§1.3 Magnification and Distortion

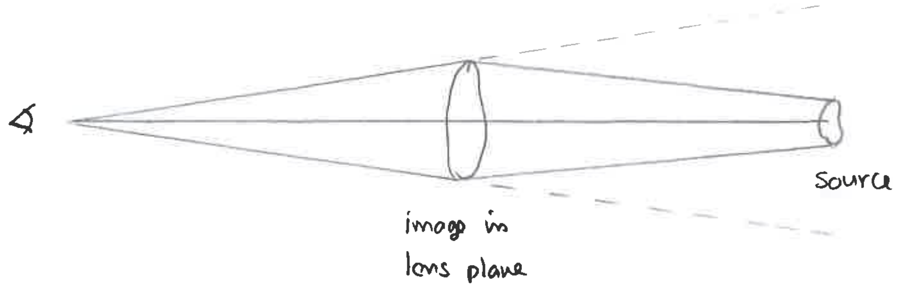
- The lens equation $\vec{\beta}(\vec{\vartheta}) = \vec{\vartheta} - \vec{\alpha}(\vec{\vartheta})$ maps image to source $\vec{\vartheta} \to \vec{\beta}$

  With $\vec{\beta}$ given, it may have several solutions $\vec{\vartheta}$, i.e. multiple images (strong lensing);

  but a given image $\vec{\vartheta}$ always has a single source $\vec{\beta}$.

- Image shape differs from source shape: distortion determined by $\vec{\beta}(\vec{\vartheta})$, or its inverse.

- Surface brightness $\left(\frac{W}{m^2 . sr}\right)$ conserved in lensing: if lensing magnifies image, it also focuses more photons towards the observer.



- If source smaller than scale over which lens properties change, linearize $\vec{\beta}(\vec{\vartheta})$ locally:

  distortion given by Jacobian matrix

$$\boxed{A(\vec{\vartheta}) \equiv \frac{\partial \vec{\beta}}{\partial \vec{\vartheta}}} \qquad \boxed{A_{ij} = \frac{\partial \beta_i}{\partial \vartheta_j} = \frac{\partial \vartheta_i}{\partial \vartheta_j} - \frac{\partial \alpha_i}{\partial \vartheta_j} = \delta_{ij} - \frac{\partial^2 \psi}{\partial \vartheta_i \partial \vartheta_j}}$$

real symmetric 2x2 matrix $\Rightarrow$ 3 independent components; name them:

$$\boxed{A(\vec{\vartheta}) \equiv \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix} = \begin{bmatrix} 1-\kappa & \\ & 1-\kappa \end{bmatrix} + \underbrace{\begin{bmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & +\gamma_1 \end{bmatrix}}_{\text{shear, a symmetric traceless 2x2 matrix}}}$$

$$\Rightarrow \gamma_1 = \frac{1}{2}(A_{22} - A_{11}) = \frac{1}{2}(\psi_{,11} - \psi_{,22})$$

$$\gamma_2 = -A_{12} = \psi_{,12}$$

$$1-\kappa = \frac{1}{2}\mathrm{tr} A = 1 - \frac{1}{2}(\psi_{,11} + \psi_{,22}) \Rightarrow \kappa = \frac{1}{2}(\psi_{,11} + \psi_{,22}) = \frac{1}{2}\nabla^2\psi$$

= the convergence defined earlier

$$\psi_{,ij} = \begin{bmatrix} \kappa+\gamma_1 & \gamma_2 \\ \gamma_2 & \kappa-\gamma_1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix}$$

real symmetric matrix: eigenvalues are real and eigenvectors are orthogonal

Eigenvalues:

$$\boxed{\begin{aligned} a_1 &= 1-\kappa+\sqrt{\gamma_1^2+\gamma_2^2} \\ a_2 &= 1-\kappa-\sqrt{\gamma_1^2+\gamma_2^2} \end{aligned}}$$

$$\det A = a_1 a_2 = (1-\kappa)^2 - (\gamma_1^2+\gamma_2^2) \qquad \mathrm{tr}\, A = a_1 + a_2 = 2(1-\kappa)$$

- If $\det A \neq 0$ the Jacobian is invertible:

