A Bound State Perspective

Phenomenology Institute

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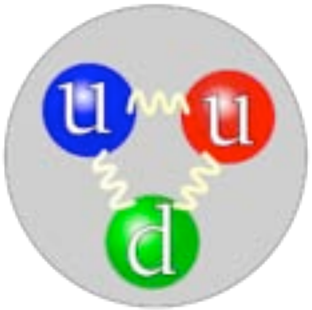
arXiv:
1106.1420

Hadrons are Ultra-Relativistic Bound States

But look much simpler than one might expect

The quark model explains the spectrum
in terms of valence quark dof's only

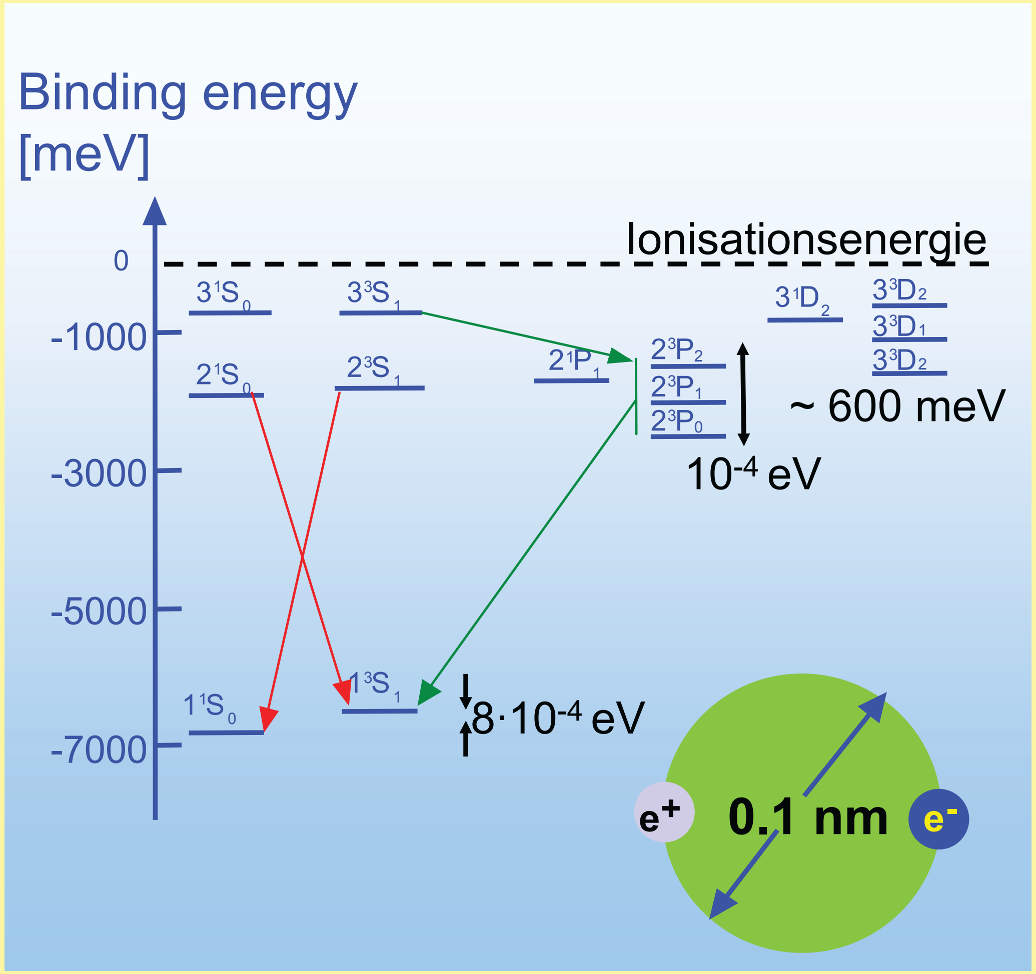


$$V(r) = c\,r - C_F\,\frac{\alpha_s}{r}$$

Where are the gluons and sea quarks?

