If you can't choose, throw dice – Random constructions in mathematical (harmonic) analysis

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Harmonic analysis

(traditionally) = Fourier analysis

= study of decomposition ('analysis') of functions / signals into basic sine waves ('harmonics')

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{i2\pi kx}$$

Harmonic analysis

(today) = study of multiple phenomena that have emerged and grown ‘around’ the classical theory of Fourier series

- **interactions of spatio-temporal and spectral information**
- **cancellations, orthogonality, oscillations**

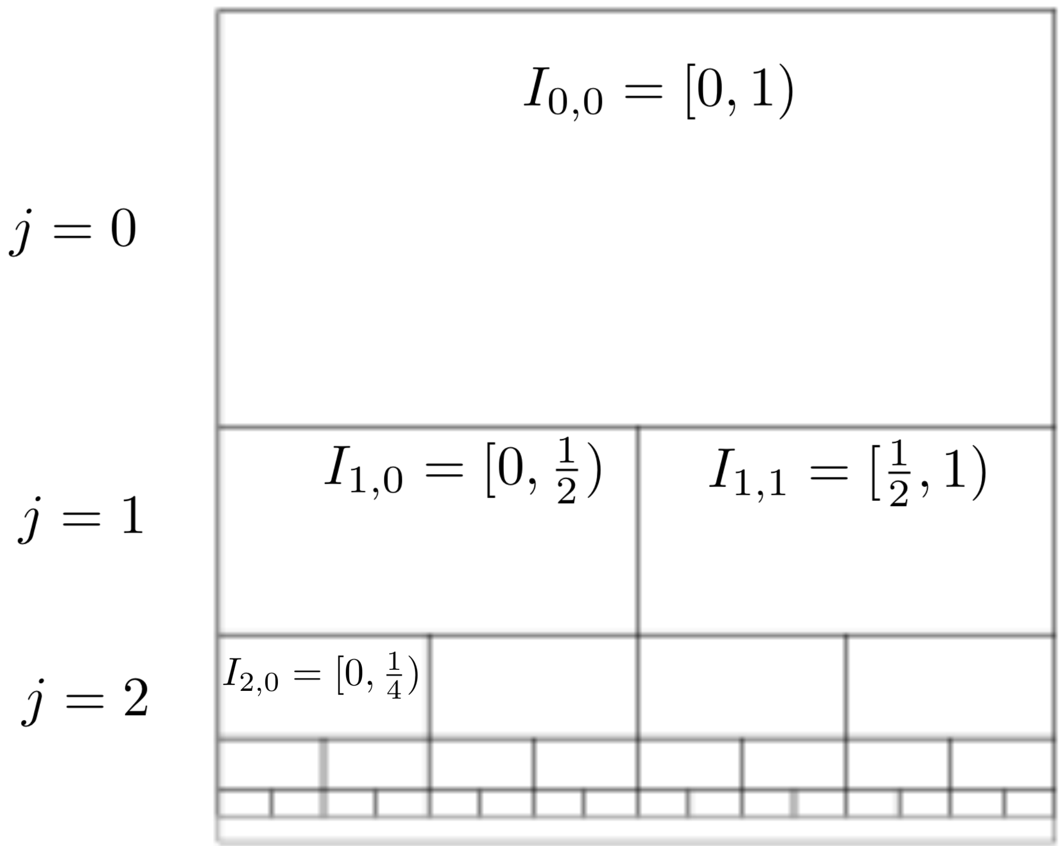
‘Modern’ block waves and wavelets: Rademacher and Haar functions

$$r_j(x) := \operatorname{sign}(\sin(2\pi \cdot 2^j x))$$

$$h_{j,k}(x) := h_I(x) := \frac{1_I(x)}{|I|^{1/2}} r_j(x)$$

$$I = I_{j,k} = [\frac{k}{2^j}, \frac{k+1}{2^j})$$

Dyadic analysis of a function $f(x)$



- **Refine grid repeatedly dividing into halves**
- **At every scale, replace *f* by its average on each dyadic interval**
- **Change from one scale to next:**

