Principles and Practice of Bound States in QED and QCD

Lectures at Università di Pavia, January 2020

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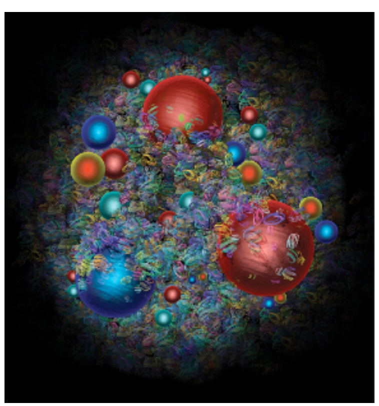
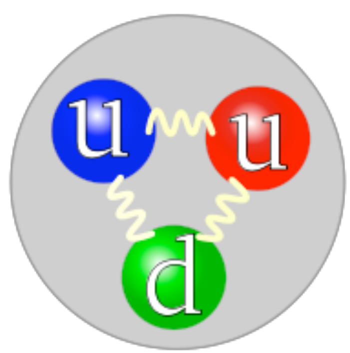
https://www.mv.helsinki.fi/home/hoyer/Pavia/Hoyer_Pavia_2020.pdf

https://www.mv.helsinki.fi/home/hoyer/Pavia/200120_Hoyer_Pavia-figs.pdf

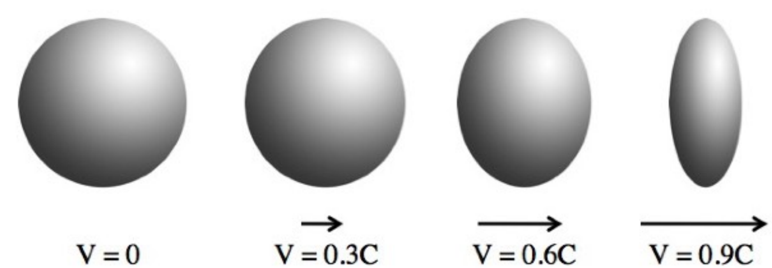
From the art to the science of bound states

Pictures of the proton



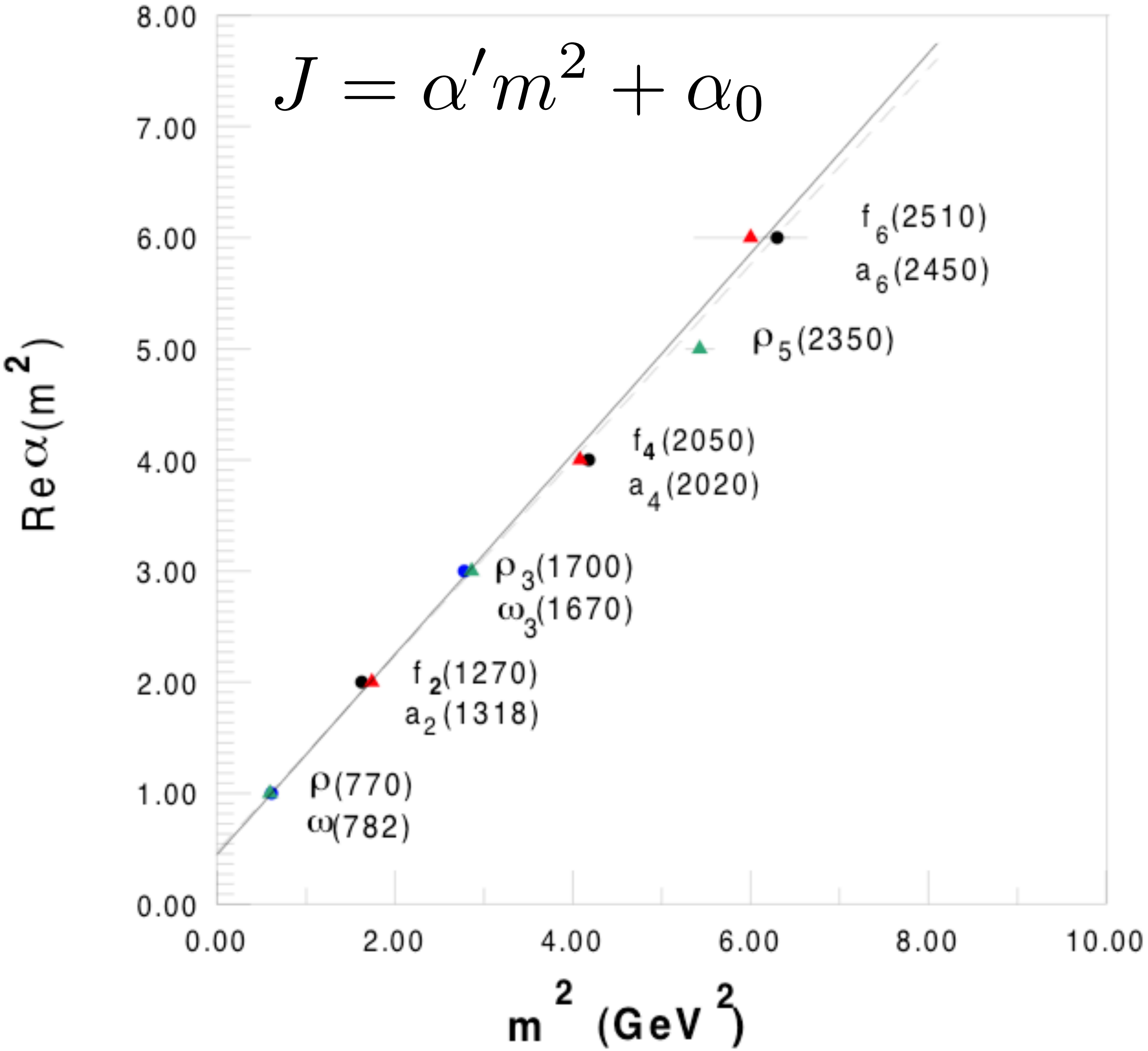


Picture of Lorentz contraction



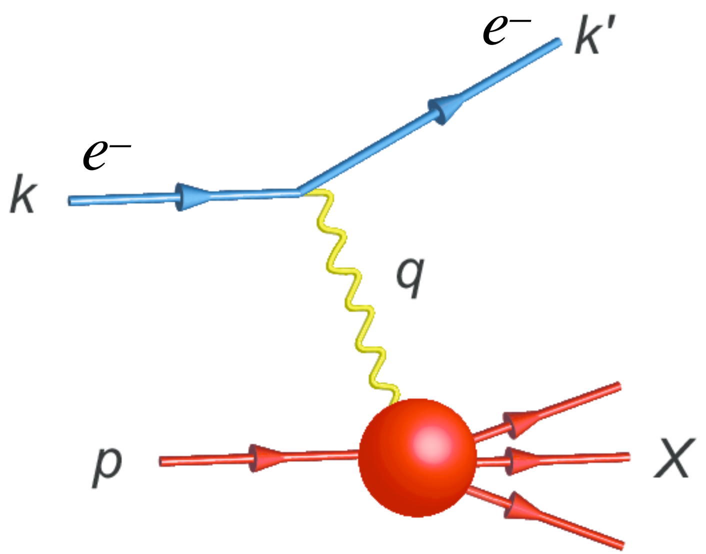
PH 1807.05598v2, 1911.07286

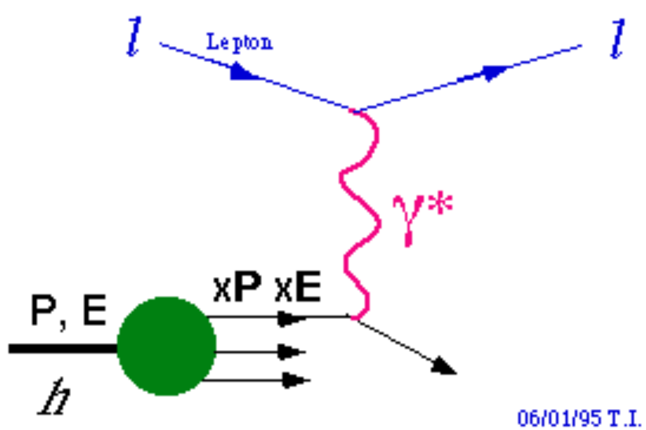
Hadrons lie on linear Regge trajectories

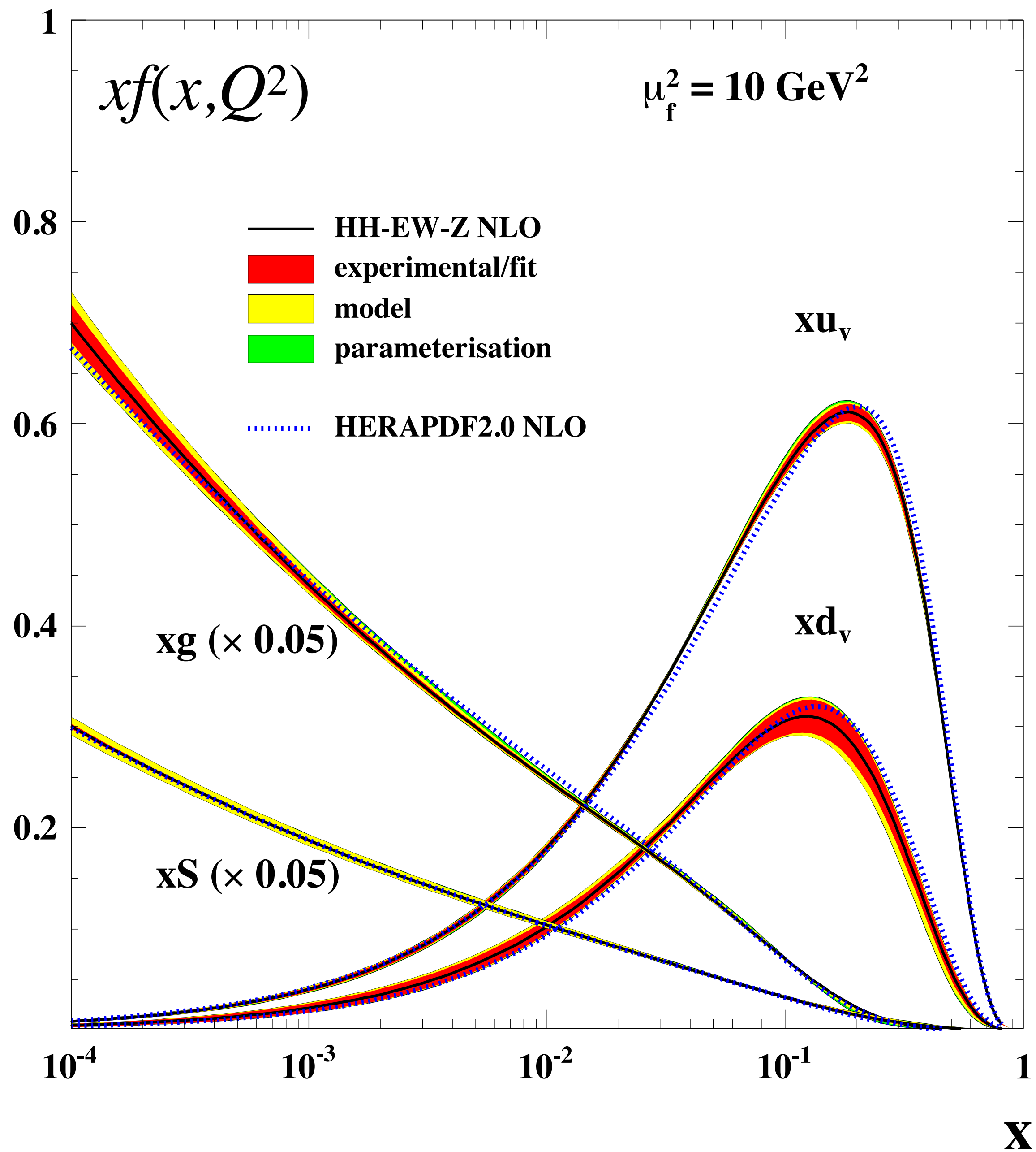


Oleg Andreev & Warren Siegel (2005) http://insti.physics.sunysb.edu/~siegel/reggepart/reggepart.html

Deep inelastic lepton scattering: gluons and sea quarks

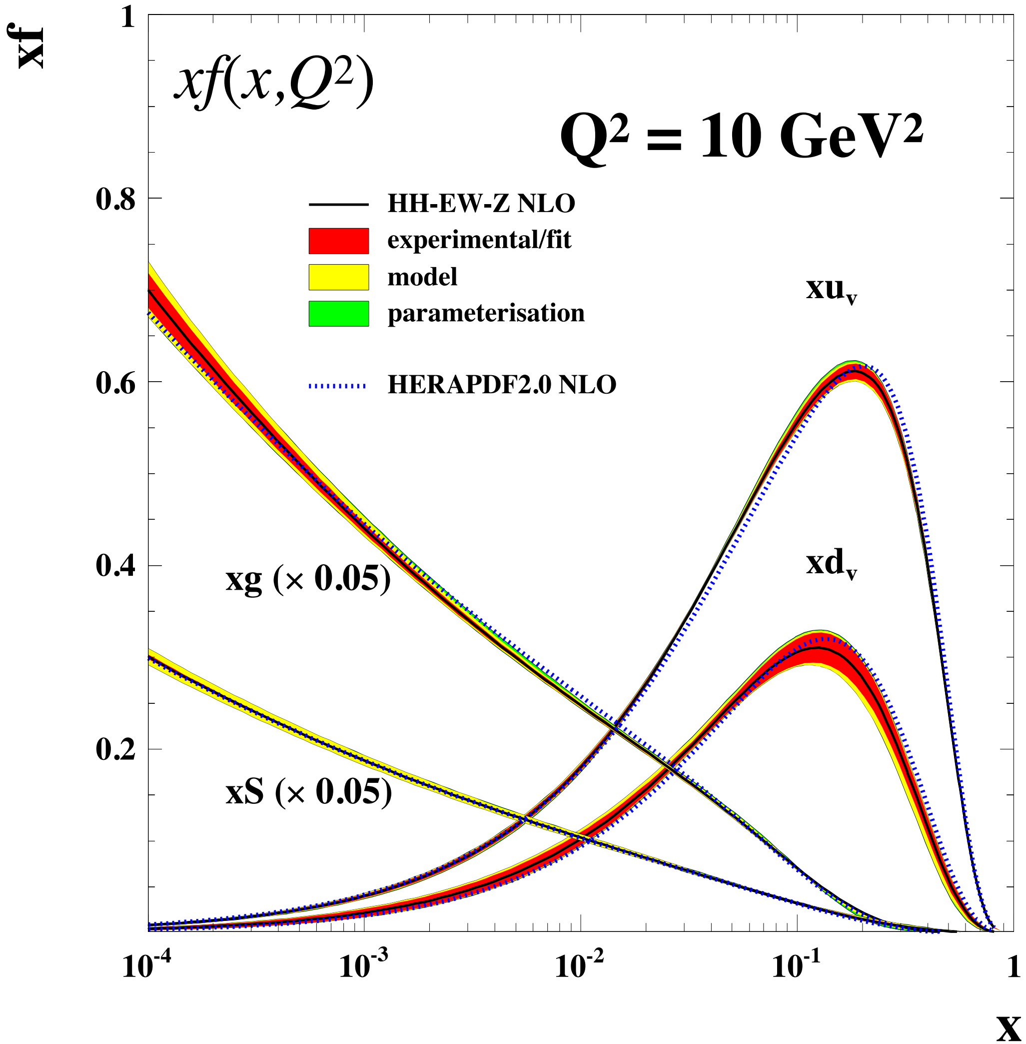


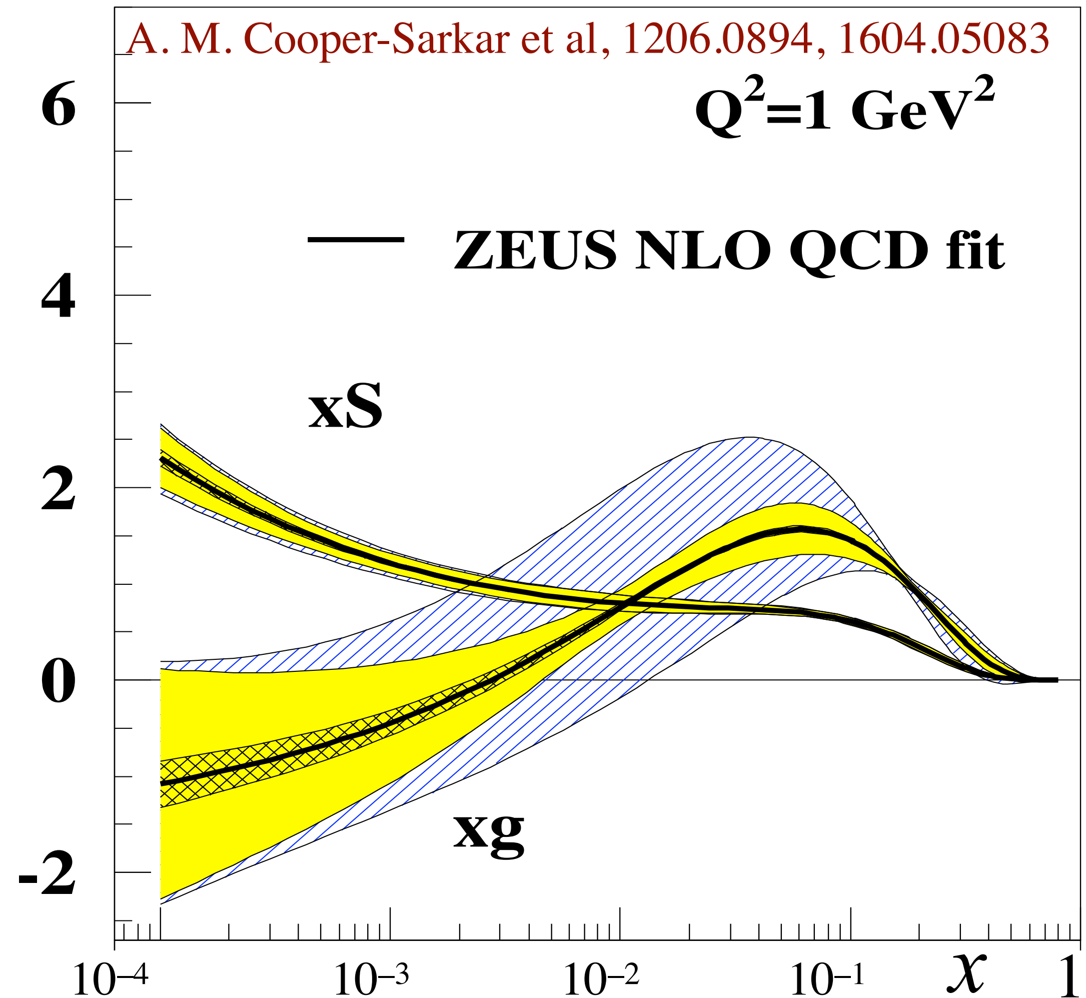




A. M. Cooper-Sarkar et al, 1206.0894, 1604.05083

Gluons at low x arise from evolution

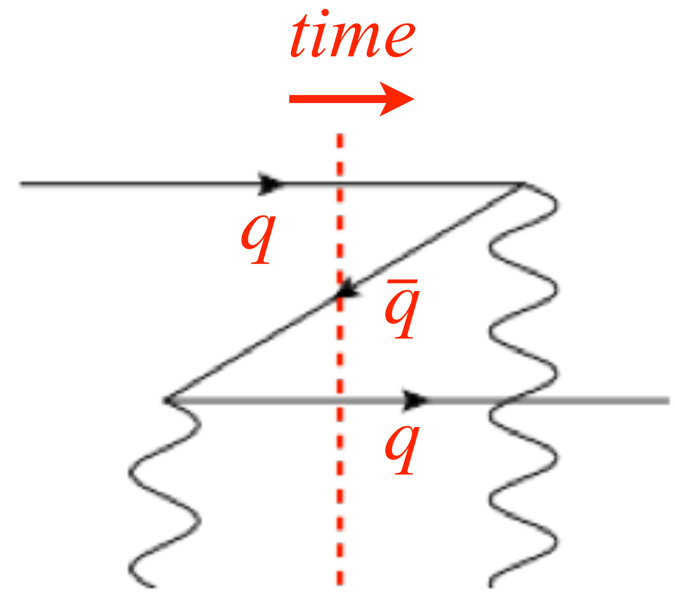




There are fewer gluons in the proton at low Q.

Sea quarks evolve more slowly, remain at hadronic scales.

Sea quarks can be described analytically



Time ordering results in $q\bar{q}$ pairs from a single quark line in a strong field.

The pairs are present in the *Dirac states*, as described by the Dirac wave function.

This is related to the “Klein paradox”.

The states corresponding to Dirac wave functions are rarely discussed.

Dirac wave functions describe a Bogoliubov transformed electron, which *in the free basis* has many pairs.

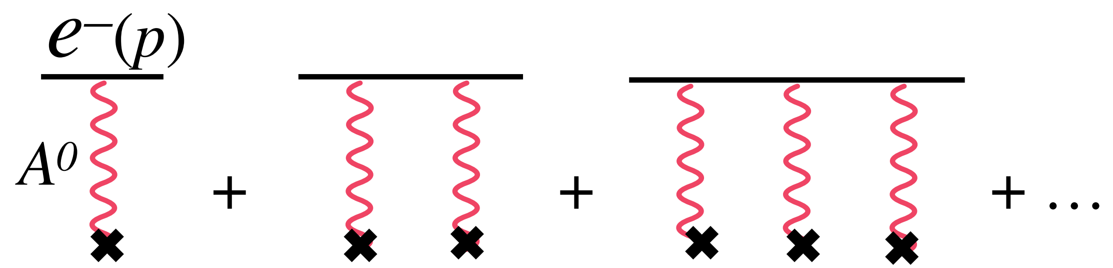
The “Dirac pairs” can qualitatively describe the sea quarks.

Dirac bound states

Electron in a strong external potential

Dirac equation from Feynman diagrams

Summing the interactions of the electron with a static A^0 field gives a pole at $p^0 = E$ if its wf Φ satisfies



$$(-i\boldsymbol{\nabla}\cdot\gamma^0\boldsymbol{\gamma} + eA^0 + m\gamma^0)\Phi(\boldsymbol{x}) = E\Phi(\boldsymbol{x})$$ Dirac equation

Note: The Dirac eq. is of Born level: There are no loop corrections.

The wave function $\Phi(\boldsymbol{x})$ has single electron quantum numbers.

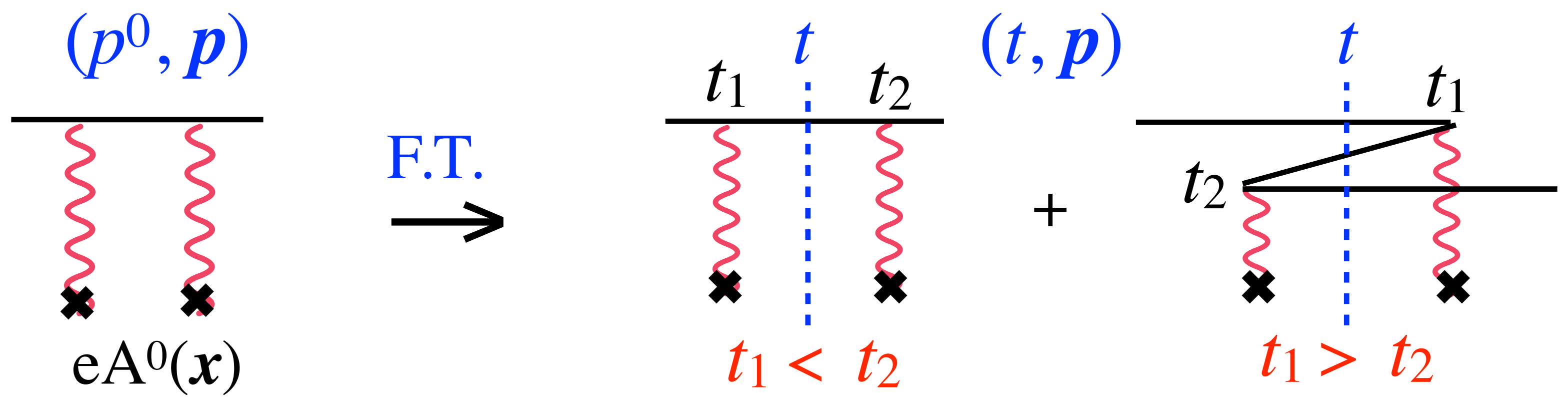
Time-ordering the vertices by Fourier-transforming the electron propagators,

$$S_F(t,\boldsymbol{p}) = \frac{1}{2E_p}\left[\theta(t)(E_p\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} + m_e)e^{-iE_pt} + \theta(-t)(-E_p\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} + m_e)e^{iE_pt}\right]$$

we find "Z-diagrams" from the negative energy components, implying, at any given time, additional e^+e^- pairs in the state.

Time ordering reveals the pair contributions

The poles of the electron Dirac propagator at $p^0 = E_p$ and $p^0 = -E_p$ give rise to two time-ordered diagrams in the Fourier transform $(p^0, \boldsymbol{p}) \to (t, \boldsymbol{p})$:



⟹ At fixed t, a Dirac state has Fock components with any number of e^+e^- pairs.

The Dirac equation: $\gamma^0\big(-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + e\not{A} + m\big)\phi(\boldsymbol{x}) = E\phi(\boldsymbol{x})$

involves the "single electron" wave function $\phi(\boldsymbol{x})$. The equation specifies the energy E and the quantum numbers of the state, but not (explicitly) its Fock components.

Determination of the Dirac state

The operator expression for the Dirac states is found by diagonalizing the Dirac Hamiltonian for a given external field $A^\mu(\boldsymbol{x})$.

J. P. Blaizot and G. Ripka: *Quantum Theory of Finite Systems*, MIT Press, Cambridge, MA (1986)

$$H = \int d^3\boldsymbol{x}\bar{\psi}(\boldsymbol{x})\big[-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + m + e\not{A}(\boldsymbol{x})\big]\psi(\boldsymbol{x})$$

$$\psi(\boldsymbol{x}) = \sum_{\boldsymbol{p},\lambda}\Big[b_{\boldsymbol{p},\lambda}u(\boldsymbol{p},\lambda)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + d^\dagger_{\boldsymbol{p},\lambda}v(\boldsymbol{p},\lambda)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\Big]$$

$$\sum_{\boldsymbol{p},\lambda} \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_p}\sum_\lambda$$

Since H is quadratic in $b, b^\dagger, d, d^\dagger$ it can be diagonalized for any $A^\mu(\boldsymbol{x})$.

