Random Matrix Universality

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Randomness and disorder in the exact sciences

Helsinki, September 2-4, 2013

Law of large numbers. Central limit theorem.

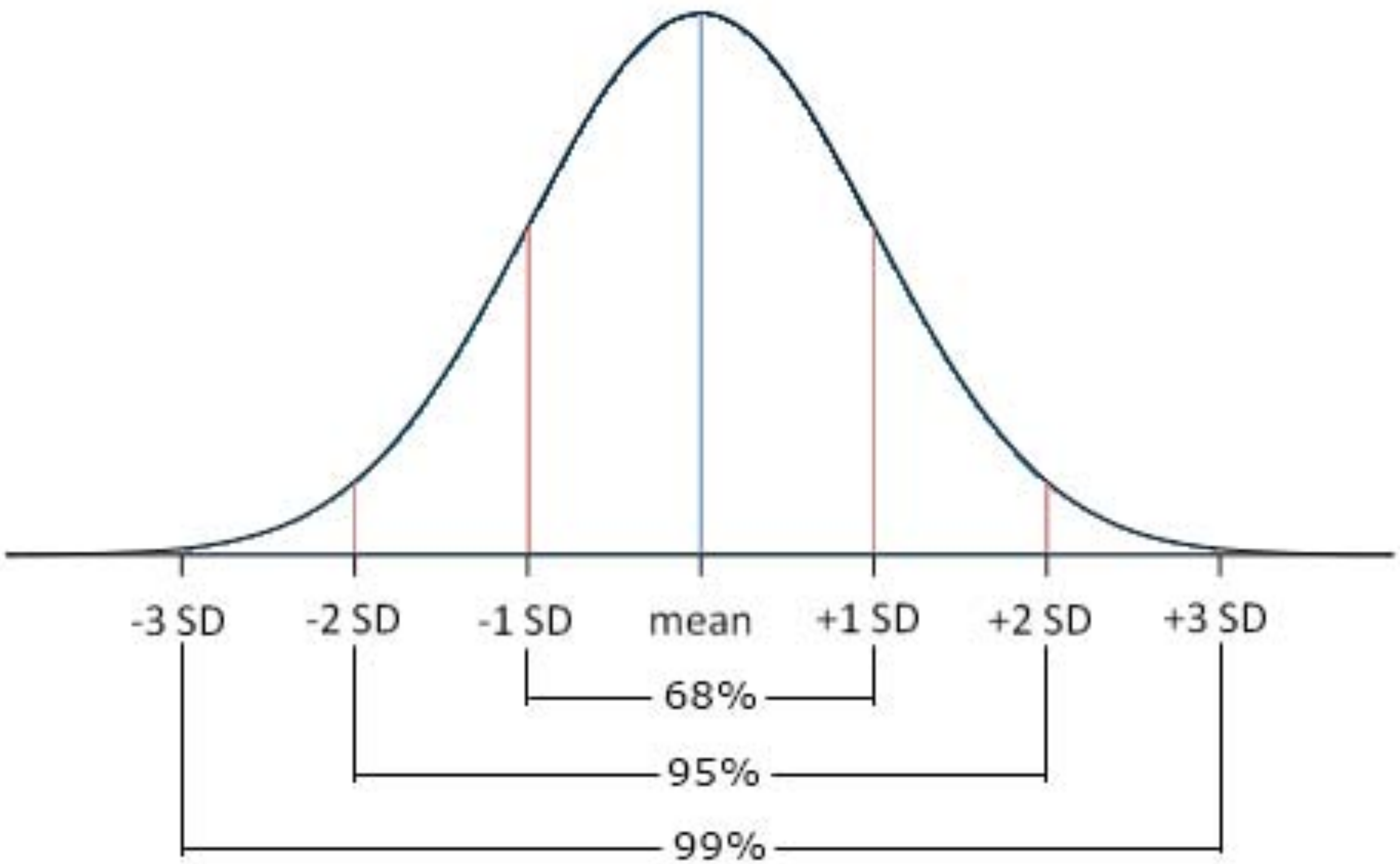
$$\overline{X}_n = \frac{X_1 + \dots + X_n}{n}$$, X_i independent random variables

$$\overline{X}_n \to m \, , n \to \infty$$ Law of large numbers

$$\overline{X}_n \approx m + \frac{1}{\sqrt{n}} Y$$, Y Gaussian

Central limit theorem. Fluctuations.

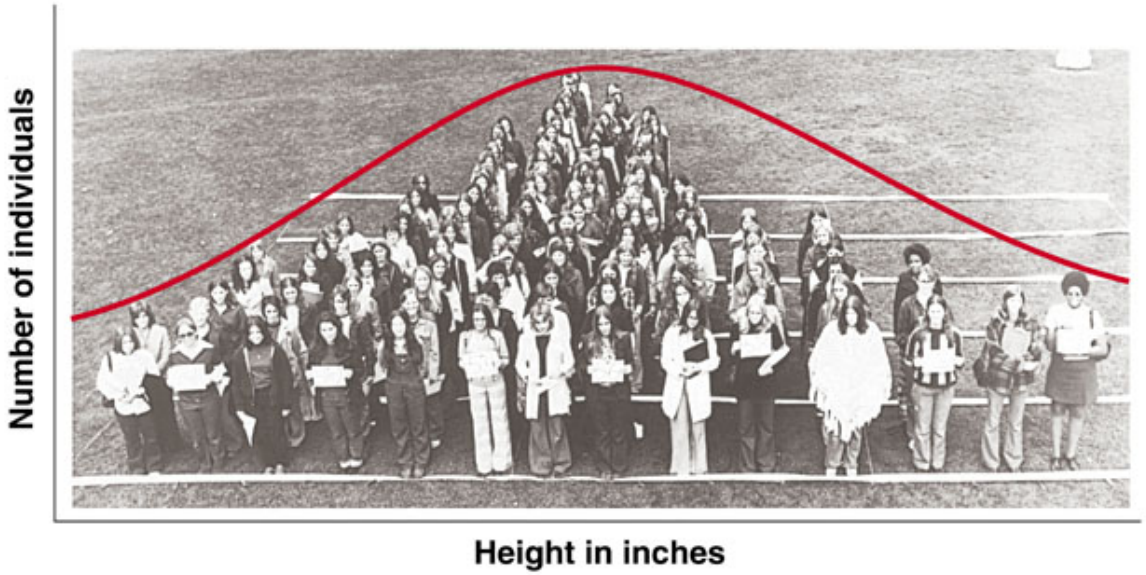
Gaussian distribution



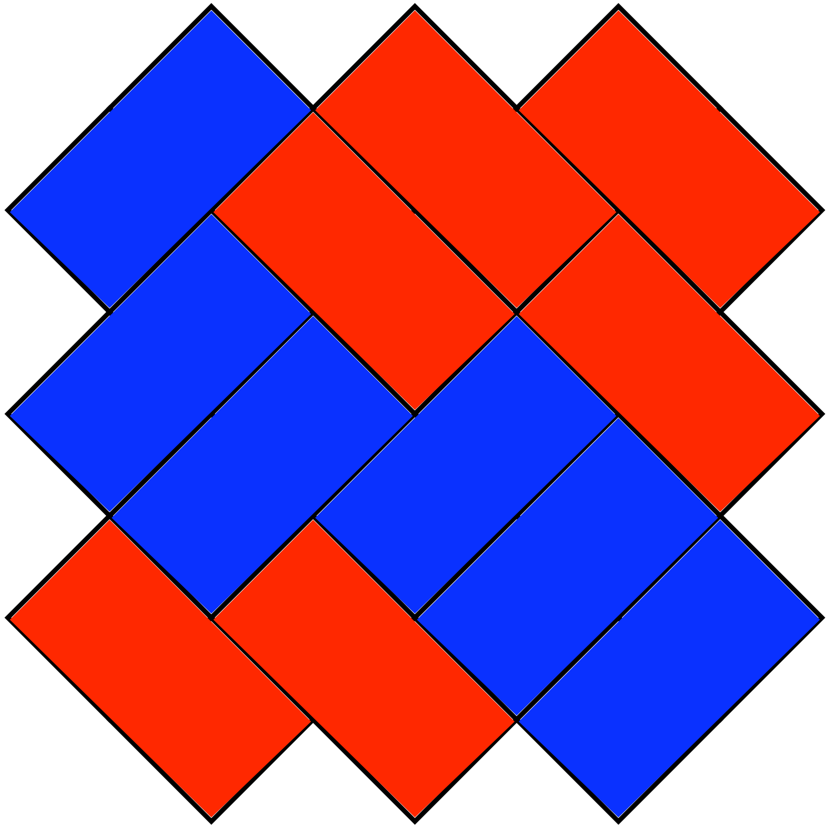
Universality of the Gaussian distribution

The Gaussian distribution is a natural limit law in many contexts.