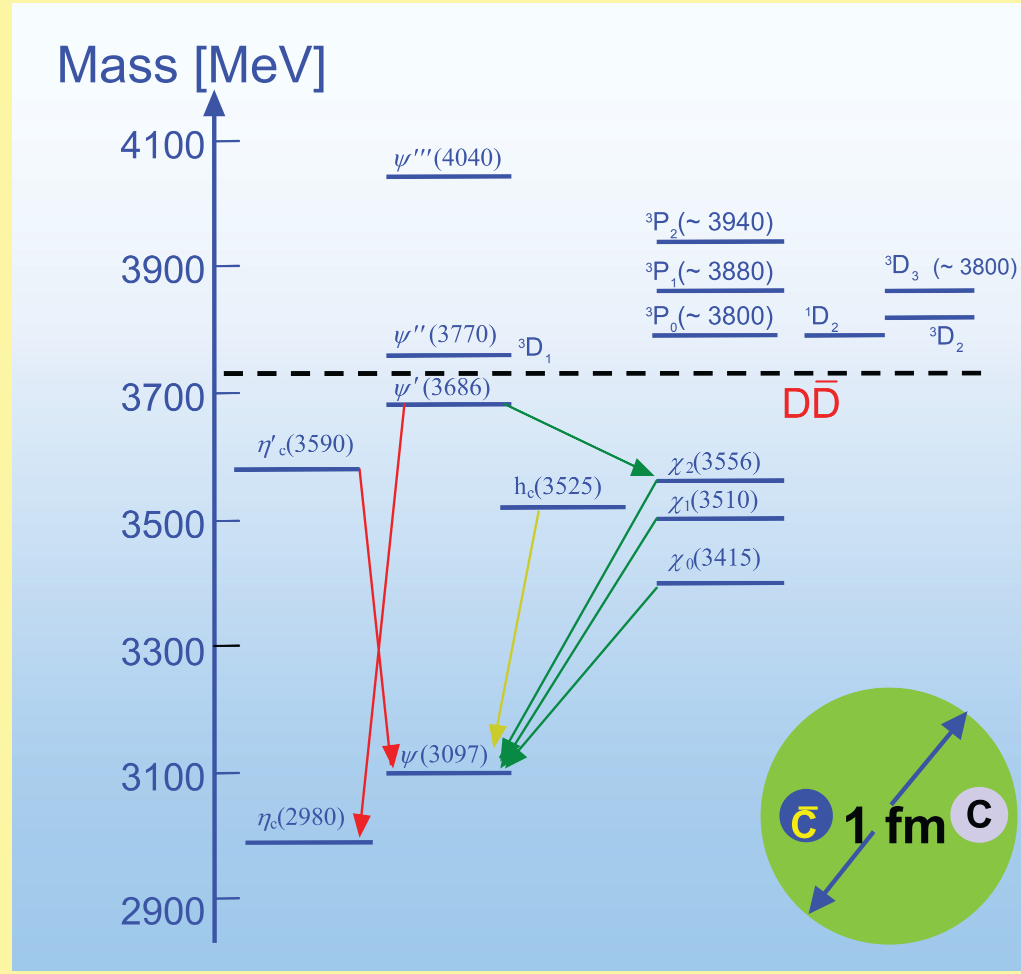
Charmonium – the Positronium of QCD

- Positronium



PQED

- Charmonium

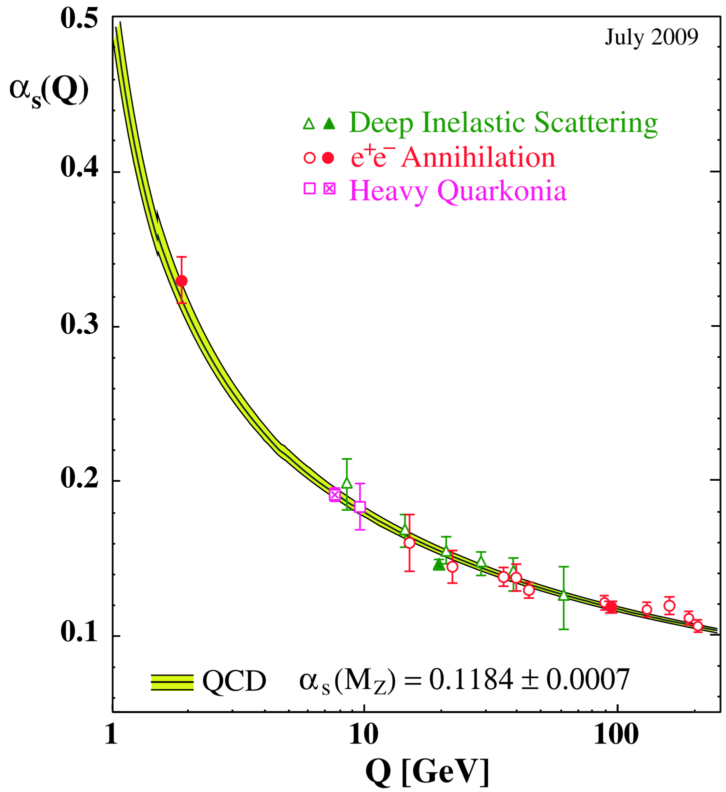


PQCD?

Can we get away with Perturbation Theory?

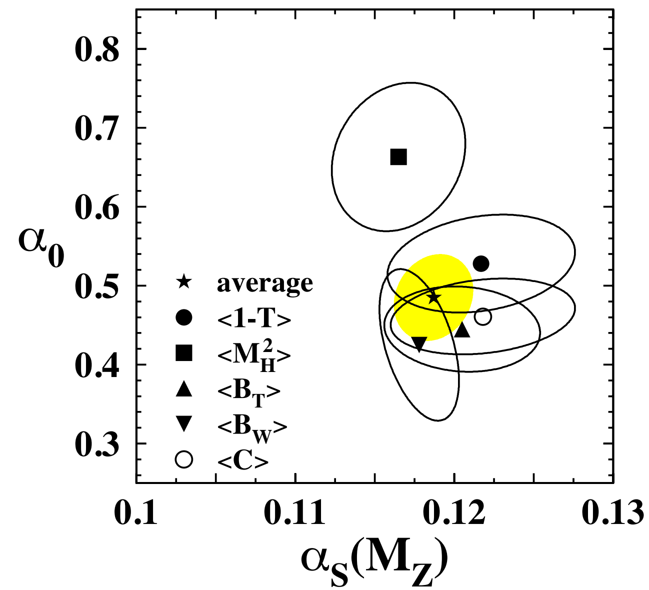
At least we should try!

There is more than the spectrum which suggests that $\alpha_s(0) < 1$



$$\alpha_0(\mu_I) \equiv \frac{1}{\mu_I}\int_0^{\mu_I} dQ\,\alpha_s(Q)$$

α_0 from e^+e^- event shapes



Dokshitzer et al

$$\alpha_s(M_Z) = 0.1153 \pm 0.0017\,(\mathrm{exp}) \pm 0.0023\,(\mathrm{th})$$
$$\alpha_0 = 0.5132 \pm 0.0115\,(\mathrm{exp}) \pm 0.0381\,(\mathrm{th})$$

T. Gehrmann et al (2010)

QCD *is about to undergo a* **faith transition**

QCD practitioners prepare themselves - slowly but steadily - to start using, in earnest, the language of ***quarks*** and ***gluons*** down into the region of small characteristic momenta - "**large distances**"

Unusual analytic properties of ***quark*** and ***gluon*** Green functions will take responsibility for what we refer to as "***colour confinement***".

