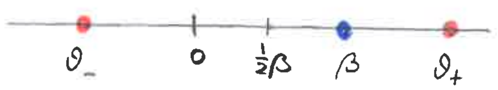
§2.2 Point-Mass Lens

Violates earlier assumptions $|\vec{\alpha}| < \alpha_{max}$ (or κ smooth)

From where we started: $\vec{\alpha} = \frac{D_{ds}}{D_s} 4GM \frac{\vec{\xi}}{\xi^2} = \frac{D_{ds}}{D_s D_d} 4GM \frac{\vec{\vartheta}}{\vartheta^2} = m \frac{\vec{\vartheta}}{\vartheta^2} = \left(\frac{\vartheta_E}{\vartheta}\right)^2 \vec{\vartheta}$

$\Rightarrow \beta = \vartheta - \alpha = \left[1 - \left(\frac{\vartheta_E}{\vartheta}\right)^2\right] \vartheta$ Choose β positive (y); ϑ may have either sign (x)

$x \equiv \frac{\vartheta}{\vartheta_E}$, $y = \frac{\beta}{\vartheta_E}$ $\hookrightarrow$ $y = \left(1 - \frac{1}{x^2}\right) x = x - \frac{1}{x}$ $\Rightarrow$ $x = \frac{1}{2}\left(y \pm \sqrt{y^2+4}\right)$



$x_+ > y, |x_-|$, $x_+ \geq 1$

$-1 \leq x_- < 0$

$\therefore$ one image each side of lens and source

For $\beta = \vartheta_E$, $\vartheta = \frac{1}{2}(1 \pm \sqrt{5})\vartheta_E = \begin{cases} 1.62\,\vartheta_E \\ -0.62\,\vartheta_E \end{cases}$

$\bar{\kappa}(\vartheta) = \frac{m}{\vartheta^2} = \left(\frac{\vartheta_E}{\vartheta}\right)^2 = \frac{1}{x^2}$

$\downarrow$ $\kappa = 0$ outside the mass distribution

magnification $\mu = \frac{1}{\det A} = \frac{1}{(1-\bar{\kappa})(1+\bar{\kappa}-2\kappa)} = \frac{1}{1-\bar{\kappa}^2} = \frac{1}{1-x^{-4}}$

$$= \pm\frac{1}{4}\left[\underbrace{\frac{y}{\sqrt{y^2+4}} + \frac{\sqrt{y^2+4}}{y}}_{>2} \pm 2\right]$$ (exercise, may be tedious)

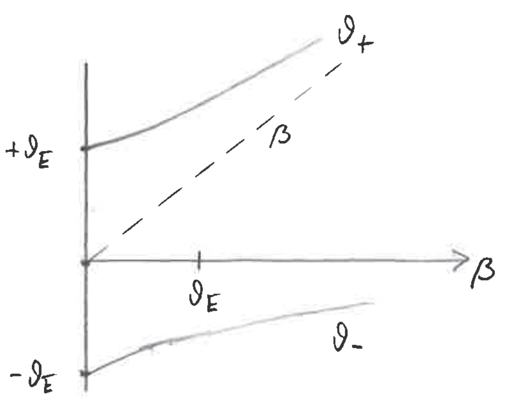
>2 ($\to 2$ as $y \to \infty$; so that $\mu_+ \to 1$ and $\mu_- \to 0$)

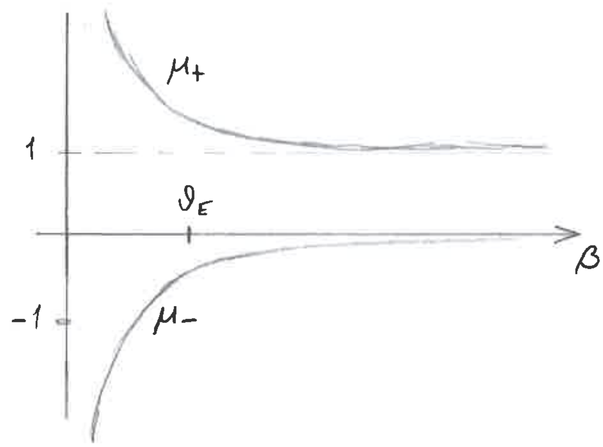
$x_+ > 1 \Rightarrow \mu_+ > 1$ this image is always magnified

