§2. The Principles of Weak Gravitational Lensing

assume κ, $|\gamma|$, $\kappa+|\gamma|$, $|g| < 1$

§2.1 Distortion of Faint Galaxy Images

- Assume size of image is small compared to the scale over which A_{ij} varies, so A_{ij} can be approximated as constant over the image. Let the center (or some other reference point) of the image and source be $\vec{\vartheta}_0$ and $\vec{\beta}_0$. Different parts of the image and source are then mapped to each other by

$$\vec{\beta}-\vec{\beta}_0 = A(\vec{\vartheta}_0)\cdot(\vec{\vartheta}-\vec{\vartheta}_0)$$

Since surface brightness is conserved in lensing, the surface brightness of the image varies as

$$I(\vec{\vartheta}) = I^S(\vec{\beta}) = I^S[\vec{\beta}_0 + A(\vec{\vartheta}_0)\cdot(\vec{\vartheta}-\vec{\vartheta}_0)]$$

where $I^S(\vec{\beta})$ is the surface brightness of the source (i.e., the image we would have w/o lensing).

$$A = \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix} = (1-\kappa)\begin{bmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{bmatrix} \qquad g_i \equiv \frac{\gamma_i}{1-\kappa} \quad \text{reduced shear}$$

define $|\gamma| \equiv \sqrt{\gamma_1^2+\gamma_2^2}$, $|g| \equiv \sqrt{g_1^2+g_2^2}$

eigenvalues of A:

$$\left.\begin{aligned} a_1 &= 1-\kappa+|\gamma| = (1-\kappa)(1+|g|) \\ a_2 &= 1-\kappa-|\gamma| = (1-\kappa)(1-|g|) \end{aligned}\right\} \quad \begin{aligned} \operatorname{tr} A &= 2(1-\kappa) \\ \det A &= (1-\kappa)^2(1-|g|^2) \end{aligned}$$

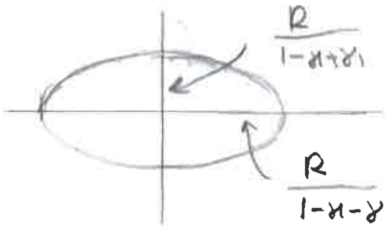
- Consider circular source w (angular) radius R. $\Rightarrow$ image is ellipse w semi-axes

$$b \equiv \frac{R}{a_1} = \frac{R}{1-\kappa+|\gamma|} = \frac{R}{(1-\kappa)(1+|g|)} \quad \text{and} \quad \frac{R}{a_2} = \frac{R}{1-\kappa-|\gamma|} = \frac{R}{(1-\kappa)(1-|g|)} \equiv a$$

$\Rightarrow$ their ratio is $\dfrac{b}{a} = \dfrac{1-|g|}{1+|g|}$ $\Rightarrow$ $(1+|g|)\,b = (1-|g|)\,a$ $\quad (b \le a)$

$\Rightarrow$ $|g|(b+a) = a-b$ $\Rightarrow$ $|g| = \dfrac{a-b}{a+b} = \dfrac{1-b/a}{1+b/a}$

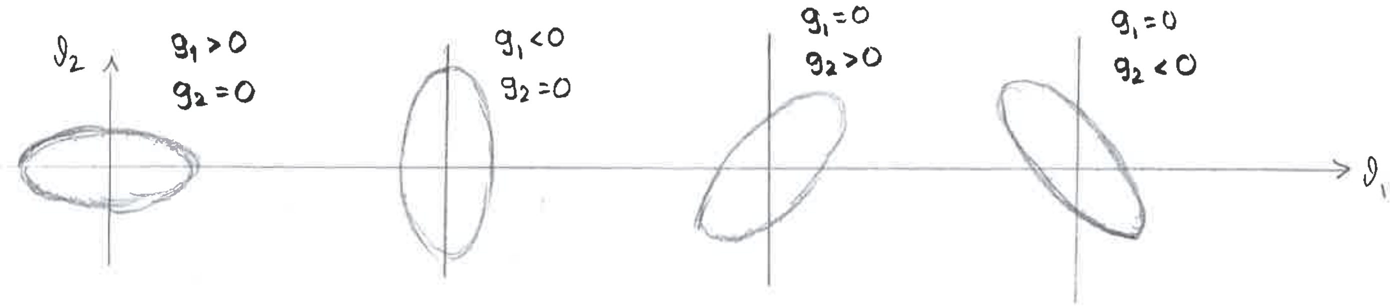
- If $g_2=0$, then the semi-axes are parallel to the cd axes:



