Estimating the Reduced Shear

- Assume we have a large number of galaxy images in a small region of the sky where we can approximate $g \simeq$ const. We can then estimate g from the observed ellipticities ε of these images, by assuming the source ellipticities ε^S are statistically isotropic random variables.

- Assume $|g| \leq 1 \quad \Rightarrow \quad \varepsilon(\varepsilon^S) = \dfrac{\varepsilon^S + g}{1 + g^* \varepsilon^S}$

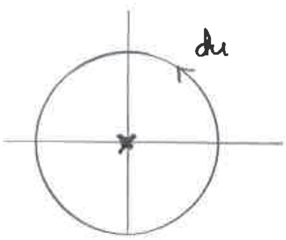
Write $\varepsilon_s \equiv y e^{i2\varphi} \equiv yu$, $\quad u = e^{i2\varphi} \Rightarrow du = 2iu\,d\varphi$ (u on unit circle of complex plane)

The expectation value $\langle \varepsilon \rangle = \int d^2\varepsilon^S \, P(\varepsilon^S)\, \varepsilon(\varepsilon^S)$

where the probability distribution $P(\varepsilon^S) = P(y) \underbrace{P(\varphi)}_{= \text{const} = \frac{1}{\pi}}$ (statistical isotropy)

$$\Rightarrow \langle \varepsilon \rangle = \int_0^1 dy\, P(y) \int_0^\pi d\varphi\, P(\varphi)\, \varepsilon(\varepsilon^S), \quad \text{where}$$

$$\int_0^\pi d\varphi\, P(\varphi)\, \varepsilon(\varepsilon^S) = \frac{1}{\pi}\int_0^\pi d\varphi \frac{\varepsilon^S + g}{1 + g^* \varepsilon^S} = \frac{1}{2\pi i} \oint \frac{du}{u} \frac{yu + g}{1 + g^* yu}$$



This is a closed integral of an analytic function in the complex plane $\Rightarrow$ calculus of residues. There are two poles:

$u_1 = 0$

$u_2 = \dfrac{-1}{g^* y} \quad \Rightarrow \quad |u_2| = \dfrac{1}{\underbrace{|g|}_{\leq 1}\underbrace{|y|}_{<1}} > 1$ outside integration contour.

$$\Rightarrow \oint = 2\pi i \,\mathrm{Res}(u=0) = 2\pi i \cdot \left.\frac{yu+g}{1+g^* yu}\right|_{u=0} = 2\pi i g, \quad \text{independent of } y$$

$$\therefore \langle \varepsilon \rangle = \int_0^1 dy\, P(y) \cdot \frac{1}{2\pi i} \cdot 2\pi i g = g \underbrace{\int_0^1 dy\, P(y)}_{1 \text{ by normalization of probability}} = g$$

Thus the image ellipticity is an unbiased (but noisy) estimator of the reduced shear!

- Exercise: Find $\langle \varepsilon \rangle$ for the case $|g| > 1$.

- Schneider p.277: "The expectation value of χ cannot be easily calculated and depends on the intrinsic ellipticity distribution of the sources. In particular, $\langle \chi \rangle$ is not simply related to g. However, in the limit $\kappa \ll 1$, $|\gamma| \ll 1$, one finds $\gamma \simeq g \simeq \langle \varepsilon \rangle \simeq \frac{1}{2}\langle \chi \rangle$."

- This simple result $\langle \varepsilon \rangle = g$ is the main reason to prefer ε over χ.

- The noise in the estimator is determined by the source ellipticity variance

$$\sigma_\varepsilon^2 \equiv \langle |\varepsilon^s|^2 \rangle$$ Schneider uses a reference value $\sigma_\varepsilon \sim 0.3$

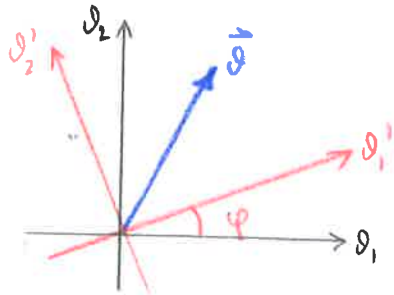
When averaging over N images, the 1σ expected error in the g estimate is $\sim \dfrac{\sigma_\varepsilon}{\sqrt{N}}$

