Fractal Structure of 2d Quantum Gravity

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Seemingly, at some point the universe had little extension.
Should we think about this as a “quantum universe” ?
Superpositions ?
ground state of a “quantum Hamiltonian” ?
what are “diffeomorphism invariant observables” ?
what is distance when geometry is fluctuating ?

2d quantum gravity is a nice laboratory to address some of these questions. It does not have propagating gravitons, but many of the conceptional questions are still there, and it is maximally quantum!

The reason is that the Einstein term is topological in 2d. Thus there is no action (except a possible cosmological term without derivatives). Formally the same as $\hbar \to \infty$: each configuration in the path integral has the same weight. No semiclassical dominant configuration.

Euclidean 2d quantum gravity

$$Z = \int \mathcal{D}[g_{\alpha\beta}]\, e^{-\Lambda \int d^2\xi \sqrt{g}} \int \mathcal{D}_g X_\mu\, e^{-\frac{1}{2}\int d^2\xi \sqrt{g} g^{\alpha\beta}\partial_\alpha X_\mu \partial_\beta X_\mu}.$$

In the case where we have no matter fields we simply have:

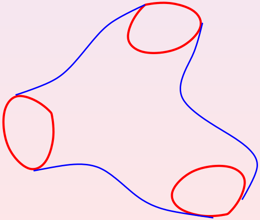
$$Z(\Lambda) = \int \mathcal{D}[g_{\alpha\beta}]\, e^{-\Lambda A(g)}, \quad A(g) = \int d^2\xi \sqrt{g}$$

$$Z(V) = \int \mathcal{D}[g_{\alpha\beta}]\delta(A(g)-V), \quad Z(\Lambda) = \int_0^\infty dV\, e^{-\Lambda V}\, Z(V).$$

More general amplitudes, where we have boundaries:

$$W(\ell_1,\ldots,\ell_n,\Lambda) = \int_{\ell_1,\ldots,\ell_n} \mathcal{D}[g_{\alpha\beta}]\, e^{-\Lambda A(g)}$$

$$W(\ell_1,\ldots,\ell_n,V) = \int_{\ell_1,\ldots,\ell_n} \mathcal{D}[g_{\alpha\beta}]\, \delta(A(g)-V)$$



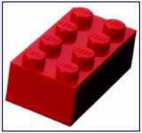
Entirely a counting problem: count number of geometries with area V and boundary lengths $\ell_1, \ldots, \ell_n$ (assuming the topology of a sphere with n boundaries)

$$W(\ell_1, \ldots, \ell_n, V) = V^{n-7/2} \sqrt{\ell_1 \cdots \ell_n}\, \mathrm{e}^{-(\ell_1 + \cdots + \ell_n)^2/V}$$

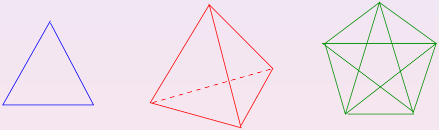
a generalized Hartle-Hawking wavefunction of 2d QG

To actually perform the calculation we need a regulatization of the set of continuous geometries: we use piecewise linear geometries constructed by gluing together identical building blocks, in the 2d case equilateral triangles with side lengths a, so-called dynamical triangulations.

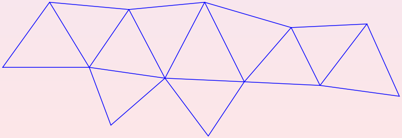
showcasing piecewise linear geometries via building blocks:



$\longrightarrow$



2d 3d 4d



$$\int \mathcal{D}[g_{\mu\nu}]\, e^{-S[g_{\mu\nu}]} \to \sum_{T_a} e^{-S[T_a]}$$

where a is the link length which serves as a UV cut off.

For 2d universes made of N_T triangles with the topology of a sphere with n boundaries of link-lengths L_i

$$V = \frac{\sqrt{3}}{4} a^2 N_T, \quad \ell_i = a \cdot L_i,$$

$$\mathcal{N}(L_1, \ldots, L_n, N_T) \propto N_T^{n-7/2} \sqrt{L_1 \cdots L_n}\, e^{-c(L_1+\cdots+L_n)^2/N_T}$$

and one has

$$\mathcal{N}(L_1, \ldots, L_n, N_T) \propto \frac{1}{a^{5n/2-7/2}} W(\ell_1, \ldots, \ell_n, V)$$

As a side remark: as long as we only use the Einstein Hilbert action also higher dimensional gravity reduces in principle to a pure counting problem if we regularize the path integral using dynamical triangulations.