Gribov **supercritical quark confinement** scenario implies all above and demands the **QCD coupling in the infrared** to exceed

$$\boxed{\frac{\alpha_{\mathrm{crit}}}{\pi} = C_F^{-1}\left[1-\sqrt{\frac{2}{3}}\right] \simeq 0.137}$$

One can well expect that in ***n*** years from now (*with* $n = \mathcal{O}(1)$) participants of Munich alpha_s meetings will be discussing the accuracy of α_s determination at scales of **1** GeV and below

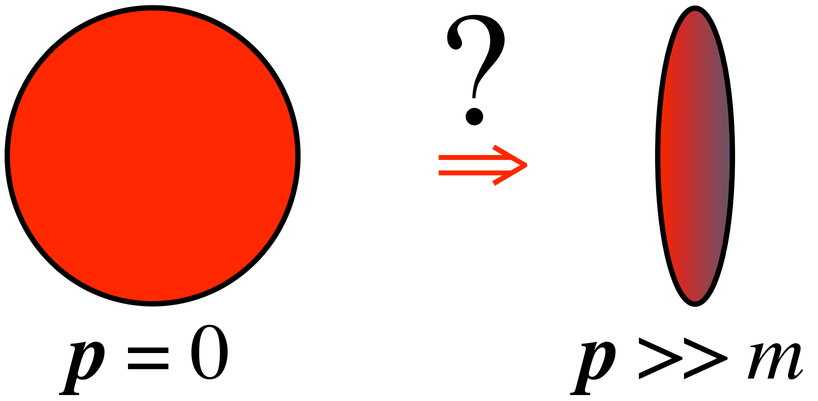
Yuri Dokshitzer (2011)

ABC of Relativistic Bound States

Questions that weren't asked, but can be answered

A. Does the Hydrogen atom wave function Lorentz contract?

Matti Järvinen, Phys. Rev. D71, 085006 (2005)



B. How does its Dirac wave function describe the infinite # of e^+e^- pairs?

PH, 0909.3045

C. Is there a Born term for bound states, as there is for scattering amplitudes?

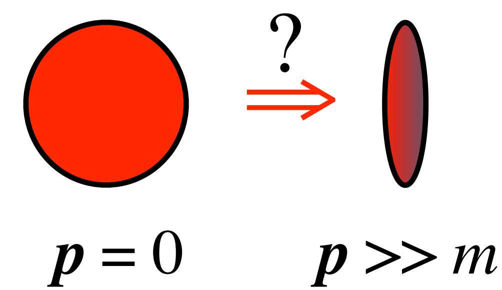
hbar expansion in QFT

S. Brodsky, PH, Phys. Rev. D83, 045026 (2011)

A. Does the Hydrogen atom wave function Lorentz contract?

In analogy to classical relativity, it is the equal time w.f. might Lorentz contract in a moving frame.

Observers measure endpoints of rigid rod at equal times in their respective frames.



$\mathrm{H}\Psi(\boldsymbol{x}) = \mathrm{E}\,\Psi(\boldsymbol{x})$ Quantization at $t = 0$

Boosts are dynamical

$\mathrm{H}_{\mathrm{LF}}\Psi_{\mathrm{LF}}(\boldsymbol{x}) = \mathrm{P}^{-}\Psi_{\mathrm{LF}}(\boldsymbol{x})$ Quantization at constant $x^{+} = t+z$

Boosts are kinematical

Ψ_{LF} is invariant

Limiting case: Infinite momentum frame - zero modes

Here: Use equal time quantization:

Theoretically more correct (space-like quantization surface)

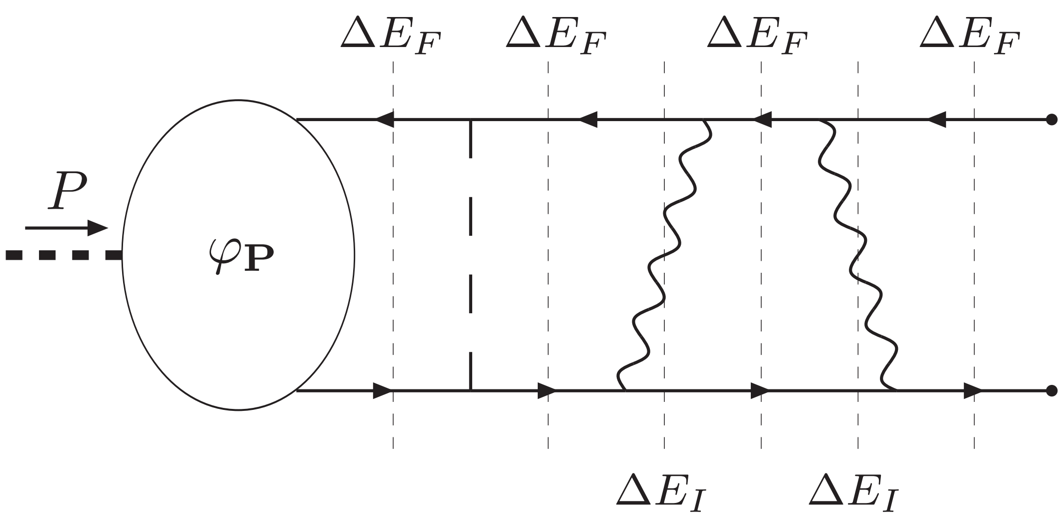
Study dynamical boosts (Yes, we can!)

A. Does the Hydrogen atom wave function Lorentz contract? II

In rest frame, A^0 dominates: Instantaneous Coulomb interaction

In boosted frame, longitudinal $\boldsymbol{A}_L$ contributes as well:

$|\, e^- p\, \gamma\rangle$ Fock state



Only Coulomb exchange does not increase Fock content (recall $|q\bar{q}\rangle$ and $|qqq\rangle$ hadrons)

The correct momentum dependence arises in a non-trivial way:

$$E(\boldsymbol{p}) = \sqrt{(m_e + m_p - E_b)^2 + \boldsymbol{p}^2 c^2}$$

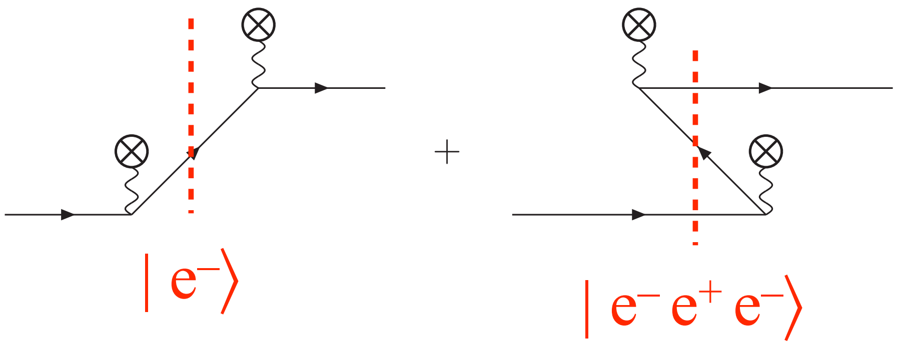
... but is guaranteed by perturbation theory: Each order in α is L-covariant

The $|\, e^- p\, \gamma\rangle$ Fock state does not Lorentz contract classically.

