Bound states and Perturbation theory

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Hadrons differ from atoms

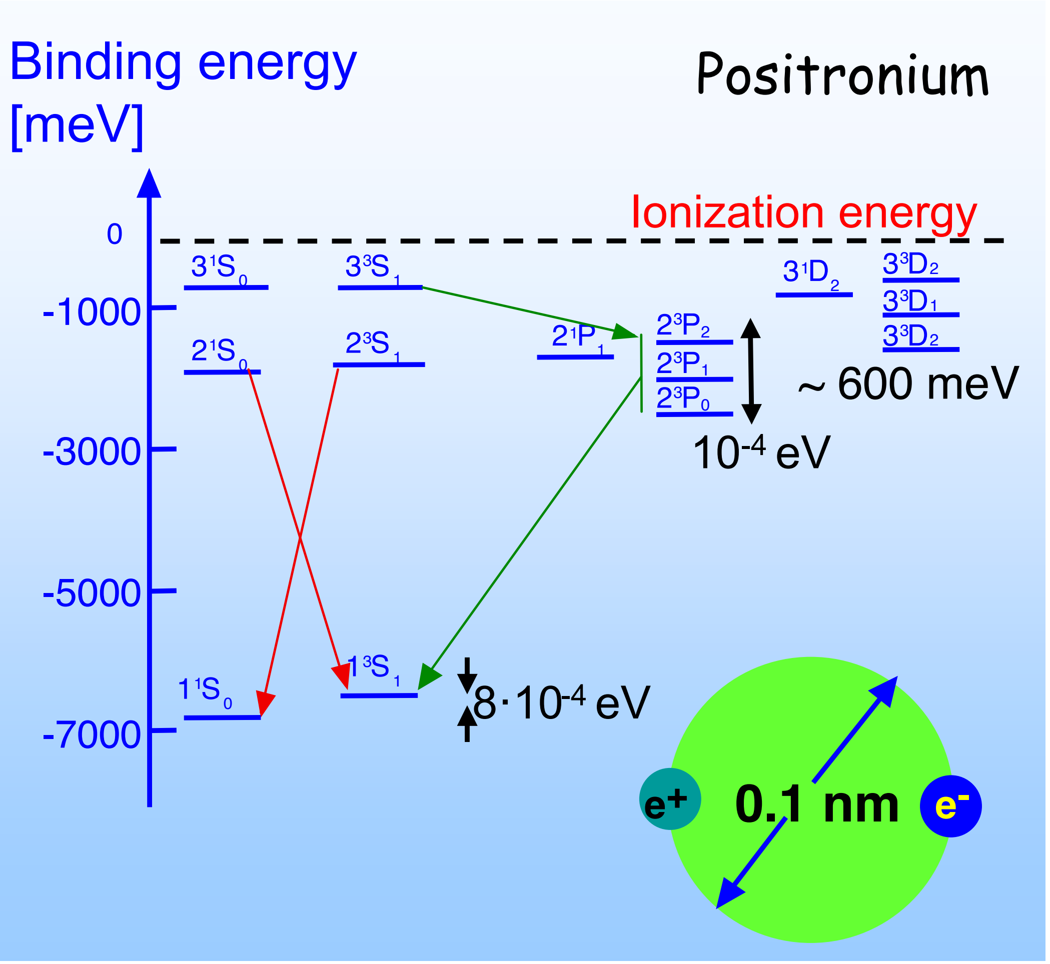
- They are strongly bound $M_p >> 2\, m_u + m_d$
- Color confining
- Chiral symmetry breaking

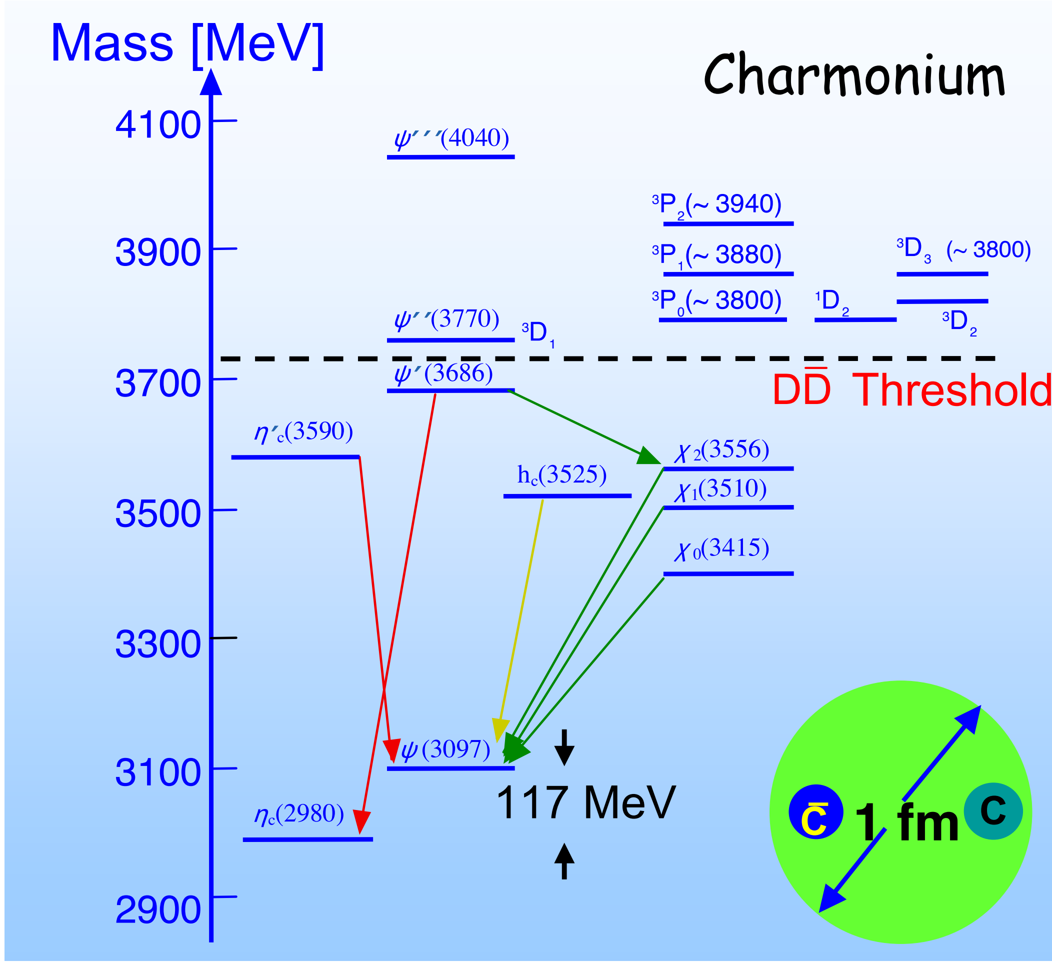
Yet:

- Heavy quarkonia are similar to atoms
- Light hadrons have $q\bar{q}$ and qqq quantum numbers (No $\bar{q}$, g dof's)
- Intriguing regularities: duality, OZI rule, …

