3. COSMOLOGY AND LENSING

§3.1 Friedmann-Robertson-Walker Universe

· The metric of a homogeneous and isotropic (FRW) universe can be written

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-Kr^2} + r^2 (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \right]$$

$$= -dt^2 + a(t)^2 \left[dw^2 + f_K(w)^2 \, (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \right]$$

where $a(t)$ is the scale factor, K is the curvature constant, and

$$f_K(w) \equiv \begin{cases} K^{-1/2} \sin(K^{1/2} w) & \text{for } K>0 \quad \text{(closed universe)} \\ w & K=0 \quad \text{(flat universe)} \\ |K|^{-1/2} \sinh(|K|^{1/2} w) & K<0 \quad \text{(open universe)} \end{cases}$$

We normalize the scale factor so that $a_0 \equiv a(t_0) = 1 \quad \Rightarrow \quad a(t) = \frac{1}{1+z}$

Two alternative radial coordinates r and w: $r = f_K(w)$

$a(t) r \, d\vartheta$: transverse (proper) distance ds at time t

$a(t) \, dw$: radial distance ds at time t

$r \, d\vartheta$: transverse comoving distance

dw : radial comoving distance

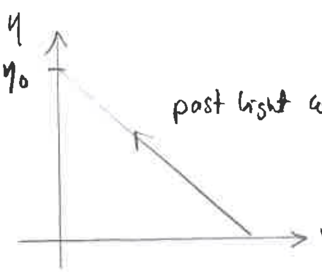
· Conformal time η: $\quad dt \equiv a \, d\eta$

$$ds^2 = a(\eta)^2 \left[-d\eta^2 + \frac{dr^2}{1-Kr^2} + r^2 (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \right]$$

$$= a(\eta)^2 \left[-d\eta^2 + dw^2 + f_K(w)^2 \, (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \right]$$

effects of expansion "eliminated": light rays $(r(\eta), \vartheta(\eta), \varphi(\eta))$ do not depend on $a(\eta)$

$\Rightarrow$ radial light ray has $\quad dw = \pm d\eta = \pm \frac{dt}{a(t)}$



past light cone $w(\eta_0 - \eta) = w(a) = w(z) = \eta_0 - \eta$

- Hubble parameter $H(t)$ and Hubble constant $H_0 \equiv H(t_0)$

$$H \equiv \frac{\dot a}{a} \equiv \frac{1}{a}\frac{da}{dt} = \frac{d\ln a}{dt} = -\frac{d\ln(1+z)}{dt} = -\frac{1}{1+z}\frac{dz}{dt} = -a\frac{dz}{dt}$$

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} = H_0^2\left[\Omega_r a^{-4} + \Omega_m a^{-3} + (1-\Omega_0)a^{-2} + \Omega_\Lambda\right]$$

$$= H_0^2\left[\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + (1-\Omega_0)(1+z)^2 + \Omega_\Lambda\right]$$

$$\Omega_0 - 1 = K H_0^{-2} \qquad H dt = \frac{da}{a} = -\frac{dz}{1+z} \Rightarrow \frac{dz}{H} = -(1+z)dt = -\frac{dt}{a} = -\frac{da}{a^2 H}$$

- Comoving distance between two objects at redshifts z_1 and z_2 along the same line of sight:

$$D^c(z_1, z_2) = \int_{w(z_1)}^{w(z_2)} dw = \int_{t(z_2)}^{t(z_1)} \frac{dt}{a(t)} = \int_{z_1}^{z_2} \frac{dz}{H(z)} = \int_{a_2}^{a_1} \frac{da}{a^2 H(a)} = w(z_2) - w(z_1) = w(z_1, z_2)$$

- Comoving distances along the same line of sight are additive:

$$D^c(z_2) \equiv D^c(0, z_2) = D^c(z_1) + D^c(z_1, z_2)$$

§3.2 Distances in Cosmology

- Here we refer to distances between objects at different redshifts along the same line of sight

Angular diameter distance D^A

$D^A(z)$ from here to z $\equiv \frac{ds}{d\theta} = a(t(z))\,r(z) = \frac{r(z)}{1+z} = \frac{f_K(w(z))}{1+z} = \frac{1}{1+z} f_K(D^C(z))$

$D^A(z_1,z_2)$ from z_1 to z_2 : move origin to object seen at z_1

$\Rightarrow$ w at z_2 is $D^C(z_1,z_2)$ $\Rightarrow$ $D^A(z_1,z_2) = \frac{1}{1+z_2} f_K\left(D^C(z_1,z_2)\right)$