$$\Delta_I f := 1_{I_{\text{left}}} \langle f \rangle_{I_{\text{left}}} + 1_{I_{\text{right}}} \langle f \rangle_{I_{\text{right}}} - 1_I \langle f \rangle_I = h_I \langle h_I, f \rangle$$

$$f = \sum_I \Delta_I f = \sum_I h_I \langle h_I \rangle f$$

More robust for ‘rough’ situations

$$\Delta_I^\mu f := 1_{I_{\text{left}}} \langle f \rangle_{I_{\text{left}}}^\mu + 1_{I_{\text{right}}} \langle f \rangle_{I_{\text{right}}}^\mu - 1_I \langle f \rangle_I^\mu = h_I^\mu \langle h_I^\mu, f \rangle$$

- **Averaging over intervals is meaningful with respect to any mass distribution (‘measure’) in place of the uniform distribution (Lebesgue measure)**

$$\langle f \rangle_I^\mu := \frac{1}{\mu(I)} \int_I f(x)\,\mathrm{d}\mu(x)$$

- **Weighted Haar functions h_I^μ remain piecewise constant, average to zero with respect to μ**

Dyadic analysis – what for?

- **Divide and conquer – estimates for general *f* reduced to simpler $\Delta_I f$.**

$$f = \sum_I \Delta_I f$$

- **Especially when estimating the norm of a linear operator *T* acting on *f***

$$|\langle Tf, g\rangle| \leq C\|f\|\|g\| \ ?$$

$$\langle Tf, g\rangle = \sum_{I,J} \langle T\Delta_I^\mu f, \Delta_J^\nu g\rangle$$

Key question: bounds for the Hilbert transform

$$Hf(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)\,\mathrm{d}y}{x-y}$$

Classical Fourier analysis: $\widehat{Hf}(\xi) = -i\,\mathrm{sign}(\xi)\hat{f}(\xi)$

$$\|Hf\|_{L^2} = \Big(\int_{-\infty}^{\infty} |Hf(x)|^2\,\mathrm{d}x\Big)^{1/2}$$

$$= \|\widehat{Hf}\|_{L^2} = \|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

But for the *weighted* Hilbert transform…

$$H(f\,\mathrm{d}\mu)(x) = \int_{-\infty}^{\infty} \frac{f(y)\,\mathrm{d}\mu(y)}{x-y} \qquad \Big|\int H(f\,\mathrm{d}\mu)g\,\mathrm{d}\nu\Big| \leq C\|f\|_{L^2(\mu)}\|g\|_{L^2(\nu)}?$$

…need dyadic analysis!

Dyadic analysis of the weighted Hilbert transform

$$\int H(\Delta_I^\mu f \,\mathrm{d}\mu)\Delta_J^\nu g \,\mathrm{d}\nu = \iint_{I\times J} \frac{1}{x-y} \Delta_I^\mu f(y) \Delta_J^\nu g(x) \,\mathrm{d}\mu(y) \,\mathrm{d}\nu(x)$$

Difficulties:

- **Division by zero if *x* = *y***
- **Discontinuity if *y* lies at centre or boundary of *I***
- **Or if *x* lies at centre or boundary of *J***

Worst case: all at once, say $x \approx y \approx \partial I \approx \partial J \quad (\partial = \text{ boundary})$

In particular: $|I| \ll |J| \quad \mathrm{dist}(I, \partial J) \ll |J|$ **– 'bad' case!**

But this should be 'rare' – more 'likely' to be in the interior than the boundary.

