

Remark 2.6 Since $\alpha \vee \beta$ is just a shorthand notation for $\neg\alpha \rightarrow \beta$, the formula $(\alpha \vee \beta) \equiv (\neg\alpha \rightarrow \beta)$ is obviously true in any SCI-model. If $\vee$ were regarded as an independent connective, this would not be the case. The same applies, of course, to other shorthand notations.

We can now define a semantic notion of consequence for SCI.

Definition 2.7 Let A and B be sets of SCI-formulas. Then B is a *semantic consequence of* A, written symbolically $A \models B$, if $\mathfrak{M} \models_h B$ for all SCI-models $\mathfrak{M}$ and for all $\mathfrak{M}$-valuations h such that $\mathfrak{M} \models_h A$.

There is a corresponding syntactic notion of consequence.In other words, the semantic notion can be completely axiomatized.

Definition 2.8 The set of *truth-functional axioms* (TFA for short) is the set of all SCI-formulas that can be obtained from the following schemas by replacing α, β and γ with arbitrary SCI-formulas:

T1 $\alpha \rightarrow (\beta \rightarrow \alpha)$;

T2 $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$;

T3 $(\neg\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \alpha)$;

It is easy to see that all these axioms are tautologies of classical propositional logic. Moreover, they form a complete set of axioms, with Modus Ponens as the only rule of inference. We omit the proof.

Definition 2.9 The set of *identity axioms* (IDA for short) is the set of SCI-formulas obtained from the following schemas:

I1 $\alpha \equiv \alpha$;

I2 $(\alpha \equiv \beta) \to (\neg\alpha \equiv \neg\beta)$;

I3 $((\alpha \equiv \beta) \wedge (\gamma \equiv \delta)) \to ((\alpha \& \gamma) \equiv (\beta \& \delta))$, where & stands for either of the connectives $\to$, $\equiv$;

I4 $(\alpha \equiv \beta) \to (\alpha \to \beta)$.

Again, it is easy to see that the identity axioms are true in every SCI-model. Together with the truth-functional axioms, they also constitute a complete axiomatization for the semantic consequence relation defined above.

Definition 2.10 Let A and B be sets of SCI-formulas. Then B is a syntactic consequence of A ($A \vdash B$ for short) if every formula in B can be inferred from the set $\mathit{TFA} \cup \mathit{IDA} \cup A$ in finitely many steps using Modus Ponens as the only rule of inference.

If A or B is a singleton, we will omit the braces around its only element.

Definition 2.11 Let A be a set of SCI-formulas. We say that A is *consistent*, if there is an SCI-formula α such that $A \nvdash \alpha$.

Theorem 2.12 *Let A and B be sets of SCI-formulas. Then $A \vdash B$ iff $A \models B$.*

Proof. (Sketch.) The implication from left to right is simple; all one needs to show is that all the axioms A1–A3 and I1–I4 are true in all SCI-models and that the Modus Ponens rule is valid.

For the implication from right to left, assume that $A \nvdash B$. Thus there is a formula $\beta \in B$ such that $A \nvdash \beta$. Hence $A \cup \{\neg\beta\}$ is consistent. By a general version of Lindenbaum's Theorem, $A \cup \{\neg\beta\}$ can be extended into a maximal consistent set S. Define a relation $\sim_S$ on L by $\gamma \sim_S \delta$ iff $(\gamma \equiv \delta) \in S$. It is easy to see that $\sim_S$ is an equivalence relation. Moreover, $\sim_S$ is a congruence on the SCI-algebra L. Now let $\mathfrak{M} = \langle L/\sim_S, S/\sim_S \rangle$, and let $h : L \to L/\sim_S$ be the canonical homomorphism. Then $\mathfrak{M}$ is an SCI-model and $\mathfrak{M} \models_h S$; thus $\mathfrak{M} \models_h A$ but $\mathfrak{M} \not\models_h B$. □

Definition 2.13 1. An *SCI-theory* is a set of SCI-formulas that is closed under $\vdash$.

2. An *invariant SCI-theory* is an SCI-theory that is closed under substitution.

The following proposition explains some of the importance of invariant SCI-theories.

