Image ellipticity

- Deviation from circular symmetry measured by $Q_{11}-Q_{22}$ and Q_{12}.

Two alternative definitions for image ellipticity:

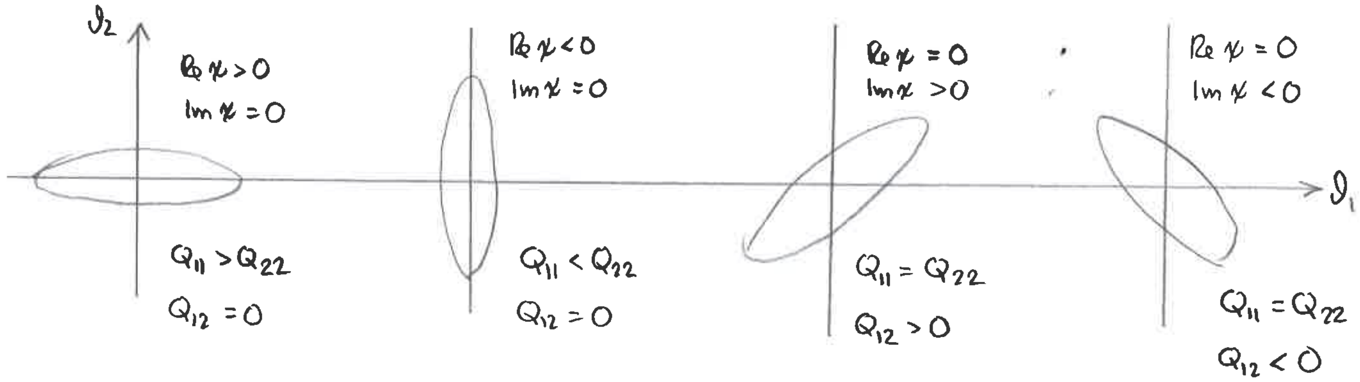
$$\chi \equiv \chi_1 + i\chi_2 \equiv \frac{Q_{11}-Q_{22}+2iQ_{12}}{Q_{11}+Q_{22}} \quad \text{and} \quad \varepsilon \equiv \varepsilon_1 + i\varepsilon_2 \equiv \frac{Q_{11}-Q_{22}+2iQ_{12}}{\underbrace{Q_{11}+Q_{22}}_{\mathrm{tr}Q}+2\sqrt{\underbrace{Q_{11}Q_{22}-Q_{12}^2}_{\det Q}}}$$

$$0 \le |\varepsilon| \le |\chi| < 1$$

they have the same phase, but different absolute value

$$\therefore\ Q = \frac{1}{2}\mathrm{tr}Q \begin{bmatrix} 1+\chi_1 & \chi_2 \\ \chi_2 & 1-\chi_1 \end{bmatrix} \qquad \det Q = \frac{1}{4}(\mathrm{tr}Q)^2(1-|\chi|^2)$$

$$\frac{\chi}{\varepsilon} = \frac{\mathrm{tr}Q + 2\sqrt{\det Q}}{\mathrm{tr}Q} = 1+\sqrt{1-|\chi|^2} \quad \Rightarrow \quad \varepsilon = \frac{\chi}{1+\sqrt{1-|\chi|^2}}, \quad \chi = \frac{2\varepsilon}{1+|\varepsilon|^2}$$



From Source to Image Ellipticity

- Likewise, define
$$Q^S_{ij} \equiv \frac{\int d^2\beta\,(\beta_i-\bar\beta_i)(\beta_j-\bar\beta_j)\,I^S(\vec\beta)}{\int d^2\beta\, I^S(\vec\beta)}$$

Since $I(\vec\vartheta) = I^S(\vec\beta)$ and $d^2\beta = \det A\cdot d^2\vartheta$,

$$Q^S_{ij} = \frac{\det A\cdot\int d^2\vartheta\, A_{ik}(\vartheta_k-\bar\vartheta_k)A_{jl}(\vartheta_l-\bar\vartheta_l)\,I(\vec\vartheta)}{\det A\cdot\int d^2\vartheta\, I(\vartheta)} = A_{ik}A_{jl}Q_{kl}$$

$$\therefore\quad Q^S = AQA^T = AQA \qquad \text{(since } A \text{ is symmetric)}$$