Denote the solutions of the Dirac equation with positive and negative energies as

$$\big(-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}+m+e\not{A}\big)\phi_n(\boldsymbol{x}) = E_n\gamma^0\phi_n(\boldsymbol{x}) \qquad E_n>0$$

$$\big(-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}+m+e\not{A}\big)\bar\phi_n(\boldsymbol{x}) = -\bar E_n\gamma^0\bar\phi_n(\boldsymbol{x}) \qquad \bar E_n>0$$

The eigenstates and their creation operators are then

$$H\,|n\rangle = E_n\,|n\rangle \qquad |n\rangle = \int d\boldsymbol{x}\,\psi_\alpha^\dagger(\boldsymbol{x})\phi_{n\,\alpha}(\boldsymbol{x})\,|\Omega\rangle \equiv c_n^\dagger\,|\Omega\rangle$$

$$H\,|\bar n\rangle = \bar E_n\,|\bar n\rangle \qquad |\bar n\rangle = \int d\boldsymbol{x}\,\bar\phi_{n\,\alpha}^\dagger(\boldsymbol{x})\psi_\alpha(\boldsymbol{x})\,|\Omega\rangle \equiv \bar c_n^\dagger\,|\Omega\rangle$$

Using $\left\{\psi_\alpha(\boldsymbol{x}),\psi_\beta^\dagger(\boldsymbol{y})\right\} = \delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y})$ we next verify that these are eigenstates of H, provided $H\,|\Omega\rangle = 0$

The pair contributions are hiding in the ground state $|\Omega\rangle$.

Determination of the Dirac state (cont.)

Verification that $|n\rangle = \int d\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\phi_n(\boldsymbol{x})\,|\Omega\rangle$ is an eigenstate of H:

$$H\,|n\rangle = \int d\boldsymbol{x}\,\left[H,\psi^\dagger(\boldsymbol{x})\right]\phi_n(\boldsymbol{x})\,|\Omega\rangle + \int d\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\phi_n(\boldsymbol{x})H\,|\Omega\rangle$$

$H = \int d\boldsymbol{y}\,\bar\psi(\boldsymbol{y})\big[-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}+m+e\not{A}(\boldsymbol{y})\big]\psi(\boldsymbol{y})$ and $\left\{\psi_\alpha(\boldsymbol{y}),\psi_\beta^\dagger(\boldsymbol{x})\right\} = \delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y})$

$$\left[H,\psi^\dagger(\boldsymbol{x})\right] = \int d\boldsymbol{y}\,\bar\psi(\boldsymbol{y})\big[-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}+m+e\not{A}(\boldsymbol{y})\big]\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

$$H\,|n\rangle = E_n\,|n\rangle + \int d\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\phi_n(\boldsymbol{x})H\,|\Omega\rangle$$

Thus we need $H\,|\Omega\rangle = 0$

Note that $H\,|0\rangle \neq 0$ since the $b^\dagger d^\dagger$ term in H creates an e^+e^- pair.

The eigenstate operators can be expressed in terms of the wf's in mom. space:

$$c_n = \sum_{\boldsymbol{p}} \phi_n^\dagger(\boldsymbol{p})\left[u(\boldsymbol{p})b_{\boldsymbol{p}} + v(-\boldsymbol{p})d_{-\boldsymbol{p}}^\dagger\right] \equiv B_{np}b_p + D_{np}d_p^\dagger$$

$$\bar{c}_n = \sum_{\boldsymbol{p}} \left[b_{\boldsymbol{p}}^\dagger u^\dagger(\boldsymbol{p}) + d_{-\boldsymbol{p}}v^\dagger(-\boldsymbol{p})\right]\bar{\phi}_n(\boldsymbol{p}) \equiv \bar{B}_{np}b_p^\dagger + \bar{D}_{np}d_p$$

The Dirac Hamiltonian is diagonalized: $H = \sum_n \left[E_n c_n^\dagger c_n + \bar{E}_n \bar{c}_n^\dagger \bar{c}_n\right]$

The ground state is $\left|\Omega\right\rangle = N_0 \exp\left[-b_p^\dagger\left(B^{-1}\right)_{pm} D_{mq}d_q^\dagger\right]\left|0\right\rangle$

and satisfies: $c_n\left|\Omega\right\rangle = \bar{c}_n\left|\Omega\right\rangle = H\left|\Omega\right\rangle = 0$

Check: $B_{np}b_p\left|\Omega\right\rangle = -B_{np}\left(B^{-1}\right)_{pm} D_{mq}d_q^\dagger\Big]\left|\Omega\right\rangle = -D_{nq}d_q^\dagger\left|\Omega\right\rangle$

The Dirac states

The case of a linear potential in D=1+1 dimensions

Example of Dirac states: $V(x) = \frac{1}{2}|x|$ in D=1+1

The Coulomb potential in D=1+1 is $V(x) = \frac{1}{2}e^2|x|$. We set $e = 1$ (scale).

The potential confines electrons, and repels positrons: $V(e^+) = -V(e^-)$
Any e^+ in the state is accelerated to large $|x|$.

To keep T+V ≃ constant, positrons have large momenta at high $|x|$:
$|p| \sim E_p \sim |x|/2$

Since we consider time independent solutions, there will also be decelerating positrons, moving towards $x = 0$.

The positron energy spectrum is continuous, whereas the electrons form bound states around $x = 0$, with discrete energies.

The relative size of the electron and e^+e^- pair components can be adjusted. However, pairs are completely absent only in the NR limit, $m \to \infty$.

This explains the observation made already in the 1930's:

AUGUST 1, 1932 *PHYSICAL REVIEW* *VOLUME 41*

The Dirac Electron in Simple Fields*

By MILTON S. PLESSET
Sloane Physics Laboratory, Yale University

(Received June 6, 1932)

The relativity wave equations for the Dirac electron are transformed in a simple manner into a symmetric canonical form. This canonical form makes readily possible the investigation of the characteristics of the solutions of these relativity equations for simple potential fields. If the potential is a polynomial of any degree in x, a continuous energy spectrum characterizes the solutions. If the potential is a polynomial of any degree in $1/x$, the solutions possess a continuous energy spectrum when the energy is numerically greater than the rest-energy of the electron; values of the energy numerically less than the rest-energy are barred. When the potential is a polynomial of any degree in r, all values of the energy are allowed. For potentials which are polynomials in $1/r$ of degree higher than the first, the energy spectrum is again continuous. The quantization arising for the Coulomb potential is an exceptional case.

See also: E. C. Titchmarsh, Proc. London Math. Soc. (3) 11 (1961) 159 and 169; Quart. J. Math. Oxford (2), 12 (1961), 227.

Example of Dirac states: $V(x) = \frac{1}{2}|x|$ in D=1+1

The Dirac matrices can be represented as 2x2 Pauli matrices

$$\gamma^0 = \sigma_3 \qquad \gamma^0\gamma^1 = \sigma_1$$

$$\Phi(x) = \left[\begin{array}{c} \varphi(x) \\ \chi(x) \end{array}\right]$$

$$-i\partial_x\varphi = (M - V + m)\chi$$
$$-i\partial_x\chi = (M - V - m)\varphi$$

We may choose the phases of $\varphi(x)$ and $\chi(x)$, and their parity $\eta = \pm 1$:

$$\varphi^*(x) = \varphi(x) = \eta\varphi(-x) \qquad \chi^*(x) = -\chi(x) = \eta\chi(-x)$$

Defining

$$f(x) \equiv [\varphi(x) + \chi(x)]e^{i\sigma}$$
$$g(x) \equiv [\varphi(x) - \chi(x)]e^{i\sigma}$$

where $\sigma \equiv (M - V)^2$

the (arbitrarily normalized) solution is, in terms of the parameter $\beta = \beta^*$:

Example of Dirac states: $V(x)=\frac{1}{2}|x|$ in D=1+1

$$f(x)=e^{i\beta}\,{}_1F_1(-i\tfrac{1}{2}m^2,\tfrac{1}{2},2i\sigma)+2ime^{-i\beta}(M-V)\,{}_1F_1(\tfrac{1}{2}-i\tfrac{1}{2}m^2,\tfrac{3}{2},2i\sigma)$$

$$g(x)=e^{-i\beta}\,{}_1F_1(\tfrac{1}{2}-i\tfrac{1}{2}m^2,\tfrac{1}{2},2i\sigma)-2ime^{i\beta}(M-V)\,{}_1F_1(1-i\tfrac{1}{2}m^2,\tfrac{3}{2},2i\sigma)$$

For $x\to\infty$, with $\delta\equiv\arg\left[\Gamma(1-i\tfrac{1}{2}m^2)/\Gamma(\tfrac{1}{2}-i\tfrac{1}{2}m^2)\right]$,

$$\varphi(x)+\chi(x)=e^{-ix^2/4}\,\frac{e^{i(\beta-\delta)}}{2\sqrt{2\pi}\,m}\Gamma(1-i\tfrac{1}{2}m^2)(\tfrac{1}{2}x^2)^{im^2/2}e^{3\pi m^2/4}$$

$$\times\left[\sqrt{1-e^{-2\pi m^2}}+e^{i(\delta-2\beta-\pi/4)}(1-e^{-\pi m^2})\right]\left[1+\mathcal{O}\left(x^{-1}\right)\right]$$

$$\varphi(x)-\chi(x)=[\varphi(x)+\chi(x)]^*$$

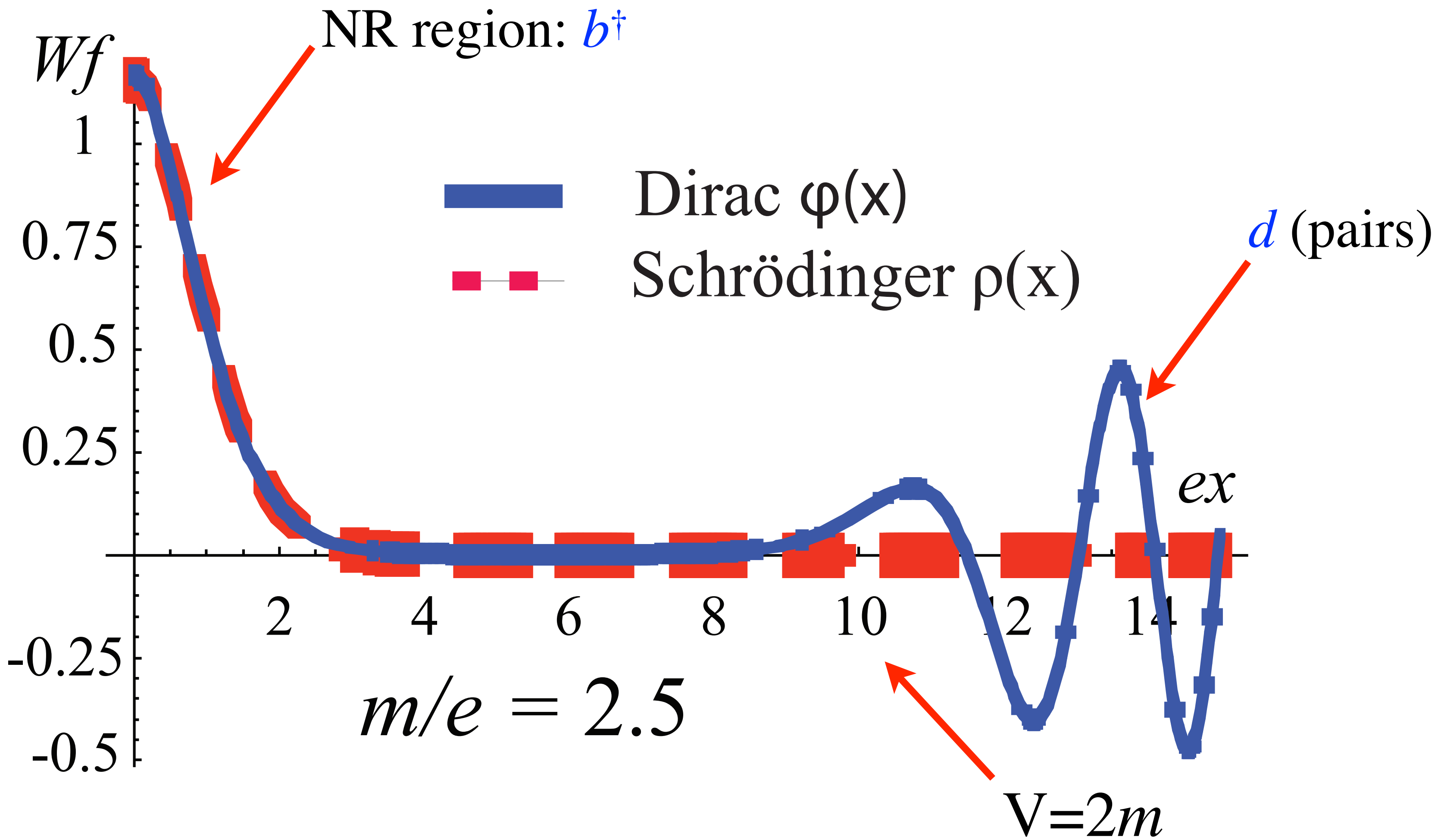
The wf's oscillate at with constant norm at large x: $\sim\exp(\pm ix^2/4)$

There is a solution for any M (the spectrum is continuous).

Adjusting β the oscillations can be made to vanish for $x\to\infty$, up to terms of $\mathcal{O}\left[\,\exp(-\pi m^2)\right]$

Example of Dirac states: $V(x) = \frac{1}{2}|x|$ in D=1+1

$m = 2.5$; generic β : Dirac $\varphi(x)$ versus the Schrödinger *Ai* solution $\varrho(x)$



$$|M \geq 0\rangle = \int \frac{dp}{2\pi 2E} \int dx \Big[b_p^\dagger u^\dagger(p) e^{-ipx} + d_p v^\dagger(p) e^{ipx} \Big] \begin{bmatrix} \varphi(x) \\ \chi(x) \end{bmatrix} |\Omega\rangle$$

$$|\Omega\rangle = N_0 \exp\Big[-b_p^\dagger \big(B^{-1}\big)_{pm} D_{mq} d_q^\dagger \Big] |0\rangle$$

Example of Dirac states: $V(x) = \frac{1}{2}|x|$ in D=1+1

As $|p| \to \infty$ the x-integrand oscillates rapidly in

$$|M \geq 0\rangle = \int \frac{dp}{2\pi 2E} \int dx \Big[b_p^\dagger u^\dagger(p) e^{-ipx} + d_p v^\dagger(p) e^{ipx}\Big] \left[\begin{array}{c} \varphi(x) \\ \chi(x) \end{array}\right] |\Omega\rangle$$

The stationary phase approximation shows that only d_p contributes:

$$|M > 0\rangle_{|p|\to+\infty} \simeq \left(C + \eta C^*\right) \int \frac{dp}{\sqrt{2|p|}} \exp(ip^2)(2p^2)^{im^2/2}\, d_p\, |\Omega\rangle$$

where

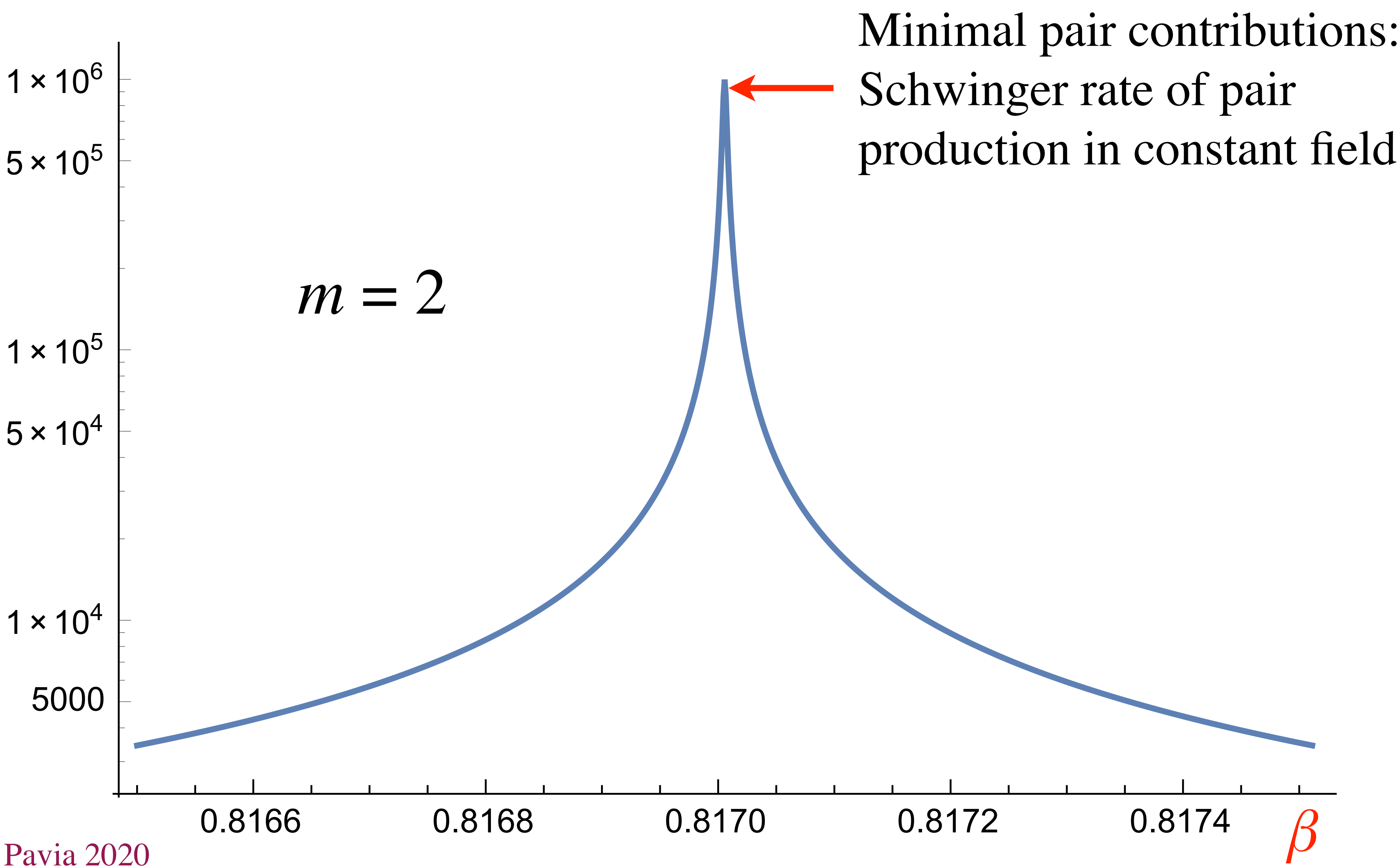
$$C = e^{i(\beta-\delta-\pi/4)} \frac{\Gamma(1-\frac{1}{2}im^2)}{4\pi m} e^{3\pi m^2/4} \Big[\sqrt{1-e^{-2\pi m^2}} + e^{i(\delta-2\beta-\pi/4)}(1-e^{-\pi m^2})\Big]$$

$E.g.$, for $m = 2$ we have $\exp(-\pi m^2) \simeq 3.49 \cdot 10^{-6}$

The term in [ ] is minimized for $\beta = 0.8170$. The ratio of the wave function at $x = 0$ to the amplitude of the oscillations depends sensitively on β :

Example of Dirac states: $V(x) = \frac{1}{2}|x|$ in D=1+1

$$R_{osc} = \frac{|\varphi(x=0)|}{\lim_{x\to\infty}|\varphi(x)+\chi(x)|}$$

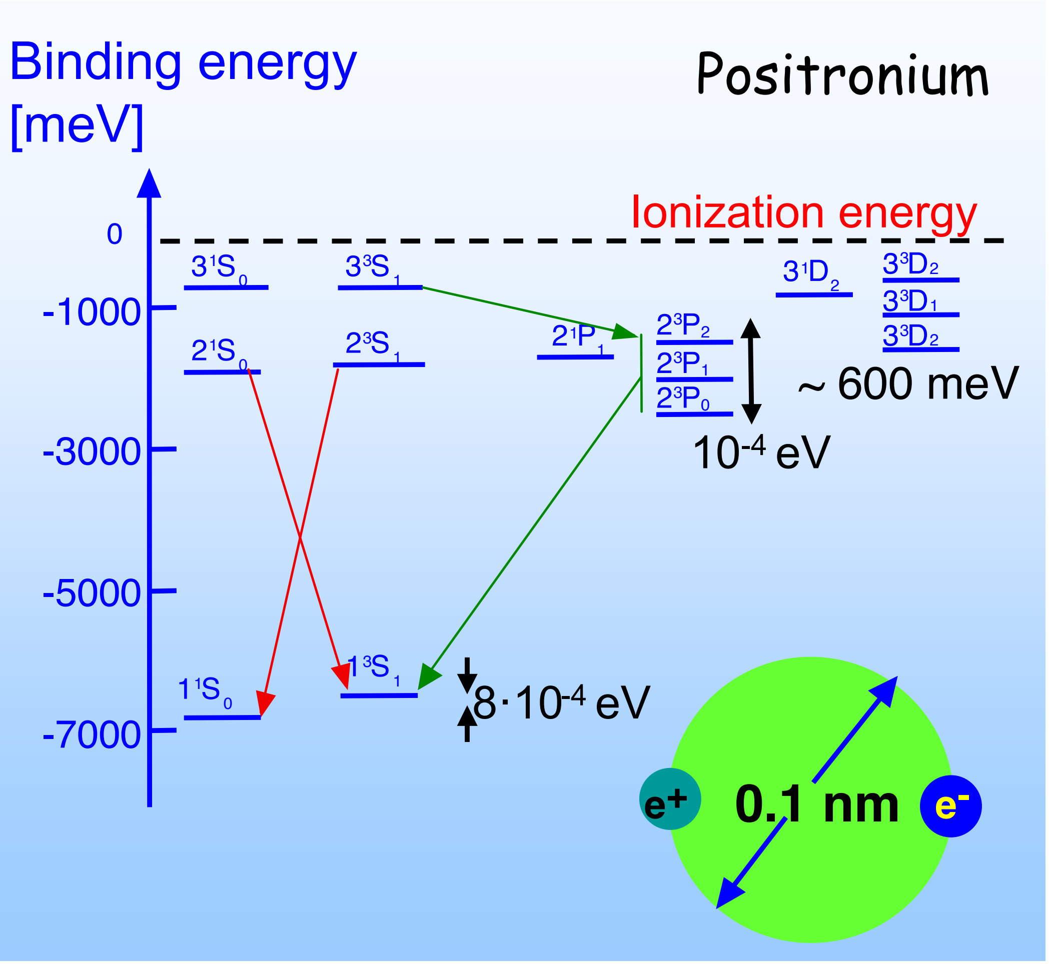


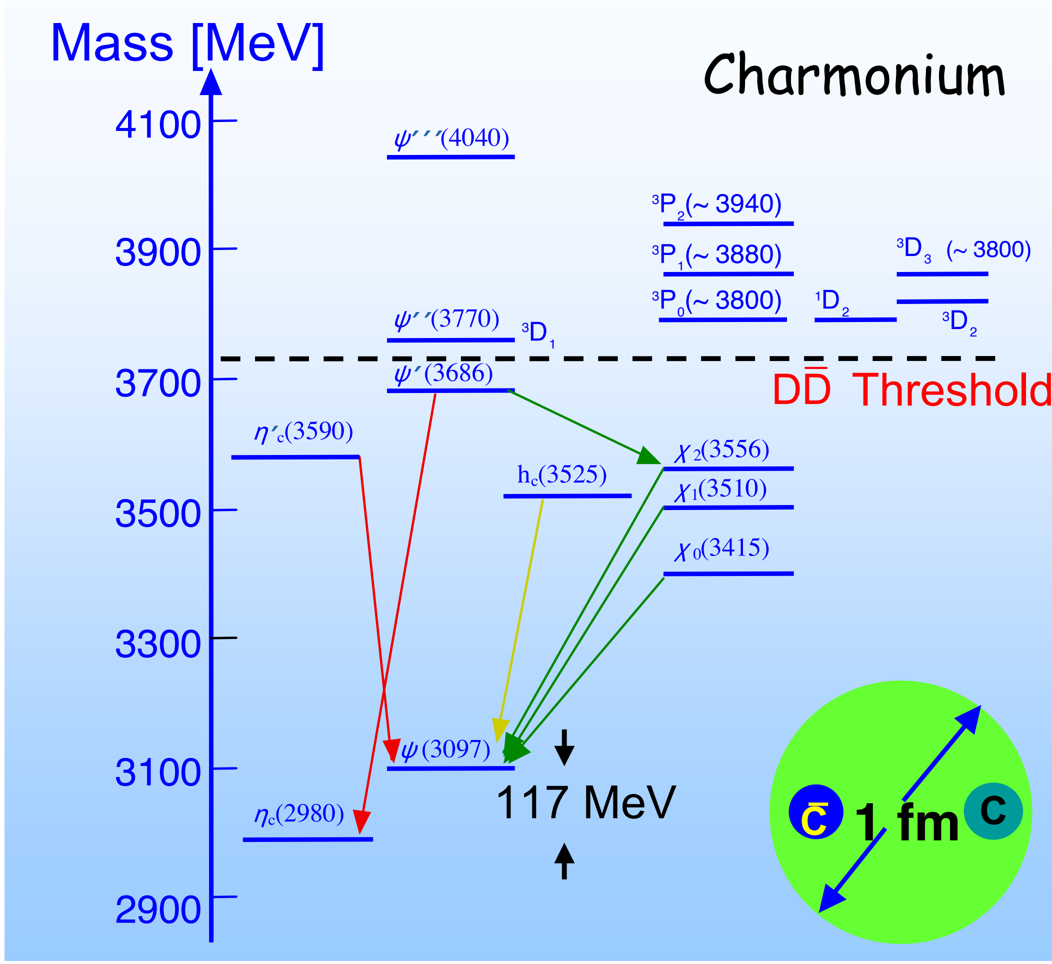
Hadron spectrum: No gluon nor sea quark dof's