The standard Einstein action has a very geometric representation on piecewise linear geometries as a sum over deficit angles of the $(D-2)$–dimensional subsimplices (Regge). Using identical building blocks it becomes really simple:

$$
\begin{aligned}
S[g] &= -\frac{1}{16\pi G}\int d^D x \sqrt{g(x)}\, R(x) + \frac{2\Lambda}{16\pi G}\int d^D x \sqrt{g} \\
S[T] &= -\kappa_{D-2} N_{D-2}(T) + \kappa_D N_D(T)
\end{aligned}
$$

$$
Z(x,y) = \sum_T e^{-S[T]} = \sum_{N_{D-2},N_D} \mathcal{N}(N_D, N_{D-2})\, x^{N_D} y^{N_{D-2}} \qquad
\begin{aligned} x &= e^{-\kappa_D} \\ y &= e^{\kappa_{D-2}} \end{aligned}
$$

Quantum field theory assumes we have a notion of distance between spacetime points, as its basic objects are correlations between fields separated a given spacetime distance.

Two layers of complication in a theory including gravity: (1): how do we define field correlators which are coordinate independent? and (2): having solved (1) in a given background geometry (where we can define distance as geodesic distance), how do we define distance if the quantum theory involves an average over the geometries used to define the distance?

But clearly, already the concept of geodesic distance is going to be complicated if geometry is a "quantum object". As we will see geodesic distance scales anomalously.

The partition function for a universe with cosmological constant Λ and where two marked points are separated a geodesic distance R is:

$$Z_R(\Lambda) = \int \mathcal{D}[g_{\alpha\beta}]\, e^{-\Lambda A(g)} \iint d^2x \sqrt{g(x)} d^2y \sqrt{g(y)}\; \delta(D_g(x,y) - R)$$

where $D_g(x,y)$ is the geodesic distance between x and y. If we use as geodesic distance in the triangulations the minimal link distance between two vertices, the calculation of $Z_R(\Lambda)$ also becomes a counting problem: counting the triangulations where two vertices are separated a given link-distance. Result:

$$Z_R(\Lambda) = \Lambda^{3/4} \frac{\cosh(\sqrt[4]{\Lambda}\, R)}{\sinh^3(\sqrt[4]{\Lambda}\, R)}$$

For a given geometry, define the “area” (length) of a spherical shell of radius R centered at x:

$$S_g(x,R) = \int d^2y \sqrt{g(y)}\, \delta(D_g(x,y) - R).$$

Define the average length of a spherical shell for a given geometry of area V as

$$S_g(R) = \frac{1}{V} \int d^2x \sqrt{g(x)}\, S_g(x,R).$$

For smooth, compact geometries we have (expressing that the geometry is two-dimensional)

$$S_g(x,R) \propto R, \quad S_g(R) \propto R, \quad R \to 0.$$

$$Z_R(V) = \int_{A(g)=V} \mathcal{D}[g_{\alpha\beta}] \iint d^2x \sqrt{g(x)} d^2y \sqrt{g(y)}\ \delta(D_g(x,y) - R)$$

$$Z_R(\Lambda) = \int_0^V dV\ e^{-\Lambda V}\ Z_R(V)$$

$$\langle S_g(R) \rangle_V = \frac{1}{Z(V)} \int_{A(g)=V} \mathcal{D}[g_{\alpha\beta}]\ S_g(R) = \frac{Z_R(V)}{VZ(V)}$$

From $Z_R(\Lambda)$ we can calculate $Z_R(V)$ and we know that $VZ(V) \propto V^{-5/2}$. Thus

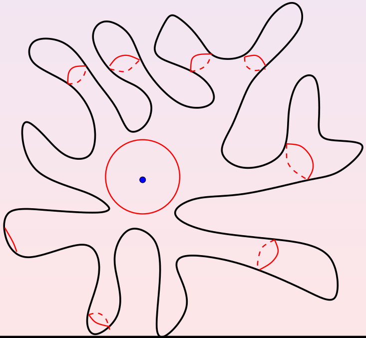
$$\langle S_g(R) \rangle_V = R^3 F\left(\frac{R}{V^{1/4}}\right), \qquad F(0) > 0.$$

Define the Hausdorff dimension by

$$\langle S_g(R) \rangle_V \propto R^{d_h - 1}, \qquad R \to 0$$

Thus we see that $d_h = 4$ in 2d quantum gravity and that geodesic distance scales anomalously.