PH 1807.05598 and refs. therein

Similarity of quarkonia and atoms



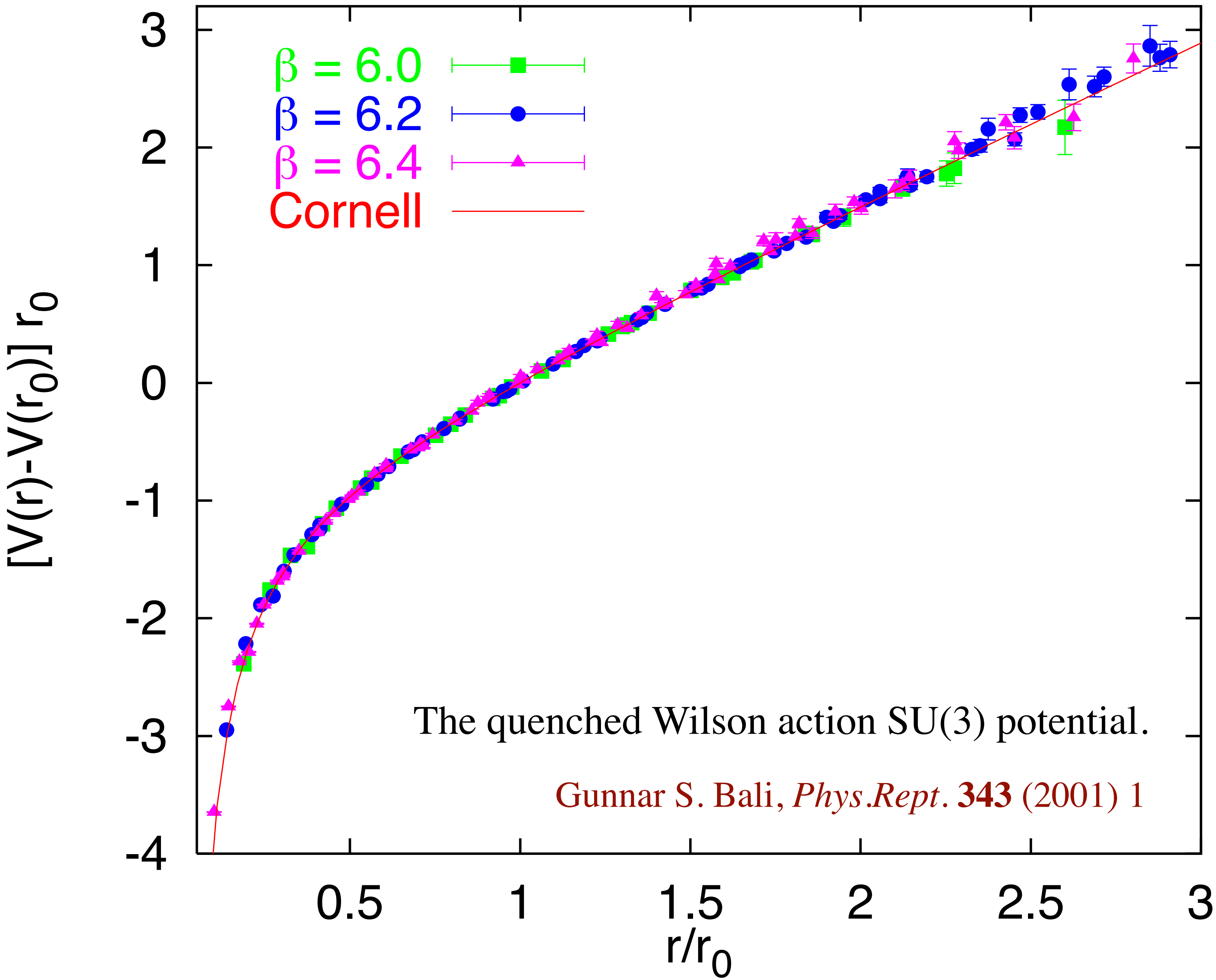


$$V(r) = -\frac{\alpha}{r}$$

$$V(r) = c\,r - \frac{4}{3}\frac{\alpha_s}{r}$$

"The J/ψ is the Hydrogen atom of QCD"

Linear Cornell potential agrees with Lattice QCD



Hadron spectrum is too(?) simple

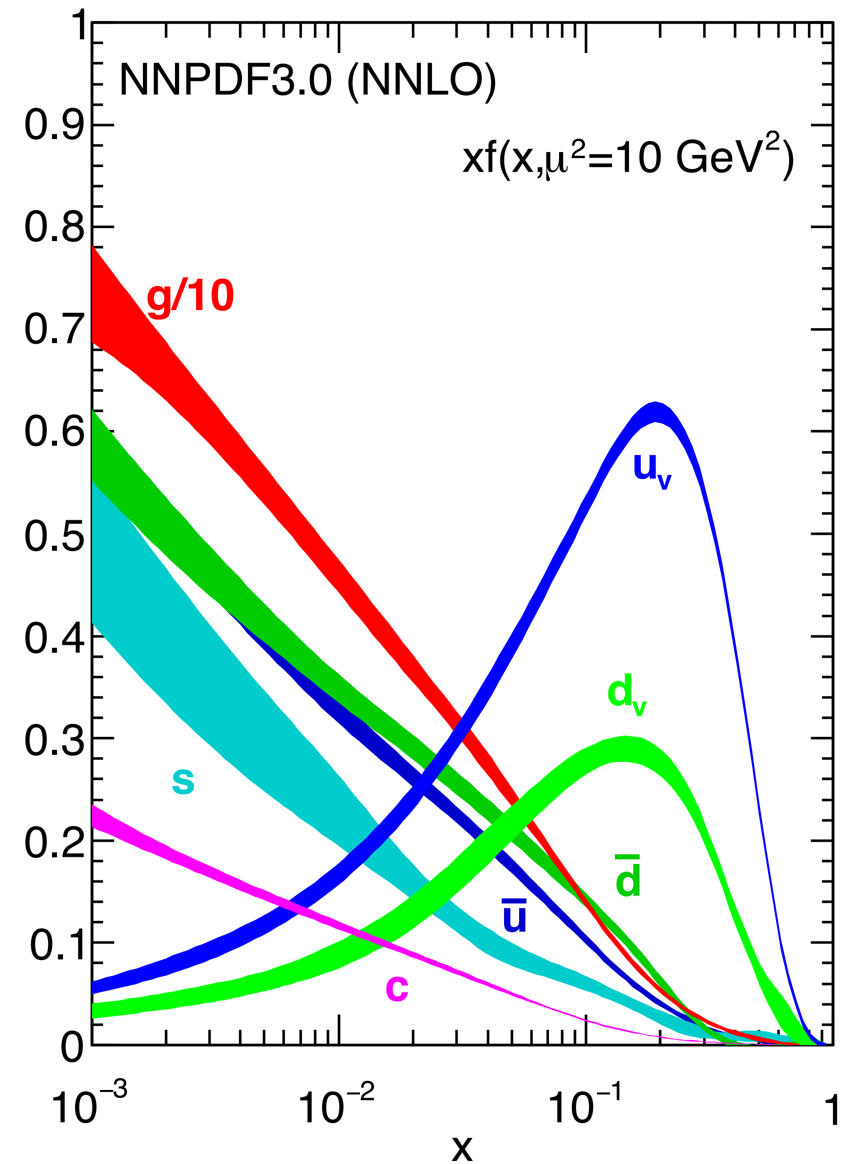
Why only $q\bar{q}$ and qqq quantum numbers?

The sea quarks and gluons are not manifest in hadron spectra

Cf. relative (rotational, vibrational) motions of atomic and nuclear constituents.