Proposition 2.14 *Let T be the set of SCI-formulas that are true in a class of SCI-models. Then T is an invariant SCI-theory.*

Proof. It is easy to see that the claim holds for a single SCI-model, and closure properties are preserved under intersections. $\square$

There are some important extensions of SCI that have been studied in the literature. We present here two.

Definition 2.15 The *Boolean SCI-theory* (WB for short) is the set of SCI-formulas that can be proved from the SCI-axioms and instances of the following schemata:

B1 $((\alpha \wedge \beta) \vee \gamma) \equiv ((\beta \vee \gamma) \wedge (\alpha \vee \gamma))$;

B2 $((\alpha \vee \beta) \wedge \gamma) \equiv ((\beta \wedge \gamma) \vee (\alpha \wedge \gamma))$;

B3 $(\alpha \vee (\beta \wedge \neg\beta)) \equiv \alpha$;

B4 $(\alpha \wedge (\beta \vee \neg\beta)) \equiv \alpha$;

Models for WB are exactly those whose elements form a Boolean algebra under the operations $\tilde{\wedge}$, $\tilde{\vee}$ and $\tilde{\neg}$, as shown by M. Omyła in [2].

Definition 2.16 The *Henlean SCI-theory* (WH for short) is the set of SCI-formulas that can be proved from the SCI-axioms, the axioms of WB and instances of the following schema:

H1 $((\alpha \equiv \beta) \equiv (\gamma \vee \neg\gamma)) \vee ((\alpha \equiv \beta) \equiv (\gamma \wedge \neg\gamma))$.

Models for WH are exactly those whose elements form a *Henle algebra* under $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\neg}$ and $\tilde{\equiv}$, that is, a Boolean algebra expanded with a binary operation $\tilde{\equiv}$ such that

$$a \mathbin{\tilde{\equiv}} b = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases}$$

This was shown by R. Suszko in [4].

Fact 2.17 *Both WB and WH are invariant SCI-theories.*

3 Expressive power of SCI

In this section we prove two small theorems about the expressive power of SCI. We find them both interesting in their own right and useful for applications.

Definition 3.1 The *first-order SCI-language* is the first-order language with vocabulary $\tau_{SCI} \cup \{P\}$, where P is a unary predicate symbol.

Theorem 3.2 *Let α be an SCI-formula. There is a universal sentence α^* in the first-order SCI-language such that $\mathfrak{M} \models \alpha$ (in the sense of Definition 2.5) iff $\mathfrak{M} \models \alpha^*$ (in the sense of ordinary first-order logic), where the interpretation of P is D and the function symbols are interpreted in the obvious way.*

Proof. For any SCI-formula β, we define a corresponding τ_{SCI}-term t_β inductively as follows:

$$\begin{aligned} t_{p_i} &= v_i, \\ t_{\neg\beta} &= \tilde{\neg} t_\beta, \\ t_{\beta\to\gamma} &= t_\beta \mathrel{\tilde{\to}} t_\gamma, \\ t_{\beta\equiv\gamma} &= t_\beta \mathrel{\tilde{\equiv}} t_\gamma. \end{aligned}$$

Now we simply let α^* be the universal closure of $P(t_\alpha)$. It is straightforward to check that the claim holds. $\Box$

Also the converse is true.

Theorem 3.3 *Let α be a universal sentence in the first-order SCI-language. There is an SCI-formula α' such that $\mathfrak{M} \models \alpha$ iff $\mathfrak{M} \models \alpha'$, where the satisfaction relations are defined as in the previous theorem.*

Proof. For any τ_{SCI}-term t we define a corresponding SCI-formula β_t inductively as follows:

$$\begin{aligned} \beta_{v_i} &= p_i, \\ \beta_{\tilde{\neg} r} &= \neg\beta_r, \\ \beta_{r \tilde{\to} s} &= \beta_r \to \beta_s, \\ \beta_{r \tilde{\equiv} s} &= \beta_r \equiv \beta_s. \end{aligned}$$

