§5.5 E-Modes and B-Modes

- As we have seen, the shear field $\gamma(\vec{\vartheta})$ cannot be an arbitrary complex (or 2-component real) field, but it satisfies certain constraint relations, which are due to it arising from a single scalar potential $\psi(\vec{\vartheta})$. The constraint can be expressed in terms of a vector field $\vec{u}_\gamma$.

- In this section we deal with lots of derivatives of $\psi(\vec{\vartheta})$; so we introduce a further simplified notation for them (dropping the comma ','):

$$\frac{\partial^2 \psi}{\partial\vartheta_i \partial\vartheta_j} \equiv \psi_{,ij} \equiv \psi_{ij} \quad ; \quad \psi_{,ijk} = \psi_{ijk} \quad \text{etc}$$

- Remember

$$\kappa = \tfrac{1}{2}\nabla^2\psi = \tfrac{1}{2}(\psi_{11}+\psi_{22}) \qquad \gamma_{1,1} = \tfrac{1}{2}(\psi_{111}-\psi_{122}) \qquad \gamma_{2,1} = \psi_{112}$$

$$\gamma_1 = \tfrac{1}{2}(\psi_{11}-\psi_{22}) \quad \gamma_2 = \psi_{12} \quad \Rightarrow \quad \gamma_{1,2} = \tfrac{1}{2}(\psi_{112}-\psi_{222}) \qquad \gamma_{2,2} = \psi_{122}$$

We define $\vec{u}_\gamma$ as $\nabla\kappa$ (which guarantees it is a vector field):

$$\nabla\kappa = \begin{pmatrix} \kappa_{,1} \\ \kappa_{,2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \psi_{111}+\psi_{122} \\ \psi_{112}+\psi_{222} \end{pmatrix} = \underline{\begin{pmatrix} \gamma_{1,1}+\gamma_{2,2} \\ \gamma_{2,1}-\gamma_{1,2} \end{pmatrix} \equiv \vec{u}_\gamma(\vec{\vartheta})} \tag{38}$$

This gives a local relation between (derivatives of) κ and γ.

$$\therefore \quad \nabla^2\kappa = \nabla\cdot\vec{u}_\gamma \qquad \text{and} \qquad \underline{\nabla\times\vec{u}_\gamma} = \nabla\times\nabla\kappa \ \underline{= 0}$$

- Thus the shear field satisfies the constraint $\boxed{\nabla\times\vec{u}_\gamma = 0}$ (39)

Another constraint relation, of the same origin, for the shear field is (from Eq. 36)

$$\int_0^\infty d\vartheta\,\vartheta\,\xi_+(\vartheta)\,J_0(l\vartheta) = \int_0^\infty d\vartheta\,\vartheta\,\xi_-(\vartheta)\,J_4(l\vartheta) \tag{40}$$

- However, the measured shear field based on galaxy image ellipticities, does not necessarily satisfy the constraints (39) or (40); and therefore also other relations from §5.4 do not have to hold for it. This may be due to a number of reasons:

  1) statistical error ("noise") due to ellipticities of sources

  2) systematic error (intrinsic correlations of source ellipticities or between source ellipticity and shear)

  3) higher order effects (Born apx not valid; Schneider mentions also "clustering of sources")

Also, e.g., instrumental effects may contribute to statistical / systematic error.

The difference between statistical and systematic error is that the former can be reduced by increasing the size of the survey (more galaxy images).