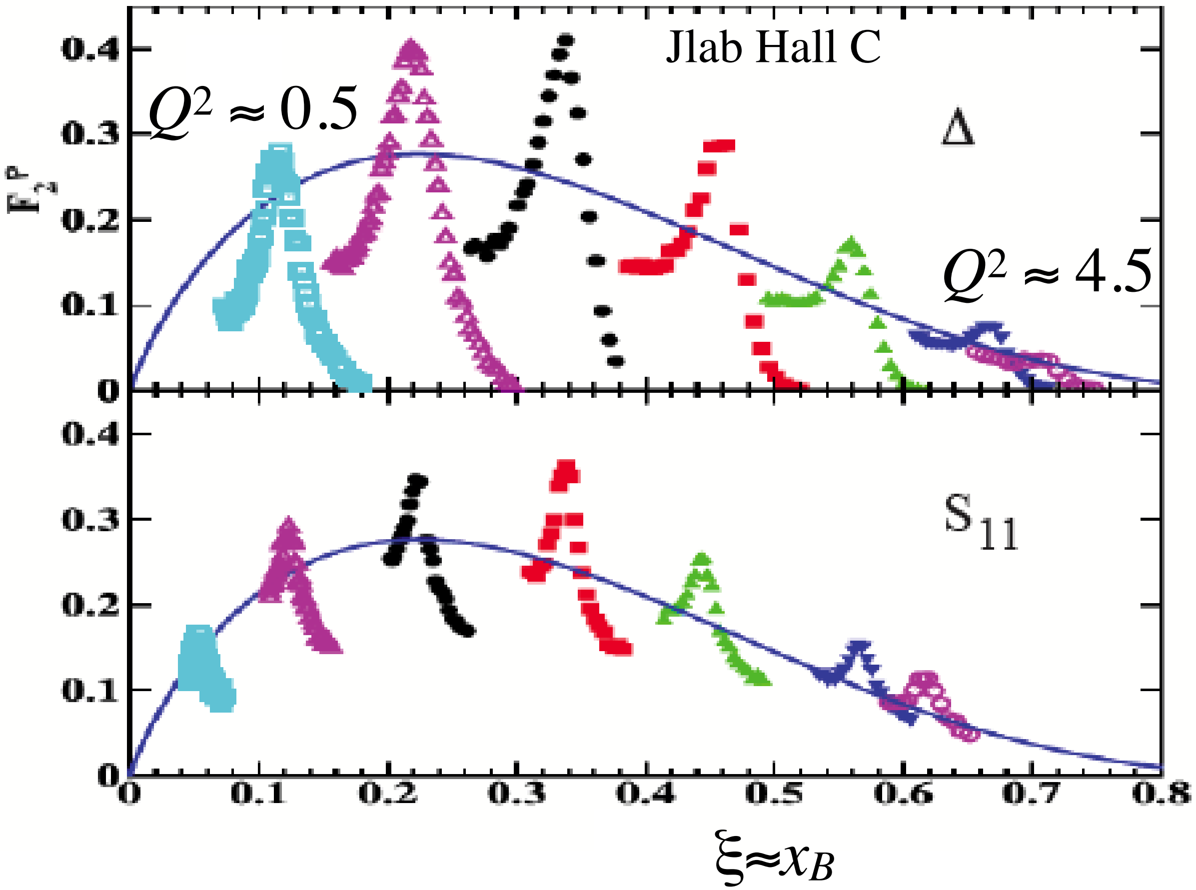
⇒ Relativistic effect
Cf. Dirac bound states

A relativistic wave function for valence quarks implies a $q\bar{q}$ sea.



Bloom-Gilman Duality

W. Melnitchouk et al, Phys. Rep. 406 (2005) 127

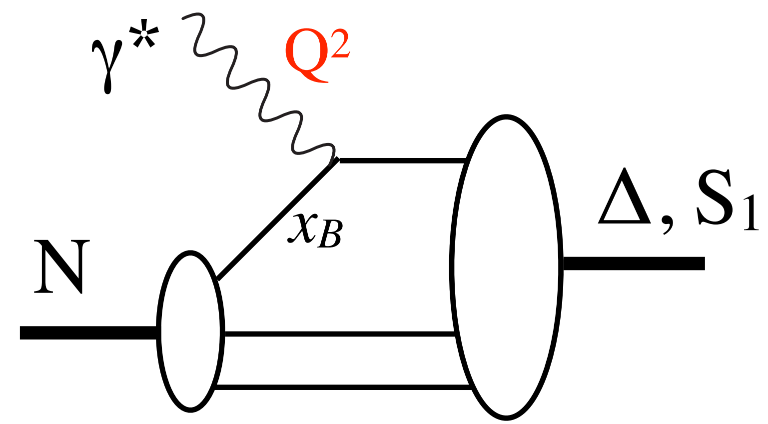


Resonance contributions

$ep \to eN^*$

build DIS scaling in

$ep \to eX$



$$m_{N^*}^2 = m_N^2 + Q^2\left(\frac{1}{x_B} - 1\right)$$

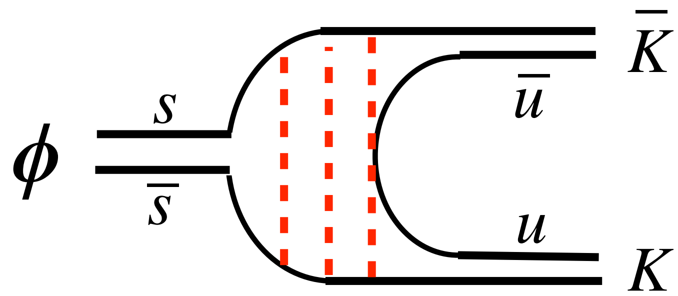
Scattering dynamics is built into hadron wave functions.

We must understand relativistic bound states in motion.

The OZI rule

Connected diagrams: Unsuppressed, string breaking from confining potential

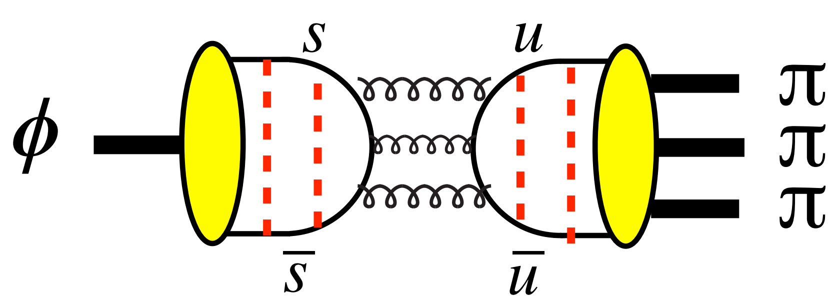
$\phi(1020) \to K\bar{K}$



ΔE	Br
26 MeV	83.1 %

Disconnected, perturbative diagrams are suppressed

$\phi(1020) \not\to \pi\pi\pi$



610 MeV 15.3 %

This suggests that perturbative corrections are small even in the soft regime.

QCD bound states

How are atoms treated in perturbative QED?

Can the same method be applied to QCD?

Use temporal gauge: $A^0 = 0$

Confinement: A novel boundary condition in Gauss' law.

Relativistic binding with PT for hadrons (not for atoms).

Chiral Symmetry Breaking: Massless bound states

QED bound states are (non)perturbative

- There is no Positronium pole in any $e^+e^- \to e^+e^-$ Feynman diagram
- Atomic binding energies do have a perturbative expansion in α

Example: Hyperfine splitting in Positronium

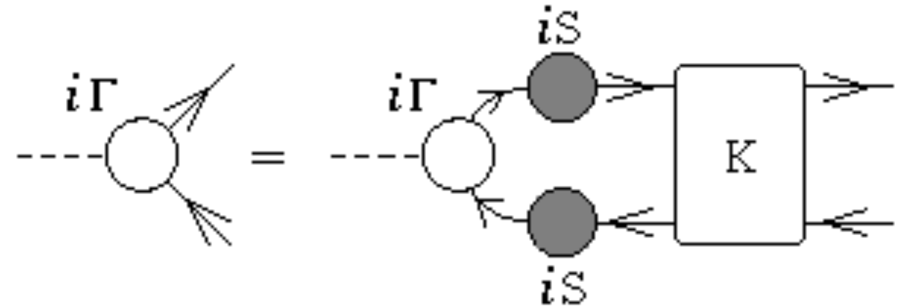
G. S. Adkins,
Hyperfine Interact. **233** (2015) 59

$$\Delta\nu_{QED} = m_e\alpha^4\left\{\frac{7}{12} - \frac{\alpha}{\pi}\left(\frac{8}{9} + \frac{\ln 2}{2}\right) + \frac{\alpha^2}{\pi^2}\left[-\frac{5}{24}\pi^2\ln\alpha + \frac{1367}{648} - \frac{5197}{3456}\pi^2 + \left(\frac{221}{144}\pi^2 + \frac{1}{2}\right)\ln 2 - \frac{53}{32}\zeta(3)\right] - \frac{7\alpha^3}{8\pi}\ln^2\alpha + \frac{\alpha^3}{\pi}\ln\alpha\left(\frac{17}{3}\ln 2 - \frac{217}{90}\right) + \mathcal{O}\left(\alpha^3\right)\right\} = 203.39169(41)\ \text{GHz}$$