$\mu_- < 0$ mirror image, may be magnified or demagnified

As $y \to 0$, $x_+ \to 1$ ($\vartheta_+ \to \vartheta_E$) and $\mu_+ \to \infty$

$x_- \to -1$ ($\vartheta_- \to -\vartheta_E$) and $\mu_- \to -\infty$

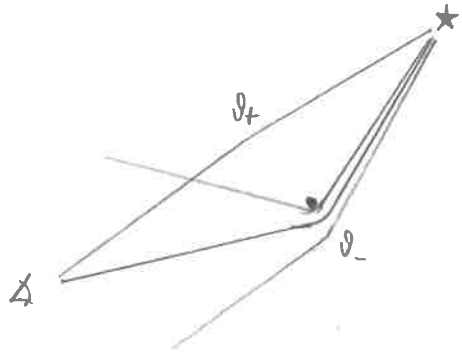




Total magnification $\mu_p \equiv \mu_+ + |\mu_-| = \dfrac{y^2+2}{y\sqrt{y^2+4}}$

For $y=1$: $\mu_+ = 1.171$, $\mu_- = -0.171$ $\Rightarrow$ $\mu_p = 1.342$

- The separation between images is $\sqrt{y^2+4}\,\vartheta_E \geq 2\vartheta_E$, but in practice not much larger, since for $y \gg 1$, $|\mu_-| \ll 1$ and this image cannot be seen.



- Odd number theorem? The above applies also to an extended mass M; but only for light rays that stay outside it. If the ϑ_- ray passes through M, it is deflected less and does not produce an image. If it passes outside M, there will be a third ray, passing through M, producing a third image. For a compact mass with $\kappa \gg 1$,

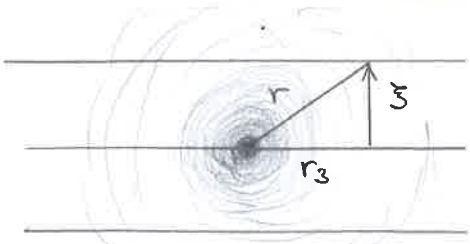
$$\mu = \frac{1}{(1-\bar{\kappa})(1+\bar{\kappa}-2\kappa)} \sim \frac{1}{(1-\kappa)^2} \ll 1$$ for this third image.

§2.3 Singular Isothermal Sphere (SIS)

- The simplest model for the density profile of galaxies and clusters

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2} \propto r^{-2}$$

produces flat rotation curves (exercise)



$\Rightarrow$ surface mass density (exercise)

$$\Sigma(\xi) = \int_{-\infty}^{\infty} dr_3\, \rho\left(\sqrt{\xi^2 + r_3^2}\right) = \frac{\sigma_v^2}{2G} \cdot \frac{1}{\xi} \propto \xi^{-1}$$

$$\kappa(\vartheta) = 2\pi \frac{D_{ds}}{D_s} \sigma_v^2 \cdot \frac{1}{|\vartheta|}$$

$$\bar{\kappa}(\vartheta) = 2\kappa(\vartheta)$$

$\Rightarrow 1 + \bar{\kappa} - 2\kappa = 1$

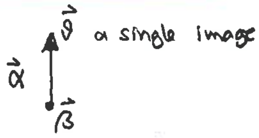
no radial critical curves

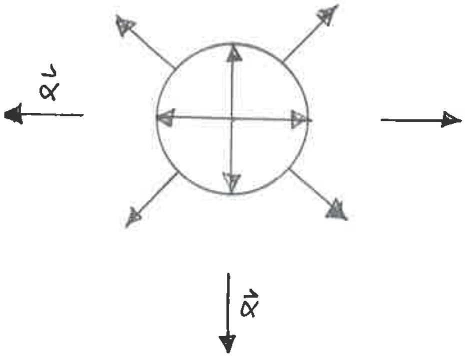
Tangential critical curve $\bar{\kappa} = 1 \Rightarrow \vartheta = \vartheta_E = 4\pi \frac{D_{ds}}{D_s} \sigma_v^2$