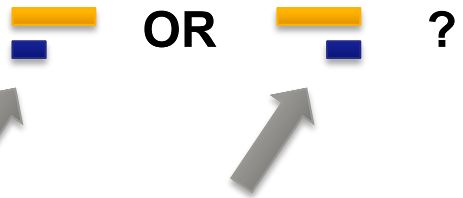
Enter probability: random dyadic cubes

(F. Nazarov, S. Treil, A. Volberg)

- Start from a given finest scale of dyadic intervals, coarser scales to be chosen

…  …

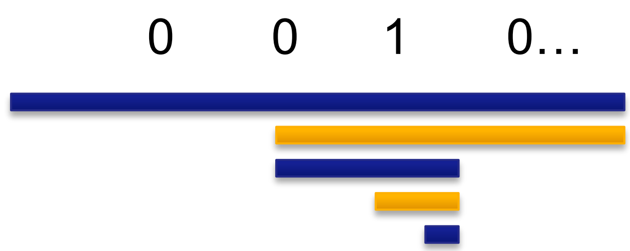
- For a representative of these intervals, need to decide if it will be the left or right half of its dyadic 'parent'



- How to decide?

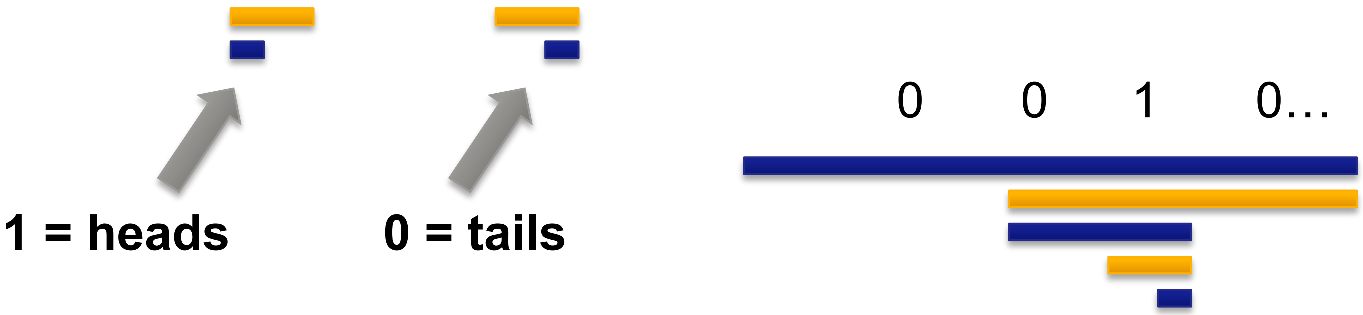
1 = heads

0 = tails



Repeat for all coarser levels.

Random dyadic cubes, continued



- **Probability space: an infinite product of coin tosses**

$$\Omega = \{0,1\}^{\mathbb{Z}} \ni \omega = (\omega_j)_{j\in\mathbb{Z}}$$

- **Random intervals = random translates of standard intervals**

$$I\dot{+}\omega := I + \sum_{\substack{j\in\mathbb{Z} \\ 2^{-j}<|I|}} 2^{-j}\omega_j \sim I + u \quad u \sim \mathrm{Unif}[0,|I|]$$

- **A random translate of *I is* 'bad' if for some much bigger *J***

$$2^r|I| < |J| \quad \mathrm{dist}(I\dot{+}\omega, \partial(J\dot{+}\omega)) \le |I|^{\gamma}|J|^{1-\gamma} = \left(\frac{|I|}{|J|}\right)^{\gamma}|J|$$

1 = heads

- **Rigorous probabilistic bound** $\mathbb{P}(I\dot{+}\omega \text{ is bad}) \le c_\gamma 2^{-r\gamma}$

Random cubes and bounds for operators

- **Splitting into good and bad parts:**

$$f = \sum_{I : I \dot{+} \omega \text{ is good}} \Delta^{\mu}_{I \dot{+} \omega} f + \sum_{I : I \dot{+} \omega \text{ is bad}} \Delta^{\mu}_{I \dot{+} \omega} f =: f_{\text{good}} + f_{\text{bad}}$$

$$\langle Tf, g \rangle = \langle Tf_{\text{good}}, g_{\text{good}} \rangle + \langle Tf_{\text{bad}}, g_{\text{good}} \rangle + \langle Tf, g_{\text{bad}} \rangle$$