Matti Järvinen, Phys. Rev. D71, 085006 (2005)

B. How does the Hydrogen's Dirac wave function describe the infinite # of e+e- pairs?

Relativistic electron scattering in a fixed external potential $A^0 = -\alpha/r$ generates states with extra pairs



or, equivalently:

$H\,|\Psi\rangle = E\,|\Psi\rangle$ in a fixed external A^0 potential requires $|\Psi\rangle$ to have an infinite number of Fock states, since H creates $e^+ e^-$ pairs.

$$H(t) = \int d^3\boldsymbol{x}\,\bar{\psi}(t,\boldsymbol{x})\left[-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + m + e\gamma^0 A^0(\boldsymbol{x})\right]\psi(t,\boldsymbol{x})$$

How is this taken into account by the Dirac wave function $\varphi(\boldsymbol{x})$,

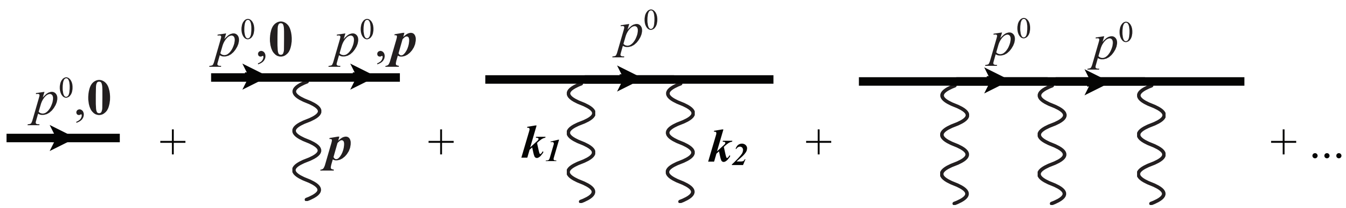
$$(-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + e\gamma^0 A^0(\boldsymbol{x}) + m)\varphi(\boldsymbol{x}) = E\gamma^0\varphi(\boldsymbol{x})$$

which apparently describes the $\boldsymbol{x}$-distribution of a single electron?

B. How does the Hydrogen's Dirac wave function describe the infinite # of e+e- pairs? II

Tree diagrams are independent of the $i\varepsilon$ prescription.

The bound state energies E_R of a fermion in a t-independent Coulomb potential

$G(p^0, \boldsymbol{p}) =$ 

may be evaluated using retarded propagators (since $p^0 \neq -E_p$)

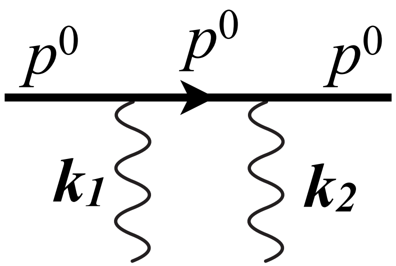
$$S_R(p^0, \boldsymbol{p}) = i\frac{\not{p} + m_e}{(p^0 - E_p + i\varepsilon)(p^0 + E_p + i\varepsilon)}$$

which only propagate forward in time,

$$S_R(t, \boldsymbol{p}) = \frac{\theta(t)}{2E_p}\left[(E_p\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} + m_e)e^{-iE_pt} + (E_p\gamma^0 + \boldsymbol{p}\cdot\boldsymbol{\gamma} - m_e)e^{iE_pt}\right]$$

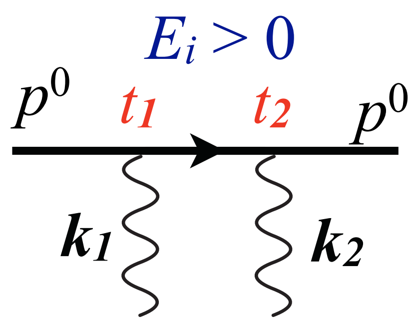
B. How does the Hydrogen's Dirac wave function describe the infinite # of e+e- pairs? III

The time-ordered diagrams, and hence also the equal-time wave functions of bound states, depend on the $i\varepsilon$ prescription,



Covariant $(p^0, \boldsymbol{p})$

$=$



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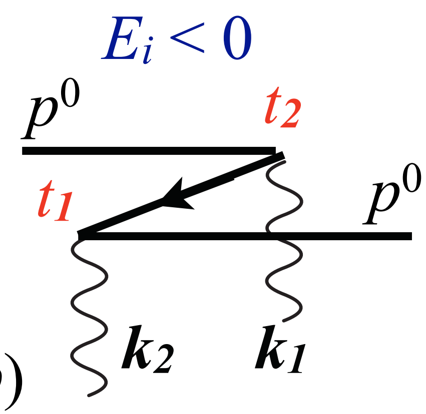
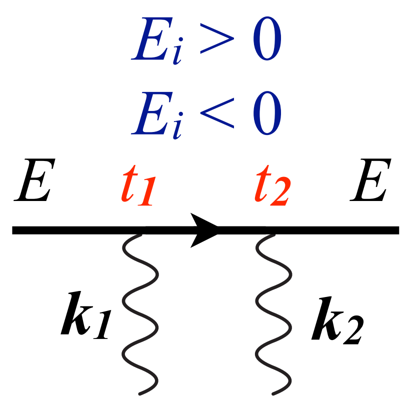


Feynman $(t, \boldsymbol{p})$

$=$



Retarded $(t, \boldsymbol{p})$

The Dirac "single particle" states with E > 0 and E < 0 are obtained with retarded propagators.