- We can provide a description of the measured, unconstrained, shear field in terms of a potential by introducing a complex potential

$$\psi = \psi^E + i\psi^B \qquad \text{where } \psi^E, \psi^B \text{ are real} \tag{41}$$

Now ψ^B provides the missing degree of freedom. Defining

$$\gamma = \tfrac{1}{2}(\psi_{,11}-\psi_{,22}) + i\psi_{,12} = \tfrac{1}{2}(\psi^E_{,11}-\psi^E_{,22}) - \psi^B_{,12} + i\left[\psi^E_{,12} + \tfrac{1}{2}(\psi^B_{,11}-\psi^B_{,22})\right] \tag{42}$$

$$\Rightarrow \quad \gamma_1 \equiv \mathrm{Re}\,\gamma = \underbrace{\tfrac{1}{2}(\psi^E_{,11}-\psi^E_{,22})}_{\gamma_1^E} - \underbrace{\psi^B_{,12}}_{\gamma_1^B} \qquad \gamma_2 \equiv \mathrm{Im}\,\gamma = \underbrace{\psi^E_{,12}}_{\gamma_2^E} + \underbrace{\tfrac{1}{2}(\psi^B_{,11}-\psi^B_{,22})}_{\gamma_2^B}$$

We also define a complex convergence $\kappa = \kappa^E + i\kappa^B$ by

$$\kappa = \tfrac{1}{2}\nabla^2\psi \quad \Rightarrow \quad \kappa^E = \tfrac{1}{2}\nabla^2\psi^E, \quad \kappa^B = \tfrac{1}{2}\nabla^2\psi^B \tag{43}$$

The complex κ and ψ are here not assumed to have any particular physical origin; their role is just to provide a description for the measured shear field.

- We define the vector field $\vec{u}_\gamma(\vec{\theta})$ in terms of the real γ_1 and γ_2; so the definition is still

$$\boxed{\vec{u}_\gamma(\vec{\theta}) \equiv \begin{pmatrix} \gamma_{1,1} + \gamma_{2,2} \\ \gamma_{2,1} - \gamma_{1,2} \end{pmatrix}} \qquad (44)$$

so that $\vec{u}_\gamma$ is real. The way γ_1 and γ_2 were defined in (42) means, that now $\vec{u}_\gamma \neq \nabla\chi$ and thus $\nabla\times\vec{u}_\gamma \neq 0$. (*) Since $\vec{u}_\gamma$ is a 2D vector field, $\nabla\times\vec{u}_\gamma$ is a scalar: $\nabla\times\vec{u}_\gamma = u_{2,1} - u_{1,2}$

- Calculating (exercise)

$$\begin{aligned} \nabla\cdot\vec{u}_\gamma &= u_{1,1} + u_{2,2} = \gamma_{1,11} + \gamma_{2,21} + \gamma_{2,12} - \gamma_{1,22} = \dots = \nabla^2\chi^E \\ \nabla\times\vec{u}_\gamma &= u_{2,1} - u_{1,2} = \dots = \nabla^2\chi^B \end{aligned} \qquad (45)$$

∴ We divide the measured shear field γ in two parts (modes):

E) The part with $\nabla\times\vec{u}_\gamma = 0$, which has the properties expected of shear, and thus provides our estimate of the true shear.

B) The part with $\nabla\cdot\vec{u}_\gamma = 0$, which presumably is due to error (systematic + statistical) in shear measurement and higher-order effects.

The error & higher-order effects seen in B presumably contribute in similar magnitude to the E mode; thus the measured B mode provides an error estimate for our estimate of the true shear.

- For small surveys, statistical error is expected to dominate the B mode.

  For larger surveys, a stronger B mode has been observed than expected from statistical error or higher-order effects; the natural conclusion is that it must be due to systematic effects, like correlations between some ellipticities, or between some ellipticity and shear.