Presumable, a precise mathematical statement is that the path integral is over continuous 2d geometries and a continuous 2d geometry is a.s. fractal with Hausdorff dimension $d_h = 4$.



$$\langle S_g(R) \rangle_V = \frac{1}{Z(V)} \int_{A(g)=V} \mathcal{D}[g] \, S_g(R)$$

R is an external parameter setting a scale

Consider now the interaction between matter and geometry in to form of 2d quantum gravity coupled to a conformal field theory with central charge $c < 1$.

$$Z(\beta)_V = \int_{A(g)=V} \mathcal{D}[g_{\alpha\beta}] Z_M(g,\beta)$$

where $Z_M(g,\beta)$ is the matter partition function coupled covariantly to geometry, defined by the metric $g_{\alpha\beta}(x)$.

What is d_h for this ensemble of geometries?

$$d_h(c) = 2\frac{\sqrt{49-c}+\sqrt{25-c}}{\sqrt{25-c}+\sqrt{1-c}}, \quad d_h(0) = 4, \quad d_h(-\infty) = 2.$$

Is the formula correct ?

The "derivation" (Watabiki): let $\Phi_n[g]$ be an operator invariant under diffeomorphisms and assume the following classical scaling $\Phi_n[\lambda g] = \lambda^{-n}\Phi[g]$ for constant λ. Then the quantum average satisfies (so-called generalized KPZ-DDK scaling)

$$\langle \Phi_n[g] \rangle_{\lambda V} = \lambda^{-\alpha_{-n}/\alpha_1} \langle \Phi[g] \rangle_V, \qquad \alpha_n = \frac{2n}{1 + \sqrt{\frac{25-c-24n}{25-c}}}$$

one now applies this to the operator

$$\Phi_1[g] = \int dx \sqrt{g}\, \Delta_g(x)\, \delta_g(x, x_0)|_{x=x_0}, \qquad \Phi_1[\lambda g] = \lambda^{-1}\Phi_1[g]$$

This operator appear when we study diffusion on a smooth manifold with metric $g_{\mu\nu}$. The diffusion kernel is

$$K(x, x_0; t) = e^{t\Delta_g} K(x, x_0; t), \qquad K(x, x_0; 0) = \delta_g(x, x_0)$$

The short distance behavior is

$$K(x,x_0;t) \sim \frac{e^{-D^2(x,x_0)/2t}}{t^{d/2}}(1+O(t)), \qquad \langle D(x,x_0;t)^2\rangle \sim t + 0(t^2)$$

The return probability is

$$\begin{aligned} P(t) &= \frac{1}{V}\int dx\sqrt{g}\, K(x,x;t) \\ &= \frac{1}{V}\int dx\sqrt{g}\,(1+t\Delta_g+\cdots)\,\delta_g(x,x_0)|_{x=x_0} \\ &= 1+t\,\Phi_1[g]+O(t^2) \end{aligned}$$

The problem with the derivation is that most likely these expansions are not true on the fractal structures encountered in 2d quantum gravity

For the Hausdorff dimension we have (declaring Dim[V] = 2)

$$\langle V \rangle_R = R^{d_h}, \qquad \mathrm{Dim}[R] = \frac{2}{d_h}$$

From the diffusion equation

$$\mathrm{Dim}[D(x,x_0)] = -\frac{1}{2}\mathrm{Dim}[\Phi_1[g]]$$

Taken the quantum average, using KPZ scaling:

$$\mathrm{Dim}[\langle D(x,x_0)\rangle] = -\frac{1}{2}\mathrm{Dim}[\langle \Phi_1[g]\rangle] = -\frac{\alpha_{-1}}{\alpha_1}$$