Bound state energies are independent of $i\varepsilon$ only in the absence of loops!

B. How does the Hydrogen's Dirac wave function describe the infinite # of e+e- pairs? IV

We now can formulate a hamiltonian description of Dirac states.
We need a boundary condition which gives retarded fermion propagators:

$$S_R(x-y) = {}_R\langle 0|\, T[\psi(x)\bar{\psi}(y)]\, |0\rangle_R$$

$$|0\rangle_R = N^{-1}\prod_{\boldsymbol{p},\lambda} d^\dagger_{\boldsymbol{p},\lambda}|0\rangle \qquad \Longrightarrow \qquad \psi(x)|0\rangle_R = 0$$

With the Dirac state defined by $\quad |\varphi, t\rangle = \int d^3\boldsymbol{x}\, \psi^\dagger_\alpha(t,\boldsymbol{x})\varphi_\alpha(\boldsymbol{x})|0\rangle_R$

the bound state condition $\quad H|\varphi\rangle = E|\varphi\rangle$

gives the Dirac equation for the c-numbered spinor $\varphi(\boldsymbol{x})$.

C. Is there a Born term for bound states, as there is for scattering amplitudes?

Born terms are defined as being of lowest order in $\hbar$.

$\hbar$ is a fundamental constant related to quantum effects. Each order in an $\hbar$ expansion must obey all symmetries of the theory.

The $\hbar$ expansion is relevant for both relativistic and nonrelativistic, scattering and bound state dynamics.

$\Rightarrow$ It defines the proper starting point for a relativistic description of bound states.

This field is plagued by a large number of suggestions for "relativistic-looking wave equations"

As the Hydrogen example (A) showed, boosts are dynamic symmetries in QFT. Boost invariance is not manifest.

$\hbar$ in the Harmonic Oscillator

$\hbar \to 0$ does not always imply classical physics. For the harmonic oscillator

$$Z = \int [dx] \exp\left[\frac{i}{\hbar}\int dt(\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega^2 x^2)\right]$$

$$\propto \int [d\tilde{x}] \exp\left[i\int dt(\tfrac{1}{2}m\dot{\tilde{x}}^2 - \tfrac{1}{2}m\omega^2 \tilde{x}^2)\right]$$

The $\hbar$ can be completely absorbed in $\tilde{x} \equiv x/\sqrt{\hbar}$

Bound states with $E_n = \hbar\omega(n+\frac{1}{2})$ have small $x \propto \sqrt{\hbar\, n}$ as $\hbar \to 0$ (with fixed n).

The classical path $x_i(t_i) \to x_f(t_f)$ is obtained when the boundary positions $x_{i,f}$ are held fixed as $\hbar \to 0$, hence $n \propto 1/\hbar$ ensuring a classical limit.

$\hbar$ expansion in QED

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - \tilde{e}A\!\!\!/ - \tilde{m})\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Dimensions: Requiring $[S] \equiv [\int d^4x\mathcal{L}] = [\hbar] = E\cdot L$ $\qquad c = \epsilon_0 = 1$

$[\psi] = E^{1/2}\ L^{-1}$ $\qquad$ also from: $\{\psi^\dagger(t,\boldsymbol{x}),\psi(t,\boldsymbol{y})\} = \hbar\,\delta^3(\boldsymbol{x}-\boldsymbol{y})$

$[A^\mu] = E^{1/2}\ L^{-1/2}$

$[\tilde{m}] = L^{-1}$ wave number! $\tilde{m} = m/\hbar$

$[\tilde{e}] = E^{-1/2}\ L^{-1/2}$

$[e] = E^{+1/2}\ L^{+1/2}$! $\qquad \alpha = \dfrac{e^2}{4\pi\hbar} = \dfrac{\tilde{e}^2\hbar}{4\pi} \simeq \dfrac{1}{137}$

(e is the classical charge) $\qquad \tilde{e} = e/\hbar$

> We shall define the $\hbar \to 0$ limit by keeping the quantities $\tilde{e},\tilde{m}$ of the "classical" action fixed

Rescaling the fields with $\hbar$

$$Z = \int [\mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A] \exp\left[\frac{i}{\hbar}\int d^4x \mathcal{L}\right] \propto \int [\mathcal{D}\tilde{\psi} \mathcal{D}\bar{\tilde{\psi}} \mathcal{D}\tilde{A}] \exp\left[i\int d^4x \tilde{\mathcal{L}}\right]$$

The rescalings $\tilde{\psi} \equiv \psi/\sqrt{\hbar}, \quad \tilde{A}^\mu \equiv A^\mu/\sqrt{\hbar}$

introduce an $\hbar$ dependence in the interaction term:

$$\tilde{\mathcal{L}} = \bar{\tilde{\psi}}(i\partial\!\!\!/ - \tilde{e}\sqrt{\hbar}\tilde{A\!\!\!/} - \tilde{m})\tilde{\psi} - \tfrac{1}{4}(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu)^2$$

$\hbar$ now appears only in the coupling: $\alpha = \dfrac{\tilde{e}^2\hbar}{4\pi} = \mathcal{O}(\hbar)$

and the perturbative (loop) expansion is equivalent to the $\hbar$ expansion.

A comment on:

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Classical Physics and Quantum Loops

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The standard picture of the loop expansion associates a factor of $\hbar$ with each loop, suggesting that the tree diagrams are to be associated with classical physics, while loop effects are quantum mechanical in nature. We discuss counterexamples wherein classical effects arise *from loop diagrams* and display the relationship between the classical terms and the long range effects of massless particles.

This paper appears to use a different definition of the limit $\hbar \to 0$

where *m* and *e* are held fixed, hence $\tilde{m} = m/\hbar \to \infty, \quad \tilde{e} = e/\hbar \to \infty$

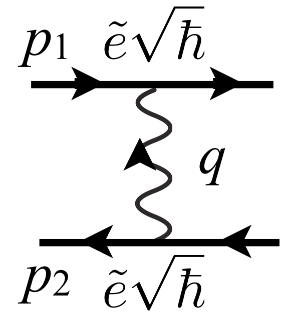
Then also $\alpha = e^2/4\pi\hbar \to \infty$ hence the $\hbar$ and loop expansions are not equivalent.