Comoving angular diameter distance D^{CA}

$$D^{CA}(z) = r(z) = f_K(w(z)) = f_K(D^C(z)) = (1+z)D^A(z)$$

$$D^{CA}(z_1,z_2) = f_K\left(D^C(z_1,z_2)\right) = (1+z_2)D^A(z_1,z_2)$$

Luminosity distance $D^L \equiv \sqrt{\frac{L}{4\pi l}}$ (L absolute luminosity, l apparent luminosity)

$$D^L(z) = (1+z)D^{CA}(z) = (1+z)^2 D^A(z)$$

surface brightness $= \frac{\text{flux density}}{\text{solid angle}} \propto \frac{D^A(z)^2}{D^L(z)^2} = (1+z)^{-4}$ decreases fast with increasing redshift

If $K \neq 0$, D^A, D^{CA}, D^L are not additive : $\sin(x+y) \neq \sin x + \sin y$

If $K=0$ (flat universe) then $D^{CA} = D^C$ and thus additive

$$\Rightarrow \quad D^{CA}(z_2) = D^{CA}(z_1) + D^{CA}(z_1,z_2)$$

$$D^A(z_2) = \frac{1+z_1}{1+z_2} D^A(z_1) + D^A(z_1,z_2)$$

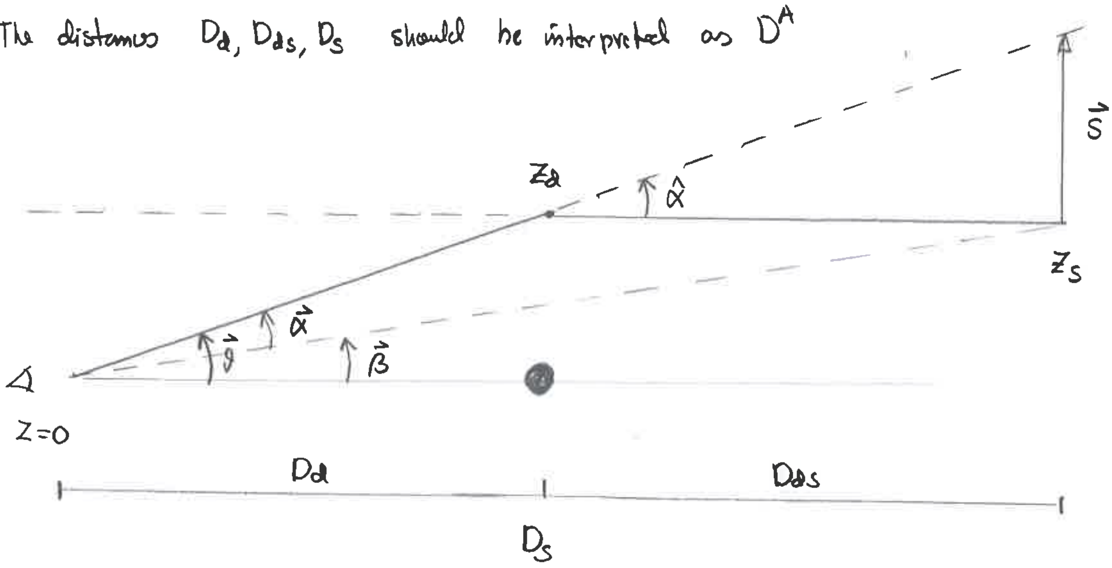
$$D^L(z_2) = \left(\frac{1+z_1}{1+z_2}\right)^2 D^L(z_1) + D^L(z_1,z_2)$$

$\Rightarrow$ $w = r$ $\Rightarrow$ $ds^2 = a(\eta)^2\left[-d\eta^2 + dw^2 + d\theta^2 + \sin^2\theta\, d\varphi^2\right] = a(\eta)^2\left[-d\eta^2 + dx^2 + dy^2 + dz^2\right]$

$\Rightarrow$ light rays travel in those coords exactly as in Minkowski space (as function of η instead of t).

§3.3 Cosmological Distances and Lensing

- The distances $D_\ell, D_{\ell s}, D_s$ should be interpreted as D^A



$$\hat{\alpha} = \frac{\vec{s}}{D^A(z_\ell, z_s)} \qquad \vec{\alpha} = \frac{\vec{s}}{D^A(z_s)} \qquad \Rightarrow \quad \vec{\alpha} = \frac{D^A(z_\ell, z_s)}{D^A(z_s)}\hat{\alpha} = \frac{D^{CA}(z_\ell, z_s)}{D^{CA}(z_s)}\hat{\alpha}$$

$$\Sigma_{cr} = \frac{1}{4\pi G}\frac{D^A(z_s)}{D^A(z_\ell)D^A(z_\ell, z_s)} = \frac{1+z_\ell}{4\pi G}\frac{D^{CA}(z_s)}{D^{CA}(z_\ell)D^{CA}(z_\ell, z_s)}$$

$$\vartheta_E = \sqrt{4GM\frac{D^A(z_\ell, z_s)}{D^A(z_\ell)D^A(z_s)}} = \sqrt{4GM(1+z_\ell)\frac{D^{CA}(z_\ell, z_s)}{D^{CA}(z_\ell)D^{CA}(z_s)}}.$$

