Bound states in perturbation theory

Paul Hoyer 30 November 2021

In view of color confinement and strong binding, hadrons have unexpected similarities with atoms. This motivates to try QED methods for atoms also on QCD hadrons. Here I summarise a Hamiltonian method for QED which is applicable in any frame and adheres to the principles of quantum field theory. Its application to hadrons appears promising. A detailed description is found in [1] and a shorter summary in [2].

The similarities of hadrons and atoms are well-known:

- Heavy ($c\bar{c}$ and $b\bar{b}$) quarkonia are well described using the Schrödinger equation with a phenomenological, linear potential [3, 4]. This "Cornell" potential turned out to agree with the potential energy of static quarks calculated using lattice QCD [5]. It is remarkable that the novel effects of confinement can be thus simply described.

- Hadron quantum numbers are determined by their valence quark constituents ($q\bar{q}$ and qqq). Yet light (u, d, s) quark states are strongly bound, with excitations energies comparable to their masses. Hadron spectra show no sign of the expected abundance, due to the strong interaction, of quark pairs and gluon constituents.

- The valence Fock states appear to consist of fundamental ("current") quarks, rather than "constituent" quarks with a cloud of gluons and ~ 300 MeV effective masses. Decays such as $\pi^+ \to \mu^+\nu_\mu$ are unsuppressed ($f_\pi = 93$ MeV $\sim \Lambda_{QCD}$), and occur only from the valence $u\bar{d}$ Fock state without gluons[1].

The prevailing view is that perturbative QCD (PQCD) methods apply only to hard (short-distance) scattering processes, not to the soft interactions that form bound states. On the other hand, atomic properties are calculated with high precision using PQED. If hadrons were essentially non-perturbative their similarity to atoms would be "accidental".

Part of this dilemma originates from the definition of "non-perturbative". Functions with an essential singularity such as $\exp(-\alpha/m\,r)$ cannot be expanded in powers of α around $r = 0$ and are in this sense truly non-perturbative. On the other hand, functions like $\exp(-\alpha m\,r)$ are non-polynomial in α yet do have a perturbative expansion at $r = 0$. The solutions of the Schrödinger equation belong to the second category. Atoms may be (and are [6]) considered "non-perturbative": Feynman diagrams lack atomic poles, and atomic wave functions describe exponentially suppressed tunneling processes. Atomic binding energies can nevertheless be expanded in powers of α and $\log\alpha$. Hadrons may be "non-perturbative" similarly as atoms, not excluding a perturbative expansion.

Bound states are rarely discussed in modern textbooks on Quantum Field Theory. The topic is considered to involve some "art" [6], in stark contrast to the systematic expansion of scattering amplitudes in terms of Feynman diagrams.

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[1] This observation is due to Stan Brodsky (private communication).

In [1] I propose a formally exact bound state method based on a perturbative expansion around a "non-perturbative" approximation. In QCD the scale parameter Λ stems from a boundary condition on the gauge constraint. This results in a confining potential, agreeing with the Cornell potential for quarkonia. The applications to bound states with light quarks appears promising. In the following I summarize the key aspects of the approach.

The initial state of the perturbative expansion

Scattering amplitudes are expanded around free, $\mathcal{O}(\alpha^0)$ *in* and *out* states. This is an obvious choice, given that the scattering matrix assumes the absence of interactions between the incoming as well as outgoing particles at asymptotic times ($t \to \pm\infty$).

Feynman diagrams lack bound state poles because the free asymptotic states have no overlap with bound states. A perturbative expansion for bound states must start from an initial (approximate) bound state which has a finite overlap with the exact state. The initial wave function is then necessarily non-polynomial in α. Some of its power corrections may be absorbed into the initial wave function, and *vice versa*. Choosing different initial states amounts to a reordering of the perturbative expansion, leaving the all orders sum unaffected. Physical judgment is required in the choice of an optimal expansion.

Instantaneous ($\Delta t = 0$) interactions in gauge theories

Positronium (at rest) is well approximated by its $|e^+e^-\rangle$ Fock state. Hadron spectroscopy suggests to similarly develop a perturbative expansion around the valence ($q\bar{q}$ or qqq) Fock state, which can be bound only by an instantaneous interaction. However, relativistic theories with a local action generally do not have instantaneous potentials. Interactions are transmitted by constituents whose speed is limited by the velocity of light[2].