C. Is there a Born term for bound states, as there is for scattering amplitudes? II

Bound state poles are generated by the divergent sum of ladder diagrams (in the non-relativistic case)

$$A = A_1 + A_2 + \ldots$$

A₁:



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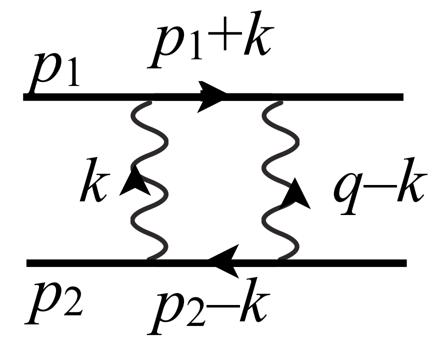
A₂ :



+ ...

For an overlap with the bound state wave function,

$$q^0 \sim \alpha^2 m_e \qquad \boldsymbol{q} \sim \alpha m_e$$

This makes all ladder diagrams of the same order in α and $\hbar$

$$A_1 \sim \frac{\alpha}{\boldsymbol{q}^2} \sim \frac{1}{\alpha}$$

$$A_2 \sim \int \overset{\alpha^2}{dk^0}\, \overset{\alpha^3}{d^3\boldsymbol{k}} \frac{\overset{\alpha^2}{\tilde{e}^4\hbar^2}}{\underset{\alpha^2}{\left[p_1^0+k^0-\sqrt{(\boldsymbol{p}_1+\boldsymbol{k})^2+m^2}+i\varepsilon\right]}\underset{\alpha^2}{\left[p_2^0-k^0-\sqrt{(\boldsymbol{p}_2-\boldsymbol{k})^2+m^2}+i\varepsilon\right]}\underset{\alpha^2}{\boldsymbol{k}^2}\underset{\alpha^2}{(\boldsymbol{q}-\boldsymbol{k})^2}}$$

C. Is there a Born term for bound states, as there is for scattering amplitudes? III

Hence ladder diagrams are special: They are the most singular ones as $\alpha \to 0$
All ladders are of the same order in α and hbar, and their sum diverges when the ***q***-distribution is given by the Schrödinger wave function.

Hence the Schrödinger bound states are Born terms: Lowest order in hbar.

The momenta in loop corrections to propagators or vertices do not scale with α and hence give true higher order corrections.

The Dirac equation turns into the Schrödinger equation in the $\alpha \to 0$ limit.
Its $\mathcal{O}(\alpha)$ relativistic contributions are of the same order as loop corrections.

Indeed, the Dirac energy levels are unphysical (complex) for $Z\alpha > 1$:

$$E_{n,j} = m\left[1 + \frac{Z^2\alpha^2}{n - j - \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 - Z^2\alpha^2}}\right]^{-1/2} \qquad (j + \tfrac{1}{2} \leq n)$$

Determination of A^0 for an $e^-\mu^+$ bound state

Define an $|\, e^- \mu^+ \rangle$ state in terms of a 4×4 wave function χ

$$|E, t=0\rangle = \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \psi_e^\dagger(t=0, \boldsymbol{y}_1)\chi(\boldsymbol{y}_1, \boldsymbol{y}_2)\psi_\mu(t=0, \boldsymbol{y}_2)|0\rangle_R$$

The QED operator equation of motion: $\partial_\mu F^{\mu\nu}(x) - e \sum_{i=e,\mu} \bar{\psi}_i(x)\gamma^\nu \psi_i(x) = 0$

determine A^0 through Gauss' law: $-\boldsymbol{\nabla}^2 A^0(t, \boldsymbol{x}) = e \sum_{f=e,\mu} \psi_f^\dagger(t, \boldsymbol{x})\psi_f(t, \boldsymbol{x})$

The actual value of $A^0(\boldsymbol{x}; \boldsymbol{y}_1, \boldsymbol{y}_2)$ depends on the positions $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ of the charged particles e^- and μ^+, hence differs for each wave function component $\chi(\boldsymbol{y}_1, \boldsymbol{y}_2)$

Consider the state component

$$|\boldsymbol{y}_1, \boldsymbol{y}_2\rangle = \psi_e^\dagger(t=0, \boldsymbol{y}_1)\psi_\mu(t=0, \boldsymbol{y}_2)|0\rangle_R$$

Determination of A^0 for an $e^-\mu^+$ bound state II

$$A^0(t,\boldsymbol{x})|\boldsymbol{y}_1,\boldsymbol{y}_2\rangle = \left[\frac{e}{4\pi}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}_1|}-\frac{1}{|\boldsymbol{x}-\boldsymbol{y}_2|}\right)+\Lambda^2\hat{\boldsymbol{\ell}}\cdot\boldsymbol{x}\right]|\boldsymbol{y}_1,\boldsymbol{y}_2\rangle$$

I added a homogeneous solution to Gauss' law: $\Lambda^2\hat{\boldsymbol{\ell}}$ is independent of $\boldsymbol{x}$,

which gives rise to an asymptotically constant energy density $\frac{1}{2}(\boldsymbol{\nabla}A^0)^2 = \frac{1}{2}\Lambda^4$

and appears as a boundary condition on the solution.

If we allow this, it has very interesting consequences (for QCD).