$n\,^{2s+1}\ell_J$	J^{PC}	I = 1 $u\bar{d},\ \bar{u}d,\ \frac{1}{\sqrt{2}}(d\bar{d}-u\bar{u})$	I = $\frac{1}{2}$ $u\bar{s},\ d\bar{s};\ \bar{d}s,\ -\bar{u}s$	I = 0 f'	I = 0 f	θ_{quad} [°]	θ_{lin} [°]
$1\,^1S_0$	0^{-+}	$\boldsymbol{\pi}$	$\boldsymbol{K}$	$\boldsymbol{\eta}$	$\boldsymbol{\eta'(958)}$	−11.3	−24.5
$1\,^3S_1$	1^{--}	$\boldsymbol{\rho(770)}$	$\boldsymbol{K^*(892)}$	$\boldsymbol{\phi(1020)}$	$\boldsymbol{\omega(782)}$	39.2	36.5
$1\,^1P_1$	1^{+-}	$\boldsymbol{b_1(1235)}$	$\boldsymbol{K_{1B}}^\dagger$	$h_1(1380)$	$\boldsymbol{h_1(1170)}$		
$1\,^3P_0$	0^{++}	$\boldsymbol{a_0(1450)}$	$\boldsymbol{K_0^*(1430)}$	$\boldsymbol{f_0(1710)}$	$\boldsymbol{f_0(1370)}$		
$1\,^3P_1$	1^{++}	$\boldsymbol{a_1(1260)}$	$\boldsymbol{K_{1A}}^\dagger$	$\boldsymbol{f_1(1420)}$	$\boldsymbol{f_1(1285)}$		
$1\,^3P_2$	2^{++}	$\boldsymbol{a_2(1320)}$	$\boldsymbol{K_2^*(1430)}$	$\boldsymbol{f_2'(1525)}$	$\boldsymbol{f_2(1270)}$	29.6	28.0
$1\,^1D_2$	2^{-+}	$\boldsymbol{\pi_2(1670)}$	$\boldsymbol{K_2(1770)}^\dagger$	$\eta_2(1870)$	$\boldsymbol{\eta_2(1645)}$		
$1\,^3D_1$	1^{--}	$\boldsymbol{\rho(1700)}$	$\boldsymbol{K^*(1680)}$		$\boldsymbol{\omega(1650)}$		
$1\,^3D_2$	2^{--}		$\boldsymbol{K_2(1820)}$				
$1\,^3D_3$	3^{--}	$\boldsymbol{\rho_3(1690)}$	$\boldsymbol{K_3^*(1780)}$	$\boldsymbol{\phi_3(1850)}$	$\boldsymbol{\omega_3(1670)}$	31.8	30.8
$1\,^3F_4$	4^{++}	$\boldsymbol{a_4(2040)}$	$\boldsymbol{K_4^*(2045)}$		$\boldsymbol{f_4(2050)}$		
$1\,^3G_5$	5^{--}	$\rho_5(2350)$	$K_5^*(2380)$				
$1\,^3H_6$	6^{++}	$a_6(2450)$			$f_6(2510)$		
$2\,^1S_0$	0^{-+}	$\boldsymbol{\pi(1300)}$	$K(1460)$	$\boldsymbol{\eta(1475)}$	$\boldsymbol{\eta(1295)}$		
$2\,^3S_1$	1^{--}	$\boldsymbol{\rho(1450)}$	$\boldsymbol{K^*(1410)}$	$\boldsymbol{\phi(1680)}$	$\boldsymbol{\omega(1420)}$		
$3\,^1S_0$	0^{-+}	$\boldsymbol{\pi(1800)}$			$\eta(1760)$		

M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018)

Quarkonia are like atoms with confinement





$$V(r) = -\frac{\alpha}{r}$$

$$V(r) = V'r - \frac{4}{3}\frac{\alpha_s}{r} \qquad (1980)$$

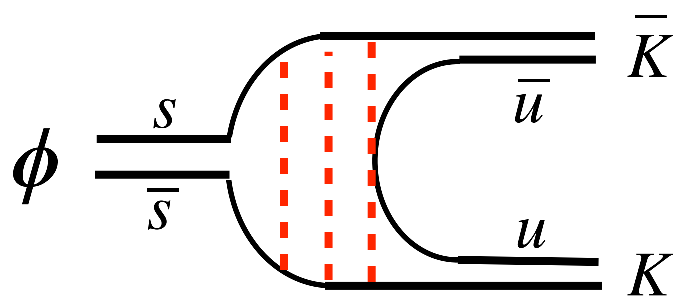
E. Eichten, S. Godfrey, H. Mahlke and J. L. Rosner, Rev. Mod. Phys. **80** (2008) 1161

"The J/ψ is the Hydrogen atom of QCD"

The OZI Rule

Connected diagrams: Unsuppressed, string breaking from confining potential

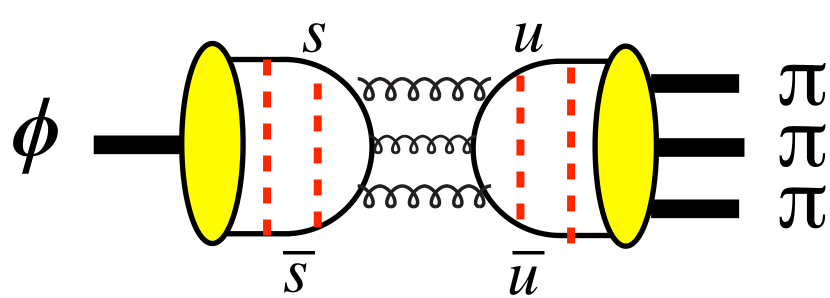
$\phi(1020) \to K\bar{K}$



ΔE	Br
26 MeV	83.1 %

Disconnected, perturbative diagrams are suppressed

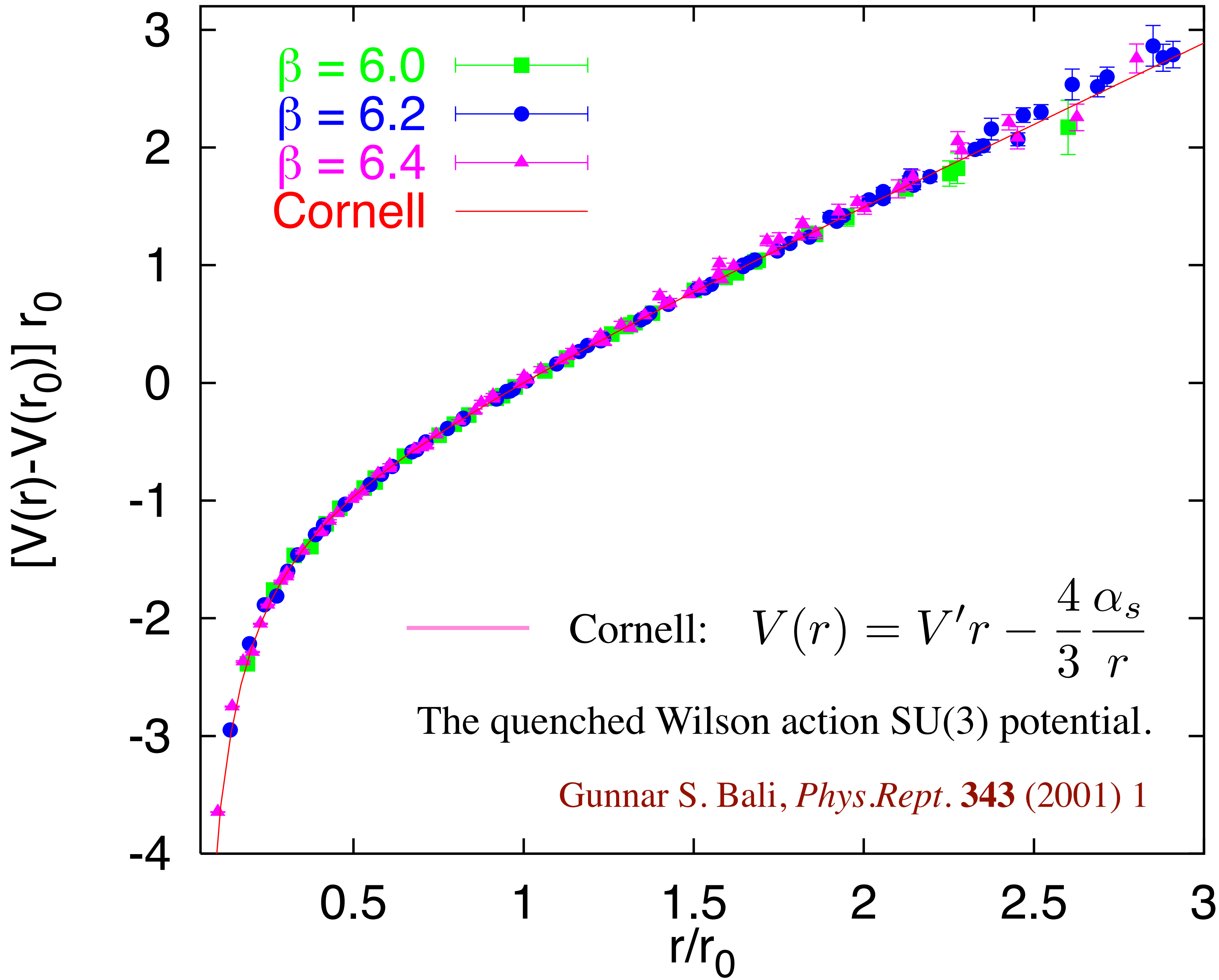
$\phi(1020) \not\to \pi\pi\pi$



610 MeV 15.3 %

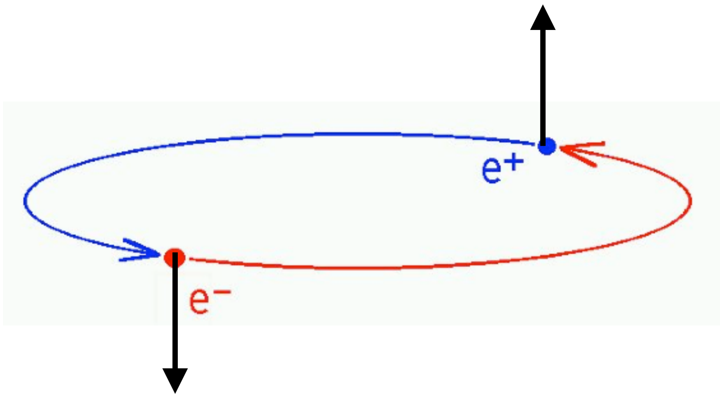
This suggests that perturbative corrections are small even in the soft regime.

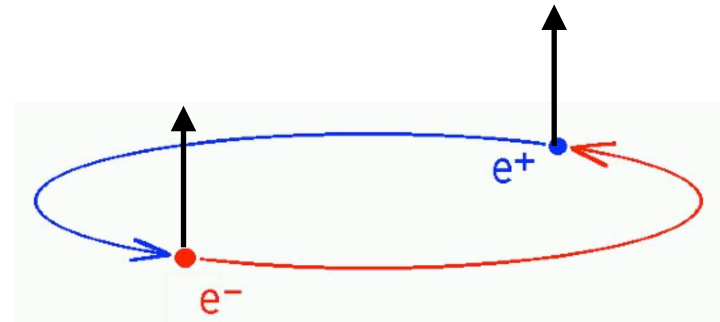
Lattice QCD agrees with the Cornell potential



The Positronium atom

Parapositronium (S = 0) $L = 0$ Orthopositronium (S = 1)





Schrödinger eq.: $$\left[-\frac{\boldsymbol{\nabla}^2}{m}+V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})=E_b\,\Phi(\boldsymbol{x})$$ $$V(\boldsymbol{x})=-\frac{\alpha}{|\boldsymbol{x}|}$$

$$\Phi_{pos}(\boldsymbol{x})=N\exp(-\alpha\, m\,|\boldsymbol{x}|/2)$$ Has all powers of α, is gauge dependent

$$E_b=-\tfrac{1}{4}\alpha^2 m$$ E_b can be expanded in powers of α, is measurable

$$E_b(ortho)-E_b(para)=\tfrac{7}{12}\,\alpha^4 m+\mathcal{O}\left(\alpha^5\right)$$

Hyperfine structure gives the 21 cm line observed in H I regions in interstellar medium

QED atoms (are not) in QFT textbooks

Bound states are not discussed in today's textbooks. The last exception:

C. Itzykson and J.-B. Zuber: Quantum Field Theory (1980)

10-3 HYPERFINE SPLITTING IN POSITRONIUM

It should not be concluded that relativistic weak binding corrections cannot be obtained for two-body systems that agree with experiment. On the contrary, the positronium states give an example of a successful agreement. This will serve to illustrate the theory. To be completely fair, we should admit that accurate predictions require some artistic gifts from the practitioner. As yet no systematic method has been devised to obtain the corrections in a completely satisfactory way.

I & Z do not derive the Schrödinger equation from the QED action.

The situation has not improved qualitatively.

The art of atoms

Review paper: Rev. Mod. Phys. **57** (1985) 723

Recoil effects in the hyperfine structure of QED bound states

G. T. Bodwin

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D. R. Yennie

Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853

M. A. Gregorio

Instituto de Fisica, Universidade Federal de Rio de Janeiro, Rio de Janeiro, Brazil

"In spite of the statement in the preceding paragraph that bound-state theory is nonperturbative, it is possible to make use of small parameters such as α and m_e/m_A (where m_A is the mass of the nucleus) to develop expressions in increasing orders of smallness. However, the nonperturbative nature of the expansion shows up in non-analytic dependence on these parameters (such as logarithms). As indicated in the preceding paragraph, there is an art in developing a theoretical expression in this manner."

The NRQED expansion in p/m = v ≃ α

Non-Relativistic QED has turned out to be the most efficient way of calculating higher order QED corrections to atoms. The Lagrangian is expanded in powers of p/m_e. Loops contract to effective vertices.

The NRQED effective Lagrangian is found to be:

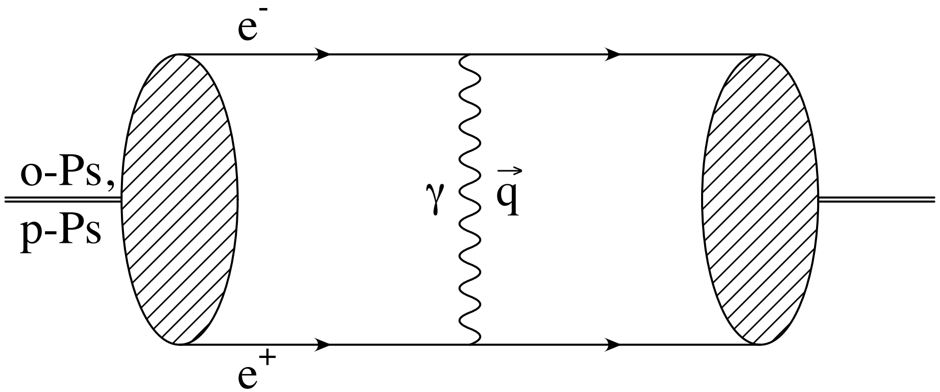
$$
\begin{aligned}
\mathcal{L}_{\rm NNRQED} \quad = \quad & -1\,/4(F^{\mu\nu})^2 + \psi^\dagger \Big\{ i\partial_t - eA_0 + \mathbf{D}^2\,/2m + \mathbf{D}^4\,/8m^3 \\
& + c_1\; e\,/2m\,\boldsymbol{\sigma}\cdot\mathbf{B} + c_2 e\,/8m^2\,\nabla\cdot\mathbf{E} \\
& + c_3\; ie\,/8m^2\,\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}) + \ldots \Big\}\psi \\
& + d_1\,/m^2\,(\psi^\dagger\psi)^2 + d_2\,/m^2\,(\psi^\dagger\boldsymbol{\sigma}\psi)^2 + \ldots \\
& + \text{ positron and positron-electron terms.}
\end{aligned}
$$

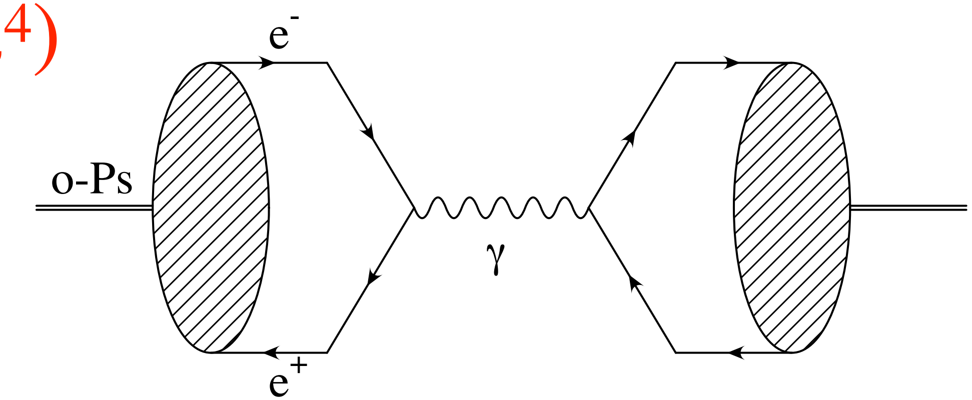
The coefficients c_1, d_1... are determined by matching with the exact theory.

T. Kinoshita and G. P. Lepage,
in *Quantum Electrodynamics (1990)*

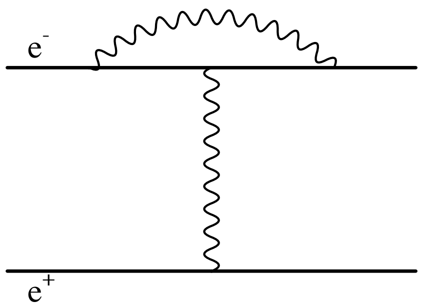
Corrections via $e^+e^- \to e^+e^-$ scattering at threshold

$O(\alpha^4)$

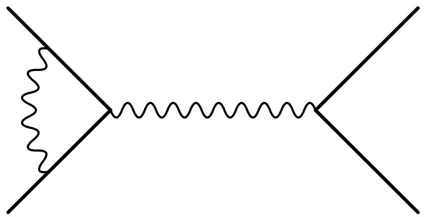




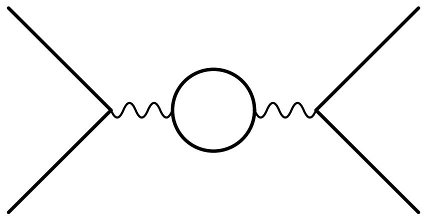
$O(\alpha^5)$



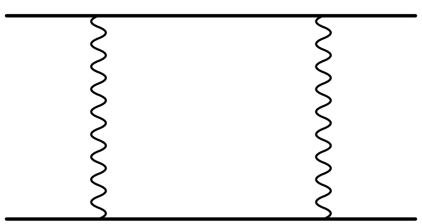
(a)



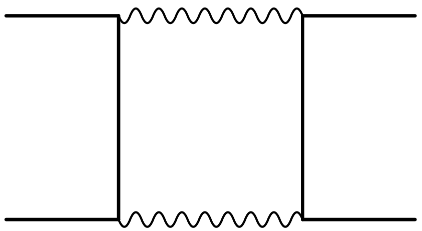
(b)



(c)



+ crossed photons



+ crossed photons

A. Czarnecki, hep-ph/9911455

State of the art: Hyperfine splitting in Positronium

Atomic calculations choose to perturb around the Schrödinger atom at rest,

with its $\mathcal{O}(\alpha^\infty)$ wave function $\Psi(\boldsymbol{x}) \sim \exp(-\alpha m r/2)$

Example: Hyperfine splitting in Positronium

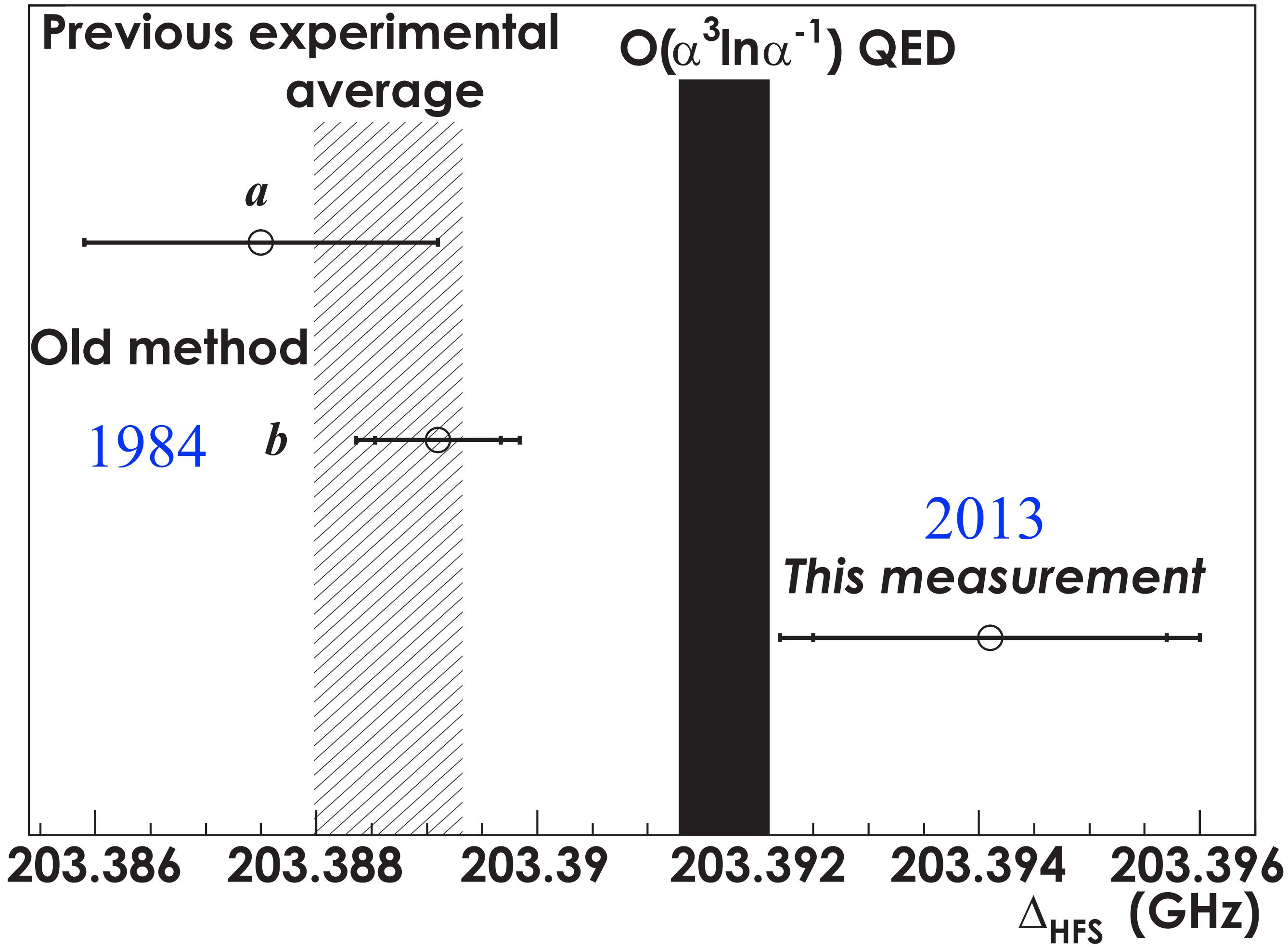
G. S. Adkins, Hyperfine Interact. **233** (2015) 59

$$\begin{aligned}\Delta\nu_{QED} \;=\; & m_e\alpha^4\left\{\frac{7}{12}-\frac{\alpha}{\pi}\left(\frac{8}{9}+\frac{\ln 2}{2}\right)\right.\\ & +\frac{\alpha^2}{\pi^2}\left[-\frac{5}{24}\pi^2\ln\alpha+\frac{1367}{648}-\frac{5197}{3456}\pi^2+\left(\frac{221}{144}\pi^2+\frac{1}{2}\right)\ln 2-\frac{53}{32}\zeta(3)\right]\\ & \left.-\frac{7\alpha^3}{8\pi}\ln^2\alpha+\frac{\alpha^3}{\pi}\ln\alpha\left(\frac{17}{3}\ln 2-\frac{217}{90}\right)+\mathcal{O}\left(\alpha^3\right)\right\}=203.39169(41)\ \text{GHz}\end{aligned}$$

$\Delta\nu_{\text{EXP}} = 203.394 \pm .002$ GHz

QED vs Data: Hyperfine splitting in Positronium

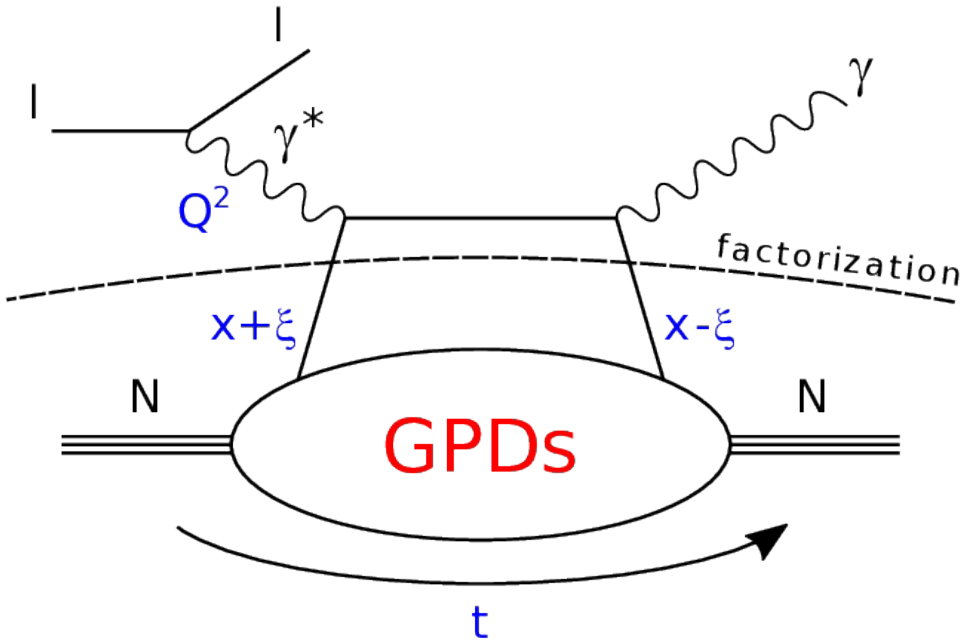
$$\Delta\nu_{QED} = 203.39169(41)\ \text{GHz}$$



$\Delta\nu_{EXP} = 203.38865(67)$ GHz (1984) M. W. Ritter et al, Phys. Rev. A30 (1984) 1331

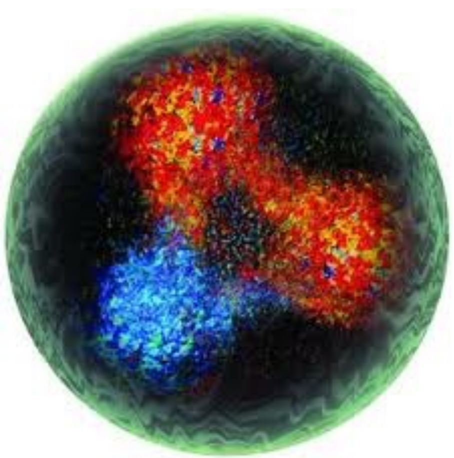
$\Delta\nu_{EXP} = 203.3941 \pm .003$ GHz (2013) A. Ishida et al, PLB 734 (2014) 338 [1310.6923]

Present understanding of hadrons in QCD



Established and successful at large Q:

Factorize the short distance, perturbative part of hard scattering from the universal parton distributions



At low Q:
hadronic scales

http://fisica.unipv.it/ricerca/LineeRic/ENG/EN_FisTeo_StrutAdrQCD.htm

In principle, the structure of the nucleon should be computed starting from the theory of Quantum Chromodynamics (QCD). In practice, the confinement of quarks and gluons within nucleons is a nonperturbative phenomenon, and QCD is extremely hard to solve in nonperturbative regimes. For this reason, despite the enormous progress of the last decades, we still have a limited knowledge of the internal structure of nucleons, which constitute more than 99% of ordinary matter.

Minority view of a perturbative approach to hadrons

Yu. Dokshitzer: *Perturbative QCD Theory (Includes our knowledge of α_s)*
Plenary talk at ICHEP 98, Vancouver. hep-ph/9812252

"To embark on such a quest one should believe in legitimacy of using the language of quarks and gluons down to small momentum scales, which implies understanding and describing the physics of confinement in terms of the standard QFT machinery, that is, essentially, perturbatively."