Now for any quantifier-free formula γ of the first-order SCI-language, we define inductively an SCI-formula $\gamma^\dagger$ as follows:

$$\begin{array}{rcl}
(P(t))^\dagger &=& \beta_t, \\
(t = u)^\dagger &=& \beta_t \equiv \beta_u, \\
(\neg\psi)^\dagger &=& \neg(\psi^\dagger), \\
(\psi \to \theta)^\dagger &=& (\psi^\dagger) \to (\theta^\dagger),
\end{array}$$

Finally, if α is of the form $\forall\bar{x}\psi$, where ψ is a quantifier-free formula, we let $\alpha' = \psi^\dagger$. Again, it is straightforward to check the claim. $\square$

Corollary 3.4 *A class of SCI-models is definable with an SCI-theory (SCI-formula) iff it is definable with a set of universal sentences (a universal sentence) in the first-order SCI-language.*

We will later use this corollary without explicitly mentioning it, transferring results freely between the two languages.

4 Model constructions

Let $n \in \mathbb{N}$. We define the model $\mathfrak{M}_n$ as follows. Let $A_n = \{0, 1, \ldots, 2n-1\}$ and $D_n = \{n, \ldots, 2n-1\}$. For $i \in A_n$, let $\tilde{\neg}(i) = 2n-1-i$, and for $i, j \in A_n$, define $i \mathbin{\tilde{\to}} j = \max(\tilde{\neg}i, j)$. Finally, for $i, j \in A_n$, let

$$i \mathbin{\tilde{\equiv}} j = \begin{cases} 2n-1, & \text{if } i = j, \\ \min(i,j,n) - 1, & \text{if } i \neq j \text{ and } \min(i,j) > 0, \\ n-1, & \text{otherwise.} \end{cases}$$

Now let

$$\mathfrak{A}_n = \langle A_n, \tilde{\neg}, \tilde{\to}, \tilde{\equiv} \rangle$$

and

$$\mathfrak{M}_n = \langle \mathfrak{A}_n, D_n \rangle.$$

Define a function $G : \mathcal{P}_{<\omega}(\mathbb{N}) \to \mathbb{N}$ by $G(S) = \sum_{i \in S} 2^i$ for $S \in \mathcal{P}_{<\omega}(\mathbb{N})$, where $\mathcal{P}_{<\omega}(\mathbb{N})$ denotes the set of finite subsets of $\mathbb{N}$. It is easy to see that G is one-to-one and onto, as every natural number has a unique representation as a sum of distinct powers of 2, that is, can be uniquely written in base 2. We will use G to code certain finite sets of natural numbers as elements of A_n.

If $n = 2^k$ for some $k \in \mathbb{N}$, define $i \mathbin{\tilde{\to}'} j = G(G^{-1}(2n-1-i) \cup G^{-1}(j))$ for $i, j \in A_n$. It is easy to check that A_n is closed under $\tilde{\wedge}'$, so this is a well-defined operation on A_n. Further, let $\tilde{\neg}'$ and $\tilde{\equiv}'$ be the same as $\tilde{\neg}$ and $\tilde{\equiv}$, respectively. Finally, let

$$\mathfrak{A}'_n = \langle A_n, \tilde{\neg}', \tilde{\to}', \tilde{\equiv}' \rangle$$

and

$$\mathfrak{M}'_n = \langle \mathfrak{A}'_n, D_n \rangle.$$

Lemma 4.1 *$\mathfrak{M}_n$ and $\mathfrak{M}'_n$ are SCI-models for all $n \in \mathbb{N}$ for which they are defined.*

Proof. Easy but somewhat tedious. □

Lemma 4.2 *$\mathfrak{M}_n$ does not satisfy the WB axioms for any $n > 1$.*

Proof. Let $n > 1$. Then $0 \mathbin{\tilde{\wedge}} \tilde{\neg} 0 = n - 1$ but $1 \mathbin{\tilde{\wedge}} \tilde{\neg} 1 = 0$. However, it follows from the WB axioms that $x \mathbin{\tilde{\wedge}} \tilde{\neg} x = y \mathbin{\tilde{\wedge}} \tilde{\neg} y$ for all $x, y \in A_n$. □