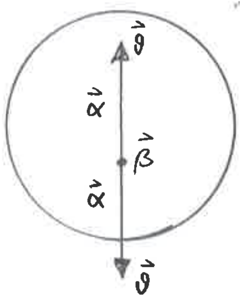
$$\therefore \quad \kappa(\vartheta) = \frac{1}{2} \frac{\vartheta_E}{|\vartheta|} \qquad \bar{\kappa}(\vartheta) = \frac{\vartheta_E}{|\vartheta|} \qquad \sqrt{\gamma_1^2 + \gamma_2^2} = |\bar{\kappa} - \kappa| = \kappa = \frac{1}{2}\frac{\vartheta_E}{|\vartheta|}$$

$$\vec{\alpha}(\vec{\vartheta}) = \bar{\kappa}(\vartheta)\vec{\vartheta} = \vartheta_E \frac{\vec{\vartheta}}{|\vec{\vartheta}|}$$

has constant magnitude ϑ_E







two images

Lens equation $\quad \beta = \vartheta - \alpha = \vartheta - \vartheta_E \cdot \frac{\vartheta}{|\vartheta|}$ or $y = x - \frac{x}{|x|}$ where $x \equiv \frac{\vartheta}{\vartheta_E}$, $y \equiv \frac{\beta}{\vartheta_E}$

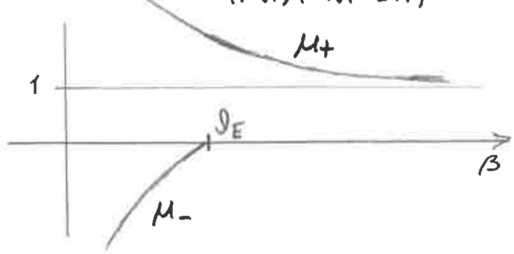
choose $y > 0$

$$\left.\begin{array}{ll} \text{for } x > 0: & y = x - 1 \Rightarrow x = y + 1 \\ \text{for } x < 0: & y = x + 1 \Rightarrow x = y - 1 \end{array}\right\} \Rightarrow \begin{array}{l} \text{two solutions } x = y \pm 1 \text{ for } 0 < y < 1 \\ \text{one solution } x = y + 1 \text{ for } y > 1 \end{array}$$

Magnification $\quad \mu = \frac{1}{(1-\bar{\kappa})(1+\bar{\kappa}-2\kappa)} = \frac{1}{1-\bar{\kappa}} = \frac{|x|}{|x|-1}$