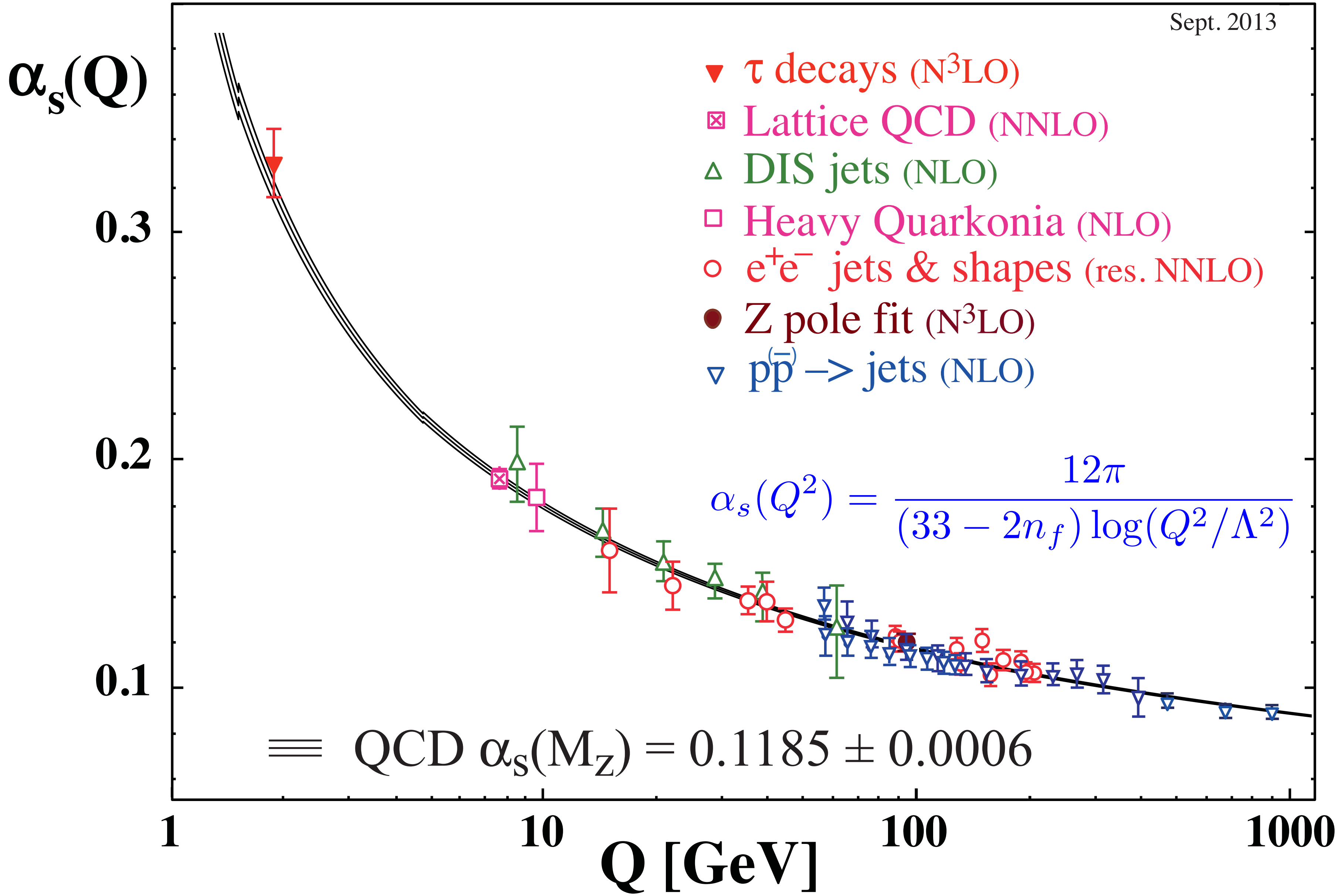
QCD *is about to undergo a* **faith transition**

QCD practitioners prepare themselves - slowly but steadily - to start using, in earnest, the language of ***quarks*** and ***gluons*** down into the region of small characteristic momenta - "**large distances**"

Yu. Dokshitzer, *Colloquium* (2011)

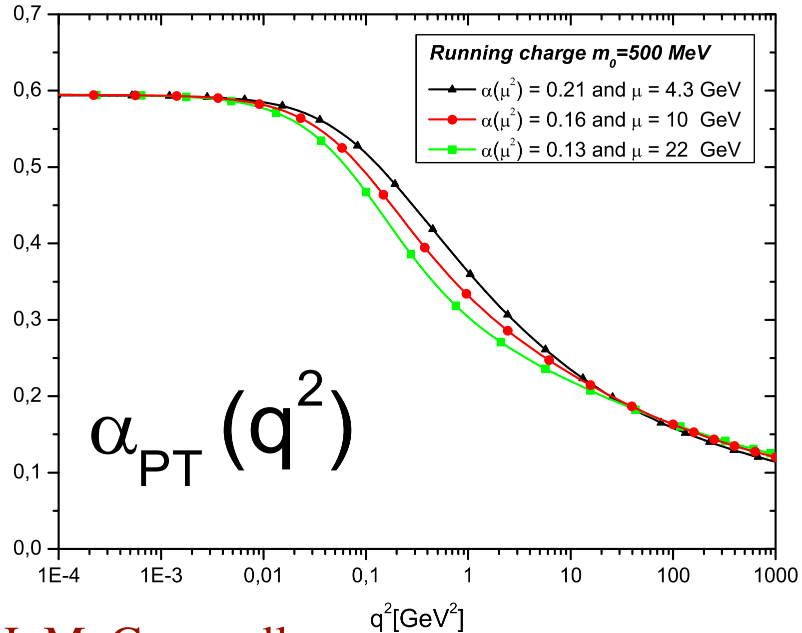
The running of α_s

α_{crit} → Gribov ★ $\alpha_s^{crit}(QCD) = \frac{\pi}{C_F}\left(1 - \sqrt{\frac{2}{3}}\right) \simeq 0.43$ (1997)



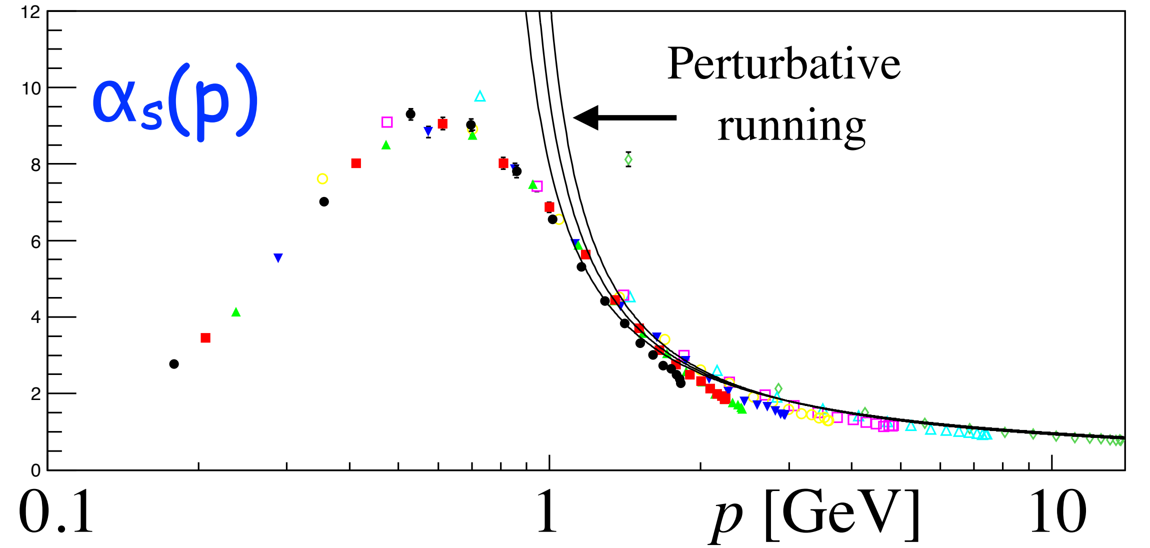
Values of α_s in the infrared

Pinch Technique



J. M. Cornwall;
A. C. Aguilar, D. Binosi, J. Papavassiliou,
J. Rodriguez-Quintero, PRD 80 (2009) 085018

Lattice QCD: SU(2) in Landau gauge



A. Maas, PRD 91 (2015) 034502
[arXiv:1402.5050v2]

Event shapes: $\frac{1}{\mu_I}\int_0^{\mu_I} dQ\,\alpha_{\rm eff}(Q^2) = \alpha_0(\mu_I)$

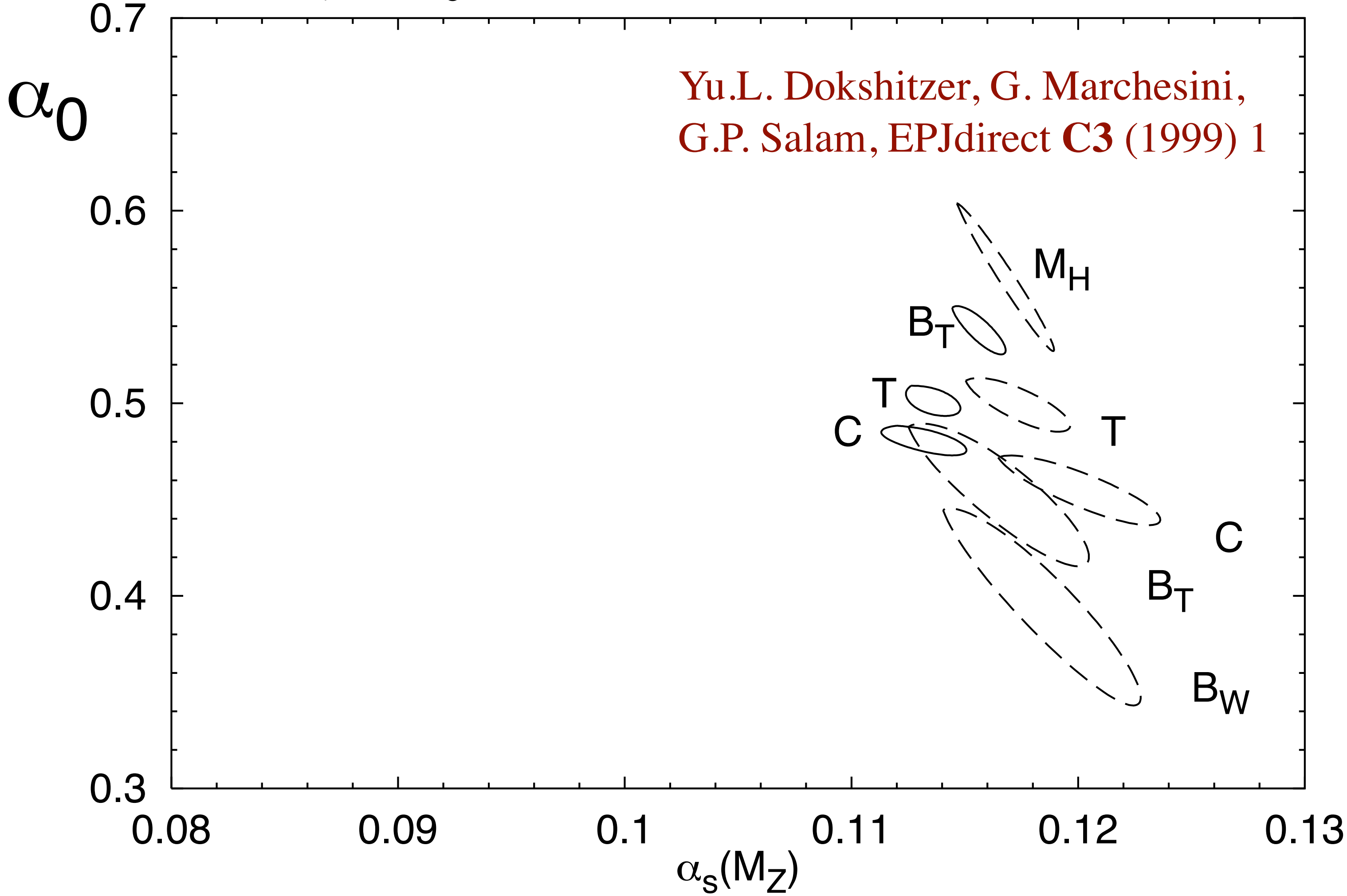
Yu.L. Dokshitzer, G. Marchesini,
G.P. Salam, EPJdirect **C3** (1999) 1

$\alpha_{0(2\ \rm GeV)} = 0.5132 \pm 0.0115(\rm exp) \pm 0.0381(\rm th)$

T. Gehrmann, M. Jaquier,
G. Luisoni,
Eur. Phys. J. C **67** (2010) 57

α_s in the infrared from event shapes

$$\frac{1}{\mu_I}\int_0^{\mu_I} \mathrm{d}Q\,\alpha_{\text{eff}}(Q^2) = \alpha_0(\mu_I)$$



$$\alpha_s(M_Z) = 0.1153 \pm 0.0017(\text{exp}) \pm 0.0023(\text{th})$$

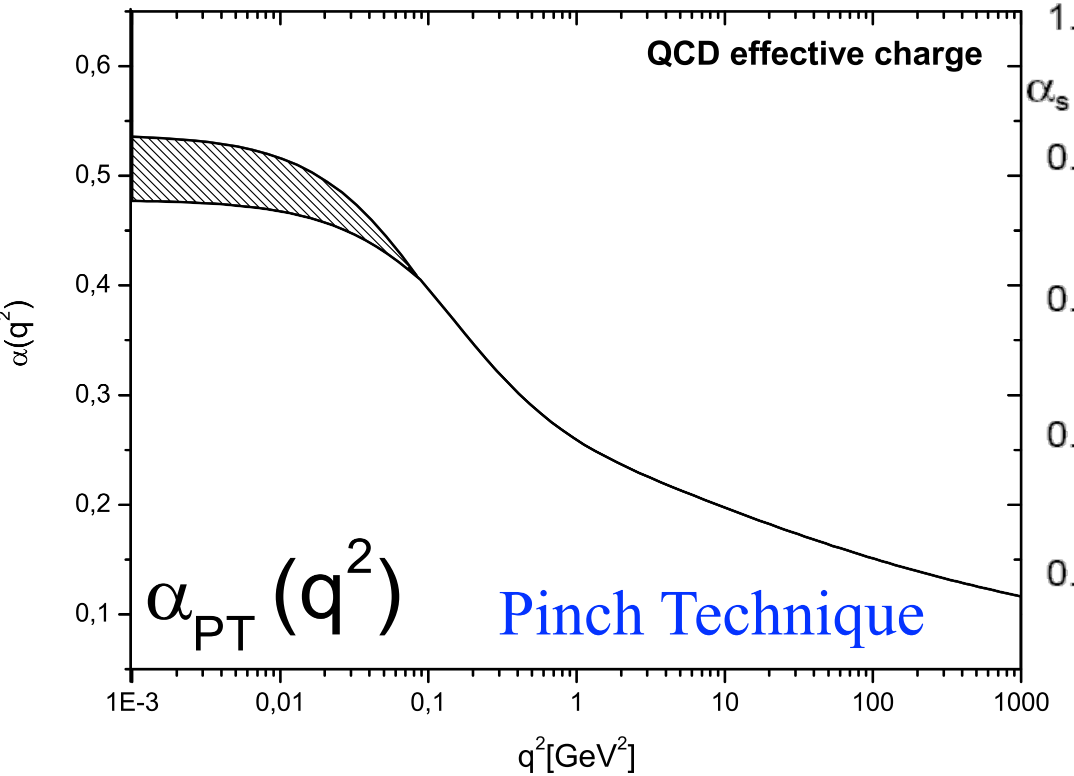
$$\alpha_0 = 0.5132 \pm 0.0115(\text{exp}) \pm 0.0381(\text{th})$$

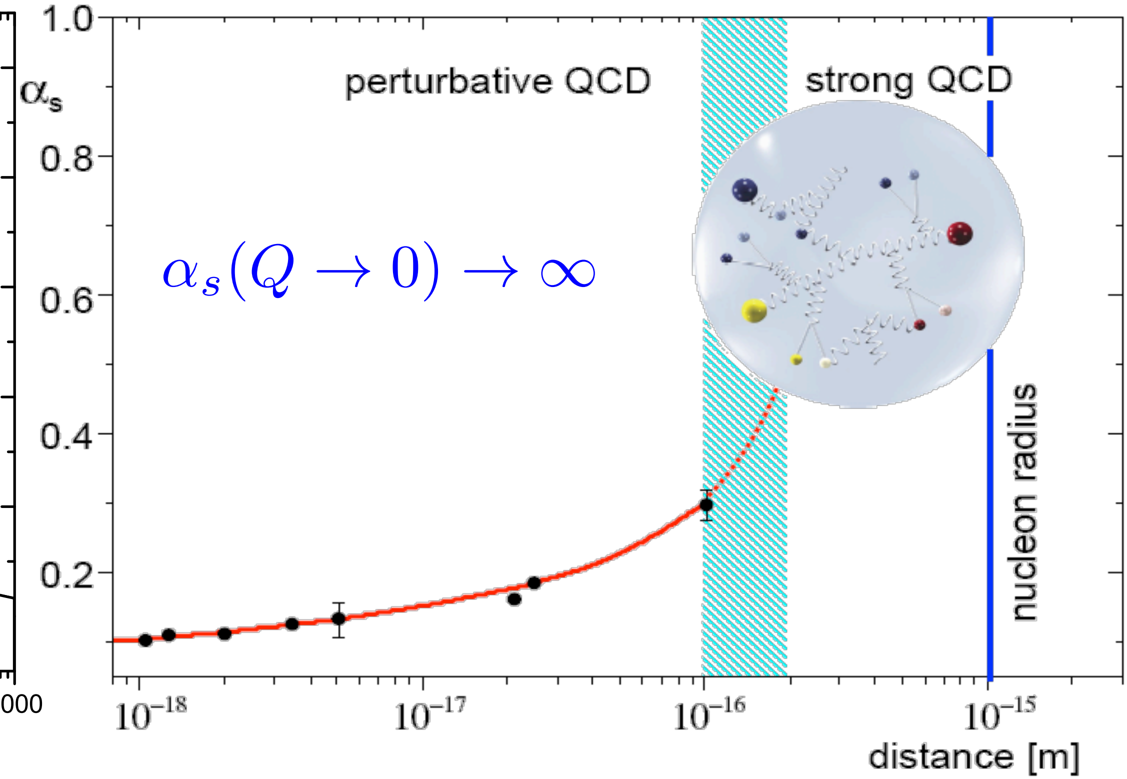
T. Gehrmann, M. Jaquier, G. Luisoni,
Eur. Phys. J. C **67** (2010) 57

$\mu_I = 2$ GeV

α_s freezes in the infrared

Theory & Phenomenology $\neq$ Popular view of confinement





D. Binosi and, J. Papavassiliou, Phys. Rept. **479** (2009) 1

J. Messchendorp, 1306.6611

$$\frac{\alpha_s(0)}{\pi} \simeq \frac{0.5}{\pi} = 0.16$$

may enable perturbative expansions for hadrons

QED bound state calculations use Feynman diagrams

The Feynman diagram expansion of scattering amplitudes $i \to f$ is defined by

$$S_{fi} = {}_{\mathbf{out}}\langle f| \left\{ \mathrm{T}\exp\left[-i\int_{-\infty}^{\infty} dt\, H_I(t)\right]\right\} |i\rangle_{\mathbf{in}}$$

where the *in* and *out* states are free, $O(\alpha^0)$ asymptotic states at $t = \pm\infty$.

The free electrons/photons get "dressed" by the H_I interactions.
The expression is formally exact provided the *in* and *out* states have a non-vanishing overlap with the physical *i* and *f* states.

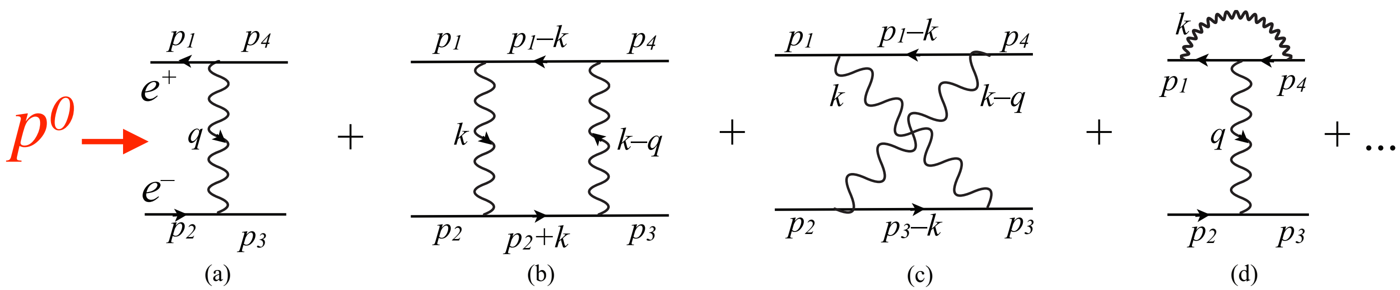
Free states have infinite size and thus zero overlap with bound states (atoms).

The technical difficulties of atomic perturbation theory originate (in part) from the inappropriate boundary conditions.

We cannot expect to describe confinement in QCD using Feynman diagrams:

The *in* and *out* states exclude confinement.

$e^+e^- \to e^+e^-$: Positronium from Feynman diagrams



$$= \frac{|\varphi_{e^+e^-}|^2}{p^0 - E + i\varepsilon} + \ldots$$

Rest frame: $E = 2m_e - \frac{1}{4}m_e\alpha^2 + \mathcal{O}\left(\alpha^4\right)$

LHS: $\sum_{n=2}^{\infty} c_n\alpha^n$

RHS: Not polynomial in α

$\Rightarrow$

Bound state poles can arise only through a divergence of the perturbative series ($n \to \infty$)

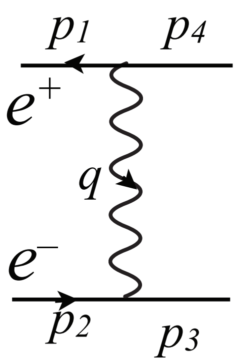
Why does the QED perturbative series diverge for atoms (at any α)?

Which diagrams cause the divergence?

Ladder diagrams (rest frame)

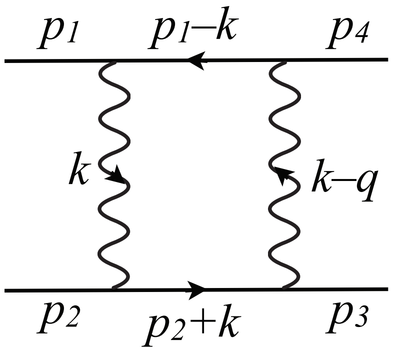
The Bohr momentum scale is $|\boldsymbol{p}| \sim \alpha m$, kinetic energy $|\boldsymbol{p}|^2/2m \sim \alpha^2 m \sim E_B$

With momenta ∝ α, the propagators bring inverse powers of α :



$$\sim \frac{e^2}{q^2} \sim \frac{\alpha}{\boldsymbol{q}^2} \sim \frac{1}{\alpha}$$

Note: $q^0 \sim \alpha^2 \ll |\boldsymbol{q}| \sim \alpha$

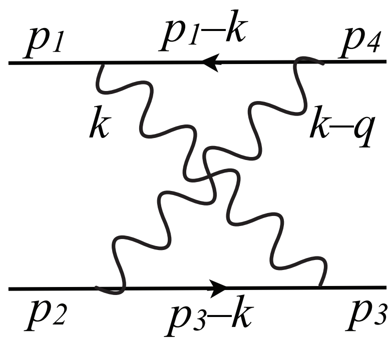


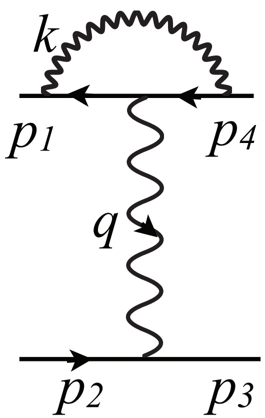
$$\sim \int dk^0\, d^3\boldsymbol{k}\, \frac{e^4}{(\boldsymbol{k}^2)^2\,(\Delta E_e)^2} \sim \alpha^2\,\alpha^3\, \frac{\alpha^2}{(\alpha^2)^2\,(\alpha^2)^2} \sim \frac{1}{\alpha}$$

All "ladder diagrams" are of order 1/α $\Longrightarrow$ Sum can diverge at bound states

Divergence is due to expanding around free states, no $V(r) = -\alpha/r$ potential

Non-ladders are suppressed by α





These diagrams have the same number of propagators and vertices as the 2-photon ladder. A similar counting would again give ~ 1/α .

However, the O(1/α) term vanishes:

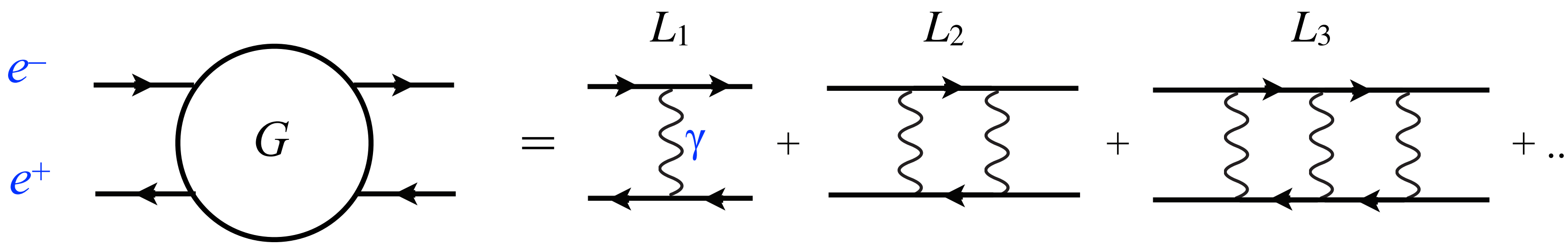
$$\propto \int \frac{dk^0}{2\pi} \frac{1}{(k^0 - a + i\varepsilon)(k^0 - b + i\varepsilon)} = 0$$

In the straight ladders the integration contour is pinched:

$$\propto \int \frac{dk^0}{2\pi} \frac{1}{(k^0 - a + i\varepsilon)(k^0 - b - i\varepsilon)} \neq 0$$

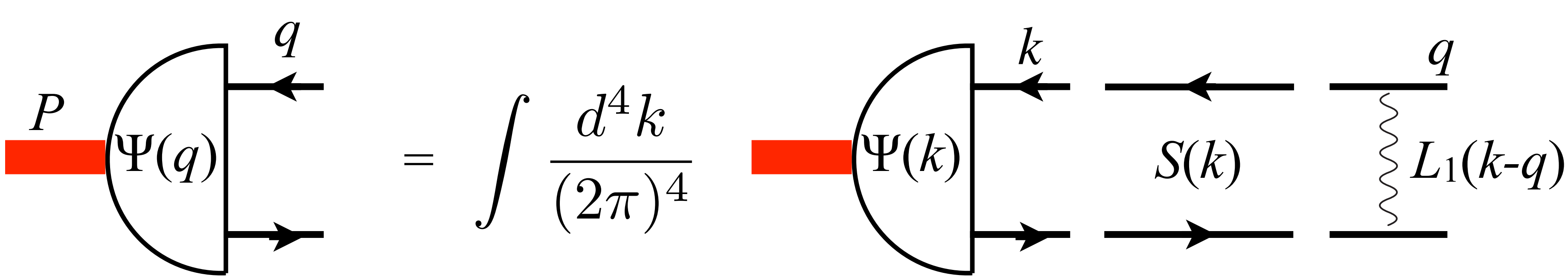
⇒ Only straight ladders are of the leading order, 1/α .