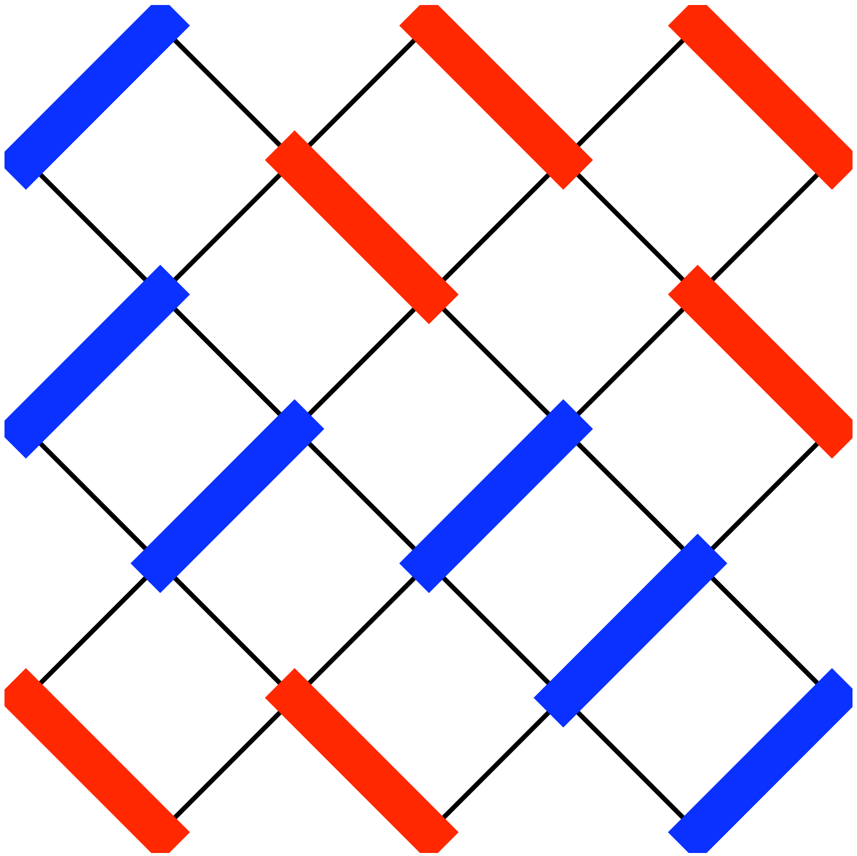
Tobin/Dusheck, Asking About Life, 2/e
Figure 16.6