Thus

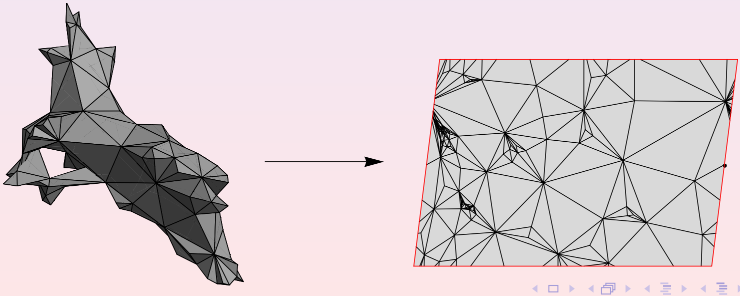
$$d_h = \frac{-2\alpha_1}{\alpha_{-1}} = 2\frac{\sqrt{49-c}+\sqrt{25-c}}{\sqrt{25-c}+\sqrt{1-c}}$$

Test the formula in the case of toroidal topology.

Virtues:

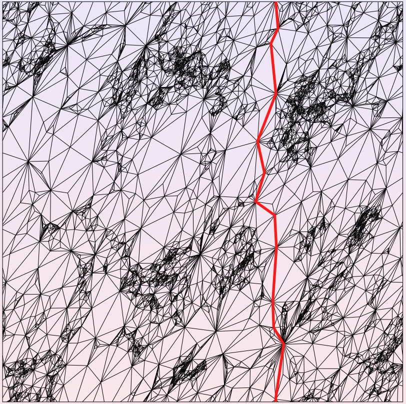
(1) the shortest non-contractable loop is automatically a geodesic curve. Thus in the discretized case we only have to look for such loops.

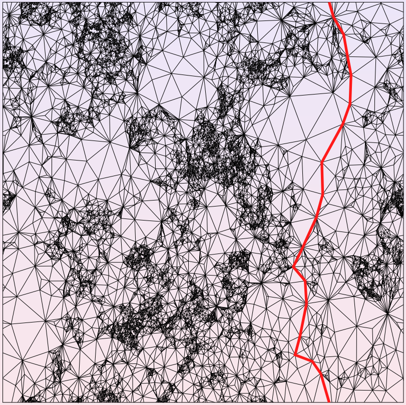
(2) If the manifold is analytic we have harmonic forms which have very nice discretized analogies, and we can use the these to construct a conformal mapping from the abstract triangulation to the complex plane.



Since the shortest contractable loop is a geodesic we expect

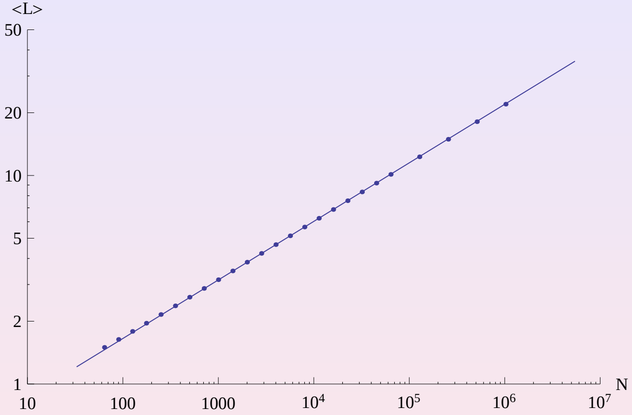
$$\langle L \rangle_N \sim N^{1/d_h(c)}$$





left figure $c = 0$, i.e. $d_h = 4$, right figure $c = -2$, $d_h = 3.56$

Quantitative check of $\langle L \rangle_N \sim N^{1/d_h}$ for $c = -2$



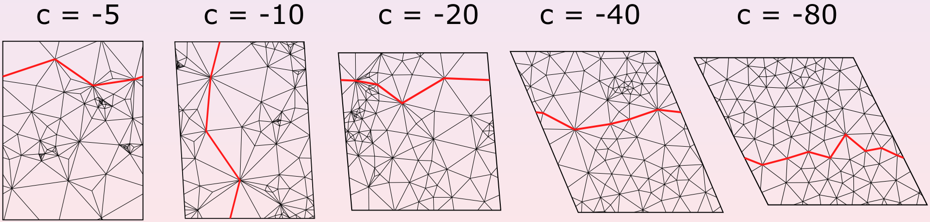
Straight line: $\langle L \rangle_n = 0.45\, N^{1/3.56}$.