$\Delta\nu_{EXP} = 203.394 \pm .002$ GHz

- Binding energy is perturbative in α and log(α) (measurable)
- Wave function $\psi(r) \propto \exp(-m\alpha r)$ is of $\mathcal{O}(\alpha^\infty)$ (gauge dependent)

Developments in bound state QED



- 1951: Salpeter & Bethe

  Perturbatively expand propagators S and kernel K
  Explicit Lorentz covariance
  No analytic solution even at lowest order in S and K

- 1975: Caswell & Lepage: BS is not unique: ∞ # of equivalent equations, $S \leftrightarrow K$

  We may choose to expand around Schrödinger atoms
  Give up explicit boost invariance

- 1986: Caswell & Lepage NRQED: Effective NR field theory

  Expand QED action in powers of ∇/m_e
  Choose to start from Schrödinger atoms

The perturbative S-matrix

Feynman diagrams are derived in the Interaction Picture:

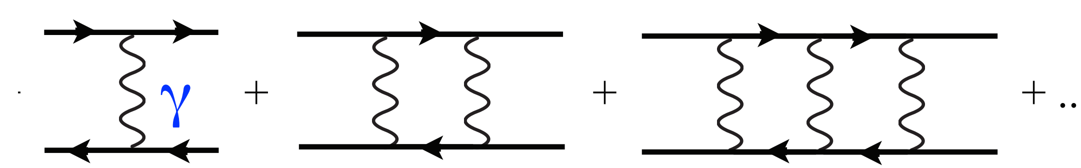
$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \qquad\qquad \mathcal{H}_0\,|i\rangle_{in} = E_i\,|i\rangle_{in}$$

$$S_{fi} = {}_{out}\langle f, t\to\infty|\left\{\mathrm{T}\exp\left[-i\int_{-\infty}^{\infty} dt\,\mathcal{H}_I(t)\right]\right\}|i, t\to -\infty\rangle_{in}$$

Formally exact expression, provided the *in*- and *out*-states have a non-vanishing overlap with the the physical i, f states.

Bound states have no overlap with free *in*- and *out*-states at $t = \pm\,\infty$

No Feynman diagram has a bound state pole.



Expanding around free states is inappropriate for bound states.

Quarks and gluons are not even physical states (*cf*. D=1+1).

Hamiltonian approach: The classical field in QED

$$\frac{\delta \mathcal{S}_{QED}}{\delta \hat{A}^0(t,\boldsymbol{x})} = 0 \quad \Rightarrow \quad -\boldsymbol{\nabla}^2 \hat{A}^0(t,\boldsymbol{x}) = e\psi^\dagger(t,\boldsymbol{x})\psi(t,\boldsymbol{x})$$

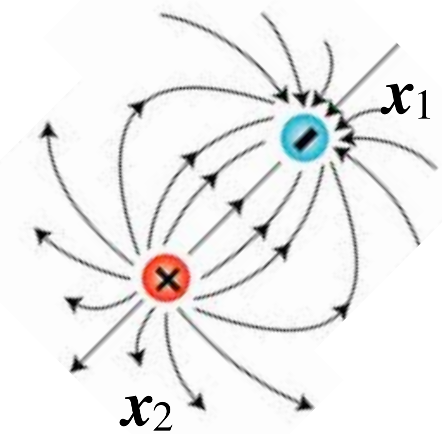
$$\hat{A}^0(t,\boldsymbol{x}) = \int d^3\boldsymbol{y}\, \frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi^\dagger\psi(t,\boldsymbol{y})$$

The classical field is the expectation value of $\hat{A}^0$ in the state

$$|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle = \bar{\psi}(t,\boldsymbol{x}_1)\psi(t,\boldsymbol{x}_2)\,|0\rangle$$

$$\frac{\langle \boldsymbol{x}_1,\boldsymbol{x}_2|\, e\hat{A}^0(\boldsymbol{x})\,|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle}{\langle \boldsymbol{x}_1,\boldsymbol{x}_2|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle} = \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_2|}$$

$$\equiv eA^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2)$$



Note: • A^0 is determined instantaneously for all $\boldsymbol{x}$

• It depends on $\boldsymbol{x}_1, \boldsymbol{x}_2$ $\Rightarrow$ The charges determine the field

• $eA^0(\boldsymbol{x}_1) = -eA^0(\boldsymbol{x}_2) = -\dfrac{\alpha}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$ is the classical $-\alpha/r$ potential

Canonical quantisation in temporal gauge: $A^0 = 0$

Avoids problem due to the missing conjugate field for A^0

$$E^i = F^{i0} = -\partial_0 A^i \qquad \text{conjugate to} \qquad A_i = -A^i \qquad (i = 1,2,3)$$

$$\left[E^i(t,\boldsymbol{x}), A^j(t,\boldsymbol{y})\right] = i\delta^{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) \qquad \left\{\psi_\alpha^\dagger(t,\boldsymbol{x}), \psi_\beta(t,\boldsymbol{y})\right\} = \delta_{\alpha\beta}\,\delta(\boldsymbol{x}-\boldsymbol{y})$$

$$H = \int d\boldsymbol{x}\left[\tfrac{1}{2}\boldsymbol{E}_L^2 + \tfrac{1}{2}\boldsymbol{E}_T^2 + \tfrac{1}{4}F^{ij}F^{ij} + \psi^\dagger(-i\alpha^i\partial_i - e\alpha^i A^i + m\gamma^0)\psi\right]$$

Gauss' operator does not vanish: $G(x) \equiv \dfrac{\delta\mathcal{S}}{\delta A^0(x)} = \partial_i E_L^i(x) - e\psi^\dagger\psi(x)$

$G(x)$ generates *time*-independent gauge transformations, allowed by $A^0 = 0$

Fix the gauge completely by constraining physical states: $G(x)\,|phys\rangle = 0$

This constrains $E_L(x)$ for each state, in effect imposing Gauss' law.