The Aztec diamond

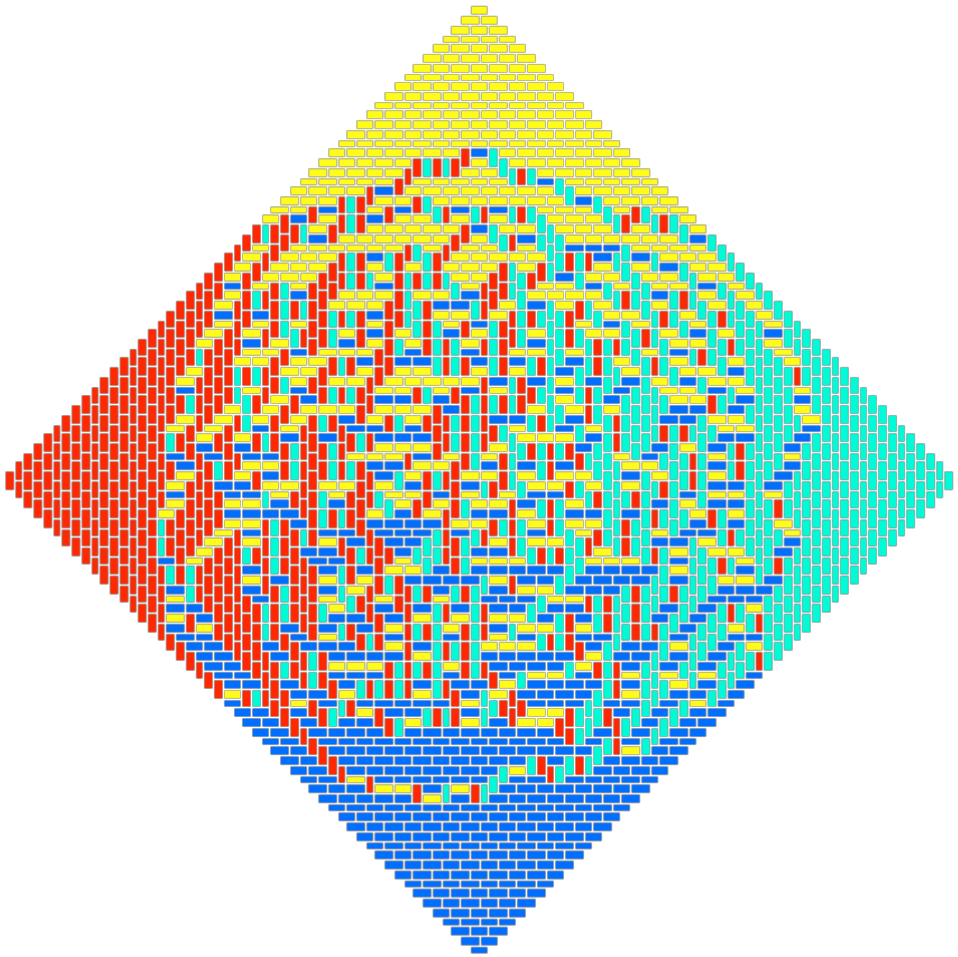


Domino tiling



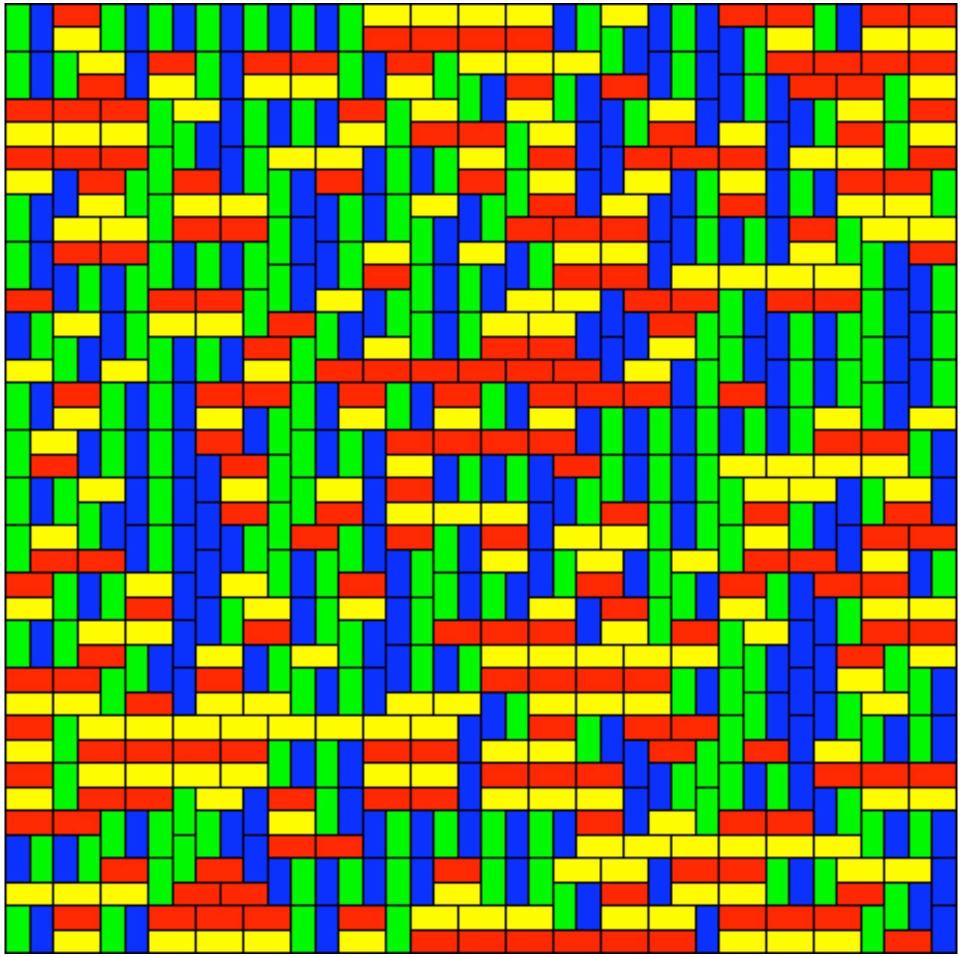
Dimer model. Perfect matching

A random tiling of an Aztec diamond

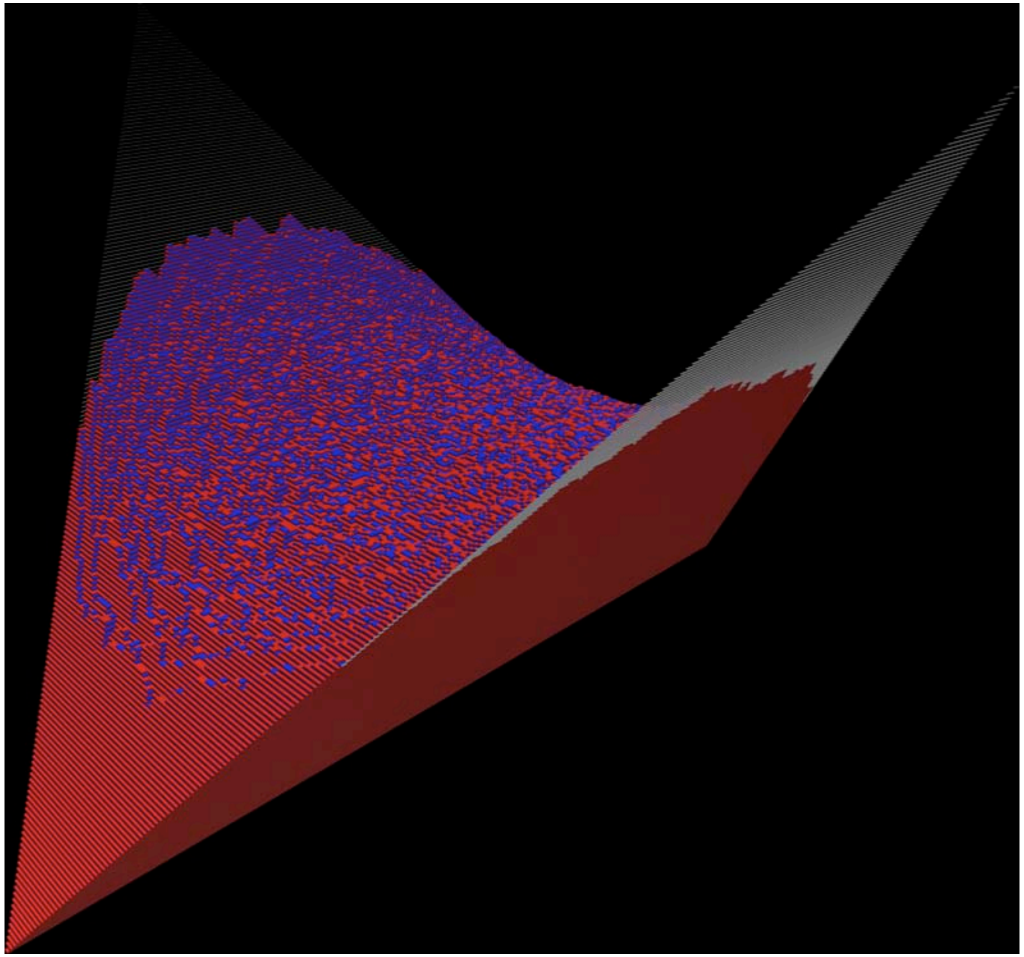


Elkies, Kuperberg, Larsen, Propp 1992

Tiling of a square



Height function of a random tiling of the Aztec diamond. A random surface.



Picture by B. Young

Random matrix theory

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$Av = \lambda v$, λ eigenvalue

Spectrum = set of all eigenvalues

Random elements a_{ij} → Random matrix

Random spectrum

A symmetric / Hermitian

$\lambda_1, \lambda_2, \ldots, \lambda_n$ real eigenvalues

A random $\rightarrow$ $\lambda_1, \ldots, \lambda_n$ random

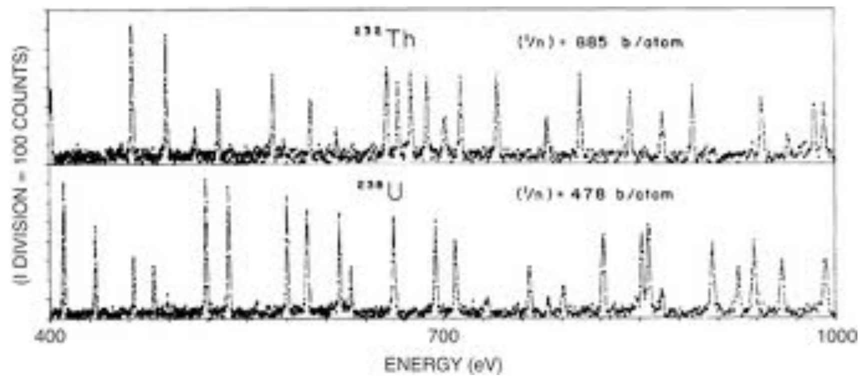
Random point process on the line

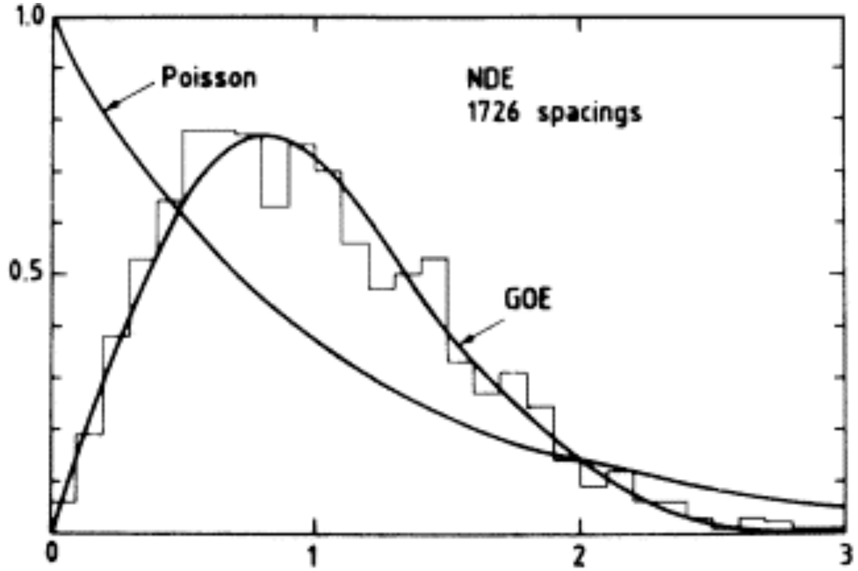
Statistical properties as $n \to \infty$?