The (regularized) bosonic string $c = d$:

$$Z(\mu) = \sum_T e^{-\mu N_T} \int \prod'_{\triangle \in T,\ \nu} dx_\nu(\triangle)\, e^{-\frac{1}{2}\sum_{\triangle,\triangle'}(x_\nu(\triangle)-x_\nu(\triangle'))^2}.$$

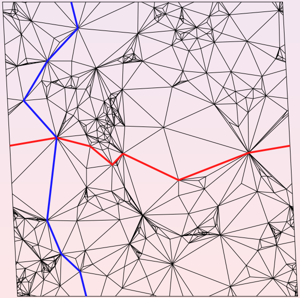
$$Z(\mu) = \sum_N e^{-\mu N_T} Z(N), \quad Z(N) = \sum_{T_N} \left(\det(-\Delta'_{T_N})\right)^{-d/2}$$

(Note that $d = -2$ is special.)



However, the situation for $c > 0$ more difficult and until recently numerical simulations could not really determine $d_h(c)$ for $c > 0$. Matter correlation functions gave agreement with Watabiki's formula, but geometric measurements agreed better with $d_h = 4$ for $0 < c < 1$.

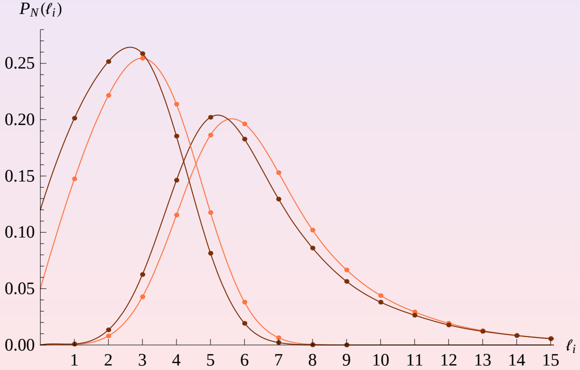
Using simulations of the DT-torus with Ising spin (c=1/2) and 3-state Pott's model (c=4/5), and analyzing the second shortest (independent) loop, one obtains data with little discretization "noise".

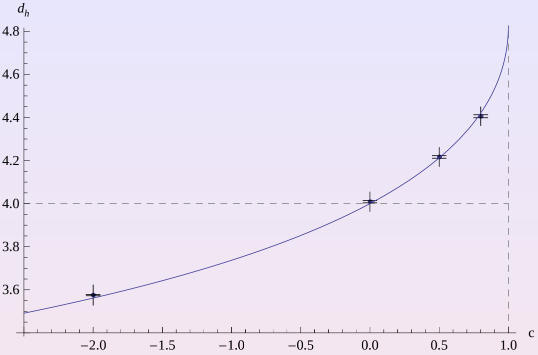


The probability distributions for homotopy classes Γ_i of simple connected, non-contractable loops:

$$P_N^{(i)}(\ell_i) = N^{1/d_h} F_i(x_i) \quad x_i = \frac{\ell_i}{N^{1/d_h}}$$

Refence loop distributions for $N = 8000$:





c	d_h (by fit)	d_h (theoretical)
-2	3.575 ± 0.003	3.562
0	4.009 ± 0.005	4.000
$1/2$	4.217 ± 0.006	4.212
$4/5$	4.406 ± 0.007	4.421

Let us finally turn to the definition of correlation functions in a theory of fluctuating geometries.

Ordinary QFT: Assume the volume V is sufficiently large and rotation and translational invariance except for boundary effects. ($S(R)$ "area" of spherical shell)

$$\langle \phi\phi(R) \rangle_V \equiv \frac{1}{V}\,\frac{1}{S(R)} \int \mathcal{D}\phi\, e^{-S[\phi]} \iint dx\, dy\; \phi(x)\phi(y)\; \delta(R - |x-y|).$$

$$\langle \phi\phi(R) \rangle_V \sim \frac{1}{R^{2\Delta_0}}, \qquad R \ll \frac{1}{m_{ph}}, \quad [\phi] = \Delta_0$$

$$\langle \phi\phi(R) \rangle_V \sim R^{-\alpha} e^{-m_{ph} R} \qquad \frac{1}{m_{ph}} \ll R \ll \frac{1}{V^{1/d}}$$