Gauge theories are an exception: Their action is local but the gauge may be fixed nonlocally. The absence of $\partial_0 A^0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{A}$ terms in the QED and QCD actions implies that the A^0 and $\boldsymbol{A}_L$ fields do not propagate. Their values are determined by the choice of gauge. Two common gauges which conserve space translation and rotation symmetry are

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}_L(t,\boldsymbol{x}) = 0 \quad \text{(Coulomb gauge)}$$

$$A^0(t,\boldsymbol{x}) = 0 \quad \text{(Temporal gauge)} \tag{1}$$

Canonical quantization

Field theories may be quantized by defining a conjugate field π_α for each field φ_α in the action $\mathcal{S}$, and imposing equal-time commutation relations between them,

$$\pi_\alpha(t,\boldsymbol{x}) \equiv \frac{\delta\mathcal{S}(\varphi,\partial\varphi)}{\delta[\partial_0\varphi_\alpha(t,\boldsymbol{x})]} \qquad\qquad [\varphi_\alpha(t,\boldsymbol{x}),\pi_\beta(t,\boldsymbol{y})]_\pm = i\delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{2}$$

[2] I am considering states defined at an instant of ordinary time t. The constituents of light-front states (defined at equal $x^+ = t + z$) can be causally connected even when spatially separated.

The conjugate field of A^0 vanishes since the action is independent of $\partial_0 A^0$. Hence quantization in Coulomb gauge requires constraints [7, 8]. Temporal gauge avoids this complication ($A^0 = \partial_0 A^0 = 0$), while quantization of the space components of the gauge field is straightforward [9, 10]. The electric field $\boldsymbol{E} = -\partial_0 \boldsymbol{A}$ is conjugate to $-\boldsymbol{A}$.

Time-independent gauge transformations

The temporal gauge condition $A^0 = 0$ allows time-independent gauge transformations, which are generated by Gauss' operator $G(x) = \delta\mathcal{S}/\delta A^0$. To ensure that physical states are invariant under such transformations one imposes the constraint

$$G(x)\,|phys\rangle \equiv \frac{\delta\mathcal{S}_{QED}}{\delta A^0(x)}\,|phys\rangle = \big[\boldsymbol{\nabla}\cdot\boldsymbol{E}_L(x) - e\,\psi^\dagger\psi(x)\big]\,|phys\rangle = 0 \tag{3}$$

Note that $G(x) = 0$ is not an operator equation of motion in temporal gauge. The constraint (3) determines $\boldsymbol{\nabla}\cdot\boldsymbol{E}_L(x)$ in terms of the charge distribution in the state. For the vacuum $\boldsymbol{E}_L(x)\,|0\rangle = 0$, whereas for an e^-e^+ Fock state $|e^-(t,\boldsymbol{x}_1)e^+(t,\boldsymbol{x}_2)\rangle \equiv \bar\psi_\alpha(t,\boldsymbol{x}_1)\psi_\beta(t,\boldsymbol{x}_2)\,|0\rangle$,

$$\boldsymbol{E}_L(t,\boldsymbol{x})\,\big|e^-(t,\boldsymbol{x}_1)e^+(t,\boldsymbol{x}_2)\big\rangle = -\frac{e}{4\pi}\boldsymbol{\nabla}_{\boldsymbol{x}}\Big(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\Big)\,\big|e^-(t,\boldsymbol{x}_1)e^+(t,\boldsymbol{x}_2)\big\rangle \tag{4}$$

is the classical, instantaneous dipole electric field generated by the electron and positron.

Temporal gauge is suitable for bound states due to the classical nature of its $\boldsymbol{E}_L$ field. In Coulomb gauge Gauss' law defines the field operator A^0 in terms of $\psi^\dagger\psi$. Then A^0 creates fermion pairs in the vacuum, which appears inconsistent with the dominance of valence Fock states in QCD. The physics of Coulomb and temporal gauge must be the same, but is more transparent in temporal gauge due to the absence of quantization constraints.

