Randomness - from micro to macro

2.9.2013

Probability in physics and mathematics

Probability entered physics in late 19th century: Maxwell, Boltzmann, Gibbs and in the 20th e.g. with Kolmogorov

Later, models from physics have had big influence on probability.

Quite recently, probabilistic ideas have spread in other fields of mathematics, especially in analysis.

Central Limit Theorem

Fluctuations of a large number of **weakly correlated** random variables $\{x_i\}_{i=1}^N$ are gaussian

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(x_i - \mathbb{E}x_i) \to x^* \quad \text{as} \quad N \to \infty$$

x^* Gaussian random variable.

Example: x_i magnetic moments of atoms in an iron bar held at low or high temperature.

Universality: details of x_i unimportant.

Universality classes

CLT breaks down at **critical temperature**

- Nontrivial **scaling exponents**
- **Nongaussian** limit laws

Universality is an experimental fact: same laws for different materials having same symmetries

For magnets and their generalizations in **two spatial dimensions** there has been much progress in finding all such limit laws by physicists and mathematicians.

Disordered systems

Physical systems such as crystals are seldom clean: they contain **impurities** at random locations that can qualitatively change their properties.

E.g. **electrical resistance** results from such randomness.

Quantum theory:

- Schrödinger equation with periodic potential due to atoms in the crystal lattice.
- Disorder modifies the potential at random locations.

Random matrices

Physics determined by the nature of the energy **spectrum**

Due to disorder this is a random set of points on the real axis.

These random numbers are expected to possess novel **universal**, non-gaussian statistical properties.

Beautiful mathematical understanding in the context of **large random matrices**.

Fluctuations in nonlinear dynamics

Laws of macroscopic physics deal with averages of microscopic degrees of freedom, e.g. fluid velocity field vs. motion of atoms.

Mathematically they emerge as **law of large numbers** from microscopic randomness and take the form of **partial differential equations**.

Fluctuations of the macroscopic quantities are governed by **stochastic partial differential equations**.

Universal Fluctuations

Example: surface growth, $h(x,t)$ hight of an interface

$$\partial_t h = \nu \Delta h + \lambda (\nabla h)^2 + \eta(x,t)$$

$\eta(x,t)$ space-time white noise.

- $\lambda = 0$ gaussian fluctuations
- $\lambda \neq 0$ non-gaussian fluctuations **"KPZ universalty class"**
- Miraculous relation to random matrix universality

Universal scale invariant non-gaussian statistics in many other domains e.g. fluctuations in a **turbulent** fluid and in financial time series

Geometry of Nature

Benoit Mandelbrot (1924-2010):

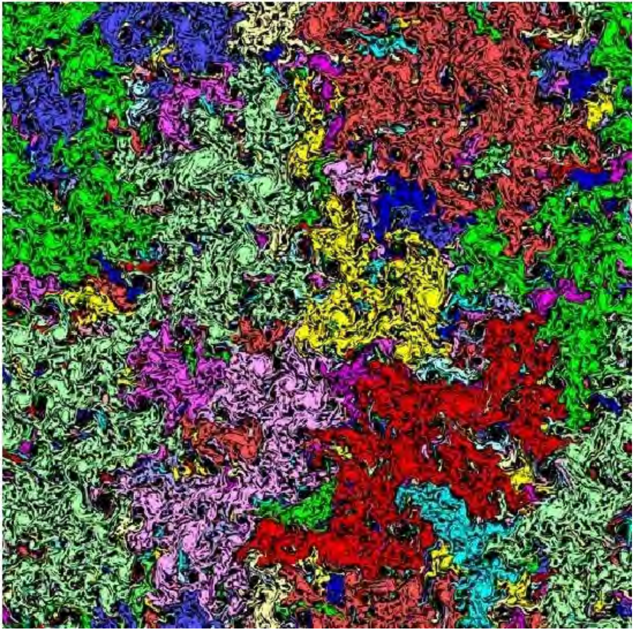
"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."

Clouds are not spheres

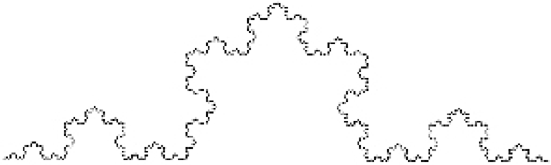
coastlines are not circles

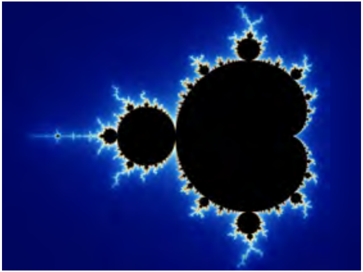
nor does lightning travel in a straight line

Turbulence



Nature's fractals are often very different from regular fractals:



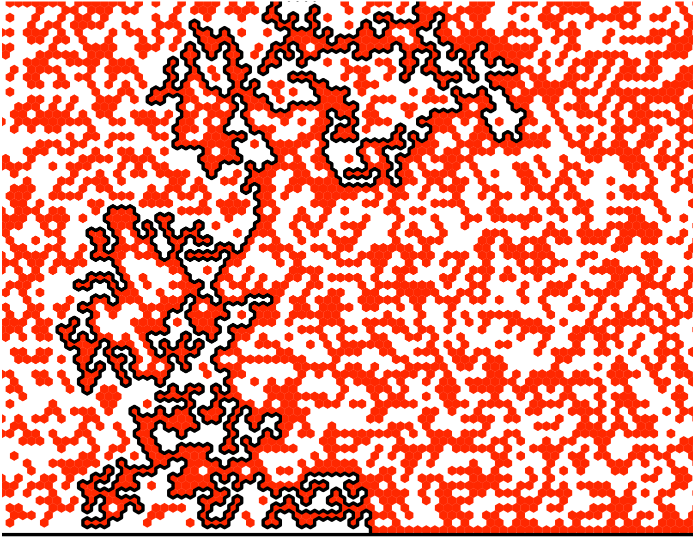


Random geometry

Traditional geometry: Study and classify geometric objects (curves, manifolds etc.) with given properties (e.g. smoothness, symmetry etc).

Random geometry: Study **ensembles** of objects together with a **probability distribution** possessing given properties (e.g. symmetry).

Percolation

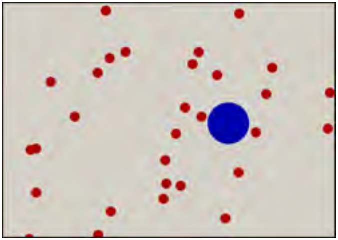


Statistically scale (and conformal) invariant!

Origins of randomness

How does randomness emerge in a deterministic world?

1st answer: the world is large



Almost surely in the initial conditions for the gas particles the blue particle's motion is a stochastic process which becomes Brownian motion in a long time **scaling limit**.

Even for an ideal gas this is still unproved!

Origins of randomness

2nd answer: the world is **non-linear**

Sensitive dependence on initial conditions i.e. **chaos** generates motions that are as random as ideal coin-toss even for systems of a **few degrees of freedom**.

What is the relative role of 1 and 2 in the randomness we see around us?

- Increase of entropy: 1?
- Viscosity of water: 2?