Model for real spectra

- Wigner introduced random matrix in physics in the 1950's to model complicated spectra.
- The Hamiltonian is modelled by a large random matrix.
- Is there any information in the spectrum or does it look like a completely random spectrum? We need a statistical model to compare with.

Nuclear spectra

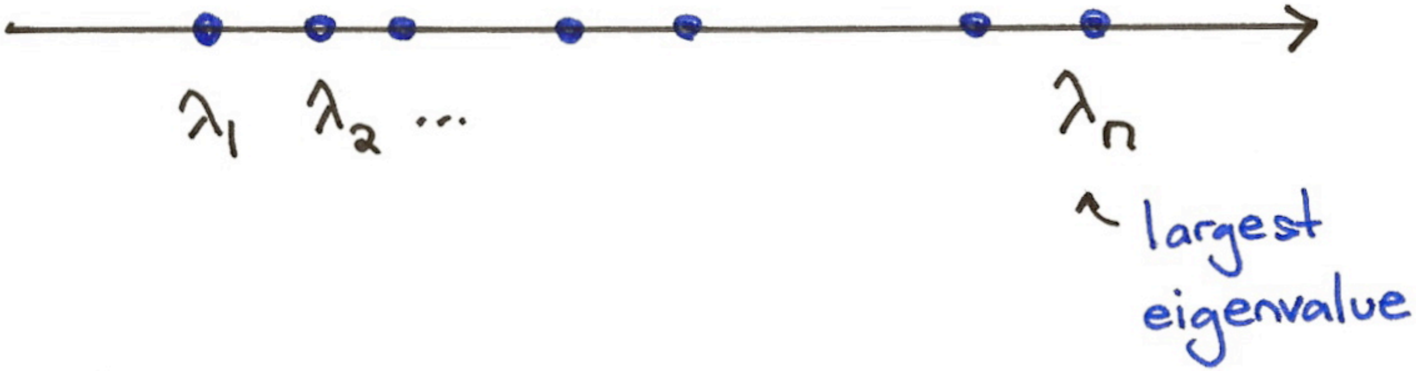




Random spectra

- Universality is important. The real Hamiltonian does not look at all like a random matrix. The Hamiltonian deterministic.
- Many applications in quantum physics, number theory, other parts of mathematics, and in statistics.
- Mathematically it is very difficult to prove theorems concerning the statistics of the eigenvalues even in simplified models.
- If we consider the random matrices themselves, there are universality theorems for various random matrix ensembles, i.e. probability distributions on elements. Pastur-Shcherbina 1997, Deift-Kriecherbauer-McLaughlin- Venakides-Zhou 1999 , J. 2001, Erdös-Schlein-Yau 2009-, Tao-Vu 2011

Largest eigenvalue



$\lambda_1, \dots, \lambda_n$ eigenvalues of A

Take A from the Gaussian Unitary Ensemble

$$\frac{1}{Z_n} e^{-\mathrm{Tr}\, M^2} dM$$

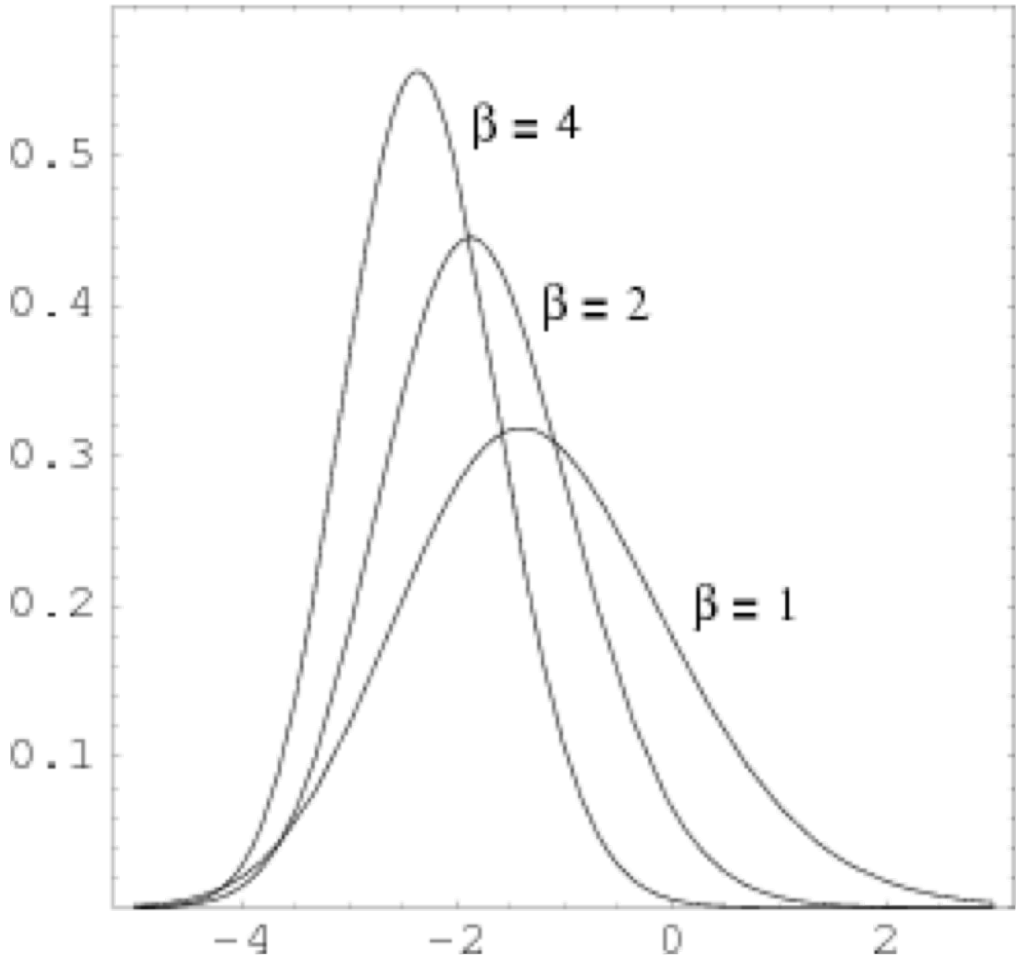
Tracy-Widom largest eigenvalue distribution