Summing ladder diagrams



$$G = L_1 + G\,S\,L_1 = L_1 + L_1\,S\,L_1 + G\,S\,L_1 S\,L_1 = \ldots$$

At a bound state pole: $G(p^0) \sim \dfrac{\Psi^\dagger \Psi}{p^0 - E}$ $\Rightarrow$ $\Psi = \Psi\,S\,L_1$

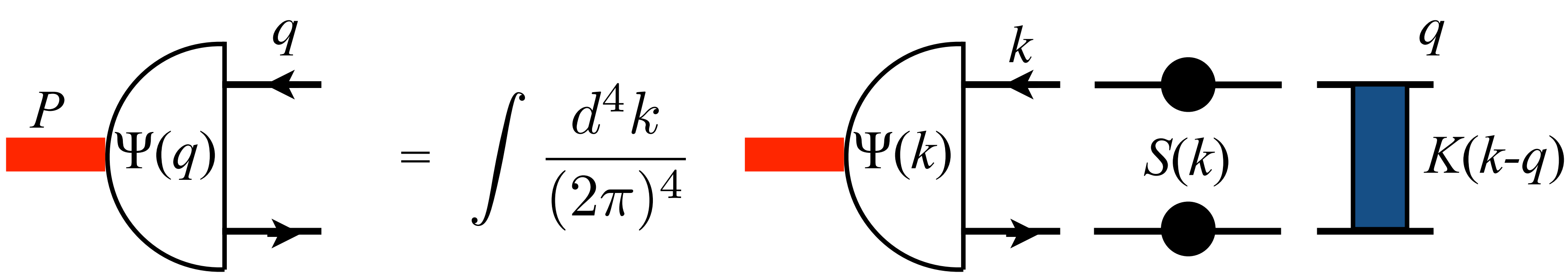


This is the Bethe-Salpeter equation for a single photon kernel L_1 .

It is valid in any frame (since Feynman diagrams are Lorentz covariant).

The Bethe-Salpeter equation (1951)

Dressing the e^- and e^+ propagators and adding all corrections to the single photon exchange (kernel) gives the formally exact B-S equation:



Expanding the propagators and kernel in α defines a perturbative expansion.

Explicit Lorentz covariance: Feynman propagators and vertices.

The B-S wave function: $\langle\Omega|\, T\left\{\bar\psi_\beta(x_2)\psi_\alpha(x_1)\right\}|P\rangle \equiv e^{-iP\cdot(x_1+x_2)/2}\,\Psi^{\boldsymbol{P}}_{\alpha\beta}(x_1-x_2)$

$$x' = \Lambda x \;:\qquad \Psi^{\boldsymbol{P}'}(x_1'-x_2') = S(\Lambda)\Psi^{\boldsymbol{P}}(x_1-x_2)S^{-1}(\Lambda)$$

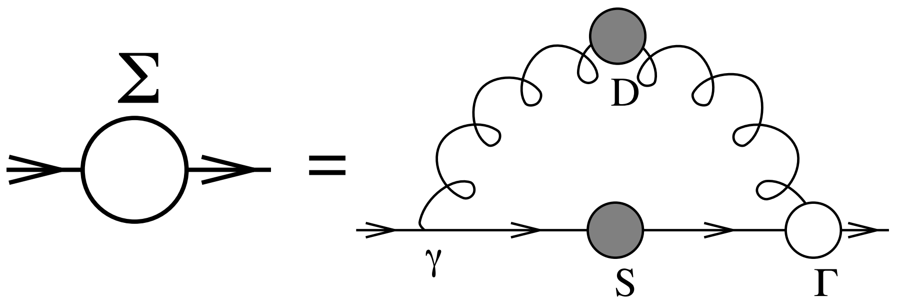
Note: Equal time is not preserved in boosts: $t_1 = t_2 \;\Rightarrow\; t'_1 \neq t'_2$

Not a Hamiltonian formalism!

Dyson-Schwinger equations

There are identities between Green functions, valid to all orders in α.

E.g., the Dyson-Schwinger equation for the quark propagator:



$$S(p) = i/(\not{p} - m - \Sigma)$$

The circles contain quark and gluon loops to all orders.

There are analogous identities for the gluon propagator and the vertices.

The D-S equations do not close. Truncations and assumptions are needed.

D-S equations are used to model hadron dynamics. C. D. Roberts, arXiv 1203.5341

Bound state constituents propagate in a field

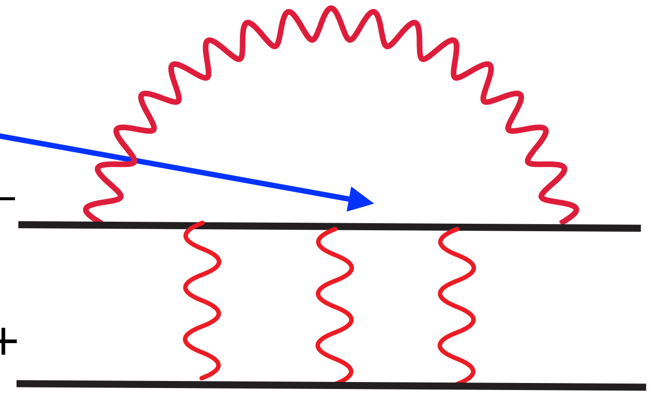
For QED lamb shift, need to calculate e^- propagator in the field of e^+

In an NR approximation, this can be described by a fixed $-\alpha/r$ potential.

QED

e^-

e^+



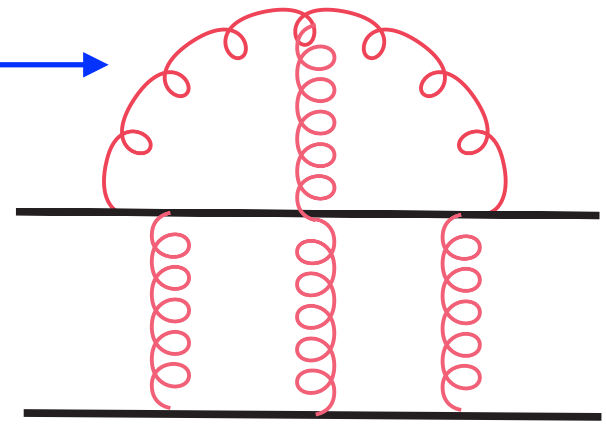
Lamb shift: $M(2S_{1/2})$-$M(2P_{1/2})$

In QCD, colored gluons interact with relativistic quarks

QCD

q

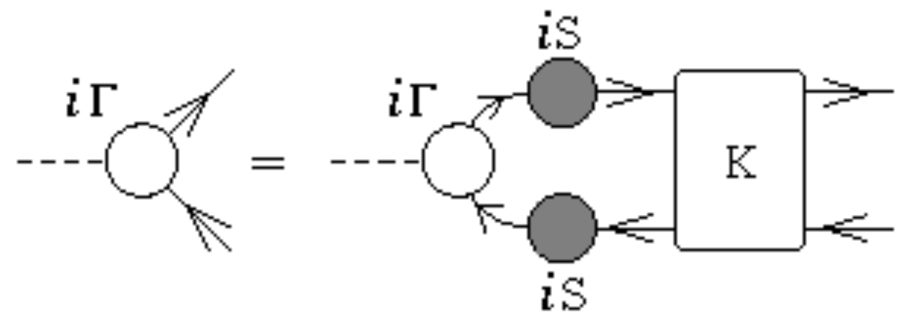
$\bar{q}$



Gluon and quark propagators depend on the state in which they propagate.

$\Rightarrow$ Cannot build bound states with constituents that have predetermined propagators.

Three developments in the theory of atoms



- 1951: Salpeter & Bethe

Expand propagators S and kernel K in powers of α
Explicit Lorentz covariance (frame dependent time separations)
No analytic solution even at lowest order in S and K

- 1975: Caswell & Lepage: BS is not unique: ∞ # of equivalent equations, $S \leftrightarrow K$

We may choose to expand around Schrödinger atoms
Give up explicit boost invariance

- 1986: Caswell & Lepage NRQED: Effective NR field theory

Expand QED action in powers of ∇/m_e
Choose to start from Schrödinger atoms (at rest)

⇒ Need a physical principle for the choice of initial wave function.

Equal time bound state condition

The S-matrix is suitable for scattering:
Start with widely separated, free states

$$S_{fi} = {}_{\textbf{out}}\langle f| \Big\{ \mathrm{T} \exp \Big[-i \int_{-\infty}^{\infty} dt\, H_I(t) \Big] \Big\} |i\rangle_{\textbf{in}}$$

Bound states are eigenstates of the Hamiltonian at an instant of time

$$\mathcal{H}_{QED}(t)\,|Pos, \boldsymbol{P}, t\rangle = E_P\,|Pos, \boldsymbol{P}, t\rangle$$

There is no need to consider the propagation of bound state constituents in time.

The time dependence of the bound state is by definition:

$$|Pos, \boldsymbol{P}, t\rangle = e^{-iE_P\,t}\,|Pos, \boldsymbol{P}, t = 0\rangle$$

Bound state calculations can be done at $t = 0$, at any order and for any bound state momentum $\boldsymbol{P}$.

Method for Positronium perturbation theory

https://www.mv.helsinki.fi/home/hoyer/Pavia/200120_Hoyer_Pavia-figs.pdf

- Expansion in α around "lowest order" bound state.
  - Not unique, since wave function is already $O(\alpha^\infty)$.

- Use Fock state expansion at an instant of time t.
  - $|Pos\rangle = \phi_{ee}\left|e^+e^-\right\rangle + \phi_{ee\gamma}\left|e^+e^-\gamma\right\rangle + \phi_\gamma\left|\gamma\right\rangle + \phi_{4e}\left|e^+e^-\,e^+e^-\right\rangle + \ldots$

- Define "lowest order" to be the valence Fock state: $\phi_{ee}\left|e^+e^-\right\rangle$

- Higher Fock states are given by the Hamiltonian: $(H_{int})^n\left|e^+e^-\right\rangle$
  - This is a perturbative expansion, since $H_{int} \propto e$.
  - Determine the ϕ_n through $H\,|Pos\rangle = E\,|Pos\rangle$, with E of $O(\alpha^n)$
  - J^{PC} and other quantum numbers of $|e^+e^-\rangle$ are conserved by H_{int} .

- Include instantaneous gauge field in each Fock state.
  - $E.g.$, gives $V = -\,\alpha/r$ for $|e^+e^-\rangle$

- Is applicable for Positronium in motion.
  - Instantaneous field is determined by instantaneous positions only.

Canonical quantization

Conjugate fields: $$\pi_n(t,\boldsymbol{x}) = \frac{\delta L(t)}{\delta[\partial_t \varphi_n(t,\boldsymbol{x})]}$$

satisfy equal *t* commutation: $$[\varphi_m(t,\boldsymbol{x}), \pi_n(t,\boldsymbol{y})]_{\mp} = i\delta_{mn}\delta(\boldsymbol{x}-\boldsymbol{y})$$

QED Lagrangian: $$\mathcal{L}_{QED}(t,\boldsymbol{x}) = \bar{\psi}(i\not{\partial} - m - e\not{A})\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Conjugate to electron field: $$\frac{\delta \int d\boldsymbol{y}\, \mathcal{L}}{\delta[\partial_t \psi_\alpha(t,\boldsymbol{x})]} = i\psi_\alpha^\dagger(t,\boldsymbol{x})$$

Anticommutation relation: $$\{\psi_\alpha(t,\boldsymbol{x}), \psi_\beta^\dagger(t,\boldsymbol{y})\} = \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y})$$

No conjugate to A^0 photon field: $$\frac{\delta \int d\boldsymbol{y}\, \mathcal{L}}{\delta[\partial_t A^0(t,\boldsymbol{x})]} = 0$$ What to do?

Temporal gauge: $A^0 = 0$

- Avoids problem with missing conjugate to A^0 .
- Maintains spatial symmetries.
  - – t is anyway special in canon. quant.
- No ghosts, no operator constraints

J. D. Bjorken, SLAC Summer Institute (1979)
G. Leibbrandt, Rev. Mod. Phys. 59, 1067 (1987)

The standard gauge for bound states is Coulomb gauge, $\nabla\cdot\boldsymbol{A} = 0$
– Perhaps because of the form of the photon propagator: $D^{00} = -i/\boldsymbol{k}^2$
– Has ghosts and operator constraints

N. H. Christ and T. D. Lee, PRD 22 (1980) 939

Temporal gauge:

$$\frac{\delta L(t)}{\delta[\partial_t A^i(t,\boldsymbol{x})]} = \partial_t A^i = -E^i$$

$$\big[E^i(t,\boldsymbol{x}), A^j(t,\boldsymbol{y})\big] = i\delta^{ij}\delta(\boldsymbol{x}-\boldsymbol{y})$$

The Hamiltonian in temporal gauge:

$$\mathcal{H}_{QED} = \int d\boldsymbol{x}\Big[\psi^\dagger(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + m\gamma^0 - e\boldsymbol{\alpha}\cdot\boldsymbol{A})\psi + \tfrac{1}{2}\boldsymbol{E}^2 + \tfrac{1}{4}F^{ij}F^{ij}\Big]$$

Time independent gauge transformations

Preserve $A^0 = 0$

Generated by Gauss' operator:

$$G(x) \equiv \frac{\delta \mathcal{S}}{\delta A^0(x)} = \partial_i E_L^i(x) - e\psi^\dagger\psi(x)$$

Unitary operator $U(t)$ with infinitesimal parameter $\Lambda(\boldsymbol{y})$

$$\mathcal{U}(t) = 1 + i\int d\boldsymbol{y}\, G(t,\boldsymbol{y})\Lambda(\boldsymbol{y})$$

Transforms the photon and electron fields as:

$$\mathcal{U}(t)A^j(t,\boldsymbol{x})\mathcal{U}^{-1}(t) - A^j(t,\boldsymbol{x}) = \partial_j\Lambda(\boldsymbol{x})$$

$$\mathcal{U}(t)\psi(t,\boldsymbol{x})\mathcal{U}^{-1}(t) - \psi(t,\boldsymbol{x}) = ie\,\Lambda(\boldsymbol{x})\psi(t,\boldsymbol{x})$$

Physical states are constrained to satisfy: and are thus invariant under $U(t)$.

$$G(t,\boldsymbol{x})\,|phys\rangle = 0$$

J. F. Willemsen, PRD 17 (1978) 574

The instantaneous electric field acts on physical states as:

$$\boldsymbol{E}_L(t,\boldsymbol{x})\,|phys\rangle = -\boldsymbol{\nabla}_x\int d\boldsymbol{y}\,\frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi^\dagger\psi(t,\boldsymbol{y})\,|phys\rangle$$

The potential energy

The instantaneous $\boldsymbol{E}$-field: $\boldsymbol{E}_L(t,\boldsymbol{x})\,|phys\rangle = -\boldsymbol{\nabla}_x \int d\boldsymbol{y}\,\frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi^\dagger\psi(t,\boldsymbol{y})\,|phys\rangle$

Contributes to the Hamiltonian: $\mathcal{H}_V \equiv \frac{1}{2}\int d\boldsymbol{x}\,\boldsymbol{E}_L^2$

$$\mathcal{H}_V\,|phys\rangle = \tfrac{1}{2}\int d\boldsymbol{x}d\boldsymbol{y}d\boldsymbol{z}\Big[\partial_i^x \frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi^\dagger\psi(\boldsymbol{y})\Big]\Big[\partial_i^x \frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{z}|}\psi^\dagger\psi(\boldsymbol{z})\Big]\,|phys\rangle$$

$$= \tfrac{1}{2}\int d\boldsymbol{x}d\boldsymbol{y}\,\frac{e^2}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\big[\psi^\dagger\psi(\boldsymbol{x})\big]\big[\psi^\dagger\psi(\boldsymbol{y})\big]\,|phys\rangle$$

H_V acts as a constraint on $|phys\rangle$ (not as an operator), so it annihilates $|0\rangle$.
Equivalently: $\boldsymbol{E}_L\,|0\rangle = 0$.

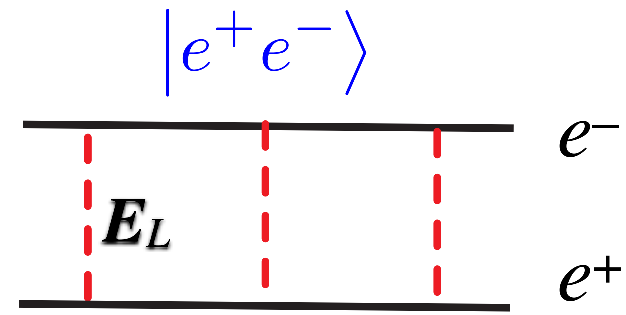
The potential energy of a state with an e^- at $\boldsymbol{x}_1$ and an e^+ at $\boldsymbol{x}_2$ is then,

$$\mathcal{H}_V\,\bar\psi_\alpha(\boldsymbol{x}_1)\psi_\beta(\boldsymbol{x}_2)\,|0\rangle = -\frac{\alpha}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}\bar\psi_\alpha(\boldsymbol{x}_1)\psi_\beta(\boldsymbol{x}_2)\,|0\rangle$$

The result of H_V acting on any other Fock state is similarly determined.

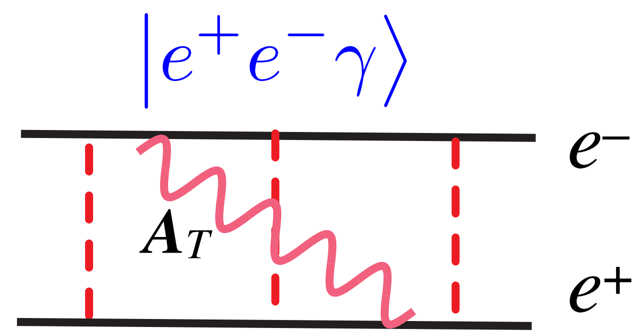
Fock state expansion for Positronium in $A^0=0$ gauge

The $|e^+e^-\rangle$ Fock state, bound by the classical field of its constituents, is taken to be of " lowest order" in α :

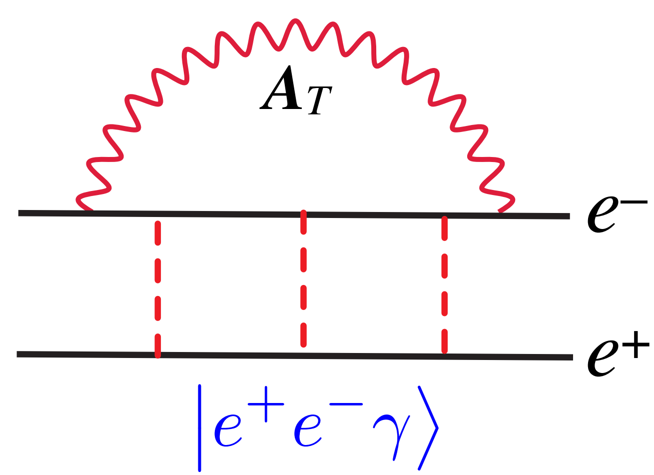


Spin dependence arises from states with a transverse photon, $|e^+e^-\gamma\rangle$.

A_T vertices give suppression by α.



The Lamb shift also arises from $|e^+e^-\gamma\rangle$.



Each Fock component of a bound state includes the instantaneous E_L field.

Schrödinger equation for Positronium

Express the valence Fock state $|e^+e^-\rangle$ at $t = 0$ in terms of a 4×4 wf. Φ,

$$|Pos\rangle = \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}_\alpha(\boldsymbol{x}_1) e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2} \Phi^{(\boldsymbol{P})}_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta(\boldsymbol{x}_2)\,|0\rangle$$

$$\psi(\boldsymbol{x}) = \int \frac{d\boldsymbol{p}}{(2\pi)^2 2E_p} \sum_{\lambda=\pm} \Big[u(\boldsymbol{p},\lambda)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} b(\boldsymbol{p},\lambda) + v(\boldsymbol{p},\lambda)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} d^\dagger(\boldsymbol{p},\lambda)\Big]$$

- For Positronium at rest we set $\boldsymbol{P} = 0$.
- For weak binding (QED) there are no Z-contributions:
  Only $b^\dagger$ in $\bar{\psi}$ and $d^\dagger$ in ψ contribute.
- For strong binding (QCD) also the b, d operators contribute (cf. Dirac)

Determine Φ from the bound state condition $\mathcal{H}_{QED}\,|Pos\rangle = (2m + E_b)\,|Pos\rangle$

For Positronium at rest we may neglect $|e^+e^-\gamma\rangle$ Fock state, *i.e.*, transverse γ

$$\mathcal{H}_{QED} = \int d\boldsymbol{x}\Big[\psi^\dagger(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + m\gamma^0)\psi + \tfrac{1}{2}\boldsymbol{E}_L^2\Big] \equiv \mathcal{H}_0 + \mathcal{H}_V$$

Schrödinger equation for Positronium (cont.)

Recalling that $\bar{u}(\boldsymbol{p},\lambda)(\boldsymbol{p}\cdot\boldsymbol{\gamma}+m)\gamma^0=\bar{u}(\boldsymbol{p},\lambda)E_p$, and only $b^\dagger$ in $\bar{\psi}$ contributes:

$$[\mathcal{H}_0,\bar{\psi}(\boldsymbol{x}_1)]=\bar{\psi}(\boldsymbol{x}_1)(-i\boldsymbol{\alpha}\cdot\overleftarrow{\boldsymbol{\nabla}}_1+m\gamma^0)=\bar{\psi}(\boldsymbol{x}_1)\sqrt{-\overleftarrow{\boldsymbol{\nabla}}_1^2+m^2}\simeq\bar{\psi}(\boldsymbol{x}_1)\Big(m-\overleftarrow{\boldsymbol{\nabla}}_1^2/2m\Big)$$

and similarly $[\mathcal{H}_0,\psi(\boldsymbol{x}_2)]\simeq\Big(m-\overrightarrow{\boldsymbol{\nabla}}_2^2/2m\Big)\psi(\boldsymbol{x}_2)$. After partial integrations,

$$\mathcal{H}_{QED}\,|Pos\rangle=\int d\boldsymbol{x}_1 d\boldsymbol{x}_2\,\bar{\psi}(\boldsymbol{x}_1)\big[m-\overrightarrow{\boldsymbol{\nabla}}_1^2/2m-\frac{\alpha}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}\big]\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)\big[m-\overleftarrow{\boldsymbol{\nabla}}_2^2/2m\big]\psi(\boldsymbol{x}_2)\,|0\rangle$$

$$=(2m+E_b)\,|Pos\rangle$$

The condition on the wave function $\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)$ is the Schrödinger equation,

$$\Big[-\frac{\boldsymbol{\nabla}^2}{m}+V(\boldsymbol{x})\Big]\Phi(\boldsymbol{x})=E_b\,\Phi(\boldsymbol{x}) \qquad V(\boldsymbol{x})=-\frac{\alpha}{|\boldsymbol{x}|}$$

All 4×4 components of Φ satisfy the same equation, but only 2×2 are leading.
The structure of the 2×2 leading components depend on the J^{PC} of the state.

Wave functions of NR e^+e^- states

Standard radial equation: $F''(r) + \frac{2}{r}F'(r) + \left[m(E_b - V) - \frac{\ell(\ell+1)}{r^2}\right]F(r) = 0$

In terms of the sph. harm. *Y* and the orbital angular momentum $\boldsymbol{L} = \boldsymbol{x} \times (-i\boldsymbol{\nabla})$

$s = 0,\ \ell = j,\ j^z = \lambda$: $\quad \Phi(\boldsymbol{x}) = (1+\gamma^0)\gamma_5\, F(r) Y_{j\lambda}(\Omega)$

$s = 1,\ \ell = j,\ j^z = \lambda$: $\quad \Phi(\boldsymbol{x}) = (1+\gamma^0)\alpha \cdot \boldsymbol{L}\, F(r) Y_{j\lambda}(\Omega)$

$s = 1,\ \ell = j\pm 1,\ j^z = \lambda$:

$$\Phi(\boldsymbol{x}) = \frac{1}{r}(1+\gamma^0)\left\{\tfrac{1}{2}\alpha\cdot\boldsymbol{x}\left[\pm(2j+1)+1\right] + i\alpha\cdot\boldsymbol{x}\times\boldsymbol{L}\right\}F(r)Y_{j\lambda}(\Omega)$$

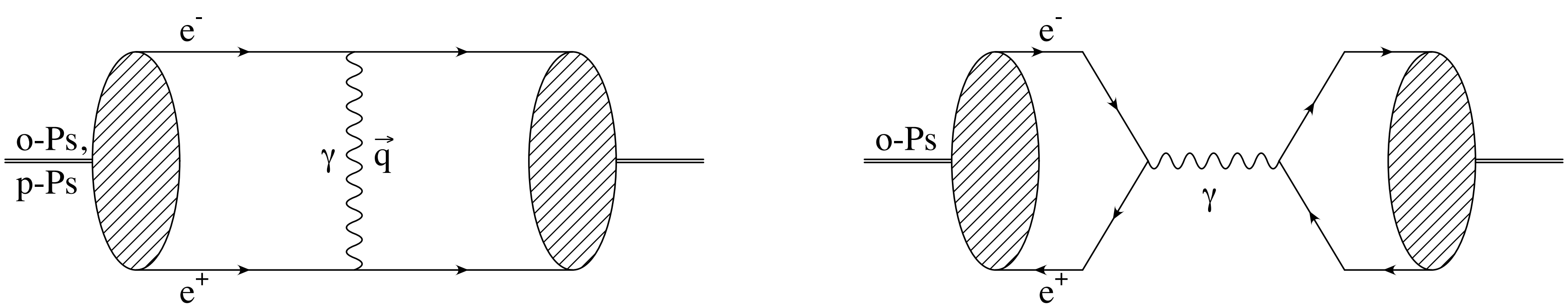
The factor $1+\gamma^0$ projects on positive energy components $b^\dagger$ of $\bar{\psi}(\boldsymbol{x}_1)$, and negative energy components $d^\dagger$ of $\psi(\boldsymbol{x}_2)$.