Lemma 4.3 *For every $k \in \mathbb{N}$, the model $\mathfrak{M}'_{2^k}$ satisfies the WB axioms.*

Proof. Clearly $\mathfrak{M}'_{2^k}$ is isomorphic to a Boolean algebra (with $\wedge$ and $\vee$ defined in the usual way in terms of $\neg$ and $\to$) consisting of the subsets of k, equipped with the principal ultrafilter generated by $k-1$, and the operation $\tilde{\leftrightarrow}'$ satisfies the required conditions. □

5 Theories

Our theories will be based on sets of formulas φ_n, for $n \in \mathbb{N} \smallsetminus \{0, 1\}$, such that φ_n expresses the condition "There are not exactly n non-facts." in the models $\mathfrak{M}_n$ and $\mathfrak{M}'_n$. Thus we can get continuum many theories with mutually distinct sets of models.

Firstly, let θ_k be the following formula, for $k \in \mathbb{N} \smallsetminus \{0\}$:

$$\left(\bigwedge_{0 \le i \le k} \neg p_i \right) \to \left(\bigvee_{0 \le i < j \le k} p_i \equiv p_j \right).$$

Its meaning is easy to describe.

Lemma 5.1 *Let* $\mathfrak{M} = \langle \mathfrak{A}, D\rangle$ *be an SCI-model and let* $k \in \mathbb{N} \smallsetminus \{0\}$*. Then* $\mathfrak{M} \models \theta_k$ *iff* $|\operatorname{dom}(\mathfrak{A}) \smallsetminus D| \leq k$*.*

Proof. Assume first that $|\operatorname{dom}(\mathfrak{A}) \smallsetminus D| \leq k$. Let $h : L \to \mathfrak{A}$ be a homomorphism. If $h(p_i) \in \operatorname{dom}(\mathfrak{A}) \smallsetminus D$ for all $i \in \{0, \dots, k\}$, then there must be some $i, j \in \{0, \dots, k\}$ such that $i < j$ but $h(p_i) = h(p_j)$. Then $\mathfrak{M} \models_h \bigvee_{0 \leq i < j \leq k} p_i \equiv p_j$. Otherwise, $\mathfrak{M} \not\models_h \bigwedge_{0 \leq i \leq k} \neg p_i$. In either case, $\mathfrak{M} \models_h \theta_k$. Since h was arbitrary, $\mathfrak{M} \models \theta_k$.

Assume then that $|\operatorname{dom}(\mathfrak{A}) \smallsetminus D| > k$. Choose a homomorphism $h : L \to \mathfrak{A}$ so that $h(i) \in \operatorname{dom}(\mathfrak{A}) \smallsetminus D$ for all $i \in \{0, \dots, k\}$ and $h(i) \neq h(j)$ for all $i, j \in \{0, \dots, k\}$ such that $i \neq j$. Then $\mathfrak{M} \not\models_h \theta_k$, so $\mathfrak{M} \not\models \theta_k$. □

For a formula η, we define

$$\begin{aligned} \eta^0 &= \eta, \\ \eta^{k+1} &= \neg \eta^k \equiv \eta^k. \end{aligned}$$

Now, for $n > 1$, let ψ_n be the SCI-formula

$$\bigwedge_{0 < i < j \leq n} p_n^i \not\equiv p_n^j.$$

Lemma 5.2 *Let* $m, n \in \mathbb{N}$*,* $m > 0$*,* $n > 1$*. Then* $\mathfrak{M}_m \models \psi_n$ *iff* $m \geq n$*. Also, if* $\mathfrak{M}'_m$ *is defined, then* $\mathfrak{M}'_m \models \psi_n$ *iff* $m \geq n$*.*

Proof. Assume first that $m < n$. Let $h : L \to \mathfrak{A}_m$ be an arbitrary homomorphism. Since there are only m non-facts in $\mathfrak{M}_m$ (and in $\mathfrak{M}'_m$, if it is defined), it cannot be the case that all the values $h(p_n^i)$ are different for $i = 1, \dots, n$, as all the p_n^i are clearly false. Therefore, $h(\psi_n) \notin D_m$. So, in fact, $\mathfrak{M}_m \models \neg \psi_n$.