The largest eigenvalue λ_n of A is a random variable

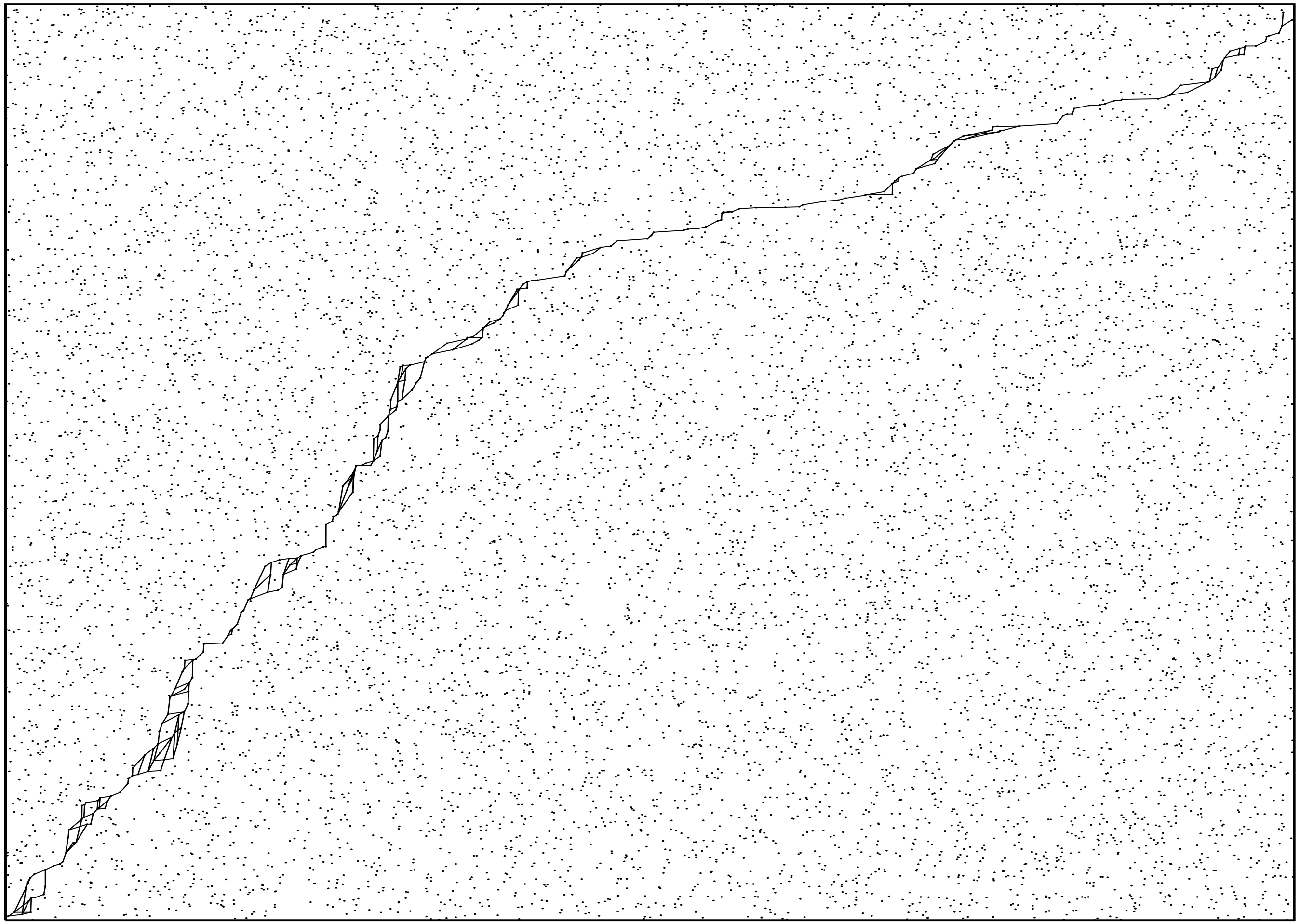
$$\sqrt{2n}\,\lambda_n \approx 2n + n^{1/3} Z \quad , \; n \to \infty$$

Z is a random variable with the Tracy-Widom distribution

Tracy-Widom distribution



10000 random points (Poisson process in the rectangle). Pick up as many points as possible in an up/right path. Random polymer. The points are sites of low energy. Expected number of points is 200 = $2\sqrt{10000}$. Fluctuations?



Fluctuations. Limit law.

ℓ_n = the number of points in an optimal path

$$\ell_n \approx 2\sqrt{n} + (\sqrt{n})^{1/3} Z$$

Z has the Tracy-Widom distribution

Baik-Deift-J. '99

Polynuclear growth model

Exactly solvable model for a random 1-dimensional interface.

h(x,t) fluctuates around a limiting shape according to the Tracy-Widom distribution

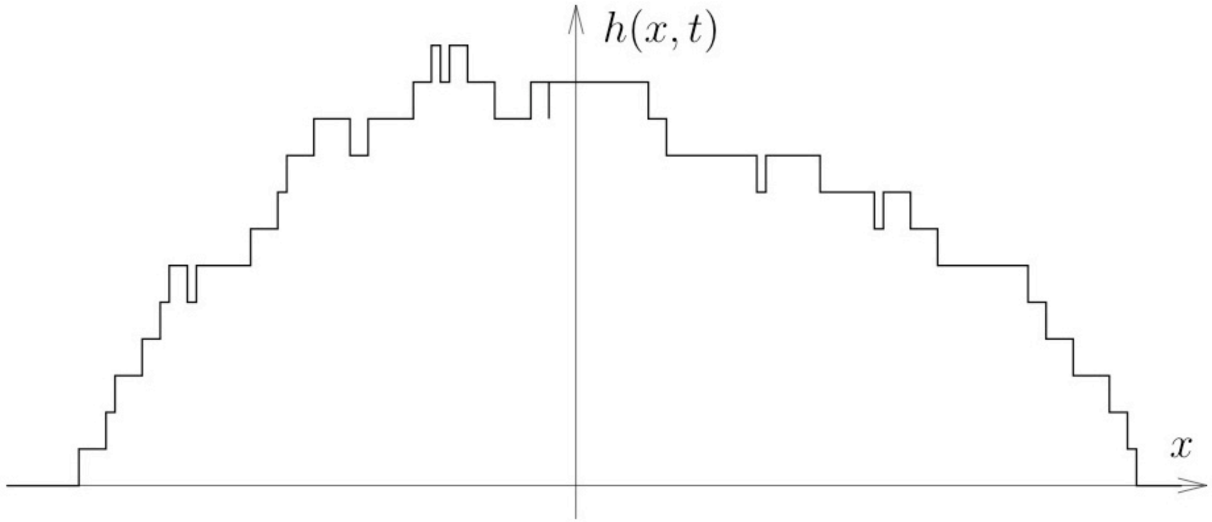
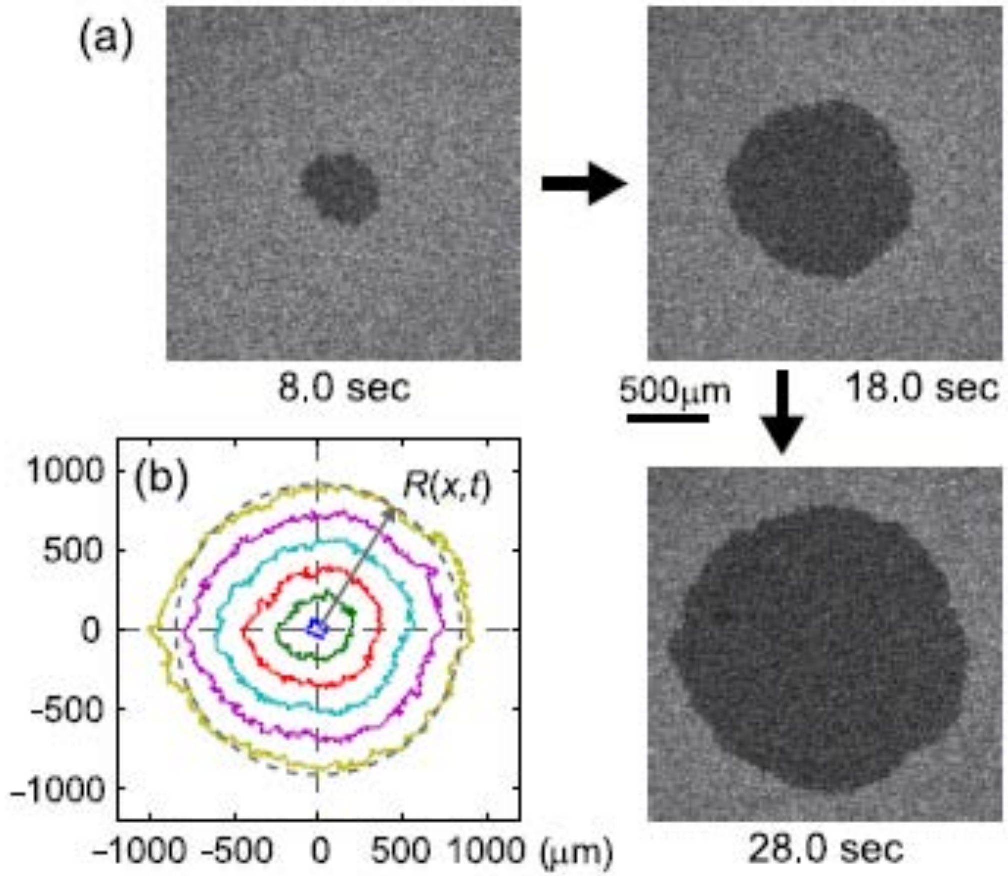