Schrödinger equation for Positronium

$$G(x)\,|phys\rangle = 0 \quad \Longrightarrow \quad \begin{aligned} &\partial_i E_L^i(t,\boldsymbol{x})\,|phys\rangle = e\psi^\dagger\psi(t,\boldsymbol{x})\,|phys\rangle \\ &E_L^i(t,\boldsymbol{x})\,|phys\rangle = -\partial_i^x \int d\boldsymbol{y}\,\frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi^\dagger\psi(t,\boldsymbol{y})\,|phys\rangle \end{aligned}$$

For the component of Positronium with an electron at $\boldsymbol{x}_1$ and a positron at $\boldsymbol{x}_2$:

$$\big|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\big\rangle = \bar\psi_\alpha(\boldsymbol{x}_1)\psi_\beta(\boldsymbol{x}_2)\,|0\rangle$$

$$E_L^i\,\big|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\big\rangle = -\partial_i^x\,\frac{e}{4\pi}\Big(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\Big)\big|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\big\rangle$$

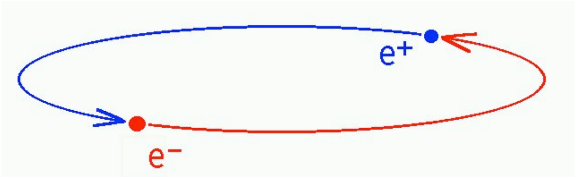
The instantaneous Hamiltonian $H_V \equiv \frac{1}{2}\int d\boldsymbol{x} E_L^i E_L^i(\boldsymbol{x})$ gives the classical potential:

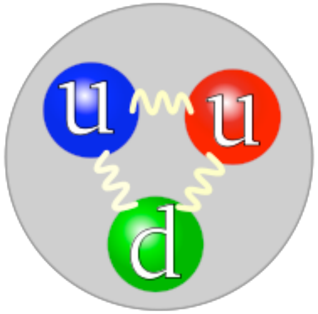
$$H_V\,\big|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\big\rangle = -\frac{\alpha}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}\big|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\big\rangle$$

The Schrödinger equation follows when the kinetic energy term in *H* is added.

A classical field in QCD?

Global gauge invariance allows a classical gauge field for neutral atoms, but not for color singlet hadrons in QCD





$$A^0 = \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_2|}$$

Positronium

QED

$$A_a^0(\boldsymbol{x}) = 0$$

Proton

QCD

However:

The classical gluon field is non-vanishing for each color component C of the state

$$A_a^0(\boldsymbol{x};C) \neq 0 \qquad \sum_C A_a^0(\boldsymbol{x};C) = 0$$

Note: This possibility does not exist for quarks and gluons.

Temporal gauge in QCD: $A_a{}^0 = 0$

Gauss' operator $$G_a(x) \equiv \frac{\delta \mathcal{S}}{\delta A_a^0(x)} = \partial_i E_a^i(x) + g f_{abc} A_b^i E_c^i - g\psi^\dagger T^a \psi(x)$$

generates time-independent gauge transformations, which keep $A_a^0 = 0$

The gauge is fully defined (in PT) by the constraint $G_a(x)\,|phys\rangle = 0$

$$\Rightarrow \qquad \partial_i E_{L,a}^i(\boldsymbol{x})\,|phys\rangle = g\big[-f_{abc}A_b^i E_c^i + \psi^\dagger T^a \psi(\boldsymbol{x})\big]\,|phys\rangle$$

In QED one solves for $\boldsymbol{E}_L$ requiring $\boldsymbol{E}_L(\boldsymbol{x}) \to 0$ for $|\boldsymbol{x}| \to \infty$

In QCD the (globally) color singlet bound states

$$|M\rangle = \sum_{A,B;\alpha,\beta} \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar\psi_\alpha^A(t=0,\boldsymbol{x}_1)\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta^B(t=0,\boldsymbol{x}_2)\,|0\rangle$$

cannot generate a classical octet field at any $\boldsymbol{x}$. Hence for a given quark color A homogeneous solutions, which do not vanish at spatial infinity, may be considered

Including a homogeneous solution for $E_L^{a,i}$

$$E_{L,a}^{i}(\boldsymbol{x})\left|phys\right\rangle = -\partial_i^x \int d\boldsymbol{y}\Big[\kappa\, \boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big]\mathcal{E}_a(\boldsymbol{y})\left|phys\right\rangle$$

where $\mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A_b^i E_c^i(\boldsymbol{y}) + \psi^\dagger T^a \psi(\boldsymbol{y})$ and $\kappa \neq \kappa(\boldsymbol{x},\boldsymbol{y})$

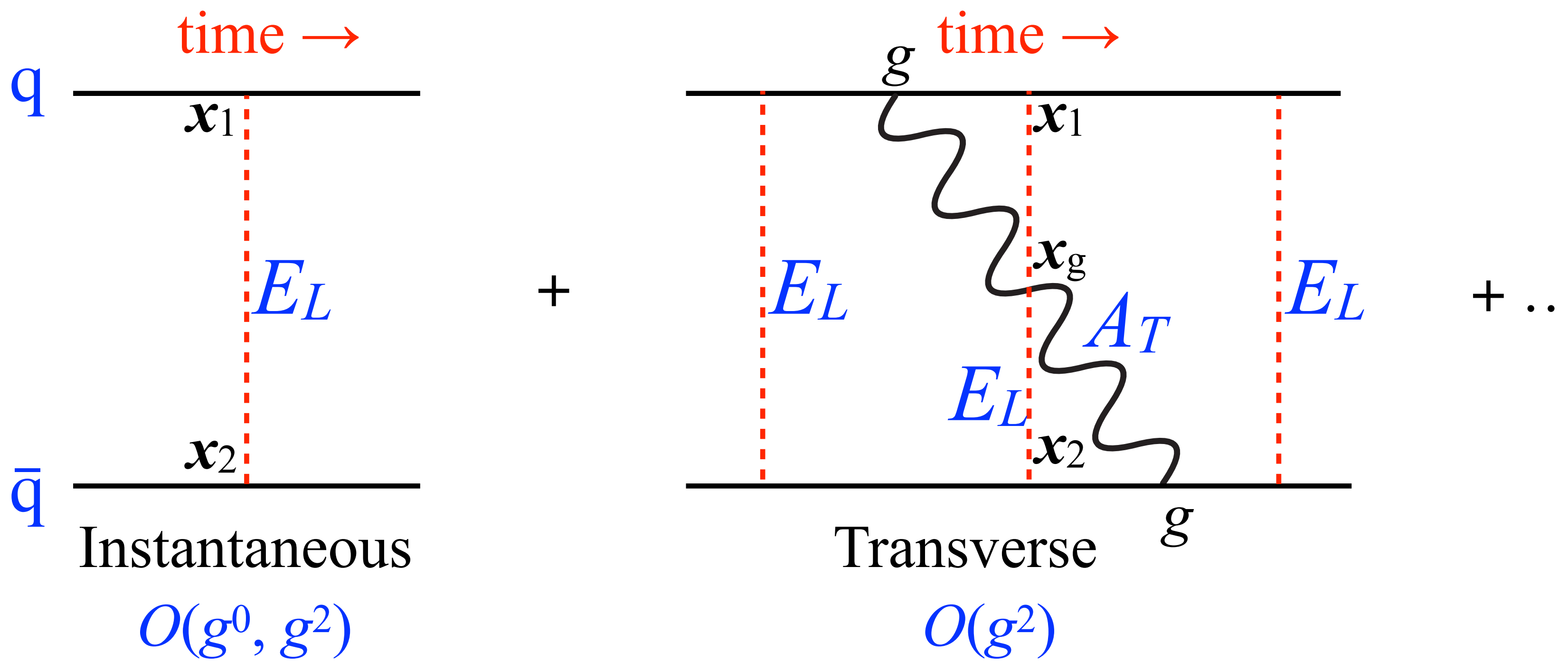
The linear dependence on $\boldsymbol{x}$ makes $\boldsymbol{E}_L$ independent of $\boldsymbol{x}$, which is required by translation invariance. The field energy density is constant and $\propto \varkappa$.

The $\boldsymbol{E}_L$ contribution to the QCD Hamiltonian is

$$H_V = \int d\boldsymbol{y} d\boldsymbol{z}\Big\{\boldsymbol{y}\cdot\boldsymbol{z}\Big[\tfrac{1}{2}\kappa^2\int d\boldsymbol{x} + g\kappa\Big] + \tfrac{1}{2}\frac{\alpha_s}{|\boldsymbol{y}-\boldsymbol{z}|}\Big\}\mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z})$$

The field energy $\propto$ volume of space is irrelevant only if it is universal. This relates the normalisation $\varkappa$ of all Fock components, leaving only an overall scale Λ as a parameter.