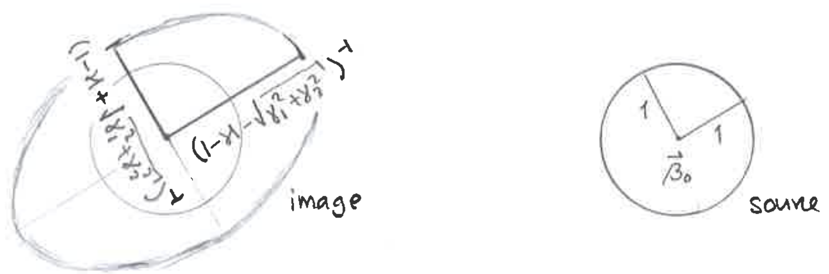
magnification tensor $M(\vec{\theta}) = A(\vec{\theta})^{-1}$ $\qquad \det M = \dfrac{1}{\det A}$

same eigenvectors; eigenvalues are

$$\mu_1 = \frac{1}{1-\kappa+\sqrt{\gamma_1^2+\gamma_2^2}}$$

$$\mu_2 = \frac{1}{1-\kappa-\sqrt{\gamma_1^2+\gamma_2^2}}$$

The image is stretched in the direction of the eigenvectors by those factors



$\therefore$ image area is magnified by $\mu \equiv \det M = \mu_1 \mu_2 = \dfrac{1}{\det A}$

Surface brightness conserved $\Rightarrow$ apparent luminosity increased by the same factor μ

- For strong lenses μ may be negative. Sign of $\mu \equiv$ parity of image

$\mu < 0 \iff$ mirror image (the two eigenvalues have opposite sign)

- Reduced shear

Rewrite A as $\quad A(\vec{\theta}) = (1-\kappa)\begin{bmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{bmatrix}$ where $g_i \equiv \dfrac{\gamma_i}{1-\kappa}$ reduced shear

distortion $\dfrac{\mu_2}{\mu_1} = \dfrac{1+\sqrt{g_1^2+g_2^2}}{1-\sqrt{g_1^2+g_2^2}}$

- Weak lensing: $\kappa, |\gamma_i| \ll 1 \Rightarrow a_1, a_2, \det A > 0$ always

Strong lensing: (some of) these are $\mathcal{O}(1)$ or greater $\Rightarrow$ may have $\det A \leq 0$

§1.4 Critical Curves and Caustics, and General Properties of Lenses

- $\det A = \text{const}$ are smooth (does this require ψ is smooth?) curves on the image plane

  critical curves: $\det A = 0$

  - cannot extend far from lens, since require strong lensing (for most of sky, lensing is weak)

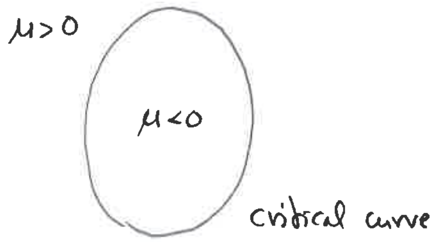
    $\Rightarrow$ they are closed curves

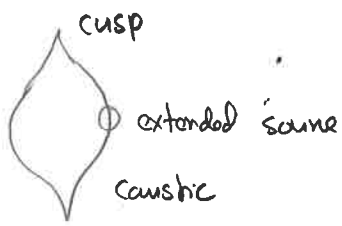
  caustics: corresponding curves $\vec{\beta}(\vec{\theta})$ on the source plane

  - not necessarily smooth, may have cusps

  $\det A = 0 \Rightarrow \mu = \pm\infty$ formally, but this applies only to point-like objects (and for these the geometrical optics (light rays) approximation breaks down); extended sources have only an infinitesimal part exactly on caustic

  $\therefore$ sources near caustics may be highly magnified





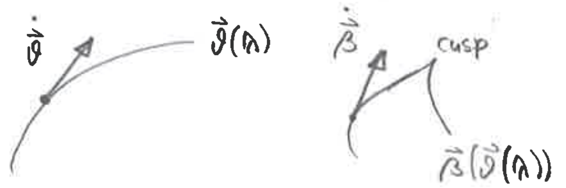
$\Rightarrow$ one of the eigenvalues a_1, a_2 is zero; image highly stretched in the direction of corresponding eigenvector