Figure 2: A sample of the PNG droplet.

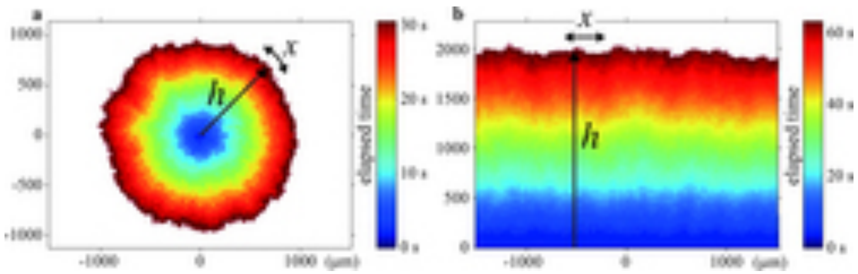
Prähofer-Spohn ’00, J. ’00

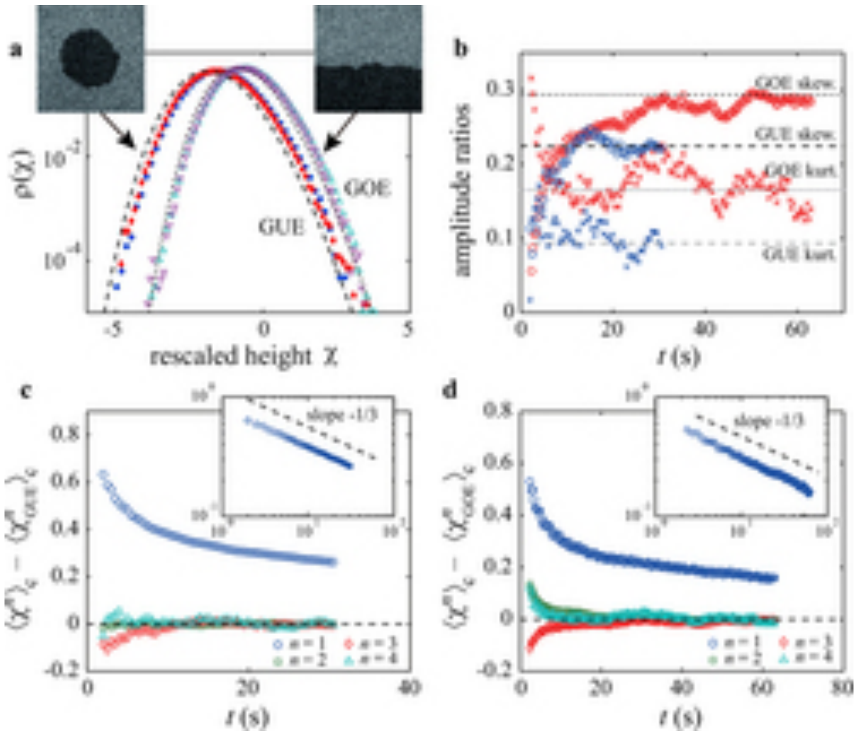
Random growth experiment



Takeuchi-Sano 2010

Random growth experiment





Kardar-Parisi-Zhang equation

$$\frac{\partial h}{\partial t} = -\lambda\left(\frac{\partial h}{\partial x}\right)^2 + \nu\frac{\partial^2 h}{\partial x^2} + \sqrt{D}\,\xi$$