Generalization to a diffeomorphism invariant, metric theory

$$\langle \phi\phi(R) \rangle_V \equiv \frac{1}{V} \int \mathcal{D}[g]\, \delta\Big(A(g) - V\Big) \int \mathcal{D}_g \phi \, e^{-S[g,\phi]}$$

$$\iint dx\, dy\, \frac{\sqrt{g(x)}\,\sqrt{g(y)}}{S_g(y,R)}\, \phi(x)\phi(y)\, \delta(R - D_g(x,y)).$$

$D_g(x,y)$ is the geodesic distance between x and y.

Thus we expect the following behavior for a conformal theory coupled 2d Euclidean QG:

$$\langle \phi\phi(R)\rangle_V = R^{-d_h\Delta}\, F\left(\frac{R}{V^{1/d_h}}\right),$$

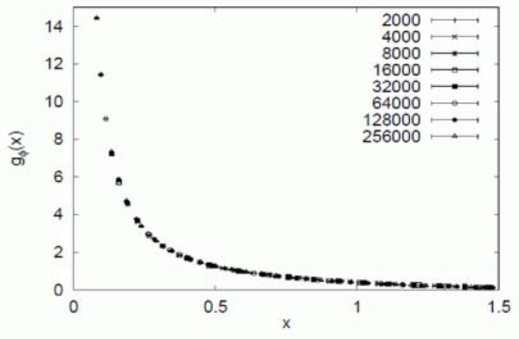
$$\langle \phi\phi(R)\rangle_V = V^{-\Delta}\, \frac{F(x)}{x^{d_h\Delta}}, \qquad x = \frac{R}{V^{1/d_h}}$$

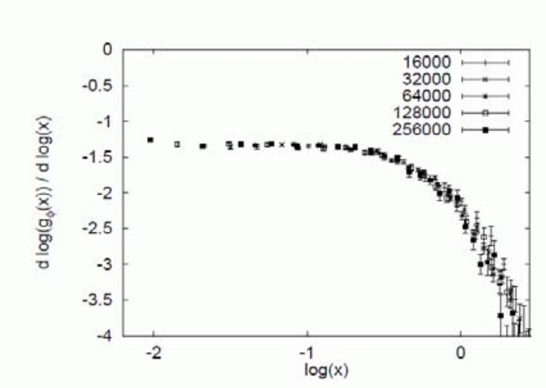
Here $F(0) = const. > 0$, and $F(x)$ falls of at least exponentially fast for $x > 1$.

$$\Delta = 2\,\frac{\sqrt{c-1+12\Delta_0}-\sqrt{c-1}}{\sqrt{25-c}-\sqrt{1-c}}, \qquad \text{KPZ} - \text{DDK scaling}.$$

Test this using dynamical triangulations: links of length a, the lattice UV cut off. $V \sim Na^2$. Geodesic distance $\ell \approx$ link distance.

$$\langle \phi\phi(\ell) \rangle_N = N^{-\Delta} \; \frac{F(x)}{x^{d_h \Delta}} \qquad x = \frac{\ell}{N^{1/d_h}} \qquad \boxed{\text{FSS!}}$$





Finite Size Scaling allows us to determine Δ and $d_h \Delta$

Theory for Ising model ($c = 1/2$):

$$\Delta_0 = \frac{1}{8} \to \Delta = \frac{1}{3}, \qquad 2\Delta_0 = \frac{1}{4} \to d_h \Delta = 1.40...$$

Conclusion

I have shown that it makes sense to apply ordinary field theoretical concepts to a theory of fluctuating geometries coupled to matter. Further, it is a theory of extreme quantum fluctuations, far from any semiclassical geometry coupled to matter.

2d quantum gravity of course differs from attempts to study higher dimensional quantum gravity using only conventional field theory methods: the theory is renormalizable. It is still an open question if these higher dimensional theories exist at all (unless one goes beyond the framework of QFT (string theory)). Some evidence (functional renormalization group methods, causal dynamical triangulations provide some hope, but nothing is settled).