Perturbative expansion = Fock state expansion



The instantaneous Hamiltonian H_V determines the potential energy.

The $q\bar{q}A_T$ *etc*. terms in H determine couplings between Fock states.

The q and A_T kinetic terms determine the time evolution.

Essential: The $O(g^0)$ potential does not create particles.

Gauss' law is a constraint, not an operator identity.

Examples: Fock state potentials (I)

$q\bar{q}$: $$H_V\,|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle = V_{q\bar{q}}\,|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle$$

$$V_{q\bar{q}} = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - C_F\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$$

$qg\bar{q}$: $$V^{(0)}_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) = \frac{\Lambda^2}{\sqrt{C_F}}\,d_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2)$$

$$d_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) \equiv \sqrt{\tfrac{1}{4}(N-2/N)(\boldsymbol{x}_1-\boldsymbol{x}_2)^2 + N(\boldsymbol{x}_g-\tfrac{1}{2}\boldsymbol{x}_1-\tfrac{1}{2}\boldsymbol{x}_2)^2}$$

$$V^{(1)}_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) = \tfrac{1}{2}\,\alpha_s\Big[\frac{1}{N}\,\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} - N\Big(\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_g|}+\frac{1}{|\boldsymbol{x}_2-\boldsymbol{x}_g|}\Big)\Big]$$

When q and g coincide: $$V^{(0)}_{qgq}(\boldsymbol{x}_1=\boldsymbol{x}_g,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| = V^{(0)}_{q\bar{q}}$$

$$V^{(1)}_{qgq}(\boldsymbol{x}_1=\boldsymbol{x}_g,\boldsymbol{x}_2) = V^{(1)}_{q\bar{q}}$$

Fock state potentials (II)

qqq :

$$V_{qqq} = \Lambda^2 d_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) - \frac{2}{3}\alpha_s\Big(\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} + \frac{1}{|\boldsymbol{x}_2 - \boldsymbol{x}_3|} + \frac{1}{|\boldsymbol{x}_3 - \boldsymbol{x}_1|}\Big)$$

$$d_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \equiv \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + (\boldsymbol{x}_2 - \boldsymbol{x}_3)^2 + (\boldsymbol{x}_3 - \boldsymbol{x}_1)^2}$$

gg :

$$V_{gg} = \sqrt{\frac{N}{C_F}}\,\Lambda^2\,|\boldsymbol{x}_1 - \boldsymbol{x}_2| - N\,\frac{\alpha_s}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$$

This agrees with the $qg\bar{q}$ potential where the quarks coincide:

$$V_{gg}(\boldsymbol{x}, \boldsymbol{x}_g) = V_{qg\bar{q}}(\boldsymbol{x}, \boldsymbol{x}_g, \boldsymbol{x})$$

It is straightforward to work out the instantaneous potential for any Fock state.

$q\bar{q}$ bound states

Express a $q\bar{q}$ bound state in terms of a wave function Φ,

$$|M\rangle = \sum_{A,B;\alpha,\beta} \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}_\alpha^A(t=0,\boldsymbol{x}_1)\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta^B(t=0,\boldsymbol{x}_2)\,|0\rangle$$

The bound state condition $H\,|M\rangle = M\,|M\rangle$ gives, with $H\,|0\rangle = 0$

and keeping only the $O(\alpha_s{}^0)$ terms in H,

$$\big[i\gamma^0\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0\big]\Phi(\boldsymbol{x}) + \Phi(\boldsymbol{x})\big[i\gamma^0\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}} - m\gamma^0\big] = \big[M - V(|\boldsymbol{x}|)\big]\Phi(\boldsymbol{x})$$

where $\boldsymbol{x} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $V(|\,\boldsymbol{x}\,|) = V'|\,\boldsymbol{x}\,| = \Lambda^2|\,\boldsymbol{x}\,|$.

In the non-relativistic limit ($m \gg \Lambda$) this reduces to the Schrödinger equation, and we may add the instantaneous gluon exchange potential.

$\Longrightarrow$ The successful quarkonium phenomenology with the Cornell potential.

Relativistic $q\bar{q}$ bound states

$$i\boldsymbol{\nabla}\cdot\left\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\right\}+m\left[\gamma^0,\Phi(\boldsymbol{x})\right]=\left[M-V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})$$

Expanding the 4×4 wave function in a basis of 16 Dirac structures $\Gamma_i(\boldsymbol{x})$

$$\Phi(\boldsymbol{x})=\sum_i\Gamma_i(\boldsymbol{x})F_i(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

we may use rotational, parity and charge conjugation invariance to determine which $\Gamma_i(\boldsymbol{x})$ may occur for a state of given j^{PC}:

0^{-+} trajectory	$[s=0,\ \ell=j]:$	$-\eta_P=\eta_C=(-1)^j$	$\gamma_5,\ \gamma^0\gamma_5,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L}$
0^{--} trajectory	$[s=1,\ \ell=j]:$	$\eta_P=\eta_C=-(-1)^j$	$\gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \boldsymbol{\alpha}\cdot\boldsymbol{L},\ \gamma^0\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$
0^{++} trajectory	$[s=1,\ \ell=j\pm 1]:$	$\eta_P=\eta_C=+(-1)^j$	$1,\ \boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$
0^{+-} trajectory	[exotic] :	$\eta_P=-\eta_C=(-1)^j$	$\gamma^0,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$

$\Rightarrow$ There are no solutions for quantum numbers that would be exotic in the quark model (despite the relativistic dynamics)

Example: 0⁻⁺ trajectory wf's

$$\Phi_{-+}(\boldsymbol{x}) = \Big[\frac{2}{M-V}(i\boldsymbol{\alpha}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0) + 1\Big]\gamma_5\, F_1(r) Y_{j\lambda}(\hat{\boldsymbol{x}})$$

$$\eta_P = (-1)^{j+1}$$

$$\eta_C = (-1)^j$$

Radial equation: $F_1'' + \Big(\frac{2}{r} + \frac{V'}{M-V}\Big)F_1' + \Big[\frac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2}\Big]F_1 = 0$

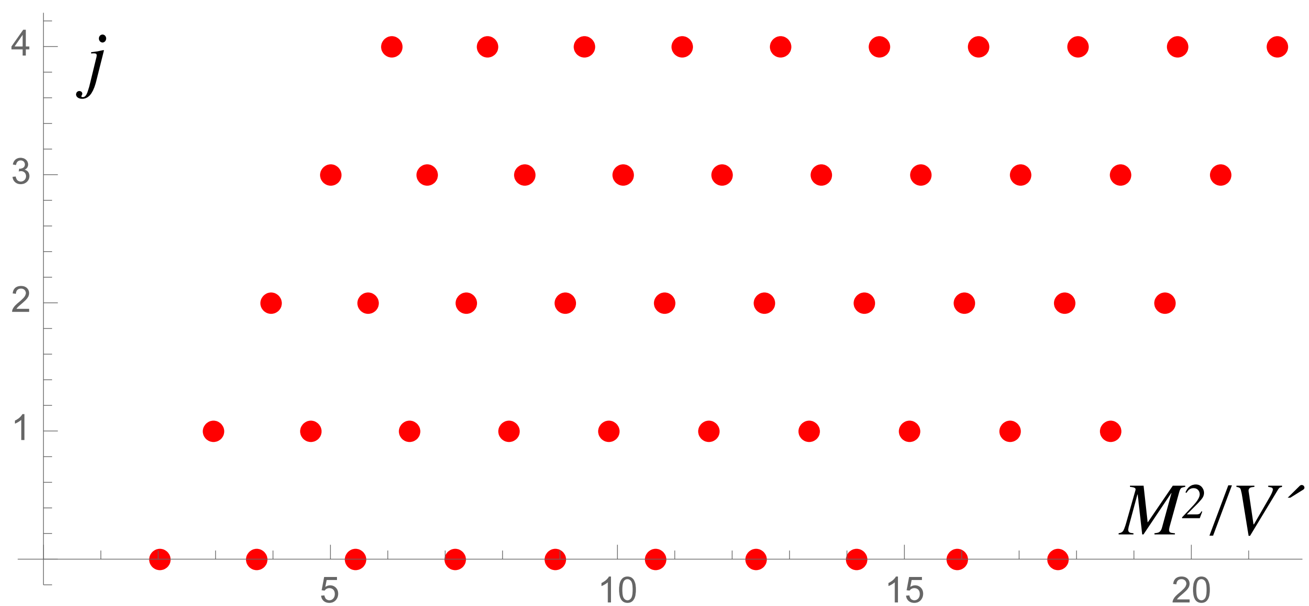
Local normalizability at $r = 0$ and at $V(r) = M$ determines the discrete M