$$\mu_+ = \frac{y+1}{y} > 0$$

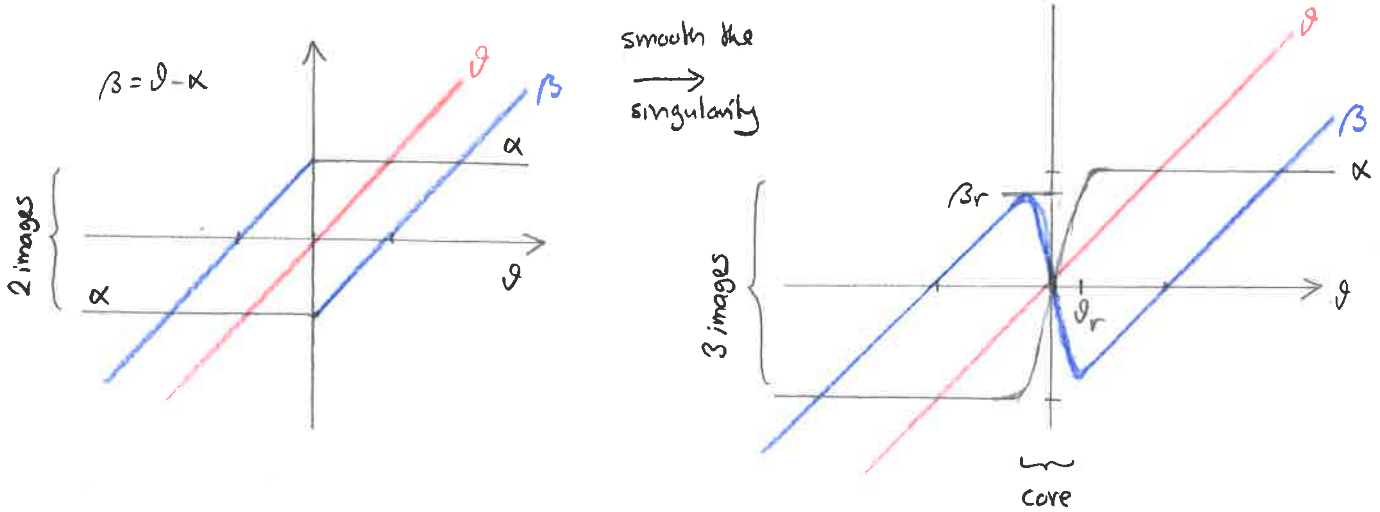
$$\mu_- = \frac{y-1}{-y} < 0$$



- Two strange features: 1) Odd-number theorem violated

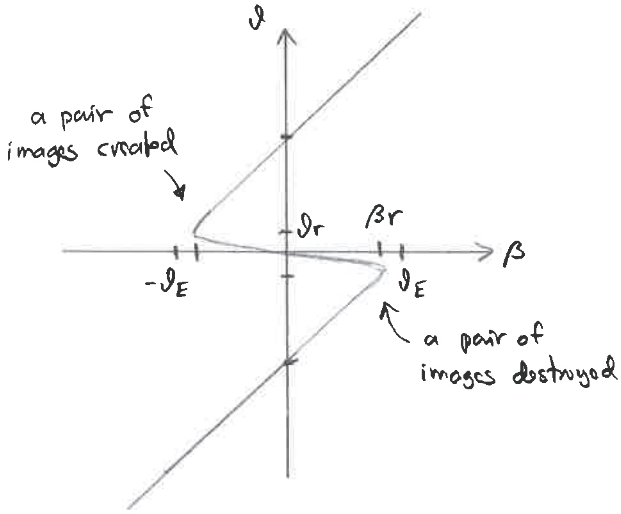
  2) # images changes by 1, when source crosses critical curve, not caustic

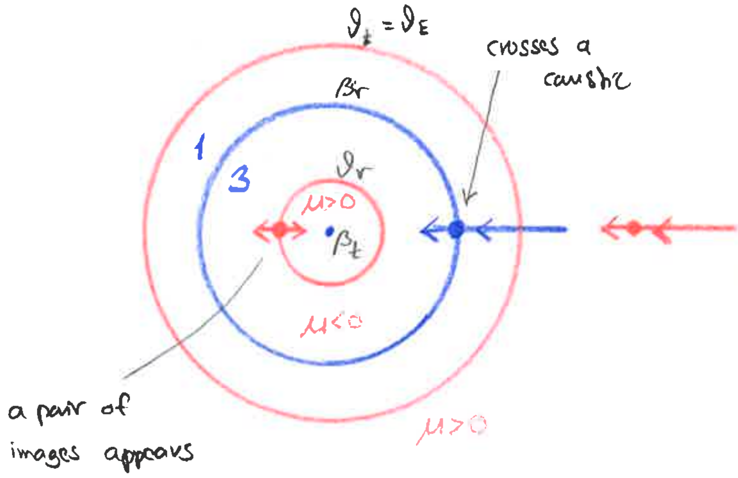
  Due to singularity $\rho(r) \to \infty$ at $r \to 0$ $\Rightarrow$ $\vec{\alpha}$ not continuous at $\vec{\vartheta} = 0$



Smooth the singularity into finite-density core $\Rightarrow$ $\vec{\alpha}$ changes continuously

$\Rightarrow$ a radial critical curve $\frac{d\beta}{d\vartheta} = 0$ and caustic appear at $\beta_r < \vartheta_E$, ϑ_r small





- If the core region is small $\Rightarrow$ $\left|\frac{d\beta}{d\vartheta}\right|$ and $\left|\frac{\beta}{\vartheta}\right| \gg 1$ $\Rightarrow$ third image strongly demagnified

  Parity of images determined by signs of $\frac{d\beta}{d\vartheta}$ and $\frac{\beta}{\vartheta}$: both negative for third image $\Rightarrow \mu > 0$

§2.4 Non-Symmetric Lenses

- Qualitative details of centrally condensed axisymmetric lenses do not depend strongly on the radial profile
- Breaking the symmetry leads to qualitatively new properties:

  central caustic point → finite curve, a source inside it may have 5 images

Many observed lenses have 4 images; probably the 5th is invisible due to strong demagnification at the center.