The O(e) term gives a stationary action for the state $|\boldsymbol{y}_1,\boldsymbol{y}_2\rangle$

We should similarly determine the unit vector ℓ so that the action is stationary under the global variation of the direction of ℓ .

$$\frac{1}{2}\int d^3\boldsymbol{x}(\boldsymbol{\nabla}A^0)^2 = \frac{1}{2}\left[\Lambda^4\int d^3\boldsymbol{x} - e\Lambda^2\,\hat{\boldsymbol{\ell}}\cdot(\boldsymbol{y}_1-\boldsymbol{y}_2)+\mathcal{O}\left(e^2\right)\right]|\boldsymbol{y}_1,\boldsymbol{y}_2\rangle$$

Hence: $\hat{\ell}\parallel\boldsymbol{y}_1-\boldsymbol{y}_2$

Bound state equation

With the QED hamiltonian

$$H(t) = \int d^3\boldsymbol{x}\, \bar\psi(t,\boldsymbol{x})\big[-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + m + e\gamma^0 A^0(\boldsymbol{x})\big]\,\psi(t,\boldsymbol{x})$$

acting on the state

$$|E, t=0\rangle = \int d\boldsymbol{y}_1 d\boldsymbol{y}_2 \psi_e^\dagger(t=0,\boldsymbol{y}_1)\chi(\boldsymbol{y}_1,\boldsymbol{y}_2)\psi_\mu(t=0,\boldsymbol{y}_2)|0\rangle_R$$

and A^0 determined by the stationarity of the action on each state $|\boldsymbol{y}_1,\boldsymbol{y}_2\rangle$
with the boundary condition $\frac{1}{2}(\boldsymbol{\nabla}A^0)^2 = \frac{1}{2}\Lambda^4$

the bound state condition $H|E\rangle = E|E\rangle$ gives the constraints:

$\Lambda \neq \Lambda(\boldsymbol{y}_1,\boldsymbol{y}_2)$ must be a universal constant

and the condition on the wave function

Bound state equation II

$$\gamma^0(-i\boldsymbol{\nabla}_1\cdot\boldsymbol{\gamma}+m_e)\chi(\boldsymbol{y}_1,\boldsymbol{y}_2) \quad - \quad \chi(\boldsymbol{y}_1,\boldsymbol{y}_2)\gamma^0(i\boldsymbol{\nabla}_2\cdot\boldsymbol{\gamma}+m_\mu)$$

$$= \quad [E-V(\boldsymbol{x}_1,\boldsymbol{x}_2)]\chi(\boldsymbol{y}_1,\boldsymbol{y}_2)$$

with $V(\boldsymbol{y}_1,\boldsymbol{y}_2)=\frac{1}{2}e\Lambda^2\left|\boldsymbol{y}_1-\boldsymbol{y}_2\right|$ + O(e^2) purely linear at O(e)!

Remarks:
- The equation is exact to O(e) and lowest order in $\hbar$
- Dirac type wave function: contains any number of pairs
- Only A^0 contributes at O(e)
- $\boldsymbol{A}$ appears at O(e^2), giving higher Fock states.
- Translation invariance requires the state to be neutral
- Rotational symmetry is manifest
- Boost invariance is dynamic (hidden)

Frame dependence ($t_1 = t_2$ in all frames!)

The wave function of a bound state with CM momentum $\boldsymbol{k}$ is

$$\chi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \exp\left[i\boldsymbol{k}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2\right]\phi(\boldsymbol{x}_1-\boldsymbol{x}_2)$$

The equation for $\phi(\boldsymbol{x})$ becomes (for $m_1 = m_2 = m$):

$$-i\boldsymbol{\nabla}\cdot[\boldsymbol{\alpha},\phi] + \tfrac{1}{2}\boldsymbol{k}\cdot\{\boldsymbol{\alpha},\phi\} + m\left[\gamma^0,\phi\right] = (E-V)\phi$$

where the solutions $\phi(\boldsymbol{x})$ and E depend on the CM momentum $\boldsymbol{k}$.

The Lorentz symmetry of QED guarantees (for a calculation correctly done to a given order in $\hbar$ and g) that the energy eigenvalues are given by

$$E = \sqrt{\boldsymbol{k}_{CM}^2 + M^2}$$

This is indeed the case for the above equation! P.H., PL B172 (1986) 101

And only holds for a purely linear potential $V(|\boldsymbol{x}|)$.

Boost covariance of the wave function

How should relativistic, equal-time wave functions transform under Lorentz boosts? The above bound state equation gives, for $\boldsymbol{k} = (0,0,k)$:

$$\gamma^0\phi_k(s) = e^{\zeta\alpha_3/2}\gamma^0\phi_{k=0}(s)e^{-\zeta\alpha_3/2}$$

for $\phi_k(\boldsymbol{s}) \equiv \phi_k(x_1{=}0,\ x_2{=}0,\ x_3(\boldsymbol{s}))$ on the z-axis and with the "invariant distance" s defined by

$$\frac{ds}{dx_3} = \tfrac{1}{2}[E - V(x_3)] \quad \text{and} \quad \tanh\zeta(s) = -\frac{k}{E-V}$$

$$s(x_3) = \tfrac{1}{2}x_3\left[E - \tfrac{1}{2}V(x_3)\right]$$

Note: For $V << E$ this reduces to standard Lorentz contraction, but in general the boost depends on the $p^0 - eA^0$.

The k-dependence of the wave function is seen also by directly boosting the bound state with an interaction-dependent boost operator.

D. Dietrich, PH, M. Järvinen, work in progress

u$\bar{\text{d}}$ meson states in in QCD

$$\mathcal{L}_{QCD} = -\tfrac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + \sum\nolimits_f \bar{\psi}_f^A(i\partial\!\!\!/ - gA\!\!\!/_a T_{AB}^a - m_f)\psi_f^B$$

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f_{abc} A_b^\mu A_c^\nu$$

$$|E, t=0\rangle = \int d^3\boldsymbol{y}_1 d^3\boldsymbol{y}_2\, \psi_u^{A\dagger}(t=0,\boldsymbol{y}_1)\chi^{AB}(\boldsymbol{y}_1,\boldsymbol{y}_2)\psi_d^B(t=0,\boldsymbol{y}_2)|0\rangle_R$$

Under time-independent gauge transformations $\psi(t,\boldsymbol{x}) \to U(\boldsymbol{x})\psi(t,\boldsymbol{x})$ the wave function transforms as

$$\chi(\boldsymbol{y}_1,\boldsymbol{y}_2) \to U(\boldsymbol{y}_1)\chi(\boldsymbol{y}_1,\boldsymbol{y}_2)U^\dagger(\boldsymbol{y}_2)$$

In a gauge where $\chi^{AB}(\boldsymbol{y}_1,\boldsymbol{y}_2) = \delta^{AB}\chi(\boldsymbol{y}_1,\boldsymbol{y}_2)$

only the diagonal color fields A_a^0 with $a = 3,8$ can be nonzero.