$\therefore$ For $\Delta g \simeq 0.01$, need $N \sim 1000$ galaxies

§4.3 Tangential and Cross Components

- Components of shear

Rotate the coord. axes by φ:



Here $\vec\vartheta, \vec\beta$ are abstract vector notation (independent of coord. system)

$\bar\vartheta, \bar\beta$ denote the column vectors of their components (depend on coord. system)

likewise, $\bar A$ is the matrix of the components of A

$$\bar\vartheta' \equiv \begin{bmatrix} \vartheta_1' \\ \vartheta_2' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}}_{R} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} = R\bar\vartheta \qquad \text{rotation matrix } R^{-1} = R^T$$

Likewise $\bar\beta' = R\bar\beta$

Lens equation $\vec\beta = A\vec\vartheta$ (choose $\vec\beta_0 = \vec\vartheta_0 = 0$) $\Rightarrow$ $\bar\beta = \bar A\bar\vartheta$ and $\bar\beta' = \bar A'\bar\vartheta'$

$$\Rightarrow R\bar\beta = \bar A' R\bar\vartheta \Rightarrow \bar\beta = R^{-1}\bar A' R\bar\vartheta = R^T\bar A' R\bar\vartheta \Rightarrow \bar A = R^T\bar A' R$$

$$\bar A' = R\bar A R^T$$

$$\bar A = (1-\kappa)\begin{bmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{bmatrix} = (1-\kappa)I + (1-\kappa)\begin{bmatrix} -g_1 & -g_2 \\ -g_2 & g_1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -g_1' & -g_2' \\ -g_2' & g_1' \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} -g_1 & -g_2 \\ -g_2 & g_1 \end{bmatrix}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} = \cdots$$

$$\Rightarrow \begin{bmatrix} -g_1' & -g_2' \\ -g_2' & g_1' \end{bmatrix} = \begin{bmatrix} -g_1\cos 2\varphi - g_2 \sin 2\varphi & g_1 \sin 2\varphi - g_2 \cos 2\varphi \\ g_1 \sin 2\varphi - g_2 \cos 2\varphi & g_1 \cos 2\varphi + g_2 \sin 2\varphi \end{bmatrix}$$

$$\therefore \quad g_1' = g_1 \cos 2\varphi + g_2 \sin 2\varphi \qquad \text{or} \qquad \begin{bmatrix} g_1' \\ g_2' \end{bmatrix} = \begin{bmatrix} \cos 2\varphi & +\sin 2\varphi \\ -\sin 2\varphi & \cos 2\varphi \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$g_2' = -g_1 \sin 2\varphi + g_2 \cos 2\varphi$$

or $\quad g_1' + i g_2' = g_1 \cos 2\varphi + g_2 \sin 2\varphi - i g_1 \sin 2\varphi + i g_2 \cos 2\varphi$

Compare to $\quad (g_1 + i g_2) e^{i2\varphi} = (g_1 + i g_2)(\cos 2\varphi + i \sin 2\varphi) = g_1 \cos 2\varphi + i g_1 \sin 2\varphi + i g_2 \cos 2\varphi - g_2 \sin 2\varphi$

Not this, but

$$(g_1 + i g_2) e^{-i2\varphi} = (g_1 + i g_2)(\cos 2\varphi - i \sin 2\varphi) = g_1 \cos 2\varphi - i g_1 \sin 2\varphi + i g_2 \cos 2\varphi + g_2 \sin 2\varphi$$

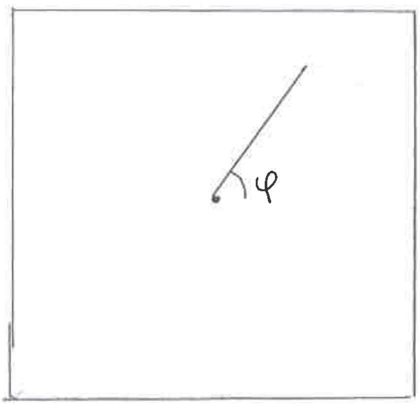
$$= g_1' + i g_2'$$

$\therefore \quad \boxed{g' = g e^{-i2\varphi}} \qquad$ Likewise $\qquad \boxed{\gamma' = \gamma e^{-i2\varphi}}$

The same transformation properties apply to the ellipticity ε, ε^S

$$\varepsilon' = \varepsilon e^{-i2\varphi}$$

- Assume now that we have a special reference direction φ, which may vary over the image plane. For example, if we have an axisymmetric lens, φ could refer to the rotation angle around the axis. (We will have other cases of a reference direction later).