- Using $A_{ij} = (1-\kappa)\begin{bmatrix}1-g_1 & -g_2\\ -g_2 & 1+g_1\end{bmatrix}$, $\det A = (1-\kappa)^2(1-|g|^2)$ we get (exercise)

$$Q^S_{11} = A_{1j}Q_{jk}A_{k1} = \ldots = (1-\kappa)^2\left[(1-g_1)^2Q_{11} - 2(1-g_1)g_2Q_{12} + g_2^2Q_{22}\right]$$

$$Q^S_{22} = A_{2j}Q_{jk}A_{k2} = \ldots = (1-\kappa)^2\left[g_2^2Q_{11} - 2(1+g_1)g_2Q_{12} + (1+g_1)^2Q_{22}\right]$$

$$\mathrm{tr}Q^S = Q^S_{11}+Q^S_{22} = \ldots = (1-\kappa)^2\cdot\mathrm{tr}Q\cdot\left(1+|g|^2-2g_1\chi_1-2g_2\chi_2\right) = (1-\kappa)^2\cdot\mathrm{tr}Q\cdot\left(1+|g|^2-2\,\mathrm{Re}\,g\chi^*\right)$$

$$\det Q^S = (\det A)^2\cdot\det Q = (1-\kappa)^4(1-|g|^2)^2\cdot\det Q$$

$$Q^S_{12} = A_{1j}Q_{jk}A_{k2} = \ldots = (1-\kappa)^2\cdot\left[-(1-g_1)g_2Q_{11} + (1-g_1^2+g_2^2)Q_{12} - (1+g_1)g_2Q_{22}\right]$$

and

$$\chi^S \equiv \frac{Q^S_{11}-Q^S_{22}+2iQ^S_{12}}{\mathrm{tr}Q^S} = \frac{\chi-2g+g^2\chi^*}{1+|g|^2-2\,\mathrm{Re}\,g\chi^*}$$

$$\boxed{\varepsilon^S = \begin{cases}\dfrac{\varepsilon-g}{1-g^*\varepsilon} & \text{for } |g|\le 1\\[2ex] \dfrac{1-g\varepsilon^*}{\varepsilon^*-g^*} & \text{for } |g|>1\end{cases}}$$

- Invert: $\varepsilon^S - g^*\varepsilon\varepsilon^S = \varepsilon - g \;\Rightarrow\; (1+g^*\varepsilon^S)\varepsilon = \varepsilon^S+g \;\Rightarrow$

$$\boxed{\begin{aligned}\varepsilon &= \frac{\varepsilon^S+g}{1+g^*\varepsilon^S} && \text{for } |g|\le 1\\ \varepsilon &= \frac{1+g\varepsilon^{S*}}{\varepsilon^{S*}-g^*} && \text{for } |g|>1\end{aligned}}$$

The same relation except w opposite sign of g

Likewise

$$\therefore\quad \chi = \frac{\chi^S+2g+g^2\chi^{S*}}{1+|g|^2+2\,\mathrm{Re}\,g\chi^{S*}}$$

· Meaning of ε and χ (*): Consider a circular source w radius R. Assume $|g|<1$.

$$\varepsilon^S = \chi^S = 0 \quad \Rightarrow \quad \varepsilon = g \quad \text{and} \quad \chi = \frac{2g}{1+|g|^2}$$

The image has semiaxes $a = \dfrac{R}{(1-\kappa)(1-|g|)}$ and $b = \dfrac{R}{(1-\kappa)(1+|g|)}$ (from p. 4-1)

$$\Rightarrow \quad r \equiv \frac{b}{a} = \frac{1-|g|}{1+|g|} = \frac{1-|\varepsilon|}{1+|\varepsilon|} \quad \Rightarrow \quad |\varepsilon| = \frac{1-r}{1+r}$$

This is the relation between the semiaxis ratio r and ellipticity $|\varepsilon|$ of an elliptic image; it holds regardless of the shape of the source.

$$|\chi| = \frac{2|\varepsilon|}{1+|\varepsilon|^2} = \dots = \frac{1-r^2}{1+r^2}$$

The phase angle φ (of ε and χ) gives the orientation of the ellipse.

*) Illustrated for the case of an image whose shape happens to be an ellipse.