*) But $\vec{u}_\gamma(\vec{\theta})$, as defined by (44), is still a vector field (exercise)

- The division of measured shear into E and B mode is simple in Fourier space:

$$\boxed{\begin{aligned}
\gamma_1^E(\vec{\ell}) &= -\tfrac{1}{2}(\ell_1^2-\ell_2^2)\,\psi^E(\vec{\ell}) &&= -\tfrac{1}{2}\ell^2\cos 2\varphi_\ell\cdot\psi^E(\vec{\ell}) &&= \cos 2\varphi_\ell\cdot\kappa^E(\vec{\ell})\\
\gamma_1^B(\vec{\ell}) &= \ell_1\ell_2\,\psi^B(\vec{\ell}) &&= \tfrac{1}{2}\ell^2\sin 2\varphi_\ell\cdot\psi^B(\vec{\ell}) &&= -\sin 2\varphi_\ell\cdot\kappa^B(\vec{\ell})\\
\gamma_2^E(\vec{\ell}) &= -\ell_1\ell_2\,\psi^E(\vec{\ell}) &&= -\tfrac{1}{2}\ell^2\sin 2\varphi_\ell\cdot\psi^E(\vec{\ell}) &&= \sin 2\varphi_\ell\cdot\kappa^E(\vec{\ell})\\
\gamma_2^B(\vec{\ell}) &= -\tfrac{1}{2}(\ell_1^2-\ell_2^2)\,\psi^B(\vec{\ell}) &&= -\tfrac{1}{2}\ell^2\cos 2\varphi_\ell\cdot\psi^B(\vec{\ell}) &&= \cos 2\varphi_\ell\cdot\kappa^B(\vec{\ell})
\end{aligned}} \tag{46}$$

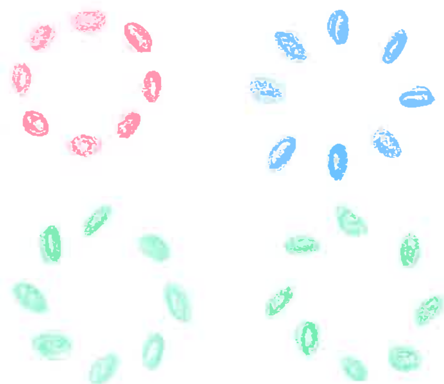
$$\Rightarrow \quad \underline{\gamma^E(\vec{\ell}) = e^{i2\varphi_\ell}\,\kappa^E(\vec{\ell})} \quad \text{and} \quad \underline{\gamma^B(\vec{\ell}) = i e^{i2\varphi_\ell}\,\kappa^B(\vec{\ell})} \tag{47}$$

$$\gamma(\vec{\ell}) = \gamma^E(\vec{\ell}) + \gamma^B(\vec{\ell}) = e^{i2\varphi_\ell}\left[\kappa^E(\vec{\ell}) + i\kappa^B(\vec{\ell})\right]$$

$$\Rightarrow \quad \boxed{\begin{aligned}
\kappa^E(\vec{\ell}) &= \mathrm{Re}\left[e^{-i2\varphi_\ell}\,\gamma(\vec{\ell})\right] &&= \cos 2\varphi_\ell\cdot\gamma_1(\vec{\ell}) + \sin 2\varphi_\ell\cdot\gamma_2(\vec{\ell})\\
\kappa^B(\vec{\ell}) &= \mathrm{Im}\left[e^{-i2\varphi_\ell}\,\gamma(\vec{\ell})\right] &&= -\sin 2\varphi_\ell\cdot\gamma_1(\vec{\ell}) + \cos 2\varphi_\ell\cdot\gamma_2(\vec{\ell})
\end{aligned}} \tag{48}$$

∴ From measured γ, obtain κ^E and κ^B using (48), and then γ^E and γ^B using (46) or (47).

- How do the E and B modes look on the sky:



E mode

B mode

Figure: E-mode and B-mode patterns of shear. The top-left pattern is caused by a mass overdensity and the top right pattern by mass underdensity. The B-mode patterns cannot be caused by gravitational lensing.