Define the tangential and cross components of shear

$$\gamma_t \equiv -\gamma_1' = -\gamma_1 \cos 2\varphi - \gamma_2 \sin 2\varphi = -\mathrm{Re}\left[\gamma e^{-i2\varphi}\right]$$

$$\gamma_\times \equiv -\gamma_2' = +\gamma_1 \sin 2\varphi - \gamma_2 \cos 2\varphi = -\mathrm{Im}\left[\gamma e^{-i2\varphi}\right]$$

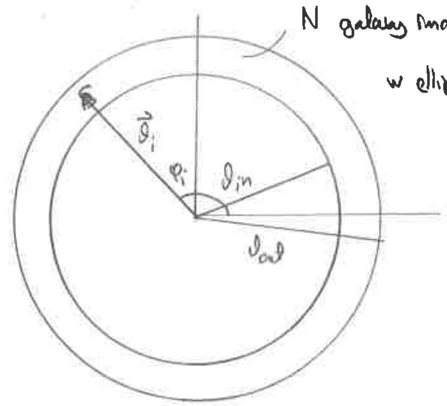
and ellipticity

$$\boxed{\begin{aligned} \varepsilon_t &= -\mathrm{Re}\left[\varepsilon e^{-i2\varphi}\right] \\ \varepsilon_\times &= -\mathrm{Im}\left[\varepsilon e^{-i2\varphi}\right] \end{aligned}}$$

- Why the $-$ sign? There are different sign conventions, but this one corresponds to positive γ_t stretching the image in the tangential direction (orthogonal to the φ direction) and negative γ_t in the radial direction. If we used the opposite sign convention, we should call γ_t the radial component. Notice how γ_t indeed corresponds to both tangential and radial shear, whereas $\gamma_\times$ corresponds to shear 45° off both

Minimum Lens Strength for its WL Detection

- Consider SIS $\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$



N galaxy images at $\vec{\vartheta}_i = \vartheta_i(\cos\varphi_i, \sin\varphi_i)$

w ellipticity ε_i

$$\varepsilon_{ti} \equiv -\mathrm{Re}\left(\varepsilon_i e^{-2i\varphi_i}\right)$$

$$\vartheta_E = 4\pi \frac{D_{ds}}{D_s} \sigma_v^2$$

BS01 §4.5 $\quad \frac{S}{N} = \frac{\vartheta_E}{\sigma_\varepsilon}\sqrt{\pi n}\sqrt{\ln\frac{\vartheta_{out}}{\vartheta_{in}}}$

n: number density of galaxies on the sky

Would have to read this to understand the S/N formula [Bartelmann & Schneider, Phys. Rep. 340, 291 (2001)]

So skip this now; but the conclusion is that clusters w $\sigma_v \gtrsim 600$ km/s can be detected by WL but individual galaxies w $\sigma_v \lesssim 200$ km/s can not

$$= 8.4\sqrt{\frac{n}{30/\text{arcmin}^2}}\left(\frac{\sigma_\varepsilon}{0.3}\right)^{-1}\left(\frac{\sigma_v}{600\text{ km/s}}\right)^2\sqrt{\frac{\ln(\vartheta_{out}/\vartheta_{in})}{\ln 10}}\underbrace{\left\langle\frac{D_{ds}}{D_s}\right\rangle}$$

cluster detection more difficult for increasing lens redshift (D_d larger $\Rightarrow$ D_{ds} smaller)

Mean Tangential Shear on Circles

- For axially symmetric lenses (§2.1)

$$\left.\begin{aligned}\gamma_1 &= (\kappa-\bar{\kappa})\cos 2\varphi \\ \gamma_2 &= (\kappa-\bar{\kappa})\sin 2\varphi\end{aligned}\right\} \quad \gamma = (\kappa-\bar{\kappa})e^{i2\varphi}$$

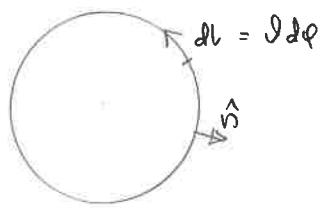
$$\gamma_t = -\mathrm{Re}\left(\gamma e^{-2i\varphi}\right) = \bar{\kappa}-\kappa$$