Assume then that $m \geq n$. Let $h : L \to \mathfrak{A}_m$ be an arbitrary homomorphism, and let $k = h(p_n^1)$. For $j \in \mathbb{N}, j > 0$, we have

$$h(p_n^{j+1}) = \begin{cases} h(p_n^j) - 1, & \text{if } h(p_n^j) > 0, \\ m - 1, & \text{otherwise.} \end{cases}$$

Thus, for $i, j \in \mathbb{N} \smallsetminus \{0\}$, it holds that $h(p_n^i) = h(p_n^j)$ iff $i \equiv j \pmod{m}$. So, $h(p_n^i) \neq h(p_n^j)$ for all $0 < i < j \leq m$, thus, in particular, for all $0 < i < j \leq n$. We may conclude that $\mathfrak{M}_m \models \psi_n$. □

Theorem 5.3 *There are continuum many different invariant SCI-theories that are independent of WB, as well as continuum many different invariant SCI-theories extending WB that are independent of WH.*

Proof. For each $A \subseteq \omega$, let S_A be the following set of SCI-formulas:

$$\{\theta_{2^{k+1}-1} \vee \psi_{2^{k+1}+1} | k \in A\}$$

Let further T_A be the closure of $S_A \cup$SCI and T'_A the closure of $S_A \cup$WB under substitution and $\vdash$. Then all the theories T_A are distinct extensions of SCI as they have different sets of models, and they are independent of WB. Likewise, all the theories T'_A are distinct extensions of WB, and they are independent of WH. $\square$

The above theorem should be contrasted with Theorem 5.9 below. However, we first need some lemmas.

Lemma 5.4 *Let A and B be models of WH and let A be finite. Then A can be embedded in B iff $|A| \leq |B|$.*

Proof. Assume $|A| \leq |B|$. Then A is a finite Boolean algebra equipped with an ultrafilter and a definable operation $\hat{\equiv}$. So, A is isomorphic to the Boolean algebra of subsets of a natural number, and the ultrafilter is principal, i.e., generated by a singleton. If B is finite, the required embedding is easy to find. If B is infinite, take a subset $S \subseteq B$ such that $|S| = |A|$. It generates a finite submodel of B, into which A can be embedded.

The "only if" part is trivial. $\square$

Corollary 5.5 *If $|A| \leq |B|$, A is finite, and φ is a universal sentence in the first-order SCI language such that $B \models \varphi$, then $A \models \varphi$.*

Lemma 5.6 *Let $\varphi(\bar{x})$ be a quantifier-free formula in the first-order SCI-language such that the sentence $\forall \bar{x} \varphi(\bar{x})$ is true in all finite models of WH. Then $\forall \bar{x} \varphi(\bar{x})$ is true in all infinite models of WH as well.*

Proof. Assume otherwise. Thus there is an infinite ultrafiltered Henle algebra H and elements $\bar{a} \in \text{dom}(H)$ such that $H \not\models \varphi(\bar{a})$. Let H' be the submodel of H generated by $\bar{a}$. Then H' is finite and $H' \not\models \varphi(\bar{a})$, hence $H' \not\models \forall \bar{x} \varphi(\bar{x})$, which is a contradiction. $\square$

Lemma 5.7 *If A is an infinite model of WH, then an SCI-formula φ is true in A iff it is true in all finite models of WH.*

Proof. Assume first that $A \models \varphi$, and let B be a finite model of WH. Then by Corollary 5.5, $B \models \varphi$.

Let then φ be an SCI-formula that is true in all finite models of WH. By Lemma 5.6, $A \models \varphi$. □

Lemma 5.8 *The class of models of an extension of WH is completely determined by the supremum of the sizes of its finite models.*

Proof. Let T be an extension of WH. Assume first that the set of sizes of finite models of T is bounded. Let n be the maximum of the sizes. Then, by Corollary 5.5, every $\varphi \in T$ is true in all models A of WH such that $|A| \leq n$. Thus a model A of WH is a model of T iff $|A| \leq n$.