$$\frac{R}{1-\kappa+\gamma_1} = \frac{R}{(1-\kappa)(1+g_1)}$$

$$\frac{R}{1-\kappa-\gamma_1} = \frac{R}{(1-\kappa)(1-g_1)}$$

- Orientation of the ellipse:



If $g_1=0$, $\quad A=(1-\kappa)\begin{bmatrix}1 & -g_2\\ -g_2 & 1\end{bmatrix}$

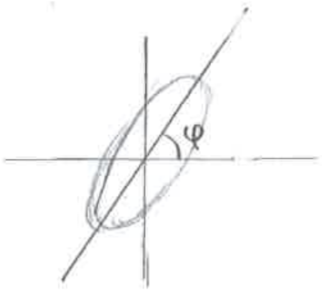
$\vec{\vartheta}-\vec{\vartheta}_0=(1,1)$ is mapped to $\vec{\beta}-\vec{\beta}_0 = A\cdot(\vec{\vartheta}-\vec{\vartheta}_0) = (1-\kappa)\begin{bmatrix}1 & -g_2\\ -g_2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$

$= (1-\kappa)\begin{bmatrix}1-g_2\\1-g_2\end{bmatrix} = (1-\kappa)(1-g_2)(1,1)$

$\vec{\vartheta}-\vec{\vartheta}_0=(1,-1)$ is mapped to $(1-\kappa)\begin{bmatrix}1 & -g_2\\ -g_2 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = (1-\kappa)\begin{bmatrix}1+g_2\\-g_2-1\end{bmatrix} = (1-\kappa)(1+g_2)(1,-1)$

For $g_2>0$ image is stretched by $\dfrac{1}{(1-\kappa)(1-g_2)}$ in the $(1,1)$ direction

and compressed by $\dfrac{1}{(1-\kappa)(1+g_2)}$ in the $(1,-1)$ direction

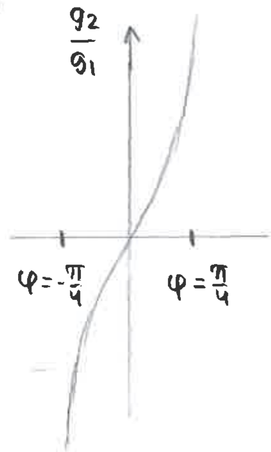


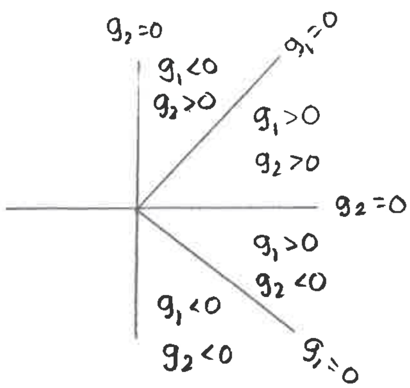
$\varphi=0$ for $g_1>0,\ g_2=0$

$\varphi=\frac{\pi}{4}$ for $g_1=0,\ g_2>0$

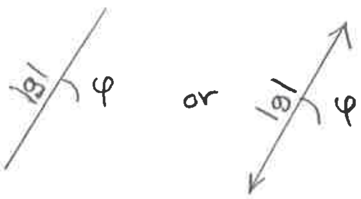
$\varphi=\frac{\pi}{2}$ for $g_1<0,\ g_2=0$

$\tan 2\varphi = \dfrac{g_2}{g_1}$





The (reduced) shear is not a vector; it is a symmetric (real) traceless 2x2 matrix $\begin{bmatrix}-g_1 & -g_2\\ -g_2 & +g_1\end{bmatrix}$; it is associated w a direction (that of the semimajor axis of the ellipse; the stretch direction): We could draw it as an arrow w/o head, or with two heads. This kind of object is sometimes called a polar, or a spin-2 field (when considered as a function of $\vec{\vartheta}$).