A similar relation holds for general matter distributions. Remember $\nabla^2\psi = 2\kappa$

$$\psi_{,11} = \kappa+\gamma_1 \qquad \psi_{,12} = \gamma_2 \qquad \psi_{,22} = \kappa-\gamma_1$$

$$\int_0^\vartheta d^2\vartheta\, \nabla^2\psi = \oint dl\, \nabla\psi\cdot\hat{n}$$

$$= \vartheta\oint d\varphi\,\frac{\partial\psi}{\partial\vartheta}$$



∇

dimensionless mass within radius ϑ: $\quad m(\vartheta) \equiv \frac{1}{\pi}\int_0^\vartheta d^2\vartheta\,\underbrace{\kappa(\vec{\vartheta})}_{\frac{1}{2}\nabla^2\psi}$

we defined this earlier for the axisymmetric case; the factor $\frac{1}{\pi}$ is to match this earlier def.

$$= \frac{\vartheta}{2\pi}\oint d\varphi\,\frac{\partial\psi}{\partial\vartheta}$$

$$\Rightarrow \quad \frac{dm(\vartheta)}{d\vartheta} = \frac{m(\vartheta)}{\vartheta} + \frac{\vartheta}{2\pi}\oint d\varphi\,\underbrace{\frac{\partial^2\psi}{\partial\vartheta^2}}$$

$\frac{\partial^2\psi}{\partial\vartheta^2} = \kappa-\gamma_t$ generally

on the ϑ_1 axis this is $\psi_{,11} = \kappa+\gamma_1 = \kappa-\gamma_t$

$$\therefore \quad \frac{dm(\vartheta)}{d\vartheta} = \frac{m(\vartheta)}{\vartheta} + \vartheta \cdot \underbrace{\frac{1}{2\pi}\oint d\varphi\,(\kappa - \gamma_t)}_{\text{mean } \kappa-\gamma_t \text{ on circle w radius } \vartheta} = \frac{m(\vartheta)}{\vartheta} + \vartheta \cdot \left[\langle\kappa(\vartheta)\rangle - \langle\gamma_t(\vartheta)\rangle\right]$$

On the other hand,

$$\bar{\kappa}(\vartheta) = \frac{1}{\pi\vartheta^2}\int d^2\vartheta\,\kappa(\vec{\vartheta}) = \frac{1}{\pi\vartheta^2}\int_0^{2\pi} d\varphi \int_0^{\vartheta} \vartheta\, d\vartheta\,\kappa(\vec{\vartheta}) = \frac{m(\vartheta)}{\vartheta^2}$$

for the axisymmetric case this gave $\frac{2}{\vartheta^2}$ (p. 1-13)

$$= \frac{1}{\pi\vartheta^2}\int_0^{\vartheta} \vartheta\, d\vartheta \underbrace{\int_0^{2\pi} d\varphi\,\kappa(\vartheta,\varphi)}_{2\pi\langle\kappa(\vartheta)\rangle}$$

$$\Leftrightarrow \quad m(\vartheta) = 2\int_0^{\vartheta} d\vartheta\,\vartheta\,\langle\kappa(\vartheta)\rangle \quad \Rightarrow \quad \frac{dm(\vartheta)}{d\vartheta} = 2\vartheta\langle\kappa(\vartheta)\rangle$$

$$\therefore \quad \cancel{2}\vartheta\langle\kappa(\vartheta)\rangle = \underbrace{\frac{m(\vartheta)}{\vartheta}}_{\bar{\kappa}(\vartheta)\vartheta} + \cancel{\vartheta\langle\kappa(\vartheta)\rangle} - \vartheta\langle\gamma_t(\vartheta)\rangle \quad \Rightarrow \quad \langle\kappa(\vartheta)\rangle = \bar{\kappa}(\vartheta) - \langle\gamma_t(\vartheta)\rangle$$

$$\therefore \quad \boxed{\langle\gamma_t\rangle = \bar{\kappa} - \langle\kappa\rangle}$$

where $\langle\,\rangle$ denotes average over circle w radius ϑ and $\bar{\kappa}$ is mean within this circle

$\therefore$ From measurements of $\langle\gamma_t\rangle$ as a function of ϑ one can determine the azimuthally (φ)-averaged mass profile of lens (even if the lens is not axisymmetric).