$h(x,t)$ height at $x \in \mathbb{R}$ at time $t \geq 0$

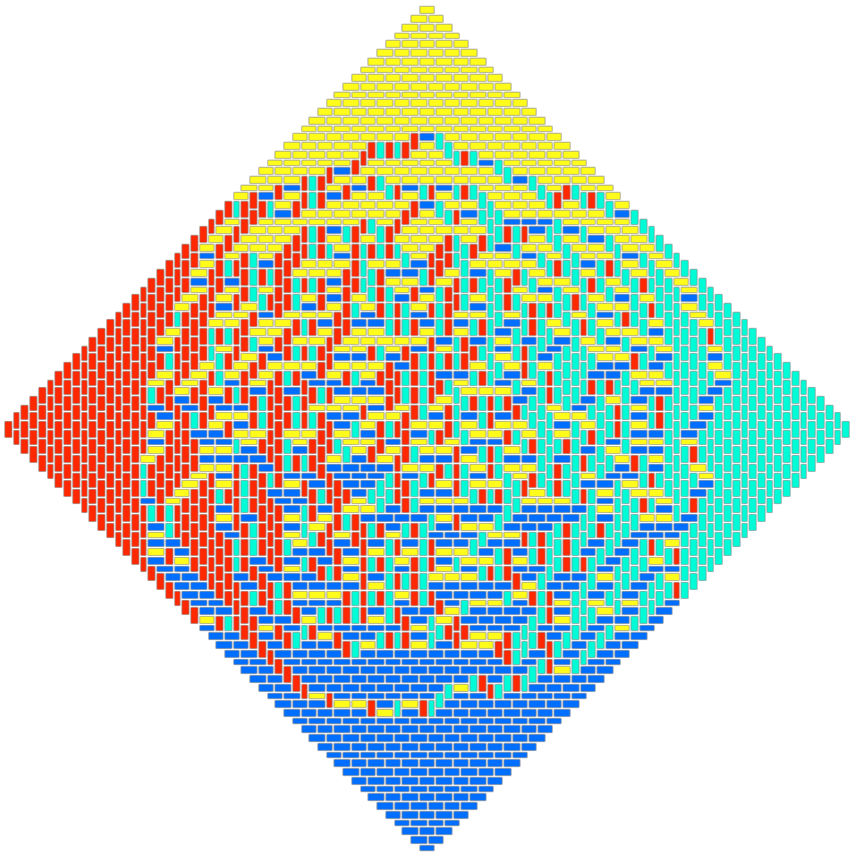
ξ space-time white noise

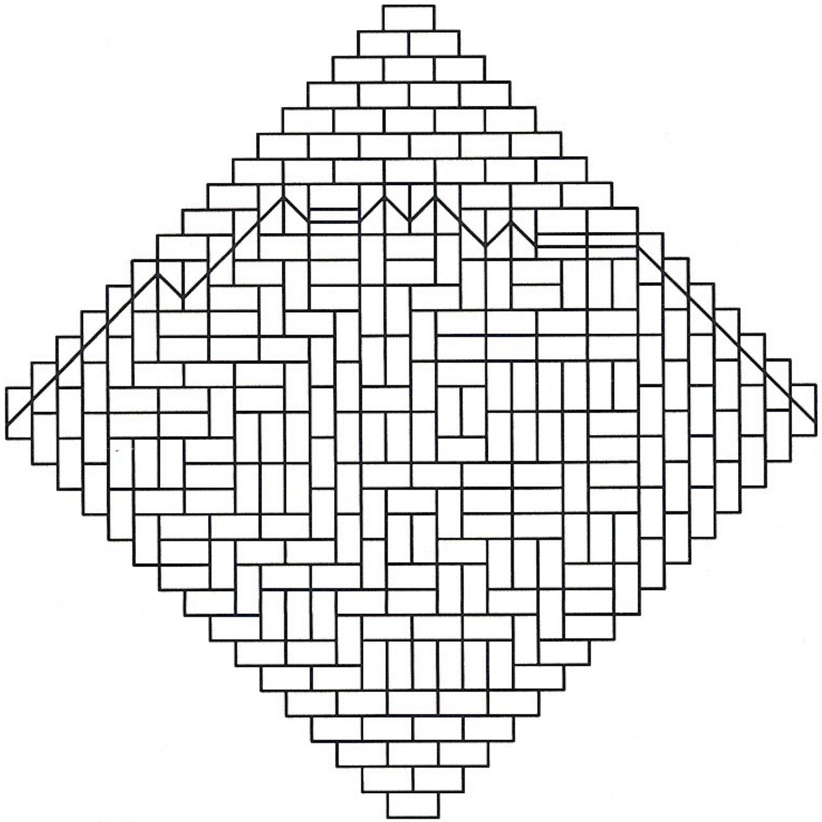
λ, ν, D physical constants

Kardar-Parisi-Zhang 1986, Tracy-Widom 2008, Amir-Corwin-Quastel 2010, Sasamoto-Spohn 2010, Borodin-Corwin 2011, Hairer 2011

A random tiling of an Aztec diamond

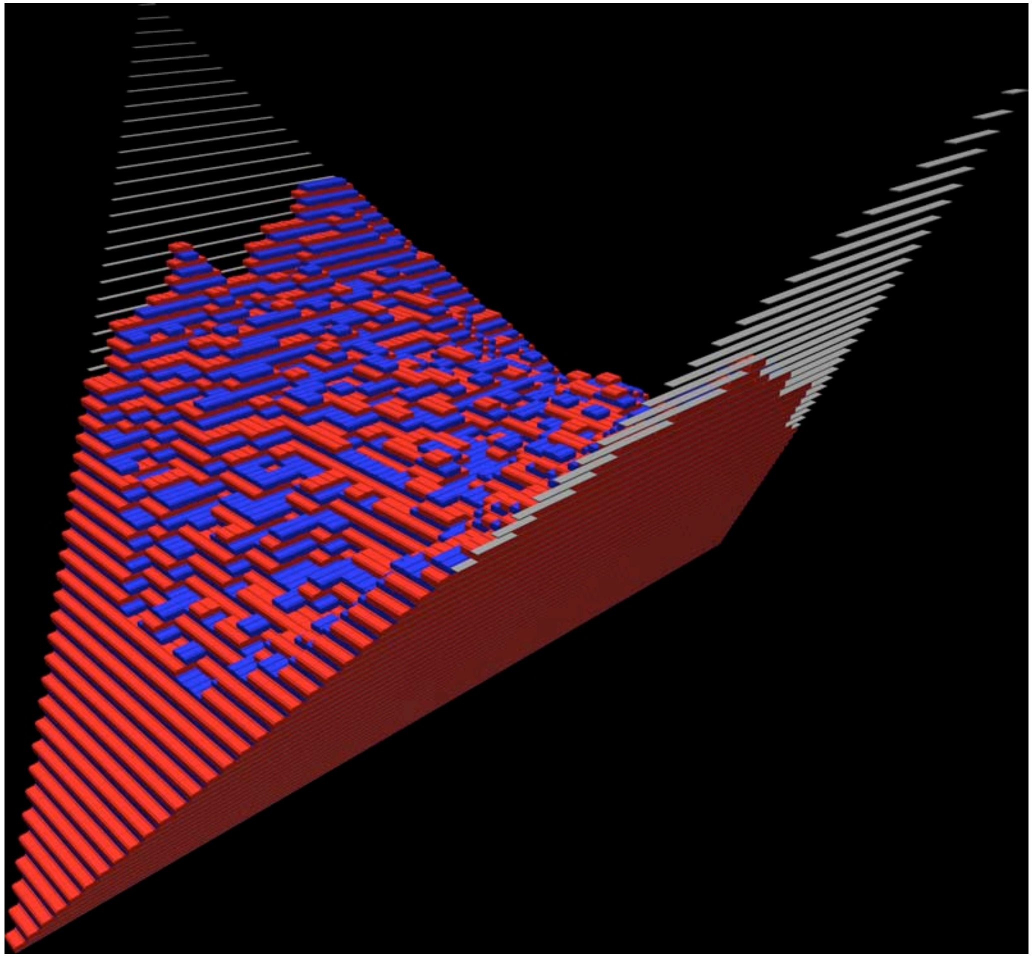
There is a limiting circular shape and the fluctuations around it are Tracy-Widom.