Hyperfine splitting of Positronium

In the rest frame $|Pos, S\rangle = \phi_{ee}^{(S)}\,|e^+e^-\rangle + \phi_{ee\gamma}^{(S)}\,|e^+e^-\gamma\rangle + \phi_{\gamma}^{(S)}\,|\gamma\rangle$

suffices for $O(\alpha^4)$ in E_b. The spin S=0 for Para- and S=1 for orthopositronium. The $\phi_{ee\gamma}$ and ϕ_γ wave functions are determined by ϕ_{ee} and $H_{int}\,|e^+e^-\rangle$, together with the stationarity requirement $\mathcal{H}\,|Pos, S\rangle = (2m + E_b)\,|Pos, S\rangle$

This corresponds to the standard evaluation in terms of Feynman diagrams:



and gives the same result at $O(\alpha^4)$ in E_b.

Also the $O(\alpha^5)$ contribution to E_b should be checked.

Positronium with P ≠ 0

At $O(\alpha^2)$ in E_b, need $\quad |Pos, P\rangle = \Psi_{ee}^{(\boldsymbol{P})}\,|e^+e^-\rangle + \Psi_{ee\gamma}^{(\boldsymbol{P})}\,|e^+e^-\gamma\rangle$

where $\quad \Psi_{ee}^{(\boldsymbol{P})}(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \exp[i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2]\Phi^{(\boldsymbol{P})}(\boldsymbol{x}_1-\boldsymbol{x}_2)$

The transverse photon contributes even at lowest order for $P \neq 0$.
The $ee\gamma$ vertex $\propto p_e$, where $p_e \propto \alpha m$ in the rest frame, but now $p_e \propto P$.

Intuitively: For $\boldsymbol{P} = (0,0,P)$ the boost turns A^0 into $\cosh\xi\, A^0 + \sinh\xi\, A^3$

Does $\Phi^{(P)}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ Lorentz contract? Note that

$$H_V|phys\rangle = \tfrac{1}{2}\int d\boldsymbol{x}d\boldsymbol{y}\,\frac{e^2}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left[\psi^\dagger\psi(\boldsymbol{x})\right]\left[\psi^\dagger\psi(\boldsymbol{y})\right]|phys\rangle$$

The instantaneous potential depends only on $|\boldsymbol{x}_1 - \boldsymbol{x}_2|$, not on P.

Detailed derivation in Appendix A of the lecture notes

The relativistic invariance of equal-time states is not trivial

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Quantum noncovariance of the linear potential in 1 + 1 dimensions

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The two-body bound states governed by the Hamiltonian $(p_a{}^2+m_a{}^2)^{1/2}+(p_b{}^2+m_b{}^2)^{1/2}+\kappa|x_a-x_b|$ in $1+1$ dimensions do not have Lorentz-invariant masses $(E_{n,P}{}^2-P^2)^{1/2}$ even to first order in P^2, if one used the standard commutation relations $[x_i,p_i]=i\hbar$. This is shown explicitly for $m_a=m_b=0$ and generalized by continuity to $m_a+m_b\neq 0$. The same is true for any other potential $V(|x_a-x_b|)$.

$$H=\sqrt{p_a^2+m_a^2}+\sqrt{p_b^2+m_b^2}+\kappa|x_a-x_b|$$

The eigenvalues *E(P)* have the wrong dependence on the CM momentum *P*.

Lorentz covariance is guaranteed in QED and QCD, if the QFT rules are obeyed.

Bound states in relative motion are needed in form factors, scattering, decays…

Positronium with P ≠ 0 (cont.)

The energy eigenvalue must be

$$E_P \equiv \sqrt{\boldsymbol{P}^2 + 4m^2}$$

$$E = \sqrt{\boldsymbol{P}^2 + (2m + E_b)^2} \simeq E_P + \frac{2m}{E_P}\,E_b \qquad \gamma = \frac{E_P}{2m} \qquad \beta = \frac{P}{E_P}$$

The $|e^+e^-\rangle$ Fock state contributes (see Appendix A of notes)

$$(\mathcal{H}_0 + \mathcal{H}_V)\left|\Psi^{(\boldsymbol{P})}\right\rangle = \left|\left[E_P - \frac{1}{m\gamma}\left(\boldsymbol{\nabla}_\perp^2 + \frac{1}{\gamma^2}\boldsymbol{\nabla}_\parallel^2\right) - \frac{\alpha}{|\boldsymbol{x}|}\right]\Psi^{(\boldsymbol{P})}\right\rangle$$

The kinetic terms scale with γ as expected for contraction: $x_\parallel \to \gamma\, x_\parallel$

The potential energy is, however, independent of γ.

Instantaneous potential: $$-\frac{\alpha}{|\boldsymbol{x}|} = -\int\frac{d\boldsymbol{q}}{(2\pi)^3}\,\frac{e^2}{\boldsymbol{q}^2}\,e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}$$

Transverse photon $\boldsymbol{q}$ exchange: $$\int\frac{d\boldsymbol{q}}{(2\pi)^3 2|\boldsymbol{q}|}\,\frac{e^2\beta^2}{\boldsymbol{q}^2}\,\frac{2|\boldsymbol{q}|\boldsymbol{q}_\perp^2}{\boldsymbol{q}_\perp^2 + q_\parallel^2/\gamma^2}\,e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}$$

Positronium with P ≠ 0 (cont.)

The instantaneous and transverse photon exchanges sum to the desired result:

$$-e^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\,\frac{1}{\boldsymbol{q}^2}\Big[1-\frac{\beta^2 q_\perp^2}{\boldsymbol{q}_\perp^2+q_\parallel^2/\gamma^2}=\frac{\boldsymbol{q}^2/\gamma^2}{\boldsymbol{q}_\perp^2+q_\parallel^2/\gamma^2}\Big]e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \;=-\frac{\alpha}{\gamma|\boldsymbol{y}|}$$

where $\boldsymbol{y}_\perp = \boldsymbol{x}_\perp \qquad \boldsymbol{y}_\parallel = \gamma \boldsymbol{x}_\parallel$

The bound state condition: $\Big[-\dfrac{1}{m}\boldsymbol{\nabla}_y^2 - \dfrac{\alpha}{|\boldsymbol{y}|}\Big]\Phi^{(\boldsymbol{P})}(\boldsymbol{x}) = E_b\,\Phi^{(\boldsymbol{P})}(\boldsymbol{x})$

is satisfied by the Lorentz contracted wave fn: $\Phi^{(\boldsymbol{P})}(\boldsymbol{x}) = \Phi^{(0)}(\boldsymbol{y})$

This works only with the correct QED potential and transverse photon couplings.

Works differently in QCD (with confinement)

Applying the QED method to QCD

Our perturbative approach to bound states was guided by hadron features:

- α_s freezes: No loop contributions at lowest ("Born") order
- Hadrons have valence quantum numbers (lowest Fock state = valence)
- Transverse gluons are perturbative, $O(\alpha_s)$ corrections to lowest order
- Sea quarks arise from Z-diagrams (Bogoliubov transformed by field)
- The E_L field is instantaneous even for relativistic constituents

How can color confinement arise?

- Quarkonia suggest that the confining potential is classical
- Gauss' law has no Λ_{QCD} scale

$\Longrightarrow$ The scale must arise from a boundary condition on Gauss' law

Temporal gauge in QCD: $A_a^0 = 0$

Gauss' operator $G_a(x) \equiv \dfrac{\delta \mathcal{S}}{\delta A_a^0(x)} = \partial_i E_a^i(x) + g f_{abc} A_b^i E_c^i - g\psi^\dagger T^a \psi(x)$

generates time-independent gauge transformations, which keep $A_a^0 = 0$

Physical states satisfy the constraint $G_a(x)\,|phys\rangle = 0$

$$\Longrightarrow \quad \partial_i E_{L,a}^i(\boldsymbol{x})\,|phys\rangle = g\big[-f_{abc}A_b^i E_c^i + \psi^\dagger T^a \psi(\boldsymbol{x})\big]\,|phys\rangle \qquad \text{(QCD)}$$

In QED we solved for $\boldsymbol{E}_L$ with a boundary condition: $\boldsymbol{E}_L(\boldsymbol{x}) \to 0$ for $|\boldsymbol{x}| \to \infty$

$$\boldsymbol{E}_L(t,\boldsymbol{x})\,|phys\rangle = -\boldsymbol{\nabla}_x \int d\boldsymbol{y}\,\frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi^\dagger\psi(t,\boldsymbol{y})\,|phys\rangle \qquad \text{(QED)}$$

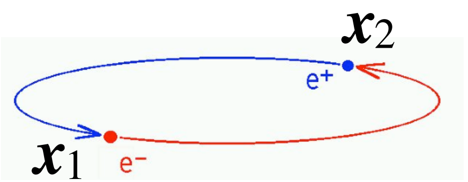
This was required to avoid long range interactions.

Is it any different in QCD?

There is a difference between QED and QCD

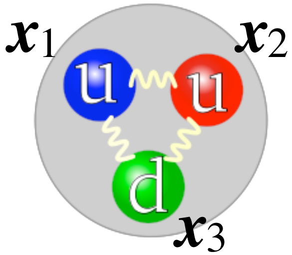
Global gauge invariance allows a classical gauge field for neutral atoms, but not a color octet gluon field for color singlet hadrons.

Positronium (QED)



$$\boldsymbol{E}_L(\boldsymbol{x}) = -\frac{e}{4\pi}\boldsymbol{\nabla}_x\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\right)$$

Proton (QCD)



$$\boldsymbol{E}_L^a(\boldsymbol{x}) = 0 \quad \text{for all } \boldsymbol{x}$$

However:
The classical gluon field is non-vanishing for each color component C of the state

$$\boldsymbol{E}_L^a(\boldsymbol{x}, C) \neq 0$$

The blue quark feels the color field generated by the red and green quarks.

An external observer sees no field:
The gluon field generated by a color singlet state vanishes.

$$\sum_C \boldsymbol{E}_L^a(\boldsymbol{x}, C) = 0$$

Including a homogeneous solution for $E^i_{L,a}$

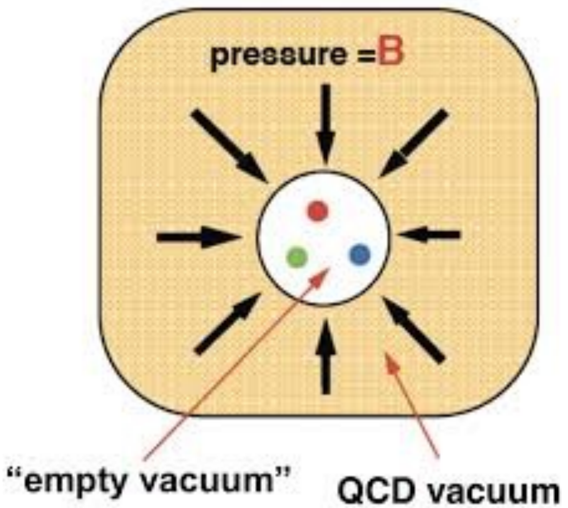
$$E^i_{L,a}(\boldsymbol{x})\,|phys\rangle = -\partial_i^x \int d\boldsymbol{y}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big]\mathcal{E}_a(\boldsymbol{y})\,|phys\rangle$$

where $\quad \mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A^i_b E^i_c(\boldsymbol{y}) + \psi^\dagger T^a \psi(\boldsymbol{y})$

$\kappa \neq \kappa(\boldsymbol{x},\boldsymbol{y})$ ensures $\partial_i \boldsymbol{E}^i(\boldsymbol{x}) = 0$ (a homogeneous solution of Gauss' law)

The linear dependence on $\boldsymbol{x}$ makes $\boldsymbol{E}_L$ independent of $\boldsymbol{x}$, as required by translation invariance: The field energy density is spatially constant.

The field energy ∝ volume of space is irrelevant only if it is universal.
This relates the normalisation $\varkappa$ of all Fock components, leaving an overall scale Λ as the single parameter.



"Bag model without a bag"

The potential energy $\mathcal{H}_V = \frac{1}{2}\int d\boldsymbol{x} \sum_a \boldsymbol{E}_L^a \cdot \boldsymbol{E}_L^a$

$$\mathcal{H}_V = \tfrac{1}{2}\int d\boldsymbol{x}\Big\{\partial_i^x \int d\boldsymbol{y}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big]\mathcal{E}_a(\boldsymbol{y})\Big\}\Big\{\partial_i^x \int d\boldsymbol{z}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{z} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{z}|}\Big]\mathcal{E}_a(\boldsymbol{z})\Big\}$$

Partial integration in $\boldsymbol{x}$, except for the $\varkappa^2$ term:

$$H_V = \int d\boldsymbol{y}d\boldsymbol{z}\Big\{\boldsymbol{y}\cdot\boldsymbol{z}\Big[\tfrac{1}{2}\kappa^2\int d\boldsymbol{x} + g\kappa\Big] + \tfrac{1}{2}\frac{\alpha_s}{|\boldsymbol{y}-\boldsymbol{z}|}\Big\}\mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z})$$

Recall: $\mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A_b^i E_c^i(\boldsymbol{y}) + \psi^\dagger T^a \psi(\boldsymbol{y})$

For the state $|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_A \bar{\psi}^A(\boldsymbol{x}_1)\,\psi^A(\boldsymbol{x}_2)\,|0\rangle$ we get:

$$\int d\boldsymbol{y}d\boldsymbol{z}\;\boldsymbol{y}\cdot\boldsymbol{z}\,\mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z})\,|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle$$

$$= (\boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 - 2\boldsymbol{x}_1\cdot\boldsymbol{x}_2)\bar{\psi}_A(\boldsymbol{x}_1)\,T^a_{AB}T^a_{BC}\psi_C(\boldsymbol{x}_2)\,|0\rangle$$

$$= C_F\,(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2\;|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle$$

Energy density is $\varkappa^2/2$ times this

Hence $\varkappa \propto 1/|\boldsymbol{x}_1 - \boldsymbol{x}_2|$

The $q\bar{q}$ potential

Define the universal scale Λ: $\kappa_{q\bar{q}} = \dfrac{\Lambda^2}{gC_F}\dfrac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$

The g$\varkappa$ term in H_V gives: $g\kappa_{q\bar{q}}C_F(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 = \Lambda^2|\boldsymbol{x}_1 - \boldsymbol{x}_2|$

Together with the $O(\alpha_s)$ term the potential agrees with the Cornell one:

$$V_{q\bar{q}}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1 - \boldsymbol{x}_2| - C_F\frac{\alpha_s}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$$

The parameter Λ is of $O(\alpha_s^0)$. Only confinement contributes at $O(\alpha_s^0)$.

Transverse gluons are calculable at $O(\alpha_s)$ (and dominate in hard processes)

The linear term was mandated by translation and rotation symmetry.

As we shall see, it also ensures boost covariance (without transverse γ's).

The qqq potential

Baryon component: $|q(\boldsymbol{x}_1)q(\boldsymbol{x}_2)q(\boldsymbol{x}_3)\rangle \equiv \sum_{A,B,C} \epsilon_{ABC}\psi^\dagger_A(\boldsymbol{x}_1)\,\psi^\dagger_B(\boldsymbol{x}_2)\,\psi^\dagger_C(\boldsymbol{x}_3)\,|0\rangle$

Action of H_V: $\int d\boldsymbol{y}d\boldsymbol{z}\;\boldsymbol{y}\cdot\boldsymbol{z}\,\mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z})\,|q(\boldsymbol{x}_1)q(\boldsymbol{x}_2)q(\boldsymbol{x}_3)\rangle$

$$\mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A^i_bE^i_c(\boldsymbol{y}) + \psi^\dagger T^a\psi(\boldsymbol{y})$$

$\boldsymbol{x}_1^2$: $\quad \epsilon_{ABC}\psi^\dagger_{A''}(\boldsymbol{x}_1)\,\psi^\dagger_B(\boldsymbol{x}_2)\,\psi^\dagger_C(\boldsymbol{x}_3)\,|0\rangle\,T^a_{A''A'}T^a_{A'A} = \frac{4}{3}\,|qqq\rangle$

$\boldsymbol{x}_1\cdot\boldsymbol{x}_2$: $\quad 2\,\epsilon_{ABC}\psi^\dagger_{A'}(\boldsymbol{x}_1)\,\psi^\dagger_{B'}(\boldsymbol{x}_2)\,\psi^\dagger_C(\boldsymbol{x}_3)\,|0\rangle\,T^a_{A'A}T^a_{B'B} = -\frac{4}{3}\,|qqq\rangle$

Sum: $\mathcal{H}_V\,|qqq\rangle = \left[\frac{1}{2}\kappa^2\int d\boldsymbol{x} + g\kappa\right]\frac{4}{3}\left[d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\right]^2|qqq\rangle + \mathcal{O}\left(\alpha_s\right)$

$$d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) \equiv \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2+(\boldsymbol{x}_2-\boldsymbol{x}_3)^2+(\boldsymbol{x}_3-\boldsymbol{x}_1)^2}$$

The normalization $\varkappa$ should give the same energy density as for $q\bar{q}$.

The qqq potential (cont.)

Universality of the energy density: $\kappa_{qqq} = \frac{\Lambda^2}{gC_F} \frac{1}{d_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)}$

fixes the qqq potential to be

$$V_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \Lambda^2 d_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) - \frac{2}{3}\alpha_s \Big(\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} + \frac{1}{|\boldsymbol{x}_2 - \boldsymbol{x}_3|} + \frac{1}{|\boldsymbol{x}_3 - \boldsymbol{x}_1|} \Big)$$

$$d_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \equiv \frac{1}{\sqrt{2}} \sqrt{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + (\boldsymbol{x}_2 - \boldsymbol{x}_3)^2 + (\boldsymbol{x}_3 - \boldsymbol{x}_1)^2}$$

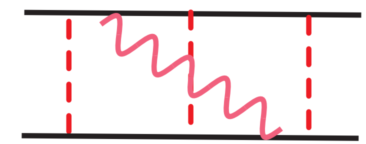
The baryon potential is completely determined, given the universal scale Λ.

When two quarks are at the same $\boldsymbol{x}$, V_{qqq} reduces to $V_{q\bar{q}}$:

$$V_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 = \boldsymbol{x}_2) = V_{q\bar{q}}(\boldsymbol{x}_1, \boldsymbol{x}_2) \qquad \text{for } N=3$$

The qgq̄ potential

A qq̄ state, after the emission of a transverse gluon:



$$|q(\boldsymbol{x}_1)g(\boldsymbol{x}_g)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_{A,B,b} \bar{\psi}_A(\boldsymbol{x}_1)\, A_b^j(\boldsymbol{x}_g)T_{AB}^b\psi_B(\boldsymbol{x}_2)\,|0\rangle$$

$$V_{qgq}^{(0)}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) = \frac{\Lambda^2}{\sqrt{C_F}}\, d_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) \qquad \text{(universal } \Lambda\text{)}$$

$$d_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) \equiv \sqrt{\tfrac{1}{4}(N-2/N)(\boldsymbol{x}_1-\boldsymbol{x}_2)^2 + N(\boldsymbol{x}_g - \tfrac{1}{2}\boldsymbol{x}_1 - \tfrac{1}{2}\boldsymbol{x}_2)^2}$$

$$V_{qgq}^{(1)}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) = \tfrac{1}{2}\,\alpha_s\Big[\frac{1}{N}\,\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} - N\Big(\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_g|}+\frac{1}{|\boldsymbol{x}_2-\boldsymbol{x}_g|}\Big)\Big]$$

When q and g coincide:

$$V_{qgq}^{(0)}(\boldsymbol{x}_1=\boldsymbol{x}_g,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| = V_{q\bar{q}}^{(0)}$$

$$V_{qgq}^{(1)}(\boldsymbol{x}_1=\boldsymbol{x}_g,\boldsymbol{x}_2) = V_{q\bar{q}}^{(1)}$$

The gg potential

A "glueball" component: $|g(\boldsymbol{x}_1)g(\boldsymbol{x}_2)\rangle \equiv \sum_a A_a^i(\boldsymbol{x}_1)\, A_a^j(\boldsymbol{x}_2)\,|0\rangle$

has the potential $V_{gg} = \sqrt{\frac{N}{C_F}}\,\Lambda^2\,|\boldsymbol{x}_1 - \boldsymbol{x}_2| - N\,\frac{\alpha_s}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$

This agrees with the qgq̄ potential where the quarks coincide:

$$V_{gg}(\boldsymbol{x}, \boldsymbol{x}_g) = V_{qg\bar{q}}(\boldsymbol{x}, \boldsymbol{x}_g, \boldsymbol{x})$$

It is straightforward to work out the instantaneous potential for any Fock state.

Quarkonium in motion

A valence, (globally) color singlet quarkonium state with momentum $\boldsymbol{P} = (0,0,P)$:

$$|q\bar{q}, \boldsymbol{P}\rangle = \frac{1}{\sqrt{N_C}} \sum_{A,B} \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}^A(\boldsymbol{x}_1) e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2} \delta^{AB} \Phi^{(\boldsymbol{P})}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \psi^B(\boldsymbol{x}_2)\,|0\rangle$$

Recall that for Positronium a contribution from transverse photons was required:

$$(\mathcal{H}_0 + \mathcal{H}_V)\,|\Psi\rangle = \Big|\Big[E_P - \frac{1}{m\gamma}\big(\boldsymbol{\nabla}_\perp^2 + \frac{1}{\gamma^2}\boldsymbol{\nabla}_\parallel^2\big) - \frac{\alpha}{|\boldsymbol{x}|}\Big]\Psi\Big\rangle \neq E(P) = \Big(E_P + \frac{1}{\gamma}E_b\Big)\,|\Psi\rangle$$

For a linear potential the bound state condition:

$$\Big[-\frac{1}{m}\big(\boldsymbol{\nabla}_\perp^2 + \frac{1}{\gamma^2}\boldsymbol{\nabla}_\parallel^2\big) + \gamma V'|\boldsymbol{x}|\Big]\Phi^{(\boldsymbol{P})}(\boldsymbol{x}) = E_b\,\Phi^{(\boldsymbol{P})}(\boldsymbol{x})$$

agrees with the Lorentz contracted rest frame wf. for $\boldsymbol{x}_\perp = 0$: $z \to \gamma z$.

There is no transverse photon contribution at $O(\alpha_s^0)$.

Since $\nabla_\perp V(\boldsymbol{x}_\perp = 0, z) = 0$ also $\nabla_\perp \Phi^{(P)}(\boldsymbol{x}_\perp = 0, z)$ Lorentz contracts. This defines a boundary condition for the PDE, allowing its (numerical) solution for all $\boldsymbol{x}_\perp$

Standard Lorentz contraction only at $\boldsymbol{x}_\perp = 0$: $\gamma|\boldsymbol{x}| = \sqrt{(\gamma\boldsymbol{x}_\perp)^2 + (\gamma z)^2}$

A perturbative approach to soft QCD?!

- The instantaneous $\mathcal{O}\left(\alpha_s^0\right)$ field binds the valence Fock states
- The higher Fock states generated by the Hamiltonian H_{QCD} are of $O(g)$
- Method seems fully defined, requires further checks.

For the approach to be viable the $\mathcal{O}\left(\alpha_s^0\right)$ dynamics must have:

Poincaré symmetry (also for relativistic binding)
Unitarity (at the hadron level)
Confinement
Chiral Symmetry Breaking (CSB)
Correct mass spectrum, up to $O(\alpha_s)$ corrections

No $\mathcal{O}\left(\alpha_s^0\right)$ contributions from gluon exchange, transverse gluons, etc.

The EM form factors, pdf's, hadron scattering, … may be evaluated in any frame

Many opportunities (if it works!)

$\mathcal{O}\left(\alpha_s^0\right)$ light $q\bar{q}$ bound states

An $\mathcal{O}\left(\alpha_s^0\right)$ meson state with $\boldsymbol{P} = 0$ and wave function Φ:

$$|M\rangle = \sum_{A,B;\alpha,\beta} \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}_\alpha^A(t=0,\boldsymbol{x}_1)\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta^B(t=0,\boldsymbol{x}_2)\,|0\rangle$$

The bound state condition $H\,|M\rangle = M\,|M\rangle$ gives, at $\mathcal{O}\left(\alpha_s^0\right)$

$$\left[i\gamma^0\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}}+m\gamma^0\right]\Phi(\boldsymbol{x})+\Phi(\boldsymbol{x})\left[i\gamma^0\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}}-m\gamma^0\right]=\left[M-V(|\boldsymbol{x}|)\right]\Phi(\boldsymbol{x})$$

where $\boldsymbol{x} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $V(|\,\boldsymbol{x}\,|) = V'|\,\boldsymbol{x}\,| = \Lambda^2|\,\boldsymbol{x}\,|$.

In the non-relativistic limit ($m \gg \Lambda$) this reduces to the Schrödinger equation.

If we add the instantaneous gluon exchange potential:

$\Rightarrow$ The quarkonium phenomenology with the Cornell potential.