Since $f_{a38} = 0$ the commutator terms do not contribute at $O(g)$.

Fock states with quarks of color C give the EOM for A_a^0

$$-\boldsymbol{\nabla}^2 A_a^0(\boldsymbol{x}) = g\, T_a^{CC}\left[\delta^3(\boldsymbol{x}-\boldsymbol{x}_1) - \delta^3(\boldsymbol{x}-\boldsymbol{x}_2)\right]$$

$$A_a^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,C) = \Lambda_a^2\,\hat{\ell}_a\cdot\boldsymbol{x} + \frac{gT_a^{CC}}{4\pi}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\right) \qquad (a=3,8)$$

$$-\frac{1}{4}\sum_a\int d^3\boldsymbol{x} F_{\mu\nu}^a F_a^{\mu\nu} = \sum_{a=3,8}\left[\frac{1}{2}\Lambda_a^4\int d^3\boldsymbol{x} + \frac{1}{3}g\Lambda_a^2\, T_a^{CC}\,\hat{\ell}_a\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2) + \mathcal{O}\left(g^2\right)\right]$$

$\Lambda^4 \equiv \sum_{a=3,8}\Lambda_a^4$ must be independent of $\boldsymbol{x}_1, \boldsymbol{x}_2$, and $\hat{\boldsymbol{\ell}}_a \parallel \boldsymbol{x}_1 - \boldsymbol{x}_2$

Determining Λ_3/Λ_8 from stationarity it turns out that
the potential is independent of the quark color C,

$$V(\boldsymbol{x}_1,\boldsymbol{x}_2) = \frac{2g\Lambda^2}{3\sqrt{3}}|\boldsymbol{x}_1-\boldsymbol{x}_2|$$

and the bound state equation for the color singlet wave function χ
has the same form as in QED.

uds baryon states in in QCD

$$|E, t=0\rangle = \int \prod_{j=1}^{3} d^3\boldsymbol{y}_j\, \psi^{A\dagger}_{u\alpha_1}(t=0,\boldsymbol{y}_1)\psi^{B\dagger}_{d\alpha_2}(t=0,\boldsymbol{y}_2)\psi^{C\dagger}_{s\alpha_3}(t=0,\boldsymbol{y}_3)\chi^{\alpha_1\alpha_2\alpha_3}_{ABC}(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3)|0\rangle_R$$

In a gauge where

$$\chi^{\alpha_1\alpha_2\alpha_3}_{ABC}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \epsilon_{ABC}\chi^{\alpha_1\alpha_2\alpha_3}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)$$

the relevant gauge fields are, for quark colors $ABC = 123$

$$A^0_3(\boldsymbol{x};\{\boldsymbol{x}_i\}, ABC=123) = \Lambda_3^2\,\hat{\ell}_3\cdot\boldsymbol{x} + \frac{g}{4\pi}\frac{1}{2}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\right)$$

$$A^0_8(\boldsymbol{x};\{\boldsymbol{x}_i\}, ABC=123) = \Lambda_8^2\,\hat{\ell}_8\cdot\boldsymbol{x} + \frac{g}{4\pi}\frac{1}{2\sqrt{3}}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} + \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|} - 2\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_3|}\right)$$

Stationarity of the $O(g)$ in the action requires:

$$\hat{\ell}_3 \parallel \boldsymbol{x}_1 - \boldsymbol{x}_2, \qquad \hat{\ell}_8 \parallel \boldsymbol{x}_1 + \boldsymbol{x}_2 - 2\boldsymbol{x}_3$$

$$\frac{\Lambda_3^2}{\Lambda_8^2} = \sqrt{3}\,\frac{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}{|\boldsymbol{x}_1 + \boldsymbol{x}_2 - 2\boldsymbol{x}_3|}$$

For different colors $ABC = 213$, *etc*., the result is given by $\boldsymbol{x}_1 \leftrightarrow \boldsymbol{x}_2$, *etc*.

When expressed in terms of the universal strength the potential obtained for stationary action is the same for all color choices ABC,

$$\Lambda^4 \equiv \sum_{a=3,8} \Lambda_a^4$$

$$V(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \frac{\sqrt{2}g\Lambda^2}{3\sqrt{3}}\sqrt{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + (\boldsymbol{x}_2 - \boldsymbol{x}_3)^2 + (\boldsymbol{x}_3 - \boldsymbol{x}_1)^2}$$

and the bound state equation for the color singlet wave function is

$$\sum_{j=1}^{3} \left[\gamma^0(-i\boldsymbol{\nabla}_j \cdot \boldsymbol{\gamma}_j + m_j)\right]\chi = (E - V)\chi$$

Comparison with the Quark Model

The quark model uses a potential $V(r) = g\Lambda^2 r - C_F \frac{\alpha_s}{r}$

where the Coulomb term (one gluon exchange) is perturbative.

In the present approach the linear (non-perturbative) term emerges as a homogenous solution of the equations of motion.

Perturbative gluon exchange is of order g^2 , hence is subdominant to the order linear term. Terms of order g^2 were dropped in the bound state equation.

This is the why boost covariance at equal time is expected to, and in fact does, hold only for a purely linear potential.

Summary of Talk

- Is there a systematic approximation of QCD which gives the quark model?
- Consider an hbar expansion for bound states: Born term at $\mathcal{O}\left(\hbar^0\right)$
- Determine A^0 from equation of motion (for each constituent configuration)
- Allow homogeneous solution: linear potential $A^0 = \boldsymbol{l} \cdot \boldsymbol{x}$
- Fix direction of $\boldsymbol{l}$ by stationarity of action (for each Fock state)
- Ignore $\mathcal{O}(g^2)$ (Coulomb exchange) – hence use purely linear potential
- Find meson and baryon states with interesting phenomenology
- Observe non-trivial Lorentz covariance for a linear potential
- Sea quarks generated implicitly, through use of retarded vacuum: no new degrees of freedom.