Assume then that T has arbitrarily large finite models. By Corollary 5.5, every $\varphi \in T$ is true in all finite models of WH, and, by Lemma 5.6, in all infinite models as well. Thus T is true in all models of WH. □

Theorem 5.9 *There are only countably many different theories between WH and WF.*

Proof. An immediate corollary of the preceding lemma, since there are only countably many suprema of subsets of $\mathbb{N}$. □

6 Open Questions

We would like to conclude by discussing some possible directions for further research.

1. We have seen that there are uncountably many different invariant axiomatic extensions of SCI. This fact in itself suggests that most of them should be regarded as curiosities, with little independent mathematical or philosophical significance. In particular, our theories T_A and T'_A have a rather artificial flavour. One can ask what criteria a non-Fregean logic should satisfy to count as mathematically or philosophically "interesting" or "natural".

2. A possible partial answer to the first question is that a natural extension of SCI should have an adequate model. An SCI-model $\mathfrak{M}$ is called *adequate* for an invariant extension T if for every SCI-formula φ such that $T \not\vdash \varphi$ there is an $\mathfrak{M}$-valuation h such that $\mathfrak{M} \models_h T$ but $\mathfrak{M} \not\models_h \varphi$. In other words, the logical consequences of T inside $\mathfrak{M}$ are the same as the logical consequences of T in general. It is known that SCI, WB and WH have adequate models (see [1], [6], [7] and [8]), while our extensions obviously do not. However, all adequate models of SCI and WB are uncountable, while WH has a countable adequate model. This may be a coincidence, but we would like to ask whether there is any connection between the number of extensions of a non-Fregean logic and the size of its smallest adequate model, if one exists.

3. If we look at the set of all invariant SCI-theories, we get a partially ordered structure, with the order determined by the relative strengths of the theories. This is a lattice, whose smallest element is SCI and largest element WF. Our results show that this lattice is uncountable (in fact, the lattice of the subsets of $\mathbb{N}$ can be embedded in it) but that WH has only countably many successors. What else can be said about this lattice? Does WB, for instance, have some special properties as an element of the lattice? Are there any non-trivial automorphisms of the lattice?

4. We have only considered extensions of SCI formed by adding axiom schemes. An extension of SCI is called *non-elementary* if it cannot be obtained from SCI by adding a set of axioms, but rather by adding new rules of inference. The existence of non-elementary extensions of SCI was proved by Bloom and Suszko in [1]. All the above questions can be extended to include the non-elementary extensions of SCI as well.

References

[1] S. L. Bloom and R. Suszko: Investigations into the sentential calculus with identity, *Notre Dame Journal of Formal Logic* **13**/3 (1972), pp. 289–308.

[2] Mieczysław Omyła: *Zarys logiki niefregowskiej*, Państwowe Wydawnictwo Naukowe, Warszawa 1986.

[3] Roman Suszko: Non-Fregean Logic and Theories, *Analele Universitatii Bucuresti, Acta Logica* **9** (1968), pp. 105–125.

[4] —: Identity Connective and Modality, *Studia Logica* **27** (1971), pp. 7–39.

[5] —: Abolition of the Fregean Axiom, in *Logic Colloquium 1972–73*, eds. A. Dold and B. Eckmann, Springer-Verlag, Berlin–Heidelberg–New York 1975, pp. 169–239.

[6] —: A note on modal systems and SCI, *Bulletin of the Section of Logic* **1**/4 (1972), pp. 42–45.

[7] —: Adequate models for the non-Fregean sentential calculus SCI, in *Logic, Language and Probability. A selection of papers of the* 4th *International Congress for Logic*, Methodology and Philosophy, Reidel Publ. Company, Dordrecht–Holland 1973, pp. 49–54.

[8] S. Zachorowski: Proof of a conjecture of R. Suszko, *Studia Logica* **34**/3 (1975), pp. 253–256.

Joanna Golińska-Pilarek

Institute of Philosophy
University of Warsaw
Krakowskie Przedmieście 3
00-047 Warszawa, Poland

`joszaG@poczta.onet.pl`

Taneli Huuskonen

Department of Mathematics
P.O. BOX 4
FIN-00014 University of Helsinki
Finland

`huuskone@cc.helsinki.fi`