$\mathcal{O}\left(\alpha_s^0\right)$ $q\bar{q}$ bound states (cont.)

$$i\boldsymbol{\nabla}\cdot\left\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\right\}+m\left[\gamma^0,\Phi(\boldsymbol{x})\right]=\left[M-V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})$$

Expanding the 4×4 wave function in a basis of 16 Dirac structures $\Gamma_i(\boldsymbol{x})$

$$\Phi(\boldsymbol{x})=\sum_i \Gamma_i(\boldsymbol{x})F_i(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

we may use rotational, parity and charge conjugation invariance to determine which $\Gamma_i(\boldsymbol{x})$ may occur for a state of given j^{PC}:

0^{-+} trajectory $[s=0,\ \ell=j]:$ $\quad -\eta_P=\eta_C=(-1)^j$ $\quad \gamma_5,\ \gamma^0\gamma_5,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L}$

0^{--} trajectory $[s=1,\ \ell=j]:$ $\quad \eta_P=\eta_C=-(-1)^j$ $\quad \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \boldsymbol{\alpha}\cdot\boldsymbol{L},\ \gamma^0\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$

0^{++} trajectory $[s=1,\ \ell=j\pm 1]:$ $\quad \eta_P=\eta_C=+(-1)^j$ $\quad 1,\ \boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$

0^{+-} trajectory [exotic] : $\quad \eta_P=-\eta_C=(-1)^j$ $\quad \gamma^0,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$

$\Rightarrow$ There are no solutions for quantum numbers that would be exotic in the quark model (despite the relativistic dynamics)

The BSE gives the radial equations for the $F_i(r)$

(There are two coupled radial equations for the 0^{++} trajectory)

Example: 0^{-+} trajectory wf's

$$\Phi_{-+}(\boldsymbol{x}) = \Big[\frac{2}{M-V}(i\boldsymbol{\alpha}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0) + 1\Big]\gamma_5\, F_1(r) Y_{j\lambda}(\hat{\boldsymbol{x}})$$

$\eta_P = (-1)^{j+1}$

$\eta_C = (-1)^j$

Radial equation: $F_1'' + \Big(\frac{2}{r} + \frac{V'}{M-V}\Big)F_1' + \Big[\frac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2}\Big]F_1 = 0$

Local normalizability at $r = 0$ and at $V(r) = M$ (!) determines the discrete M

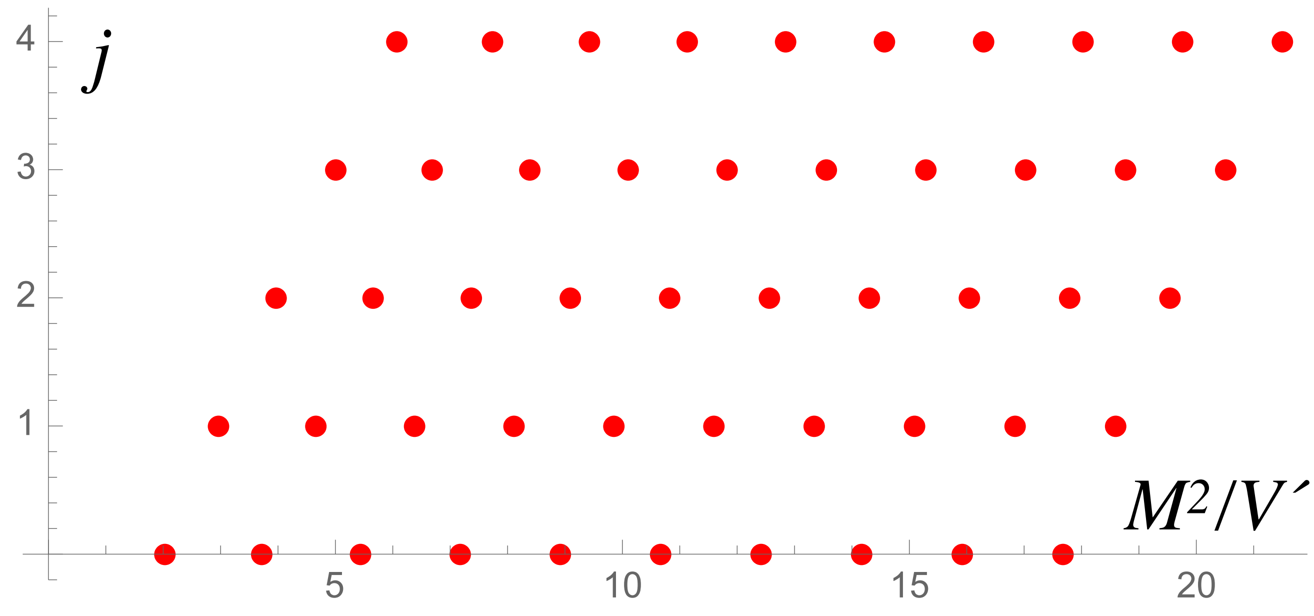
C.f.: Dirac eq.: Has continuous spectrum

$m = 0$

Mass spectrum:

Linear Regge trajectories with daughters

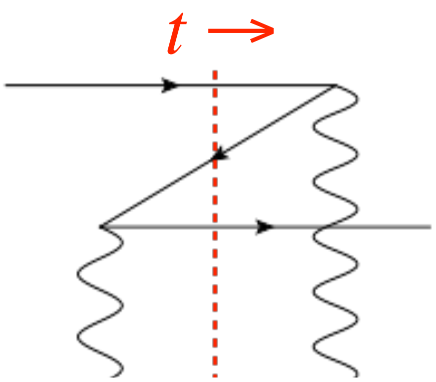
Spectrum similar to dual models



Sea quark contributions

Quark states in a strong field have E<0 components

Bogoliubov transformation, cf. Dirac states.



In time-ordered PT, these correspond to Z-diagrams, and interpreted as contributions from $q\bar{q}$ pairs.

This effect is manifest in the behavior of the wave function Φ for large $V = V' |\boldsymbol{x}|$:

$$\lim_{\boldsymbol{x}\to\infty} |\Phi(\boldsymbol{x})|^2 = const.$$

The asymptotically constant norm apparently reflects, via duality, pair production as the linear potential $V(|\,\boldsymbol{x}\,|)$ increases.

Radial wave function for r → ∞

The radial equation was, with $V(r) = V'r$:

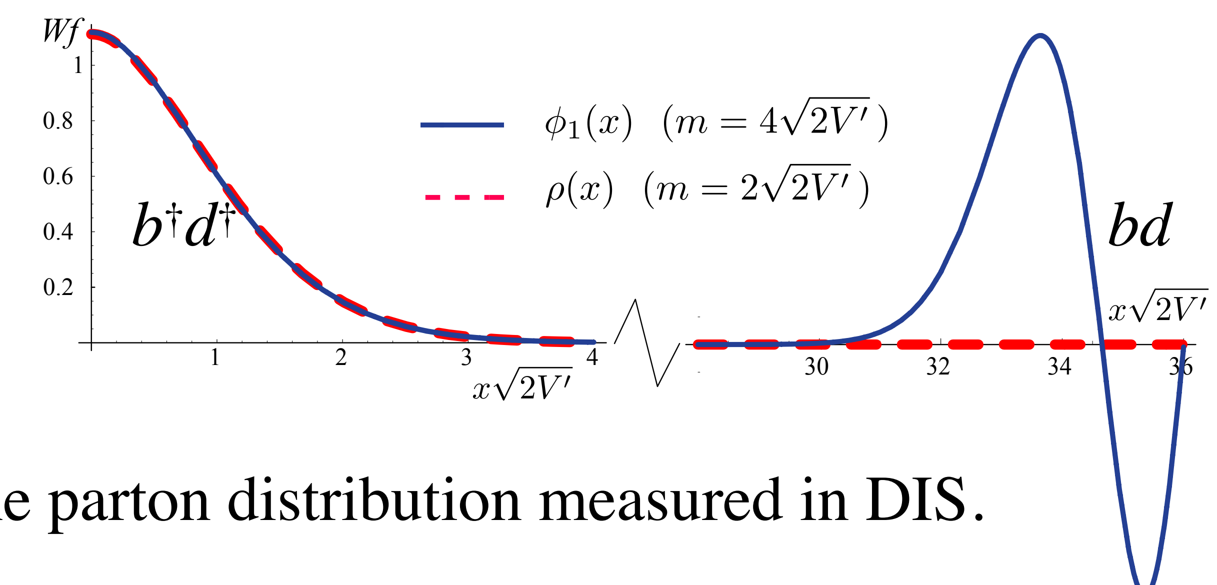
$$F_1'' + \left(\frac{2}{r} + \frac{V'}{M-V}\right)F_1' + \left[\tfrac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2}\right]F_1 = 0$$

The leading terms at large r : $\quad F''(r) + \tfrac{1}{4}V'^2 r^2 F(r) = 0$

which implies: $\quad F(r\to\infty) \sim \exp\left[\pm \tfrac{i}{4}V' r^2\right]$

This is similar to the Dirac equation with a linear potential, but now the spectrum is discrete.

Plot of one component of the D = 1+1 wf.:



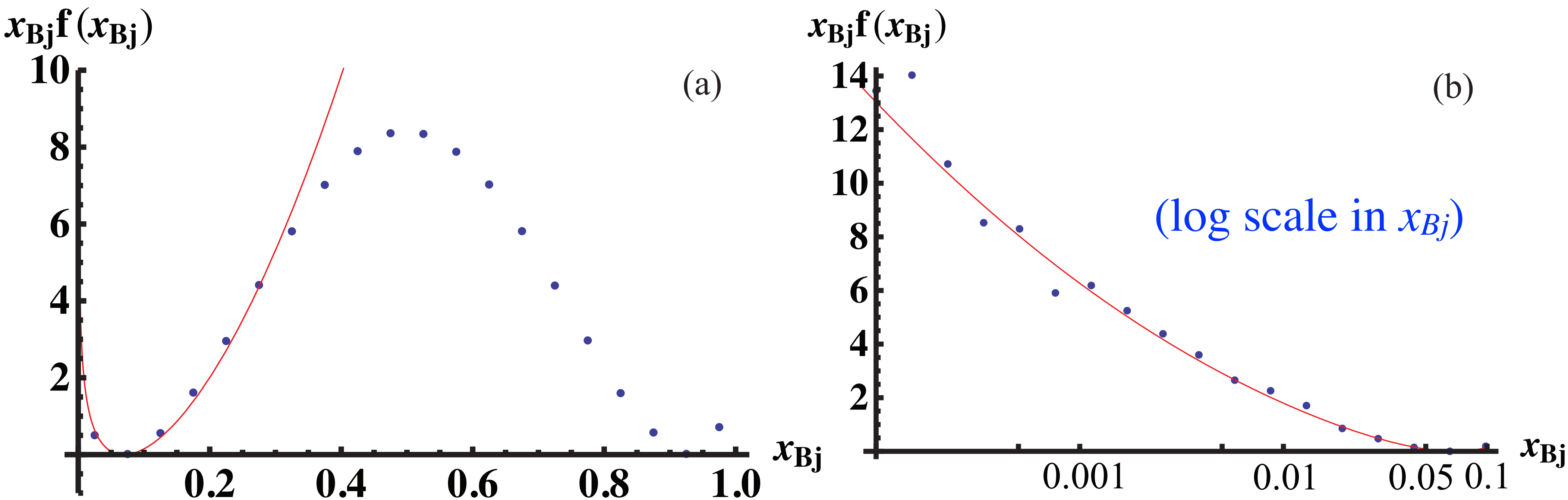
The sea quarks show up in the parton distribution measured in DIS.

Parton distributions have a sea component

In D=1+1 dimensions a sea component appears at low m/e :

$m/e = 0.1$

D. D. Dietrich, PH, M. Järvinen
arXiv 1212.4747



The red curve is an analytic approximation, valid in the $x_{Bj} \to 0$ limit.

Note: Enhancement at low x is due to bd (sea), not to $b^\dagger d^\dagger$ (valence) component.

To be calculated in D=3+1 (and in various frames!)

Example: Glueball spectrum

$$|gg\rangle \equiv \int d\boldsymbol{x}_1 d\boldsymbol{x}_2 \big[A^i_{a,T}(\boldsymbol{x}_1)A^j_{a,T}(\boldsymbol{x}_2)\Phi^{ij}_{AA}(\boldsymbol{x}_1-\boldsymbol{x}_2)$$

$$+A^i_{a,T}E^j_{a,T}\Phi^{ij}_{AE} + E^i_{a,T}A^j_{a,T}\Phi^{ij}_{EA} + E^i_{a,T}E^j_{a,T}\Phi^{ij}_{EE}\big]\,|0\rangle$$

All components have the same instantaneous potential: $V_{gg}(r) = \frac{3}{2}\Lambda^2 r$

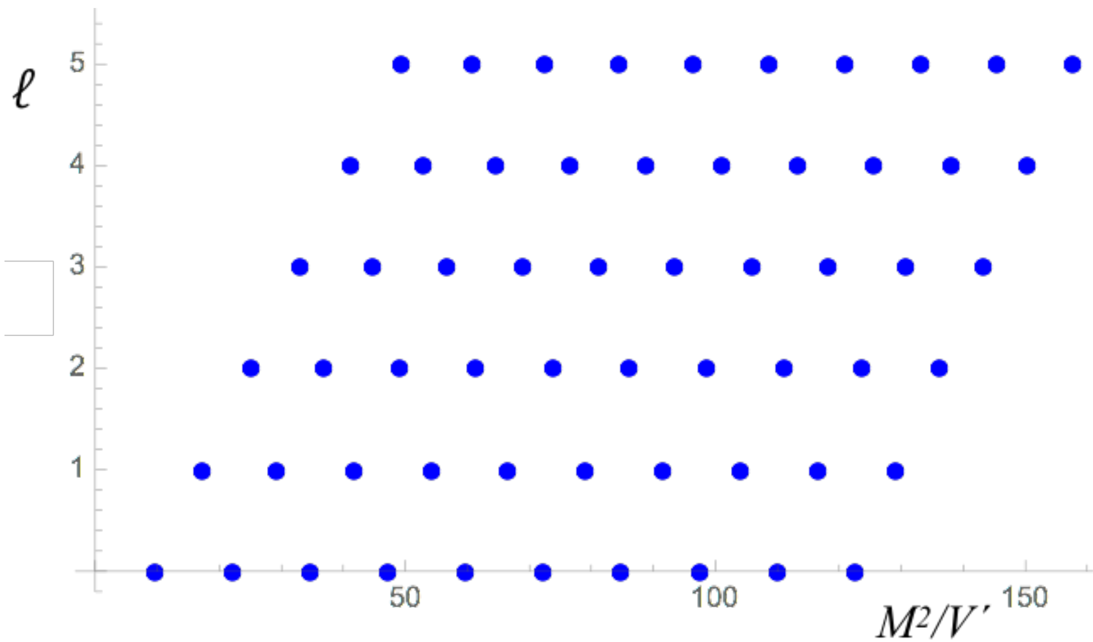
Imposing $H_{QCD}\,|gg\rangle = M\,|gg\rangle$ relates Φ_{AA} and $\Phi_{AE} = \Phi_{EA}$ to $\Phi_{EE} = F(r)Y_{\ell\lambda}(\Omega)$,

where $F''(r) + \Big(\frac{2}{r} - \frac{V_g'}{M-V}\Big)F'(r) + \Big[\frac{1}{4}(M-V)^2 - \frac{V_g'}{r(M-V)} - \frac{\ell(\ell+1)}{r^2}\Big]F(r) = 0$

The spectrum (M) is determined by local normalizability at $r = 0$ and $V(r) = M$.

With $\Lambda^2 = 0.18$ GeV^2 the lowest glueball state is

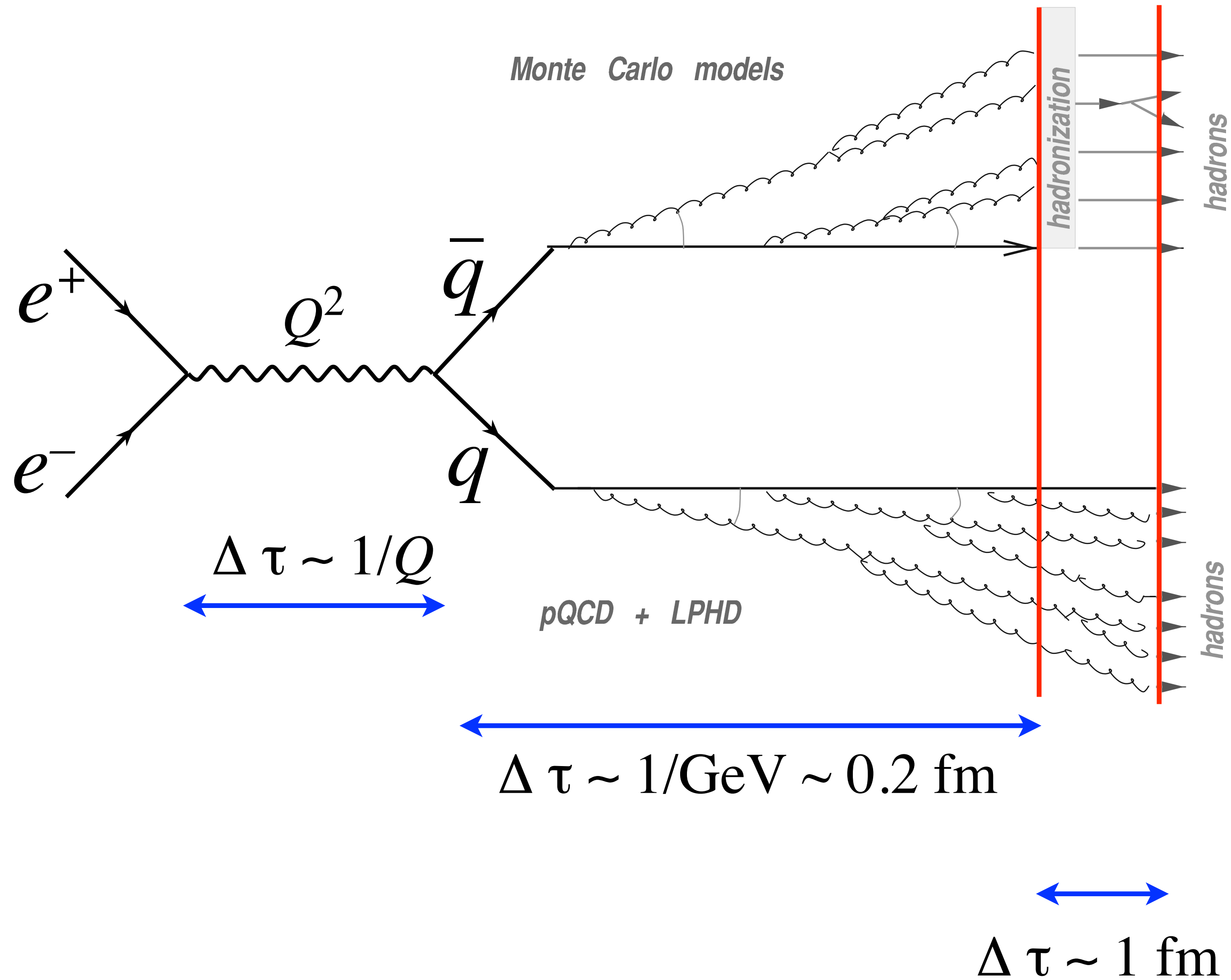
$M(\ell = 0, n=1) \simeq 1.6$ GeV.



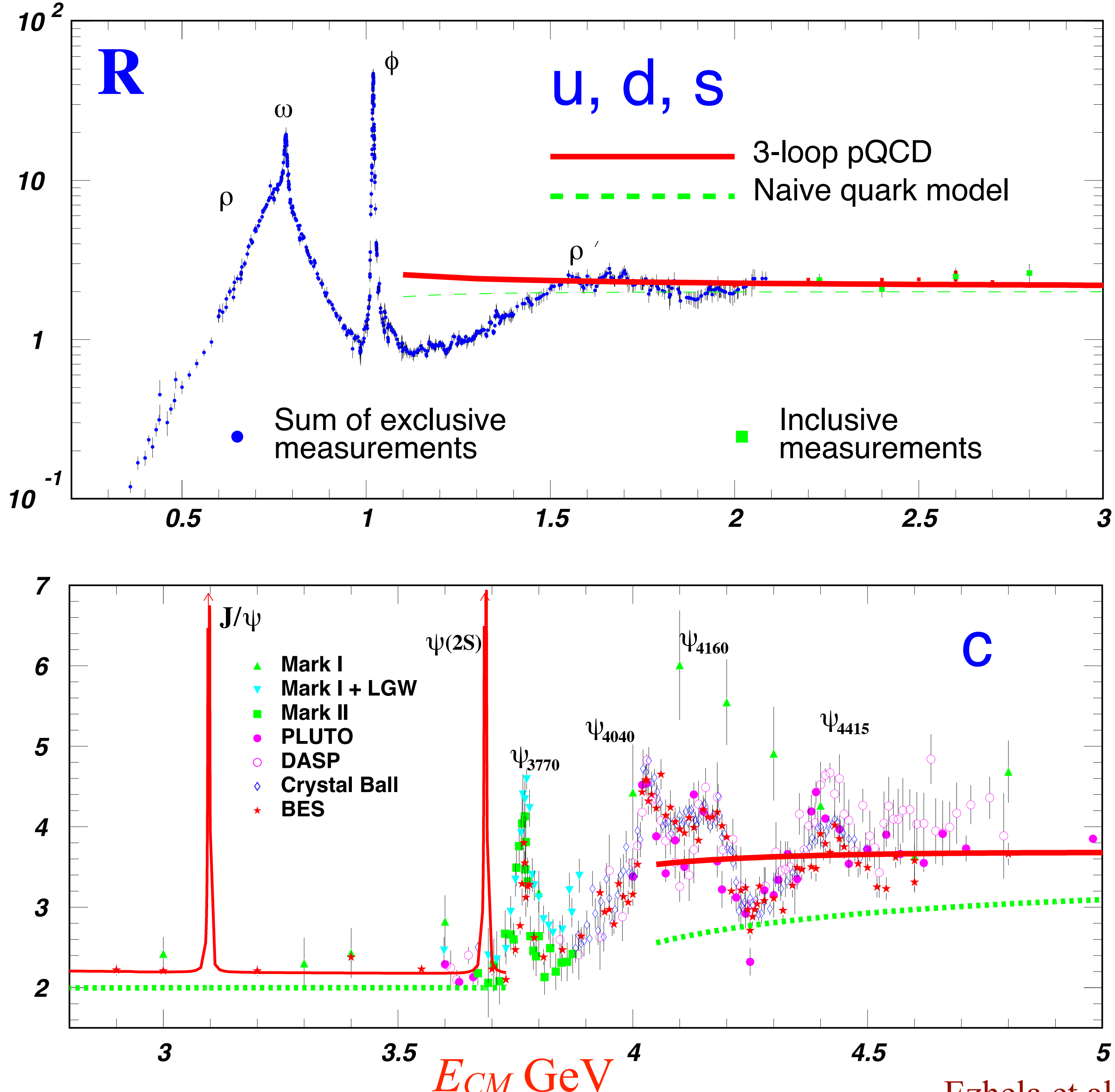
Time evolution in e+e- → of hadrons

Final state evolves in (proper) time τ with decreasing virtuality and decreasing energy uncertainty ΔE

Evolution is unitary:
Measured cross section in energy interval $E_{CM} \pm \Delta E$ must average to (parton) cross section at $\tau \sim 1/\Delta E$

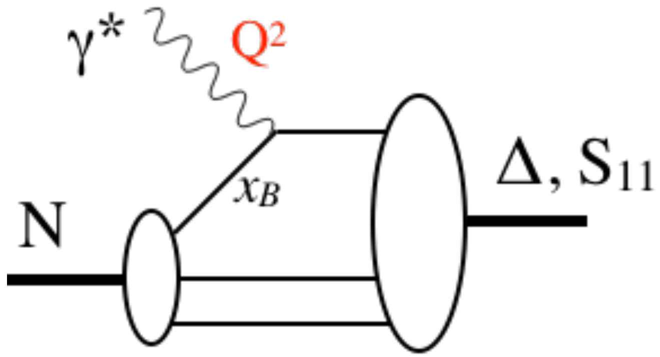


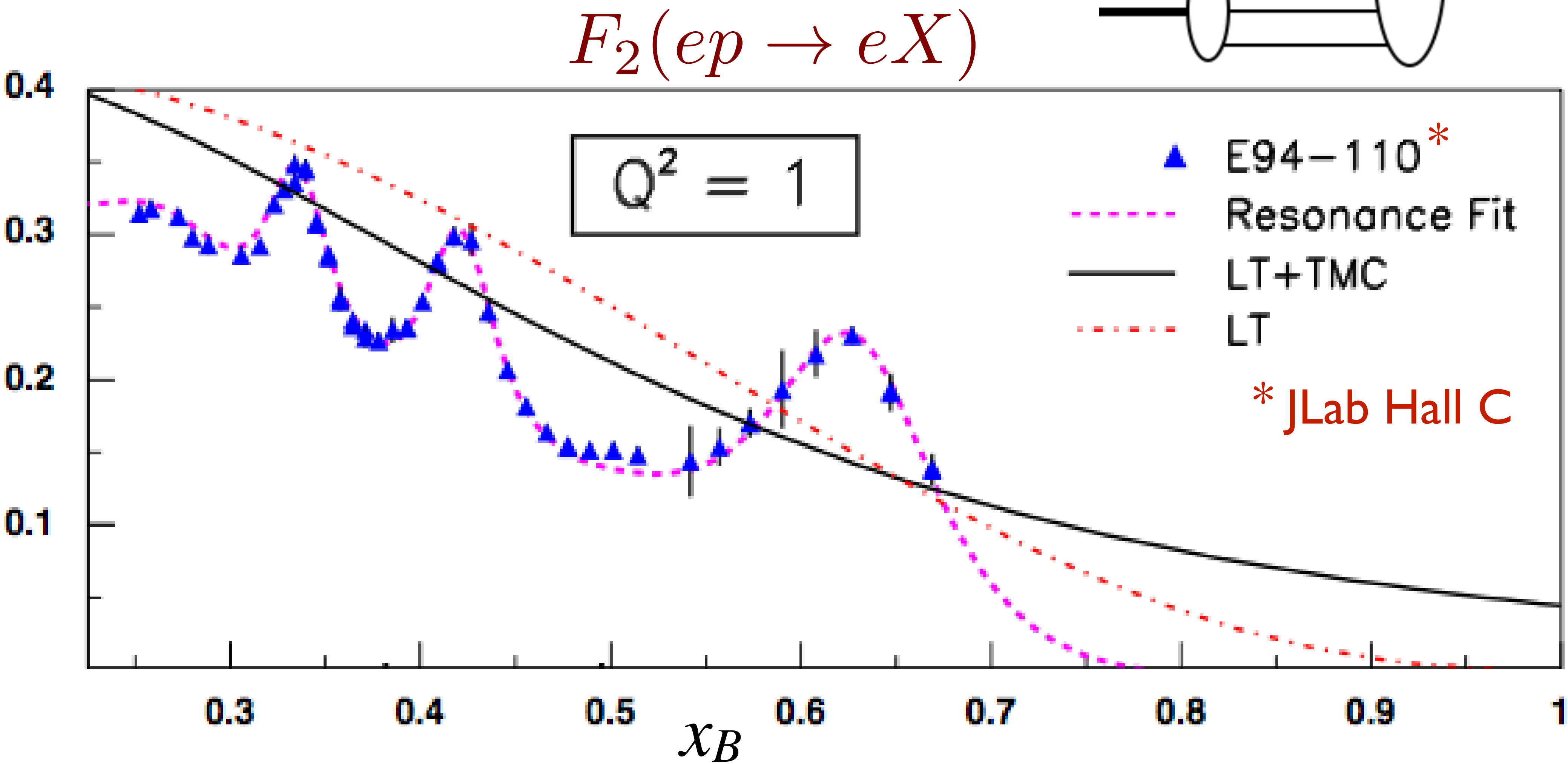
Duality in e+e- → hadrons



Ezhela et al, hep-ph/0312114

Bloom-Gilman Duality





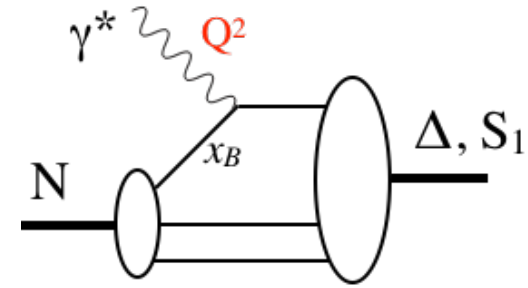
Resonances average scaling (large Q^2) curve. This holds at all $Q^2 \geqq 1$ GeV^2