- Observations have so far not been able to distinguish our universe from the $K=0$ case
  $\Rightarrow$ K must be small $\Rightarrow$ the difference between D^{CA} and D^C must be small
  $\Rightarrow$ We can approximate $D^{CA} \simeq D^C$, unless we want to do accurate work, e.g., trying to measure K.

- $|\Omega_0 - 1| < 0.005$ (Planck+BAO 2015) $\Rightarrow$ $|K| < 0.005\, H_0^2$

  $R_{curv} = \frac{1}{\sqrt{|K|}} > 14\, H_0^{-1}$

§3.4 Time Delay

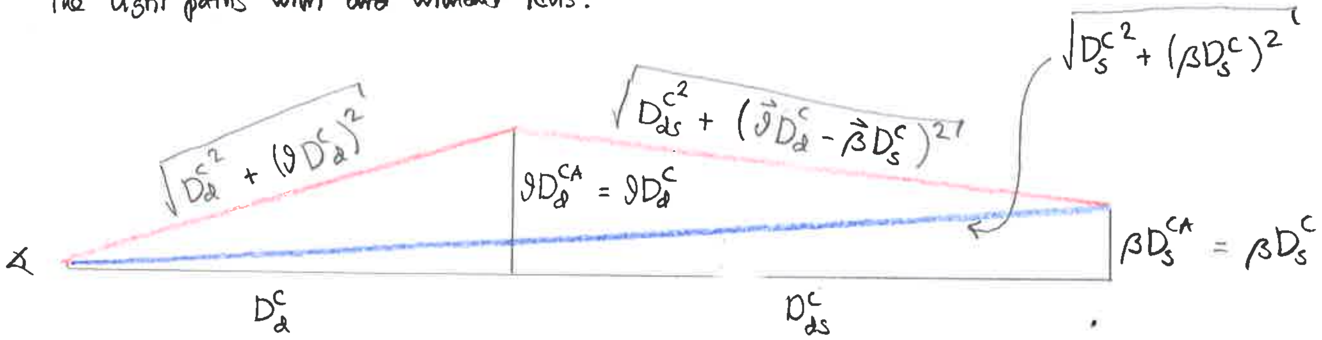
- The light signal from the source to the observer is delayed by the gravitational lens, compared to absence of the lens. The delay can be divided into two parts:

1) "geometric" delay: due to different length of light path in the background FRW geometry

2) "potential" delay: occurs at the lens and is due to local perturbation of the metric by the lens. This is also a geometrical effect, but in perturbation theory the metric perturbation is mapped into potentials.

Geometric delay in the flat (K=0) universe

The light paths with and without lens:



The time delay $\Delta\eta_{geo}$ is given by the difference in the path lengths (in $c=1$ units; using comoving distances and conformal time)

$K=0 \Rightarrow$ Two simplifications: 1) In the transverse distances ϑD_d^{CA} and βD_s^{CA}, $D^{CA} = D^c$ (which is additive); 2) Can use Euclidean trigonometry to calculate the sides of the triangle

$$\therefore\ \Delta\eta_{geo} = \sqrt{D_d^{c\,2} + (\vartheta D_d^c)^2} + \sqrt{D_{ds}^{c\,2} + (\vec{\vartheta} D_d^c - \vec{\beta} D_s^c)^2} - \sqrt{D_s^{c\,2} + (\beta D_s^c)^2}$$

$$= \dots \simeq \frac{1}{2}\frac{D_d^c D_s^c}{D_{ds}^c}(\vec{\vartheta} - \vec{\beta})^2$$ (exercise; use small-angle approximation)

- With $K \neq 0$ one needs spherical/hyperbolical trigonometry. The result is the same, except has D^{CA} instead of D^c:

$$\Delta\eta_{geo} \simeq \frac{1}{2}\frac{D_d^{CA} D_s^{CA}}{D_{ds}^{CA}}(\vec{\vartheta} - \vec{\beta})^2$$

Potential delay in the flat ($K=0$) universe

- Happens locally at the lens. The gravity of the lens perturbs the spacetime geometry locally. In relativistic (cosmological) perturbation theory we find that the metric can be written as (*)