The Bound Fock expansion

Bound states may be calculated perturbatively, starting from any state which has an overlap with the exact state. Valence Fock states are bound by the instantaneous electric field $\boldsymbol{E}_L$, with no mixing of Fock states. The QED Hamiltonian in temporal gauge is

$$\mathcal{H}^{QED}(t) = \int d\boldsymbol{x}\big[\tfrac{1}{2}\boldsymbol{E}^2 + \tfrac{1}{4}F^{ij}F^{ij} + \psi^\dagger(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - e\boldsymbol{\alpha}\cdot\boldsymbol{A} + m\gamma^0)\psi\big] \tag{5}$$

where $\boldsymbol{\alpha} \equiv \gamma^0\boldsymbol{\gamma}$ and $\boldsymbol{E}^2 = \boldsymbol{E}_L^2 + \boldsymbol{E}_T^2$ has both the instantaneous longitudinal and propagating transverse electric field. From (3) the $\boldsymbol{E}_L$ contribution is

$$\mathcal{H}_V^{QED} \equiv \tfrac{1}{2}\int d\boldsymbol{x}\,\boldsymbol{E}_L^2(\boldsymbol{x}) = \tfrac{1}{2}\int d\boldsymbol{x}d\boldsymbol{y}\,\frac{e^2}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\big[\psi^\dagger\psi(\boldsymbol{x})\big]\big[\psi^\dagger\psi(\boldsymbol{y})\big] \tag{6}$$

This gives the Coulomb potential $-\alpha/|\boldsymbol{x}_1-\boldsymbol{x}_2|$ for the $|e^-(t,\boldsymbol{x}_1)e^+(t,\boldsymbol{x}_2)\rangle$ Fock state.

A Positronium valence state is (in the rest frame) defined by a c-number wave function $\Phi_{\alpha\beta}$,

$$\big|e^-e^+,t\big\rangle \equiv \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\,\bar\psi_\alpha(t,\boldsymbol{x}_1)\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta(t,\boldsymbol{x}_2)\,|0\rangle \tag{7}$$

When the $\mathcal{O}(e)$ interaction term $e\,\boldsymbol{\alpha}\cdot\boldsymbol{A}$ in $\mathcal{H}^{QED}$ (5) is neglected, *i.e.*, at lowest order in the perturbative expansion, the eigenstate condition $\mathcal{H}\,|e^-e^+,t\rangle = M\,|e^-e^+,t\rangle$ gives a bound state equation for $\Phi_{\alpha\beta}$. In the non-relativistic limit this reduces to the Schrödinger equation, where the binding energy $E_b = -\frac{1}{4}m\alpha^2$ is defined by $M = 2m + E_b$.

At the next order of the perturbative expansion one takes into account one $e\,\boldsymbol{\alpha}\cdot\boldsymbol{A}$ interaction in $\mathcal{H}$. This creates a Fock state with a transverse photon $|e^-e^+\gamma\rangle$, as well as a state with a single (virtual) photon $|\gamma\rangle$. Neglecting Fock states with more photons and e^+e^- pairs gives a coupled equation which brings a correction of $\mathcal{O}(m\alpha^4)$ to E_b. Higher Fock states similarly appear at higher orders of the perturbative expansion. The potential energy $\mathcal{H}_V$ (6) contributes to each Fock state, suggesting the name "Bound Fock expansion".

Temporal gauge in QCD

The QED bound state approach may be applied to QCD. The temporal gauge constraint on physical states corresponding to (3) is now,

$$\boldsymbol{\nabla}\cdot E_L^a(\boldsymbol{x})\,|phys\rangle = g\big[-f_{abc}A_b^iE_c^i + \psi^\dagger T^a\psi(\boldsymbol{x})\big]\,|phys\rangle \tag{8}$$

In solving the QED constraint (3) for the classical electric field $\boldsymbol{E}_L$ (4) we implicitly assumed, on physical grounds, that $\boldsymbol{E}_L(\boldsymbol{x}\to\infty) = 0$. In QCD this assumption is not compelling. Color singlet states cannot generate a color octet gluon field $\boldsymbol{E}_L^a(\boldsymbol{x})$ at any position $\boldsymbol{x}$. While each color component of the state (say, a red-antired $q\bar{q}$ state of a meson) can be bound by an $\boldsymbol{E}_L^a$ field, we must have $\boldsymbol{E}_L^a(\boldsymbol{x}) = 0$ after the sum over quark colors. An external observer cannot detect color singlet hadrons through their classical gluon field. A non-vanishing boundary condition for the $\boldsymbol{E}_L^a$ field introduces a scale Λ.