- **Good part: 'direct' estimates, valid for any ω**

$$|\langle Tf_{\text{good}}, g_{\text{good}} \rangle| \leq C \|f\| \|g\|$$

- **Bad part: 'indirect' estimate, valid only on average**

$$\mathbb{E} |\langle Tf_{\text{bad}}, g_{\text{good}} \rangle + \langle Tf, g_{\text{bad}} \rangle| \leq \epsilon \|T\| \|f\| \|g\|$$

$$\text{if } |\langle Tf, g \rangle| \leq \|T\| \|f\| \|g\|$$

- **Synthesis:**

$$\|T\| \leq C + \epsilon \|T\| \qquad \Rightarrow \qquad \|T\| \leq \frac{C}{1 - \epsilon}$$

Achievements based on random dyadic intervals:

Nazarov-Treil-Volberg's characterization (~ 2000) of the measures for which:

$$|\langle H(f\,\mathrm{d}\mu), g\,\mathrm{d}\mu\rangle| \leq C\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}$$

In 2D (the complex plane), this led to the solution of Painlevé's problem:

Which planar sets E have the **removability** ***property that***

if a function f is bounded and analytic outside E, then it has an extension to E that remains bounded and analytic?

Solution to Painlevé's problem by X. Tolsa (2003):

A compact set *E* is non-removable for bounded analytic functions

if and only if

it supports a positive Radon measure with linear growth and finite curvature

$$\mu(D(x,r)) \leq Cr \qquad \iiint_{\mathbb{C}\times\mathbb{C}\times\mathbb{C}} \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z)}{R(x,y,z)^2} < \infty$$

Solution to the two-weight problem by M. Lacey (2013):

For a pair of measures (μ,ν)

(with no common point masses)

$$|\langle H(f\,\mathrm{d}\mu), g\,\mathrm{d}\nu\rangle| \leq C\|f\|_{L^2(\mu)}\|g\|_{L^2(\nu)}$$

if and only if

it holds whenever either $f = 1_I$ **or** $g = 1_I$

(Sawyer-type 'testing condition') and

$$\sup_{(x,t)\in\mathbb{R}^2_+} P(\mu)(x,t)\cdot P(\nu)(x,t) < \infty$$

(the 'Poisson A_2 condition').

Beyond Euclidean spaces:

- **Principle of ‘dyadic’ decomposition is very general**

Metric measure space (X,d,μ)

- **a set X equipped with**
- **‘distance’ d and**
- **‘measure’ μ (‘mass’ / ‘volume’ / etc.)**

Example: X = ‘the Internet’ (= all devices connected to it)

$d(x,y)$ = time it takes to transfer 1 Mb of data from x to y

$\mu(E)$ = total data storage capacity of all devices $x \in E$

Metric space 'dyadic cubes' of M. Christ (1990)

- Built from 'centres' and 'parent-child' relation between 'cubes' of different generation

- Randomization by T.H. & H. Martikainen (2012): for every cube, pick a child (randomly), let its centre be the centre of the new cube

Abstract extensions of many results above

Smoother wavelets

$$f = \sum_{j,k} \psi_{j,k} \langle \psi_{j,k}, f \rangle$$

$$\psi_{j,k} \text{ `smoothly' localized around } I_{j,k}$$

- **Often preferred over Haar in analysis on the Euclidean space with Lebesgue measure**

- **What about more general spaces?**

General construction of wavelets from a multiresolution analysis (Y. Meyer)

$$\{0\} \subseteq \ldots \subseteq V_{j-1} \subseteq V_j \subseteq V_{j+1} \subseteq \ldots \subseteq L^2(\mu)$$