The linear polarization of electromagnetic radiation is a similar mathematical object.

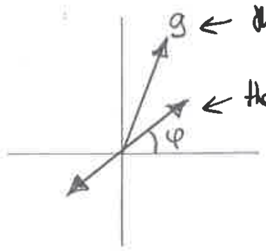
Complex Notation

- The shear is commonly represented by a complex number

$$\gamma \equiv \gamma_1 + i\gamma_2 \equiv |\gamma| e^{i2\varphi}, \qquad g \equiv g_1 + i g_2 \equiv |g| e^{i2\varphi}$$

Note that the phase of this complex number is denoted by 2φ !

This is so that φ gives the orientation angle of the image (ellipse) of a circular source, i.e., the stretch direction. $\varphi \to \varphi + \pi$ maps the ellipse back to itself, $\gamma \to \gamma$, $g \to g$



← the complex number makes a full circle while

← the polar turns a half circle

$$g = |g| e^{i2\varphi} = (\underbrace{|g|\cos 2\varphi}_{g_1}, \underbrace{|g|\sin 2\varphi}_{g_2}) \Rightarrow \tan 2\varphi = \frac{g_2}{g_1}$$

∴ Observed shape of circular source provides a measurement of reduced shear.

Problem: galaxies are not circular. We would need to know the true shape of the galaxy (as projected onto the source plane) so that its observed shape would provide this measurement.

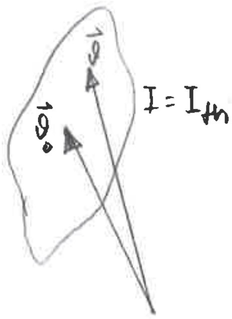
Solution: If we can observe (sufficiently) many galaxies within a region of the sky over which the reduced shear does not vary significantly, we can assume that the true shapes are randomly oriented $\Rightarrow$ the expected shape of their superposition is circular.

§2.2 Measurement of Shapes and Shear

- A galaxy image may have a complicated shape. We would like to fit an ellipse to it. How?

$I(\vec{\vartheta})$ brightness distribution of the image.

Define the center $\vec{\vartheta}_0 = (\bar{\vartheta}_1, \bar{\vartheta}_2)$ as
$$\vec{\vartheta}_0 \equiv \frac{\int d^2\vartheta \, \vec{\vartheta} \, I(\vec{\vartheta})}{\int d^2\vartheta \, I(\vec{\vartheta})}.$$



In practice integrate only over a region where $I > I_{th}$ (threshold brightness)

Define tensor of second brightness moments
$$Q_{ij} = \frac{\int d^2\vartheta \, (\vartheta_i - \bar{\vartheta}_i)(\vartheta_j - \bar{\vartheta}_j) \, I(\vec{\vartheta})}{\int d^2\vartheta \, I(\vec{\vartheta})}$$

Q_{11} measures extent in ϑ_1 direction

Q_{22} measures extent in ϑ_2 direction

Q_{12} — " — in the diagonal direction

For a circularly symmetric image $Q_{12} = 0$ (integrand is odd)

$Q_{11} = Q_{22}$

$$\mathrm{tr}\, Q = Q_{11} + Q_{22} = \frac{\int d^2\vartheta \left[(\vartheta_1 - \bar{\vartheta}_1)^2 + (\vartheta_2 - \bar{\vartheta}_2)^2\right] I(\vec{\vartheta})}{\int d^2\vartheta \, I(\vec{\vartheta})} = \frac{\int d^2\vartheta \, (\vec{\vartheta} - \vec{\vartheta}_0)^2 \, I(\vartheta)}{\int d^2\vartheta \, I(\vartheta)}$$

measures the size of the image.

The traceless part $\begin{bmatrix} \frac{1}{2}(Q_{11} - Q_{22}) & Q_{12} \\ Q_{12} & \frac{1}{2}(Q_{22} - Q_{11}) \end{bmatrix}$ contains the ellipticity information.