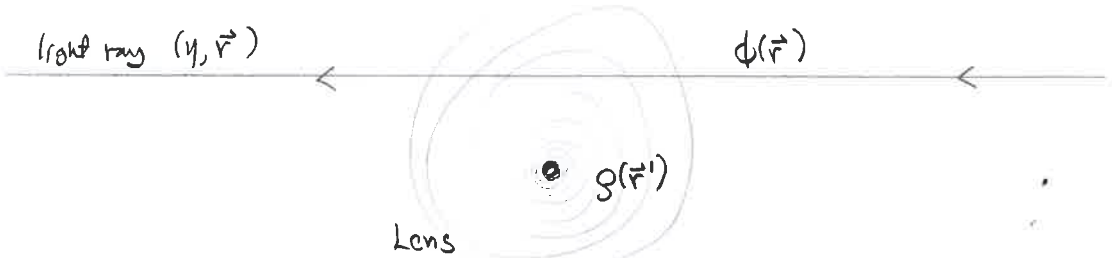
$$ds^2 = a(\eta)^2 \left[-(1+2\phi)\, d\eta^2 + (1-2\psi) \underbrace{(dx^2 + dy^2 + dz^2)}_{r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)} \right]$$

where the Bardeen potentials $\phi(\eta, \vec{r})$ and $\psi(\eta, \vec{r})$ are small, $|\phi| \ll 1$, $|\psi| \ll 1$

In the Newtonian limit $\phi = \psi$ and is equal to the Newtonian gravitational potential

$$\phi(\vec{r}) = -G \int d^3r'\, \rho(\vec{r}\,') \frac{1}{|\vec{r} - \vec{r}\,'|} \quad (**)$$

we can assume this stays constant in time while the light ray passes



For the light ray $ds^2 = 0 \Rightarrow (1+2\phi)\, d\eta^2 = (1-2\psi)\, dx^2$

$$\Rightarrow d\eta = \sqrt{\frac{1-2\psi}{1+2\phi}}\, dx \simeq (1-2\phi)\, dx = (1-2\phi)(1+z_d)\, dr_3$$

(dx: comoving distance; dr_3: physical distance)

For an overdensity, $\phi < 0 \Rightarrow$ the perturbation causes a time delay

$$\Delta\eta_{pot} \simeq -2(1+z_d) \int \phi\, dr_3$$

We use this to compare different light rays passing thru the same lens. The integral should extend far enough from the lens so that the difference in ϕ becomes negligible.

(But not $\int_{-\infty}^{\infty}$; we assume that $\phi = 0$ (no perturbation) on average far from the lens.)

(**) Here r is physical distance, not comoving distance

(*) scalar perturbations, Newtonian gauge

- Define the lensing potential $\psi(\vec{\vartheta}) = \frac{D_{ds}^A}{D_d^A D_s^A} \cdot 2 \int \phi(D_d^A \vec{\vartheta}, r_3)\, dr_3$

$$\therefore \quad \Delta y_{pot} = -(1+z_d) \frac{D_d^A D_s^A}{D_{ds}^A} \psi(\vec{\vartheta}) = - \frac{D_d^{CA} D_s^{CA}}{D_{ds}^{CA}} \psi(\vec{\vartheta})$$

We derived this for $K=0$; but the result holds also for $K \neq 0$

- This lensing potential $\psi(\vec{\vartheta})$ is the same as the deflection potential $\psi(\vec{\vartheta})$ defined earlier (we skip the proof).

$$\therefore \quad \Delta y = \Delta y_{geo} + \Delta y_{pot} = \frac{D_d^{CA} D_s^{CA}}{D_{ds}^{CA}} \left[\frac{1}{2}(\vec{\vartheta} - \vec{\beta})^2 - \psi(\vec{\vartheta}) \right]$$

$$= \frac{D_d^{CA} D_s^{CA}}{D_{ds}^{CA}} \tau(\vec{\vartheta}; \vec{\beta}) = \Delta t$$

(at the observer, since for her $\Delta t = a_0 \Delta y = \Delta y$)

τ is the Fermat potential

- What can be observed (if some observable event happens at the source), is the time delay difference between two images

$$\Delta t(\vec{\vartheta}_1, \vec{\vartheta}_2) = \frac{D_d^{CA} D_s^{CA}}{D_{ds}^{CA}} \left[\tau(\vec{\vartheta}_1; \vec{\beta}) - \tau(\vec{\vartheta}_2; \vec{\beta}) \right]$$

If we understand the lens structure well enough to estimate $\tau(\vec{\vartheta}_1; \vec{\beta}) - \tau(\vec{\vartheta}_2; \vec{\beta})$, the time delay can be used to determine cosmological parameters, especially H_0, on which $D^{CA}(z_d)$, $D^{CA}(z_s)$, $D^{CA}(z_d, z_s)$ depend.