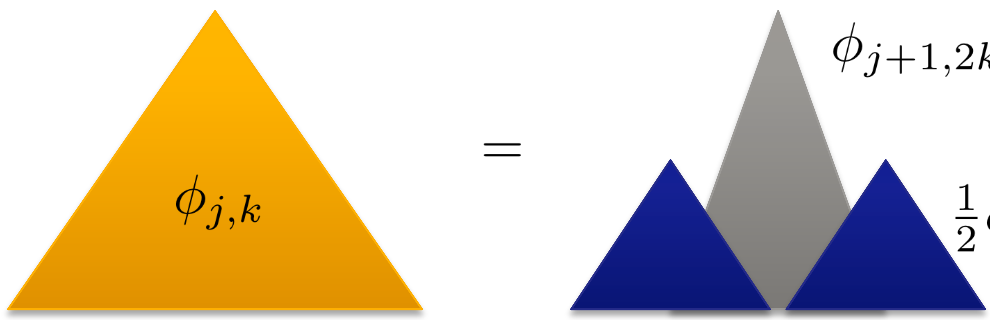
$$V_j = \operatorname{span}\{\phi_{j,k} : k \in \mathcal{K}_j\}$$

Then we can find wavelets with same regularity

$$V_{j+1} = V_j \oplus W_j \qquad W_j = \operatorname{span}\{\psi_{j,k} : k \in \mathcal{K}'_j\}$$

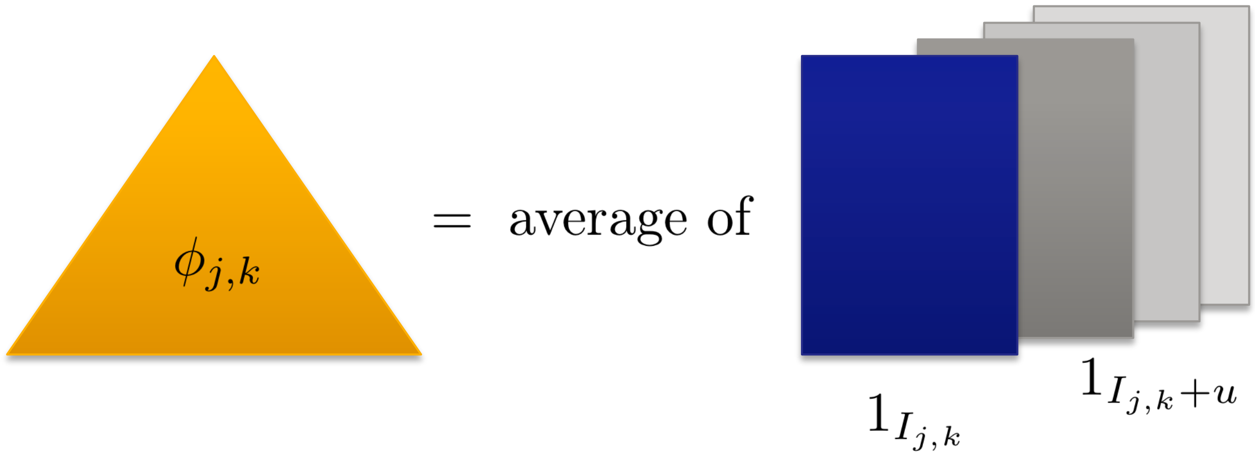
Example: piecewise linear splines

$$V_j = \{f \text{ linear on each } [\frac{k}{2^j}, \frac{k+1}{2^j}], k \in \mathbb{Z}\} = \operatorname{span}\{\phi_{j,k} : k \in \mathbb{Z}\}$$



Interpretation via probability:

$$\phi_{j,k}(x) = 2^j \int_0^{2^{-j}} 1_{I_{j,k}+u}(x)\,\mathrm{d}u = \mathbb{E}_\omega 1_{I_{j,k}+\omega}$$



Using the abstract random dyadic cubes:

first continuous splines & wavelets in metric measure spaces by P. Auscher & T.H. (2013)

$$\phi_{j,k}(x) := \mathbb{E}_\omega 1_{I_{j,k}(\omega)}$$

Thank you!