TMC = Target Mass Correction

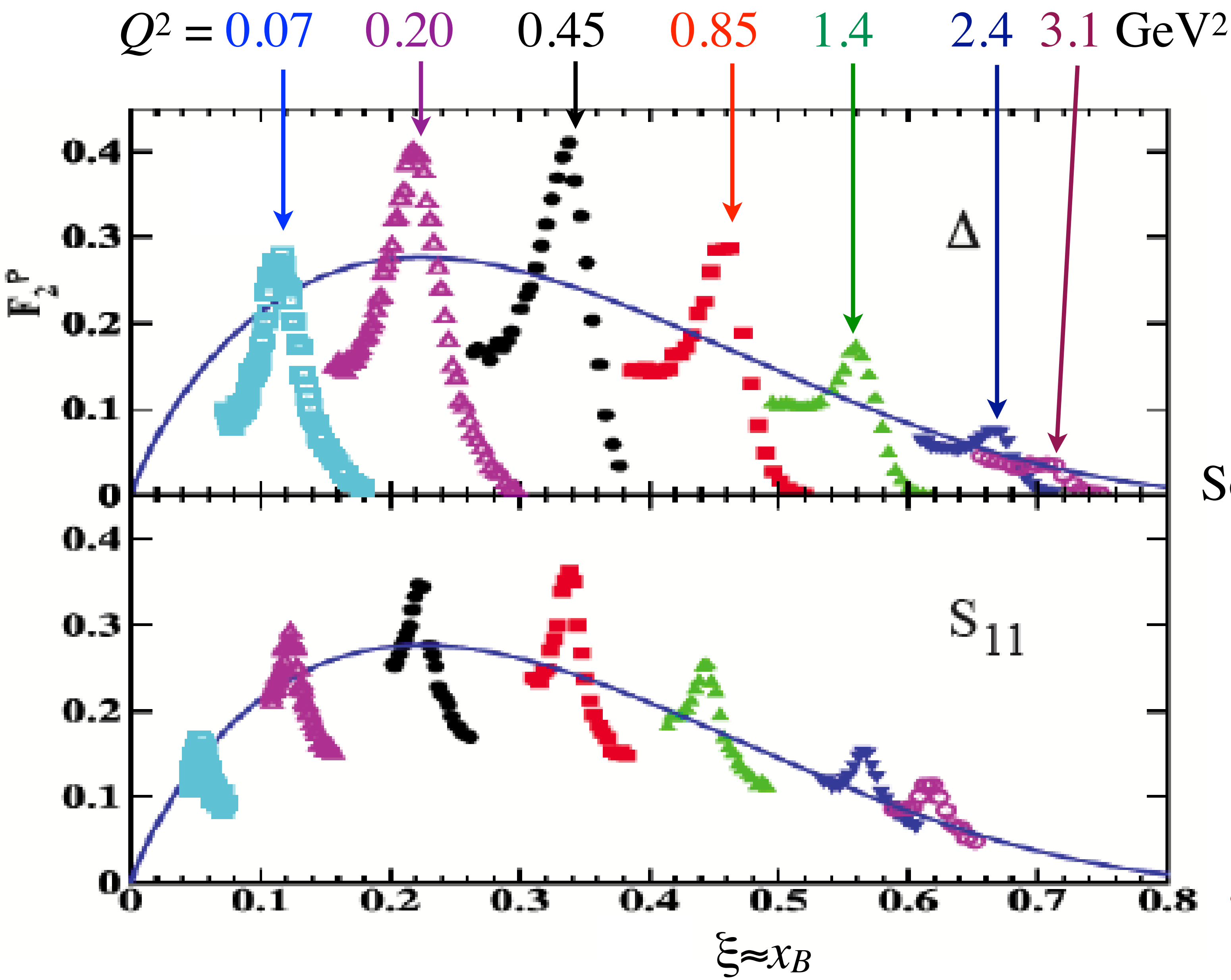
Bloom & Gilman (1970) W. Melnitchouk (2010)

Resonances slide on the scaling curve

$$W^2 = M_{N*}^2 = M_N^2 + \frac{(1-x_B)Q^2}{x_B}$$



C.S. Armstrong et al, PRD **63** (2005) 094008



$1.2 < W^2 < 1.9$ GeV^2

"Δ"

Solid curve: Large Q^2

Jlab Hall C

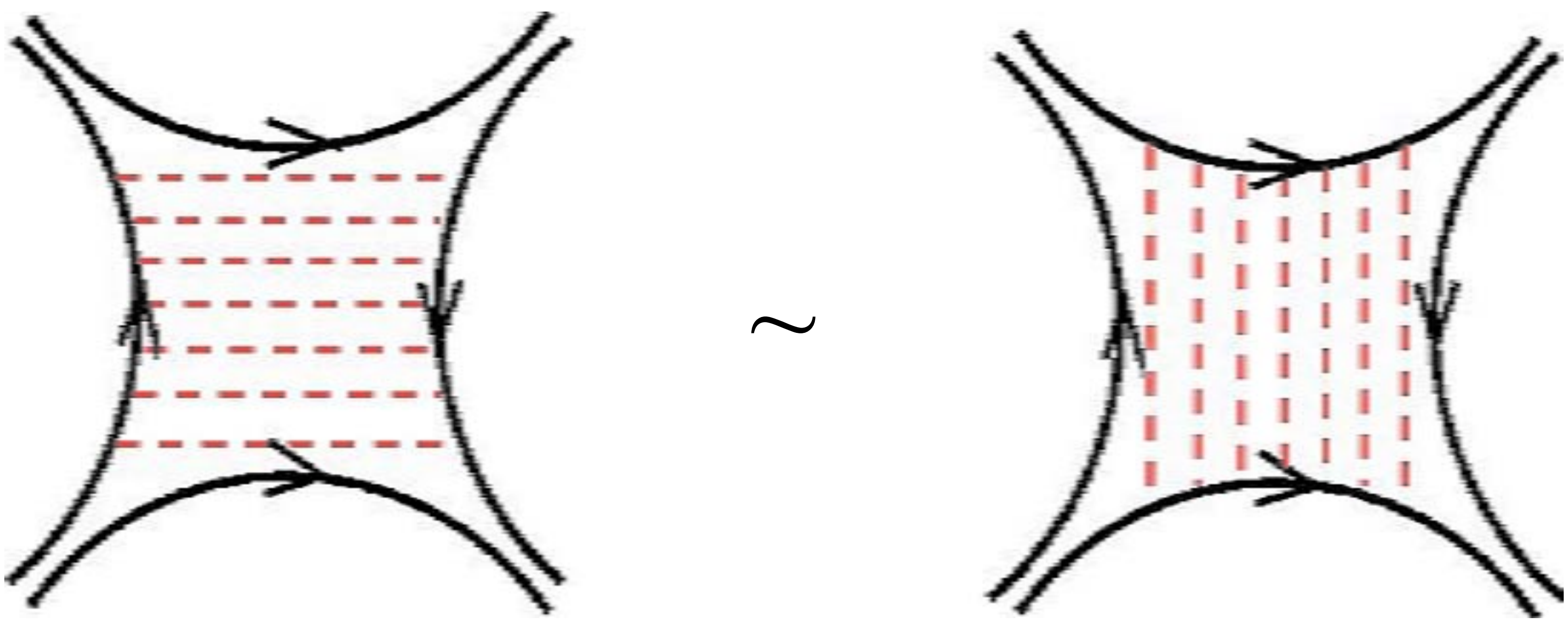
$1.9 < W^2 < 2.5$ GeV^2

"S_{11}"

$$\xi = \frac{2x_B}{1+\sqrt{1+4M_p^2 x_B^2/Q^2}}$$

Implications of duality

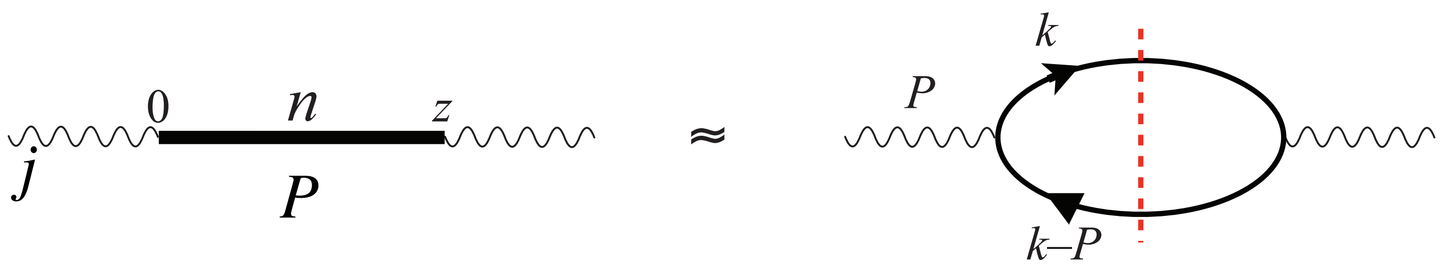
- Resonances build scattering: the two must be considered together.
- The masses, spins and couplings of all bound states are related.
- Dual diagrams are relevant.



Kopeliovich et al, arXiv:0811.2024

Plane waves in bound states

In the parton picture, high energy quarks can be treated as free constituents. They are momentum eigenstates, described by plane waves. How does this fit into the bound state wave functions?



Consider a highly excited state (P=0) in D = 1+1: $M \to \infty,\ \mathrm{V}(x) << M$

$\sigma = (M\text{-}V)^2 \approx M^2 - 2MV \to \infty$

$$\Phi(\sigma \to \infty) \sim \exp(\pm i\sigma/2) = e^{\pm iM^2} \exp(\mp ix\, M/2)$$

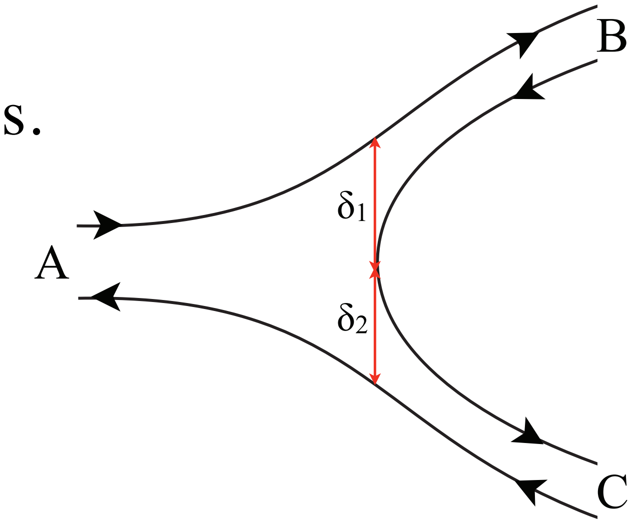
Thus oscillations of the wf at large σ gives a plane wave with $p = \pm M/2$

The state agrees, in this limit, with the parton picture:

$$|M, P=0\rangle = \frac{\sqrt{2\pi}}{2M}\big(b^\dagger_{M/2} d^\dagger_{-M/2} + b^\dagger_{-M/2} d^\dagger_{M/2}\big)|\Omega\rangle$$

Only "*valence*" particles appear (no *b* or *d* operators).

Decays and hadron loops



The bound state equation determines zero-width states.

There is an $\mathcal{O}\left(1/\sqrt{N_C}\right)$ coupling between the states: string breaking

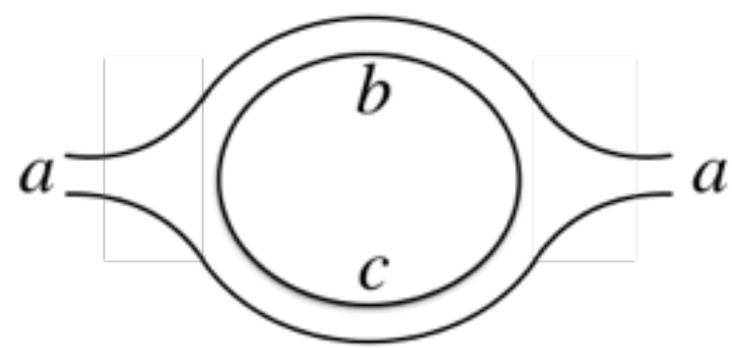
$$\langle B, C|A\rangle = -\frac{(2\pi)^3}{\sqrt{N_C}}\delta^3(\boldsymbol{P}_A - \boldsymbol{P}_B - \boldsymbol{P}_C)$$

$$\times \int d\boldsymbol{\delta}_1 d\boldsymbol{\delta}_2\, e^{i\boldsymbol{\delta}_1\cdot\boldsymbol{P}_C/2 - i\boldsymbol{\delta}_2\cdot\boldsymbol{P}_B/2}\mathrm{Tr}\left[\gamma^0\Phi_B^\dagger(\boldsymbol{\delta}_1)\Phi_A(\boldsymbol{\delta}_1+\boldsymbol{\delta}_2)\Phi_C^\dagger(\boldsymbol{\delta}_2)\right]$$

The overlap suggests that hadron A is an "average" (dual) description of B+C.

When squared, this gives a $1/N_C$

hadron loop unitarity correction:



Unitarity should be satisfied

at hadron level at each order of $1/N_C$.

Bound states in motion

An $\mathcal{O}\left(\alpha_s^0\right)$ $q\bar{q}$ bound state with CM momentum $\boldsymbol{P}$ may be expressed as

$$|M,\boldsymbol{P}\rangle = \int dx_1\, dx_2\, \bar\psi(t=0,x_1)\, e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2}\, \Phi^{(\boldsymbol{P})}(x_1-x_2)\, \psi(t=0,x_2)\,|0\rangle$$

The instantaneous potential is $\boldsymbol{P}$-independent, $V(\boldsymbol{x}) = V'|\boldsymbol{x}|$, hence the BSE:

$$i\boldsymbol{\nabla}\cdot\left\{\boldsymbol{\alpha},\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\right\} - \tfrac{1}{2}\boldsymbol{P}\cdot\left[\boldsymbol{\alpha},\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\right] + m\left[\gamma^0,\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\right] = \left[E - V(\boldsymbol{x})\right]\Phi^{(\boldsymbol{P})}(\boldsymbol{x})$$

The solution for $\Phi^{(P)}(\boldsymbol{x})$ is not simply Lorentz contracting in $\boldsymbol{x}$.

There is an analytic solution in D = 1+1 dimensions.

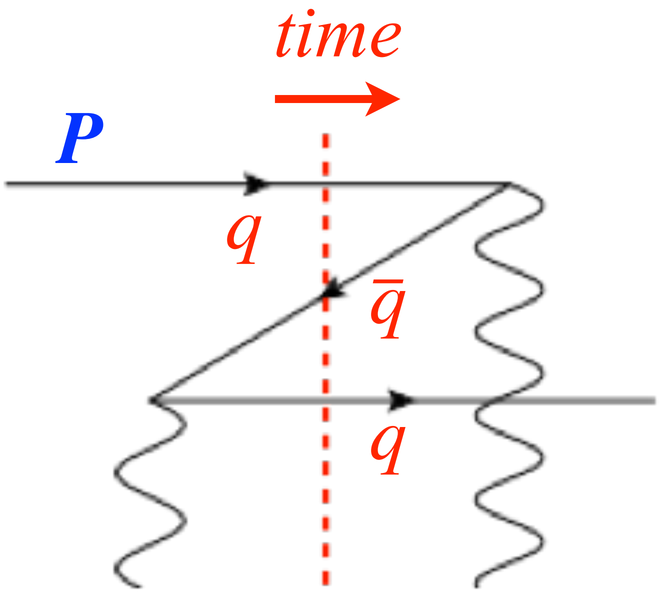
In D = 3+1 dimensions there is a boundary condition at $\boldsymbol{x}_\perp = 0$

States with general $\boldsymbol{P}$ are needed for:

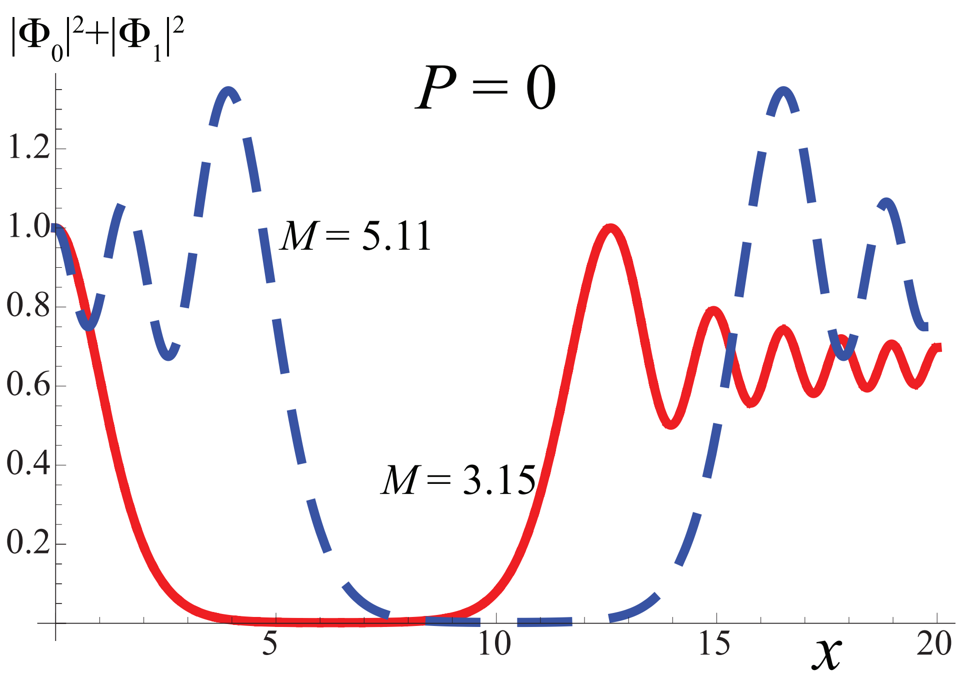
- $\boldsymbol{P}$-dependence of angular momentum ($P \to \infty$ frame).
- EM form factors (gauge invariance has been verified)
- Parton distributions
- Hadron scattering
- . . .

Z-contributions dilate in x

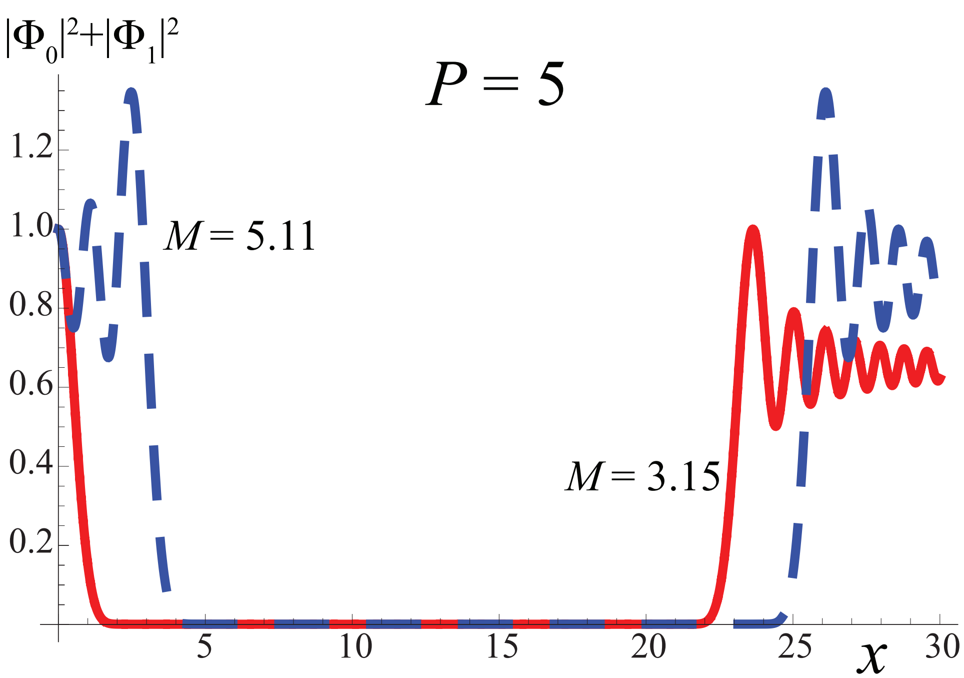
The energy of the $q\bar{q}$ pairs increase with P. Hence their production requires a larger potential $V(x)$, *ie*., $|x|$ grows with P.



This is seen in the D=1+1 analytic wf's:



(a)



(b)

EM form factor at $O(\alpha_s^0)$

Matrix element of EM current between states A, B of momenta P_a, P_b

$$|A, \boldsymbol{P}_a\rangle = \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}(\boldsymbol{x}_1) e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2} \Phi_A(\boldsymbol{x}_1 - \boldsymbol{x}_2)\psi(\boldsymbol{x}_2)\,|0\rangle$$

EM current $\quad j^\mu(z) = e\bar{\psi}(z)\gamma^\mu\psi(z) = e^{i\hat{P}\cdot z} j^\mu(0) e^{-i\hat{P}\cdot z}$

$$F^\mu_{AB}(z) \equiv \langle B, \boldsymbol{P}_b | j^\mu(z)\,|A, \boldsymbol{P}_a\rangle$$

$$= e\,(1 - C_A C_B) e^{i(P_b - P_a)\cdot z} \int d\boldsymbol{x}\, e^{i(\boldsymbol{P}_b - \boldsymbol{P}_b)\cdot \boldsymbol{x}/2}$$

$$\times \mathrm{Tr}\left[\Phi^\dagger_B(\boldsymbol{x})\gamma^\mu\gamma^0\Phi_A(\boldsymbol{x})\right]$$

C_A, C_B : Charge conjugation

Can show: $\quad \dfrac{\partial}{\partial z^\mu} F^\mu_{AB}(z) = 0 \quad$ Current conservation, for any P_a, P_b

States with M = 0

PRELIMINARY

We required the wave function to be normalizable at $r = 0$ and $V'r = M$

For $M = 0$ the two points coincide. Regular, massless solutions are found.

The massless 0^{++} meson "σ"

$$|\sigma\rangle = \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar{\psi}(\boldsymbol{x}_1)\, \Phi_\sigma(\boldsymbol{x}_1 - \boldsymbol{x}_2)\, \psi(\boldsymbol{x}_2)\, |0\rangle \equiv \hat{\sigma}\, |0\rangle$$

For $m = 0$ and $V' = 1$:

$$\Phi_\sigma(\boldsymbol{x}) = N_\sigma \Big[J_0(\tfrac{1}{4}r^2) + \boldsymbol{\alpha}\cdot\boldsymbol{x}\frac{i}{r}\, J_1(\tfrac{1}{4}r^2)\Big]$$

J_0 and J_1 are Bessel functions.

$$\hat{P}^\mu\, |\sigma\rangle = 0$$

State has *vanishing four-momentum* in any frame.
It may mix with the perturbative vacuum.
This spontaneously breaks chiral invariance.

A chiral condensate (m = 0)

PRELIMINARY

A chiral condensate vacuum ansatz:

$$|\chi\rangle = \exp(\hat{\sigma})\,|0\rangle \quad \text{for which} \quad \langle\chi|\bar{\psi}\psi\,|\chi\rangle = 4N_\sigma$$

An infinitesimal chiral rotation of the condensate generates a pion

$$U_\chi(\beta) = \exp\left[i\beta\int d\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\gamma_5\psi(\boldsymbol{x})\right] \qquad U_\chi(\beta)\,|\chi\rangle = (1-2i\beta\,\hat{\pi}\,|\chi\rangle$$

where $\hat{\pi}$ is the massless 0^{-+} state with wave function $\Phi_\pi = \gamma_5\Phi_\sigma$

⇒ An explicit realisation of the features we expect for a chiral condensate.

Bound state comments (1)

The mantra: Hadrons are non-perturbative could be unfounded.

– It's important: PQCD and LQCD are the main first principle tools in the SM.
– Based(?) on confinement and CSB being absent from Feynman diagrams.
– Feynman diagrams assume free quarks and gluons as boundary conditions.

The wave fn's of QED atoms have non-perturbative features: $O(\alpha^\infty)$.
Nonetheless, atomic binding energies are calculated with high accuracy

Example: Hyperfine splitting in Positronium

G. S. Adkins,
Hyperfine Interact. **233** (2015) 59

$$\begin{aligned}\Delta\nu_{QED} \;=\; & m_e\alpha^4\left\{\frac{7}{12}-\frac{\alpha}{\pi}\left(\frac{8}{9}+\frac{\ln 2}{2}\right)\right. \\ & +\frac{\alpha^2}{\pi^2}\left[-\frac{5}{24}\pi^2\ln\alpha+\frac{1367}{648}-\frac{5197}{3456}\pi^2+\left(\frac{221}{144}\pi^2+\frac{1}{2}\right)\ln 2-\frac{53}{32}\zeta(3)\right] \\ & \left.-\frac{7\alpha^3}{8\pi}\ln^2\alpha+\frac{\alpha^3}{\pi}\ln\alpha\left(\frac{17}{3}\ln 2-\frac{217}{90}\right)+\mathcal{O}\left(\alpha^3\right)\right\} = 203.39169(41)\ \text{GHz}\end{aligned}$$

$\Delta\nu_{EXP} = 203.394 \pm .002$ GHz

Bound state comments (2)

Hadrons are bound states of QCD (as shown by lattice calculations).

– The only strongly bound states in Nature (need not mean $\alpha_s \gg 1$)
– Should consider states with any ***P*** (form factors, scattering)
– Relativistic description only if the rules of QFT are obeyed

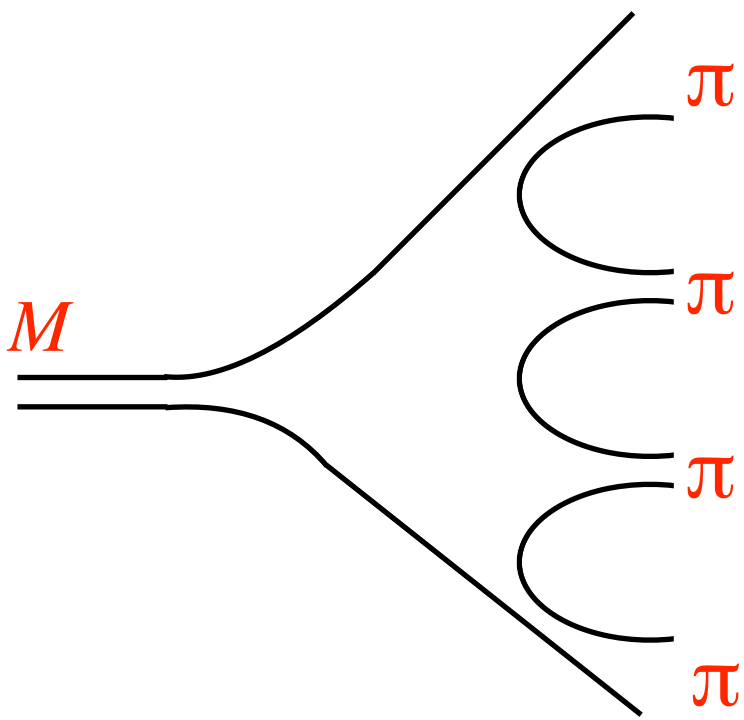
Steven Weinberg, in Preface to Vol. I of "The Quantum Theory of Fields":

> The point of view of this book is that quantum field theory is the way it is because (aside from theories like string theory that have an infinite number of particle types) it is the only way to reconcile the principles of quantum mechanics (including the cluster decomposition property) with those of special relativity.

Bound state comments (3)

Hadron data invites a perturbative approach

- Hadrons have valence quark quantum numbers (cf. QED in D = 1+1)
- Atomic features of quarkonia (Cornell potential, incl. confinement)
- Selection rules such as OZI $\phi(1020) \to K\bar{K}, \not\to \pi\pi\pi$
- Duality and dual diagrams



Only quark lines

Confining interaction

No transverse gluons