General Properties of Axisymmetric Lenses

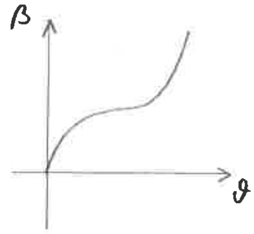
- Assumptions:
  - A) $\alpha \to 0$ as $\vartheta \to \infty$
  - B) α is bounded: $|\alpha| \le \alpha_{max}$
  - C) α is differentiable $\Rightarrow$ $\alpha(0) = 0$, since $\alpha(-\vartheta) = -\alpha(\vartheta)$

Schneider does not require that κ is smooth; but I needed $\kappa \to$ finite $\kappa(0)$ as $\vartheta \to 0$ for property 6. We also assume $\kappa, \bar{\kappa} \ge 0$.

1) For large enough β, $\exists$ only a single image at $\vartheta \simeq \beta$ (seems natural; proof in SEF)

2) A lens can produce multiple images $\Leftrightarrow$ $1 + \bar{\kappa} - 2\kappa = \dfrac{d\beta}{d\vartheta} < 0$ somewhere

"can produce": there's such a location in the source plane



Proof: If $\dfrac{d\beta}{d\vartheta} \ge 0$ everywhere then $\beta(\vartheta)$ is monotonic

and can be inverted to give a unique image at ϑ for each β

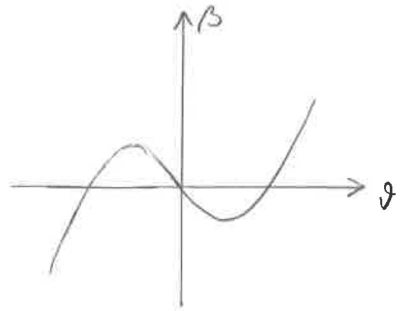
(if $\dfrac{d\beta}{d\vartheta} = 0$ the image is stretched but not split). This proves "$\Rightarrow$"

$\alpha(-\vartheta) = -\alpha(\vartheta)$ $\Rightarrow$ $\beta(-\vartheta) = -\vartheta - \alpha(-\vartheta) = -\beta(\vartheta)$, so also $\beta(0) = 0$

α and β are odd $\Rightarrow$ $\dfrac{d\alpha}{d\vartheta}$ and $\dfrac{d\beta}{d\vartheta}$ are even

If $\dfrac{d\beta}{d\vartheta} < 0$ somewhere then $\exists$ pair of values ϑ where $\dfrac{d\beta}{d\vartheta} = 0$

since $\dfrac{d\beta}{d\vartheta} \to 1$ (and $\beta \to \pm\infty$) as $\vartheta \to \pm\infty$



$\Rightarrow$ β has a local maximum and a local minimum,

and between those values produces at least 3 images

This proves "$\Leftarrow$". $\square$

$\therefore$ $\exists$ radial critical curve

3) $\kappa > \frac{1}{2}$ somewhere is necessary for multiple images

Proof: $\frac{d\beta}{d\theta} = 1 + \bar{\kappa} - 2\kappa < 0 \quad \Rightarrow \quad \kappa > \frac{1}{2}(1+\bar{\kappa}) > \frac{1}{2}$ (* □

4) $\kappa > 1$ somewhere is sufficient for multiple imaging

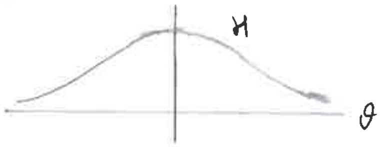
Already proven on p. 1-12 for general lenses. For the axisymmetric case it is simpler: Assume $\kappa > 1$ somewhere.

Since $\alpha(\theta) = \bar{\kappa}(\theta)\theta \to 0$ as $\theta \to \pm\infty \Rightarrow \bar{\kappa}(\theta) \to 0$, κ must have a maximum at some θ_m (**

Thus $\kappa(\theta_m) > 1$ and $\kappa(\theta_m) \geq \bar{\kappa}(\theta_m)$

$$\Rightarrow \frac{d\beta}{d\theta}(\theta_m) = 1 + \bar{\kappa}(\theta_m) - 2\kappa(\theta_m) = \underbrace{[1-\kappa(\theta_m)]}_{<0} + \underbrace{[\bar{\kappa}(\theta_m) - \kappa(\theta_m)]}_{\leq 0} < 0 \quad \square$$

· Centrally condensed lenses $\equiv \kappa'(\theta) \leq 0$ for $\theta \geq 0$



5) These can produce multiple images $\Leftrightarrow \kappa(0) > 1$

Sufficiency shown already. Necessity: If $\kappa(0) \leq 1$ then $\kappa \leq 1$ everywhere

For $\theta \geq 0$: $\kappa'(\theta) \leq 0 \Rightarrow \bar{\kappa}'(\theta) \leq 0$ and we have then

$$\frac{d\beta}{d\theta} = \underbrace{1 - \bar{\kappa}}_{\geq 0} \underbrace{- \bar{\kappa}'\theta}_{\geq 0} \geq 0 \text{ everywhere} \Rightarrow \text{no multiple images} \quad \square$$

6) These can produce multiple images $\Leftrightarrow \frac{d\alpha}{d\theta}(0) > 1 \qquad \left(\Leftrightarrow \frac{d\beta}{d\theta}(0) < 0\right)$

Assuming $\lim_{\theta \to 0} \kappa = \kappa(0)$ is finite $\Rightarrow \bar{\kappa}(0) = \kappa(0)$ and $\frac{d\alpha}{d\theta}(0) = 2\bar{\kappa}(0) - \kappa(0) = \kappa(0)$ □

**) I am not sure that $\bar{\kappa} \to 0$ as $\theta \to \pm\infty$ proves that κ must have a maximum; but it would require rather pathological behavior of κ for it not to have a maximum.

*) Note that here we assume $\kappa, \bar{\kappa} > 0$. In the cosmological context they may be negative, as density refers to density perturbation.