$m = 0$

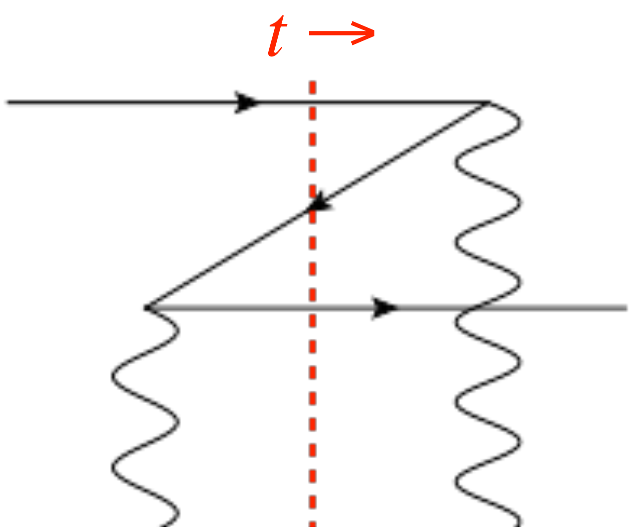
Mass spectrum:

Linear Regge trajectories with daughters

Spectrum similar to dual models



Sea quark contributions



On general grounds, and as seen in PT, relativistic interactions imply contributions from *Z*-diagrams, implying higher Fock states.

This effect is included in the wave function Φ, It is manifest in the behavior, for V = V´|***x***| ,

$$\lim_{\boldsymbol{x}\to\infty} |\Phi(\boldsymbol{x})|^2 = const.$$

The Dirac wave function has the same behaviour for a linear potential.

The asymptotically constant norm reflects, via duality, pair production as the linear potential *V*(| ***x*** |) increases.

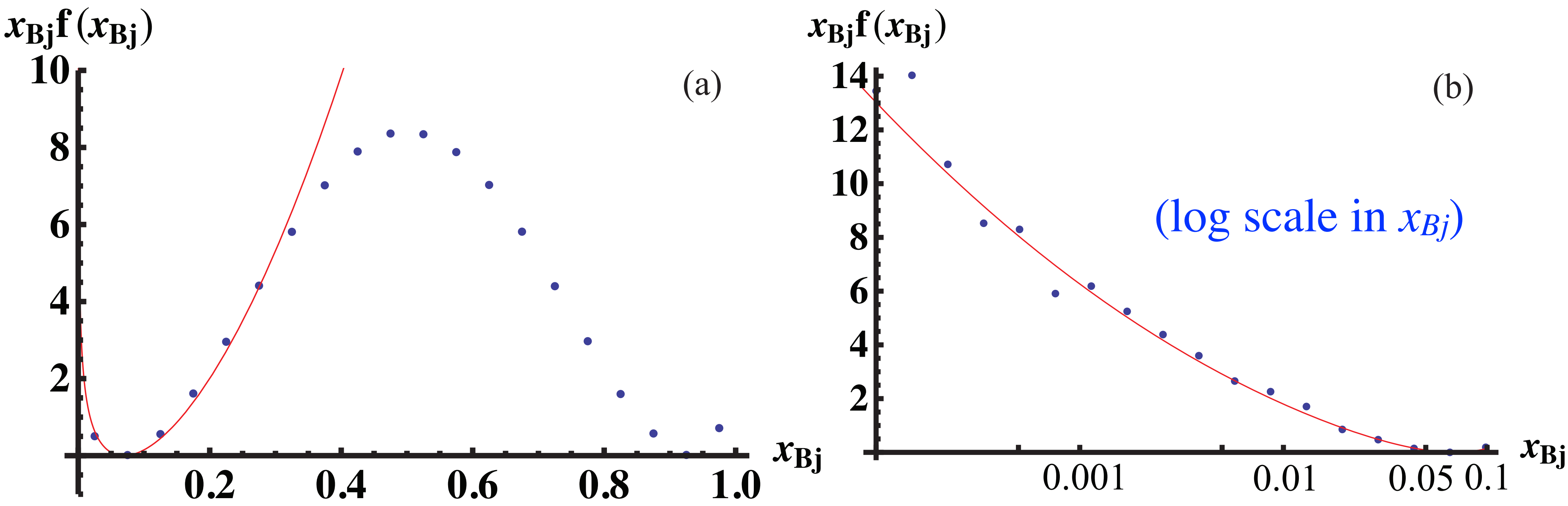
These sea quarks show up in the parton distribution measured in DIS.

Parton distributions have a sea component

In D=1+1 dimensions the sea component is prominent at low m/e :

$$m/e = 0.1$$

D. D. Dietrich, PH, M. Järvinen
arXiv 1212.4747

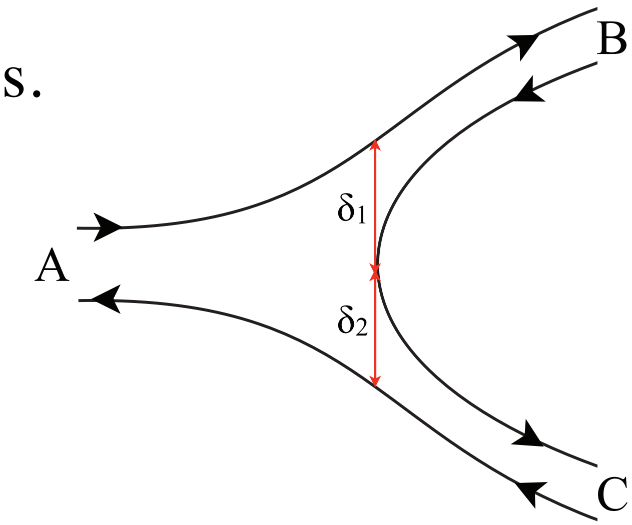


The red curve is an analytic approximation, valid in the $x_{Bj} \to 0$ limit.

Note: Enhancement at low x is due to bd (sea), not to $b^\dagger d^\dagger$ (valence) component.

Decays and hadron loops

The bound state equation determines zero-width states.