- Moving across critical curve changes sign of μ

- Show that caustics may not be smooth:

critical curve $\vec{\vartheta}(\lambda)$ $\Rightarrow$ caustic $\vec{\beta}(\vec{\vartheta}(\lambda)) = \vec{\vartheta}(\lambda) - \vec{\alpha}(\vec{\vartheta}(\lambda))$

tangent vector $\dot{\vec{\vartheta}}(\lambda) = \frac{d\vec{\vartheta}}{d\lambda}$ and $\frac{d\vec{\beta}}{d\lambda} = \frac{\partial \beta_i}{\partial \vartheta_j} \frac{\partial \vartheta_j}{\partial \lambda} = A_{ij} \dot{\vartheta}_j = A \cdot \dot{\vec{\vartheta}}$



which vanishes if $\dot{\vec{\vartheta}} \parallel$ eigenvector w zero eigenvalue

$\Rightarrow$ cusp

source near cusp $\Rightarrow$ image an arc along critical curve

- Number of images:

Assume lens is such that $\vec{\alpha} \to 0$ as $\vec{\vartheta} \to \infty$ and $|\alpha| \leq \alpha_{max}$ (excludes point masses)

$\Rightarrow$ at large distances a single image with $\vec{\vartheta} \simeq \vec{\beta}$

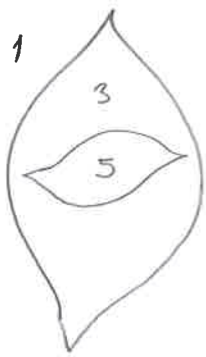
$\det A \neq 0$ $\Rightarrow$ $\vec{\beta}(\vec{\vartheta})$ invertible $\Rightarrow$ # images cannot change

source crosses caustic $\Rightarrow$ a pair of images is created or destroyed (see SEF Fig 5.1 on p.167)

$\therefore$ number of images (of point sources) is odd

"A source close to, and on the inner side (the side with more images) of a caustic possesses a pair of images with high and almost equal magnification, with opposite parity, on either side of the critical curve, in addition to any other images."

- Once caustics are known on the source plane, we know the number of images



Fermat potential: $\tau(\vec{\vartheta};\vec{\beta}) \equiv \frac{1}{2}(\vec{\vartheta}-\vec{\beta})^2 - \psi(\vec{\vartheta})$

considered as a function of the lens plane $\vec{\vartheta}$, with $\vec{\beta}$ kept as a fixed parameter

(i.e., here $\vec{\beta}$ is not assumed to be the $\vec{\beta}(\vec{\vartheta}) = \vec{\vartheta} - \alpha(\vec{\vartheta})$ of the lens equation)

$$\nabla\tau = \frac{\partial\tau}{\partial\vartheta_i} = \partial_i\tau = \frac{1}{2}\partial_i\left[(\vartheta_j-\beta_j)(\vartheta_j-\beta_j)\right] - \partial_i\psi = \delta_{ij}(\vartheta_j-\beta_j) - \partial_i\psi$$

$$= \vec{\vartheta} - \vec{\beta} - \nabla\psi = \vec{\vartheta} - \vec{\beta} - \vec{\alpha} \qquad (\partial_i\vartheta_j = \delta_{ij},\ \partial_i\beta_j = 0)$$

$\Rightarrow$ Stationary points of τ, $\nabla\tau = 0$ give the images of $\vec{\beta}$

- Turns out that $\tau(\vec{\vartheta};\vec{\beta})$ gives the light travel time $(= a\tau + b)$ from $\vec{\beta}$ via $\vec{\vartheta}$ to observer. Fermat principle

- Is $\nabla\tau = 0$ minimum, maximum, or saddle point?

$$\partial_i\partial_j\tau = \partial_i(\vartheta_j - \beta_j - \partial_j\psi) = \delta_{ij} - \partial_i\partial_j\psi = A_{ij}$$

Types of Images: classify according to nature of stationary point at $\tau(\vec{\theta};\vec{\beta})$

$\partial_i\partial_j\tau = A_{ij}$ $\Rightarrow$ determined by signs of a_1, a_2

$a_1, a_2 > 0$: minimum image, $\det A > 0$, $\mathrm{tr}\, A > 0$ $\Rightarrow$ $\kappa < 1$

$a_1, a_2 < 0$: maximum image, $\det A > 0$, $\mathrm{tr}\, A < 0$ $\Rightarrow$ $\kappa > 1$

different signs : saddle point image, $\det A < 0$

Odd-Number Theorem (Burke 1981): Assume κ smooth and decreases faster than $|\theta|^{-1}$ as $\vec{\theta}\to\infty$ and source not on caustic. Then

extremum images $(\det A > 0)$ = # saddle point images $(\det A < 0)$ + 1

$\therefore$ total # images is odd

$\therefore$ minimum # images is 1 and this is an extremum image (minimum, see below)

Any source must have a minimum image.