- We define the E and B mode power spectra as

$$\langle \gamma^E(\vec{\ell})^* \gamma^E(\vec{\ell}')\rangle = \frac{1}{4}\ell^4 \langle \psi_E^* \psi_E\rangle \equiv (2\pi)^4 \delta_D^2(\vec{\ell}-\vec{\ell}')\, P_E(\ell)$$

$$\langle \gamma^B(\vec{\ell})^* \gamma^B(\vec{\ell}')\rangle = \frac{1}{4}\ell^4 \langle \psi_B^* \psi_B\rangle \equiv (2\pi)^4 \delta_D^2(\vec{\ell}-\vec{\ell}')\, P_B(\ell) \quad (49)$$

$$\langle \gamma^E(\vec{\ell})^* \gamma^B(\vec{\ell}')\rangle = \frac{1}{4}\ell^4 \langle \psi_E^* \psi_B\rangle \equiv (2\pi)^4 \delta_D^2(\vec{\ell}-\vec{\ell}')\, P_{EB}(\ell) = 0$$

The cross spectrum $P_{EB}(\ell)$ vanishes for parity-symmetric shear fields – we assume this.

Thus $\langle \psi_E^* \psi_B\rangle = 0 \Rightarrow$ all $\langle \gamma_i^{E*} \gamma_j^B\rangle = 0$ (50)

- For shear correlations we get

$$\langle \gamma_1(\vec{\ell})^* \gamma_1(\vec{\ell}')\rangle = \langle [\gamma_1^E(\vec{\ell}) + \gamma_1^B(\vec{\ell})]^* [\gamma_1^E(\vec{\ell}') + \gamma_1^B(\vec{\ell}')]\rangle = \langle \gamma_1^{E*}\gamma_1^E\rangle + \langle \gamma_1^{B*}\gamma_1^B\rangle$$

etc.

$$\Rightarrow P_{\gamma_1}(\vec{\ell}) = P_{\gamma_1^E}(\vec{\ell}) + P_{\gamma_1^B}(\vec{\ell})$$

$$P_{\gamma_2}(\vec{\ell}) = P_{\gamma_2^E}(\vec{\ell}) + P_{\gamma_2^B}(\vec{\ell})$$

- From (46) $\Rightarrow$

$$\boxed{\begin{aligned} P_{\gamma_1^E}(\vec{\ell}) &= \cos^2 2\varphi_\ell\, P_E(\ell) \qquad & P_{\gamma_2^E}(\vec{\ell}) &= \sin^2 2\varphi_\ell\, P_E(\ell) \\ P_{\gamma_1^B}(\vec{\ell}) &= \sin^2 2\varphi_\ell\, P_B(\ell) \qquad & P_{\gamma_2^B}(\vec{\ell}) &= \cos^2 2\varphi_\ell\, P_B(\ell) \end{aligned}} \quad (51)$$

Shear Correlation Functions with E and B modes

- We have now to repeat the calculations from §5.4, including now also the B modes. The actual calculations are largely left as an exercise. If one has already done the calculations in §5.4, these provide the E-mode part, and only the B-mode part requires new calculation. Since we assumed parity symmetry, there will be no EB cross terms. The outline and results:

$$\xi_1(\vec\vartheta) \equiv \langle \gamma_1(\vec\vartheta_0)\,\gamma_1(\vec\vartheta_0+\vec\vartheta)\rangle = \frac{1}{(2\pi)^2}\int d^2L\, \underbrace{P_{\gamma_1}(\vec L)}_{P_{\gamma_1 E}+P_{\gamma_1 B} \,=\, \cos^2 2\varphi_L\cdot P_E(L) + \sin^2 2\varphi_L\cdot P_B(L)}\, e^{i\vec L\cdot\vec\vartheta} = \ldots$$

$$= \frac{1}{4\pi}\int_0^\infty L\,dL\,\Big\{P_E(L)\big[J_0(L\vartheta)+(\cos^2 2\varphi-\sin^2 2\varphi)J_4(L\vartheta)\big] + P_B(L)\big[J_0(L\vartheta)+(\sin^2 2\varphi-\cos^2 2\varphi)J_4(L\vartheta)\big]\Big\}$$