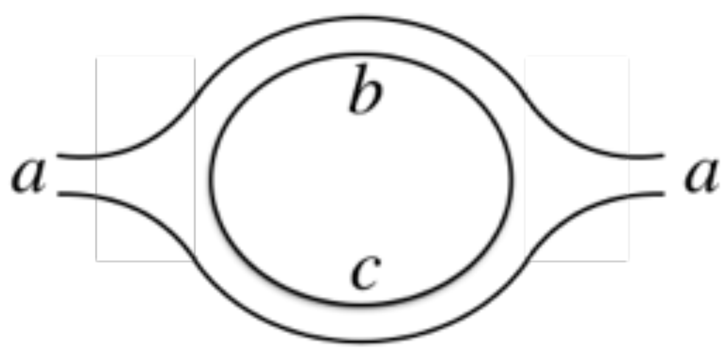


There is an $\mathcal{O}\left(1/\sqrt{N_C}\right)$ coupling between the states: string breaking

$$\langle B,C|A\rangle =$$

$$-\frac{(2\pi)^3}{\sqrt{N_C}}\delta^3(\boldsymbol{P}_A-\boldsymbol{P}_B-\boldsymbol{P}_C)\int d\boldsymbol{\delta}_1 d\boldsymbol{\delta}_2\, e^{i\boldsymbol{\delta}_1\cdot\boldsymbol{P}_C/2-i\boldsymbol{\delta}_2\cdot\boldsymbol{P}_B/2}\mathrm{Tr}\left[\gamma^0\Phi_B^\dagger(\boldsymbol{\delta}_1)\Phi_A(\boldsymbol{\delta}_1+\boldsymbol{\delta}_2)\Phi_C^\dagger(\boldsymbol{\delta}_2)\right]$$

When squared, this gives a $1/N_C$ hadron loop unitarity correction:



Unitarity should be satisfied at hadron level at each order of $1/N_C$.

Bound states in motion

A $q\bar{q}$ bound state with CM momentum $\boldsymbol{P}$ may be expressed as

$$|M,\boldsymbol{P}\rangle = \int dx_1\, dx_2\, \bar{\psi}(t=0,x_1)\, e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2}\, \Phi^{(\boldsymbol{P})}(x_1-x_2)\, \psi(t=0,x_2)\,|0\rangle$$

The potential is $\boldsymbol{P}$-independent, $V(\boldsymbol{x}) = V'|\boldsymbol{x}|$ so the BSE becomes

$$i\boldsymbol{\nabla}\cdot\left\{\boldsymbol{\alpha},\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\right\} - \tfrac{1}{2}\boldsymbol{P}\cdot\left[\boldsymbol{\alpha},\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\right] + m\left[\gamma^0,\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\right] = \left[E - V(\boldsymbol{x})\right]\Phi^{(\boldsymbol{P})}(\boldsymbol{x})$$

$\boldsymbol{P}$ breaks rotational symmetry: angular & radial dependence does not separate.

The solution for $\Phi^{(P)}(\boldsymbol{x})$ in D = 1+1 dimensions is not simply Lorentz contracting.

It provides a boundary condition at $\boldsymbol{x}_\perp = 0$ on $\Phi^{(P)}(\boldsymbol{x})$ in D = 3+1 dimensions.

This works only for a linear potential.

States with M = 0

We required the wave function to be normalizable at $r = 0$ and $V'r = M$

For $M = 0$ the two points coincide. Regular, massless solutions are found.

The massless 0^{++} meson "σ"

$$|\sigma\rangle = \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar{\psi}(\boldsymbol{x}_1)\, \Phi_\sigma(\boldsymbol{x}_1 - \boldsymbol{x}_2)\, \psi(\boldsymbol{x}_2)\, |0\rangle \equiv \hat{\sigma}\,|0\rangle$$

For $m = 0$ and $V' = 1$:

$$\Phi_\sigma(\boldsymbol{x}) = N_\sigma \Big[J_0(\tfrac{1}{4}r^2) + \boldsymbol{\alpha}\cdot\boldsymbol{x}\frac{i}{r}\, J_1(\tfrac{1}{4}r^2)\Big]$$

J_0 and J_1 are Bessel functions.

$\hat{P}^\mu\,|\sigma\rangle = 0$ State has *vanishing four-momentum* in any frame.
It may mix with the perturbative vacuum.
This spontaneously breaks chiral invariance.

A chiral condensate (m = 0)

Since | σ ⟩ has vacuum quantum numbers and zero momentum it can mix with the perturbative vacuum without violating Poincaré invariance

Ansatz: $|\chi\rangle = \exp(\hat{\sigma})\,|0\rangle$ implies $\langle\chi|\bar{\psi}\psi\,|\chi\rangle = 4N_\sigma$

An infinitesimal chiral rotation of the condensate generates a pion

$$U_\chi(\beta) = \exp\Big[i\beta\int d\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\gamma_5\psi(\boldsymbol{x})\Big] \qquad U_\chi(\beta)\,|\chi\rangle = (1-2i\beta\,\hat{\pi}\,|\chi\rangle$$

where $\hat{\pi}$ is the massless 0^{-+} state with wave function $\Phi_\pi = \gamma_5\Phi_\sigma$

Small quark mass: m > 0

When $m \neq 0$ the massless ($M_\sigma = 0$) sigma 0^{++} state has wave function

$$\Phi_\sigma(\boldsymbol{x}) = f_1(r) + i\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\,f_2(r) + i\,\boldsymbol{\gamma}\cdot\boldsymbol{x}\,g_2(r)$$

Radial functions are Laguerre fn's

An $M_\pi > 0$ pion 0^{-+} state has rest frame wave function

$$\Phi_\pi(\boldsymbol{x}) = \big[F_1(r) + i\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\,F_2(r) + \gamma^0\,F_4(r)\big]\gamma_5$$

$$F_4(0) = \frac{2m}{M}F_1(0)$$

$$F_1'' + \Big(\frac{2}{r} + \frac{1}{M-r}\Big)F_1' + \big[\tfrac{1}{4}(M-r)^2 - m^2\big]F_1 = 0$$

$$\langle\chi|j_5^\mu(x)\hat{\pi}\,|\chi\rangle = iP^\mu f_\pi\,e^{-iP\cdot x} \quad\Rightarrow\quad F_4(0) = \tfrac{1}{4}iM_\pi f_\pi$$

$$\langle\chi|\bar{\psi}(x)\gamma_5\psi(x)\,\hat{\pi}\,|\chi\rangle = -i\,\frac{M^2}{2m}\,f_\pi\,e^{-iP\cdot x} \quad\Rightarrow\quad F_1(0) = i\,\frac{M^2}{8m}\,f_\pi$$

CSB relations are satisfied for any P.

Final remarks

QED bound states have both perturbative and non-perturbative aspects.

Positronium may be derived from QED using temporal gauge: $A^0 = 0$

The solution of Gauss' law in QCD allows a homogeneous solution for color singlet states. This gives an $\mathcal{O}\left(\alpha_s^0\right)$ linear potential for $q\bar{q}$ states.

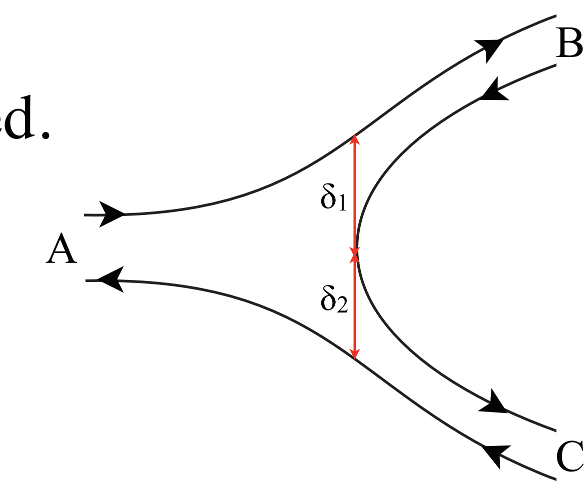
The quarkonium phenomenology is derived.

One obtains relativistic states with exact $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$.

Massless bound states allow spontaneous chiral symmetry breaking.

The perturbative and Fock expansions are closely related.

String breaking effects (decays, *etc*.) are calculable:



Hadron-level unitarity should be checked.

Many interesting studies & tests are waiting to be done!