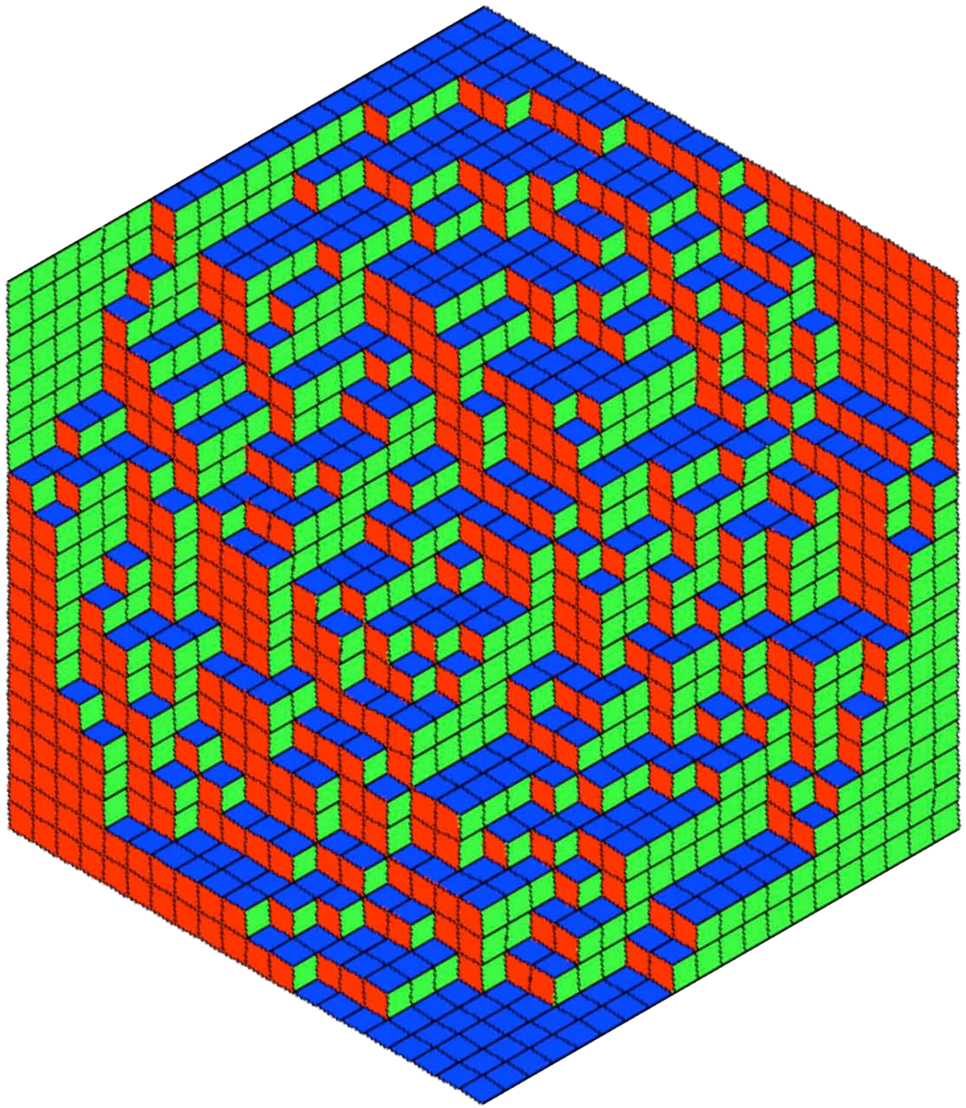


Propp-Jockush-Shor 1998, J. 2005

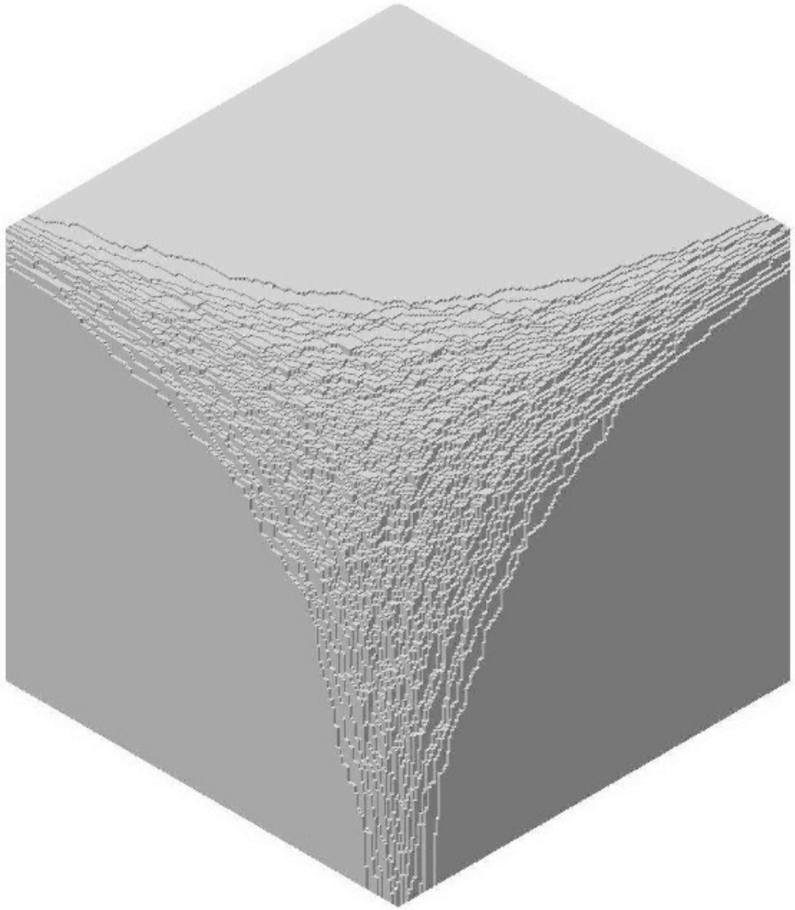
Surface representation of a tiling of the Aztec diamond. Curved part and flat parts, facets.



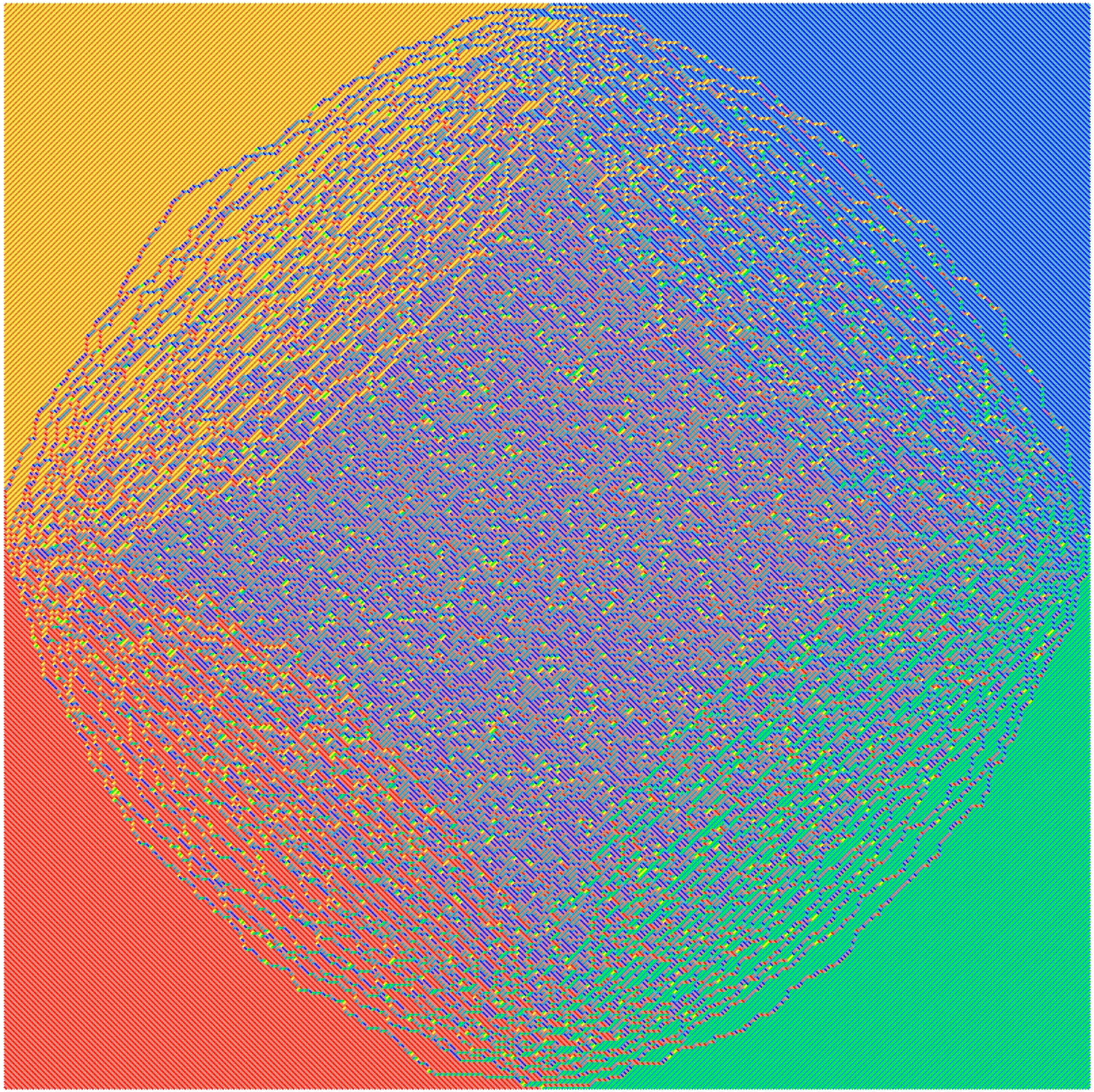
Random tiling of a hexagon by lozenges.
Cubes stacked in a corner. Crystal with facets.



Crystal corner



Okounkov-Reshetikhin 2003, Ferrari-Spohn 2003



Picture by
S. Chhita