$$\xi_2(\vec\vartheta) = \ldots = \frac{1}{4\pi}\int_0^\infty L\,dL\,\Big\{P_E(L)\big[J_0(L\vartheta)+(\sin^2 2\varphi-\cos^2 2\varphi)J_4(L\vartheta)\big] + P_B(L)\big[J_0(L\vartheta)+(\cos^2 2\varphi-\sin^2 2\varphi)J_4(L\vartheta)\big]\Big\}$$

$$\xi_{12}(\vec\vartheta) = \ldots = \frac{1}{4\pi}\int_0^\infty L\,dL\,\big[P_E(L)-P_B(L)\big]\cdot 2\sin 2\varphi\cos 2\varphi\; J_4(L\vartheta) \qquad (52)$$

- With $\gamma_t = -\gamma_1\cos 2\varphi - \gamma_2 \sin 2\varphi$, $\quad \gamma_\times = \gamma_1 \sin 2\varphi - \gamma_2\cos 2\varphi$

we then get

$$\langle\gamma_t\gamma_t\rangle(\vartheta) = \cos^2 2\varphi\cdot\xi_1(\vec\vartheta) + 2\sin 2\varphi\cos 2\varphi\cdot\xi_{12}(\vec\vartheta) + \sin^2 2\varphi\cdot\xi_2(\vec\vartheta) = \ldots$$

$$= \frac{1}{4\pi}\int_0^\infty L\,dL\,\Big\{\big[P_E(L)+P_B(L)\big]J_0(L\vartheta) + \big[P_E(L)-P_B(L)\big]J_4(L\vartheta)\Big\} \qquad (53)$$

$$\langle\gamma_\times\gamma_\times\rangle(\vartheta) = \ldots = \frac{1}{4\pi}\int_0^\infty L\,dL\,\Big\{\big[P_E(L)+P_B(L)\big]J_0(L\vartheta) - \big[P_E(L)-P_B(L)\big]J_4(L\vartheta)\Big\}$$

$$\therefore\quad \boxed{\begin{aligned} \xi_+(\vartheta) &\equiv \langle\gamma_t\gamma_t\rangle + \langle\gamma_\times\gamma_\times\rangle = \frac{1}{2\pi}\int_0^\infty L\,dL\,\big[P_E(L)+P_B(L)\big]J_0(L\vartheta) \\ \xi_-(\vartheta) &\equiv \langle\gamma_t\gamma_t\rangle - \langle\gamma_\times\gamma_\times\rangle = \frac{1}{2\pi}\int_0^\infty L\,dL\,\big[P_E(L)-P_B(L)\big]J_4(L\vartheta) \end{aligned}} \qquad (54)$$

Since ξ_+ depends on P_E+P_B and ξ_- on P_E-P_B, they can no longer be obtained from each other, but are independent.

- Using the Bessel function closure relation (35)

$$2\pi \int_0^\infty d\theta\, \theta\, \xi_+(\theta) J_0(L\theta) = P_E(L) + P_B(L)$$

$$2\pi \int_0^\infty d\theta\, \theta\, \xi_-(\theta) J_4(L\theta) = P_E(L) - P_B(L)$$

$$\Rightarrow \quad \boxed{\begin{aligned} P_E(L) &= \pi \int_0^\infty d\theta\, \theta \left[\xi_+(\theta) J_0(L\theta) + \xi_-(\theta) J_4(L\theta) \right] \\ P_B(L) &= \pi \int_0^\infty d\theta\, \theta \left[\xi_+(\theta) J_0(L\theta) - \xi_-(\theta) J_4(L\theta) \right] \end{aligned}} \qquad (55)$$

$\therefore$ The two power spectra $P_E(L)$ and $P_B(L)$ can be obtained from the measured correlation functions $\xi_+(\theta)$ and $\xi_-(\theta)$.

However, this involves integrating $\int d\theta$ over infinite range. For practical work, there are methods (e.g. "aperture measures") designed for finite surveys; but we don't have time to cover them in this course (this year, maybe they can be included when the course is lectured next time).

THE END