Proof: $\tau \equiv \frac{1}{2}(\vec{\theta}-\vec{\beta})^2 - \psi(\vec{\theta}) \sim \frac{1}{2}|\vec{\theta}|^2$ as $\vec{\theta}\to\infty$, i.e., increases at large $\vec{\theta}$

Since τ is smooth, it must have a minimum somewhere. $\square$

$\left[\psi(\vec{\theta}) \equiv \frac{1}{\pi}\int d^2\theta'\, \kappa(\vec{\theta}')\, \underbrace{\ln|\vec{\theta}-\vec{\theta}'|}_{\text{for large } \vec{\theta}, \text{ this is } \sim \ln|\vec{\theta}|} \right.$ so cannot change the large-$\vec{\theta}$ behaviour $]$

Magnification Theorem (Schneider 1984): A minimum image is magnified, provided that $\kappa \geq 0$

Proof: Magnification $\mu = \dfrac{1}{\det A} = \dfrac{1}{(1-\kappa)^2 - (\gamma_1^2+\gamma_2^2)} > 0$, since $\det A > 0$ for minimum

$(1-\kappa)^2 - (\gamma_1^2+\gamma_2^2) \leq (1-\kappa)^2 \leq 1$ since $0 \leq \kappa \leq 1$ $(\mathrm{tr}\, A > 0 \Rightarrow \kappa < 1)$

$\Rightarrow$ $\mu > 1$ $\square$

Conditions for Multiple Imaging

1) Isolated transparent lens can produce multiple images $\Leftrightarrow$ $\exists \vec{\vartheta}$ such that $\det A(\vec{\vartheta}) < 0$

Proof: If $\det A > 0$ everywhere, $\vec{\beta}(\vec{\vartheta})$ is invertible everywhere $\Rightarrow$ no multiple images.

If $\det A(\vec{\vartheta}_0) < 0$ then a source at $\vec{\beta}(\vec{\vartheta}_0)$ has a saddle point image at $\vec{\vartheta}_0$

$\Rightarrow$ by odd-number theorem it has at least 3 images. $\square$

2) A sufficient but not necessary condition for multiple images: $\exists \vec{\vartheta}$ such that $\kappa(\vec{\vartheta}) > 1$

Proof: If $\kappa(\vec{\vartheta}_0) > 1$, then source at $\vec{\beta}(\vec{\vartheta}_0)$ has a non-minimum image at $\vec{\vartheta}_0$

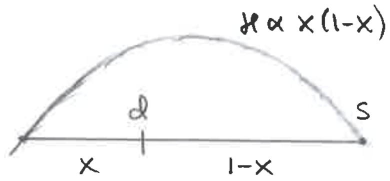
$\Rightarrow$ there has to be another (minimum) image. $\square$

Final note: For a given lens, $\kappa(\vec{\vartheta}) = \dfrac{\Sigma(\vec{\vartheta})}{\Sigma_{cr}}$, where $\Sigma_{cr} = \dfrac{1}{4\pi G} \dfrac{D_s}{D_d D_{ds}}$

depends on distance to source.

Σ_{cr} is large if D_{ds} is small and becomes smaller $\left(\rightarrow \dfrac{1}{4\pi G D_d}\right)$ for more distant sources

$\therefore$ A lens may be weak for sources closer to it and strong for more distant sources



• Fix source distance D_s and move lens (with fixed Σ)

$$\kappa = 4\pi G \frac{D_d D_{ds}}{D_s} \Sigma = 4\pi G D_s \Sigma \cdot x(1-x)$$

$\Rightarrow$ lens of fixed surface density is the most effective when placed half-way