Space translation and rotation symmetry restricts the homogeneous solutions of the gauge constraint (8). The only possibility is

$$\boldsymbol{E}_L^a(\boldsymbol{x})\,|phys\rangle = -\boldsymbol{\nabla}_x\int d\boldsymbol{y}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big]\mathcal{E}_a(\boldsymbol{y})\,|phys\rangle$$

$$\mathcal{E}_a(\boldsymbol{y}) \equiv -f_{abc}A_b^iE_c^i(\boldsymbol{y}) + \psi^\dagger T^a\psi(\boldsymbol{y}) \tag{9}$$

The $\kappa\,\boldsymbol{x}\cdot\boldsymbol{y}$ term is a homogeneous solution of (8) when $\kappa \neq \kappa(\boldsymbol{x})$, since it vanishes in $\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{E}_L^a(\boldsymbol{x})$. The contribution of $\boldsymbol{E}_L^a$ to the QCD Hamiltonian is

$$\mathcal{H}_V \equiv \tfrac{1}{2}\int d\boldsymbol{x}\,\big(\boldsymbol{E}_L^a\big)^2 = \int d\boldsymbol{y}d\boldsymbol{z}\Big\{\boldsymbol{y}\cdot\boldsymbol{z}\Big[\tfrac{1}{2}\kappa^2\!\int d\boldsymbol{x} + g\kappa\Big] + \tfrac{1}{2}\frac{\alpha_s}{|\boldsymbol{y}-\boldsymbol{z}|}\Big\}\mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z}) \tag{10}$$

The $\mathcal{O}(\kappa^2)$ term is proportional to the volume of space, $\int d\boldsymbol{x}$. Hence the homogeneous solution contributes a constant energy density throughout space. This infinite energy may be subtracted only provided it is the same for all Fock states of all bound states. The normalization κ of any Fock state can thus be determined in terms of a universal scale Λ, which is related to the magnitude of the constant energy density.

The meson potential

As an example, we may consider the instantaneous potential of a color singlet $q\bar{q}$ Fock state. With an implicit sum over the quark colors A,

$$\mathcal{H}_V\,\bar\psi_A^\alpha(t,\boldsymbol{x}_1)\psi_A^\beta(t,\boldsymbol{x}_2)\,|0\rangle = V(|\boldsymbol{x}_1-\boldsymbol{x}_2|)\bar\psi_A^\alpha(t,\boldsymbol{x}_1)\psi_A^\beta(t,\boldsymbol{x}_2)\,|0\rangle$$
$$V(|\boldsymbol{x}_1-\boldsymbol{x}_2|) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - C_F\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} \tag{11}$$

The homogeneous contribution in (9) gives a linear confining potential for mesons. The potential has the same form as the Cornell potential for quarkonia [3, 4], and applies also to relativistic, light quark states. The spatially constant energy density is

$$E_\Lambda = \frac{\Lambda^4}{2g^2C_F} \tag{12}$$

Analogous confining potentials are found for color singlet qqq, $q\bar{q}g$ and gg Fock states [1].

Remarks

Bound states must, at each order of the perturbative expansion, have exact symmetries such as Poincaré covariance. There are few studies of how equal-time bound states transform under boosts. It is common to assume that the states Lorentz contract as in classical relativity, and to depict them as ellipses. The transformation is actually more subtle, even for Positronia of lowest order [11].

The confining potential in (11) is of $\mathcal{O}(\alpha_s^0)$. Neglecting the $\mathcal{O}(\alpha_s)$ term in the potential gives QCD states bound only by confinement. For quark masses $m \ll \Lambda$ the mesons are strongly bound, yet have only the valence $|q\bar{q}\rangle$ Fock state shown in (11). They lie on nearly linear Regge trajectories and overlap states with meson pairs, much as depicted by dual diagrams. More details are given in [1]. Many aspects remain to be explored, including unitarity and chiral symmetry breaking.

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