Power Spectra

- Statistical homogeneity $\Rightarrow$ different Fourier modes are uncorrelated

$$\langle \psi(\vec{l})^* \psi(\vec{l}') \rangle = (2\pi)^2 \delta_D^2(\vec{l}-\vec{l}')\, P_\psi(\vec{l}) \tag{22}$$

Using (19),

$$\langle \kappa(\vec{l})^* \kappa(\vec{l}') \rangle = \tfrac{1}{4} l^4 \langle \psi(\vec{l})^* \psi(\vec{l}') \rangle \Rightarrow P_\kappa(\vec{l}) = \tfrac{1}{4} l^4 P_\psi(\vec{l})$$

$$\langle \gamma_1(\vec{l})^* \gamma_1(\vec{l}') \rangle = \tfrac{1}{4} (l_1^2 - l_2^2)^2 \langle \psi(\vec{l})^* \psi(\vec{l}') \rangle \Rightarrow P_{\gamma_1}(\vec{l}) = \tfrac{1}{4}(l_1^2-l_2^2)^2 P_\psi(\vec{l})$$

$$\langle \gamma_2(\vec{l})^* \gamma_2(\vec{l}') \rangle = l_1^2 l_2^2 \langle \psi(\vec{l})^* \psi(\vec{l}') \rangle \Rightarrow P_{\gamma_2}(\vec{l}) = l_1^2 l_2^2 P_\psi(\vec{l})$$

$$\therefore\ P_{\gamma_1}(\vec{l}) + P_{\gamma_2}(\vec{l}) = \tfrac{1}{4}\left[(l_1^2-l_2^2)^2 + 4 l_1^2 l_2^2\right] P_\psi(\vec{l}) = \tfrac{1}{4} l^4 P_\psi(\vec{l}) = P_\kappa(\vec{l})$$

- ψ and κ are scalar quantities. Statistical isotropy $\Rightarrow$ $P_\psi(\vec{l}) = P_\psi(l)$ and $P_\kappa(\vec{l}) = P_\kappa(l)$

γ is a polar quantity. It's components γ_1 and γ_2 depend on the orientation of the crd. system. This dependence on the crd. system breaks the isotropy of γ_1 and γ_2, and thus $P_{\gamma_1}(\vec{l})$ and $P_{\gamma_2}(\vec{l})$ depend on the direction of the 2D wave vector $\vec{l}$.

So far we have worked only with Fourier transforms and power spectra of real quantities, to be on more familiar ground. But we can also define a power spectrum for the complex $\gamma = \gamma_1 + i\gamma_2$. Do first the correlator

$$\langle \gamma_1(\vec{l})^* \gamma_2(\vec{l}') \rangle = \tfrac{1}{2} l_1 l_2 (l_1^2 - l_2^2) \langle \psi(\vec{l})^* \psi(\vec{l}') \rangle \Rightarrow \gamma_1(\vec{l}) \text{ and } \gamma_2(\vec{l}) \text{ are}$$

correlated; this correlator is real (since $\langle \psi(\vec{l})^* \psi(\vec{l}') \rangle$ is)

$$\therefore\ \langle \gamma(\vec{l})^* \gamma(\vec{l}') \rangle = \langle [\gamma_1(\vec{l})^* - i\gamma_2(\vec{l})^*][\gamma_1(\vec{l}') + i\gamma_2(\vec{l}')] \rangle$$

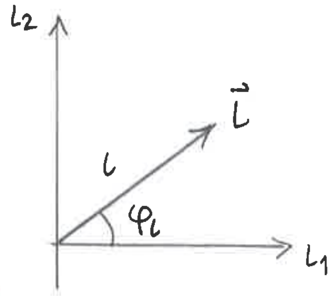
$$= \langle \gamma_1(\vec{l})^* \gamma_1(\vec{l}') \rangle + \underbrace{i \langle \gamma_1(\vec{l})^* \gamma_2(\vec{l}') \rangle - i \langle \gamma_2(\vec{l})^* \gamma_1(\vec{l}') \rangle}_{} + \langle \gamma_2(\vec{l})^* \gamma_2(\vec{l}') \rangle$$

these are equal (complex conjugate of a real quantity), so they cancel

Defining $P_\gamma(\vec{L})$ by $\langle \gamma(\vec{L})^* \gamma(\vec{L}') \rangle \equiv (2\pi)^2 \delta_D^2(\vec{L}-\vec{L}') P_\gamma(\vec{L})$ we have thus

$$P_\gamma(\vec{L}) = P_{\gamma_1}(\vec{L}) + P_{\gamma_2}(\vec{L}) = P_\kappa(L) \qquad (23) \qquad \therefore P_\gamma(L) \text{ is isotropic}$$

- In terms of length and direction of the 2D wave vector $\vec{L}$



$$L_1 = L\cos\varphi_L \qquad L_2 = L\sin\varphi_L$$

$$P_{\gamma_1}(\vec{L}) = \frac{(L_1^2-L_2^2)^2}{L^4} P_\kappa(L) = (\cos^2\varphi_L - \sin^2\varphi_L) P_\kappa(L) = \cos^2 2\varphi_L \cdot P_\kappa(L)$$

$$P_{\gamma_2}(\vec{L}) = 4\frac{L_1^2 L_2^2}{L^4} P_\kappa(L) = 4\cos^2\varphi_L \sin^2\varphi_L \, P_\kappa(L) = \sin^2 2\varphi_L \cdot P_\kappa(L)$$

(24)

- Or directly for $\gamma(\vec{L}) = \gamma_1(\vec{L}) + i\gamma_2(\vec{L})$:

$$\gamma(\vec{L}) = \frac{L_1^2 - L_2^2 + 2iL_1L_2}{L^2} \kappa(\vec{L}) = \frac{(L_1+iL_2)^2}{L^2} \kappa(\vec{L}) = e^{i2\varphi_L} \kappa(\vec{L}) \qquad (25)$$

$$\Rightarrow \quad P_\gamma(\vec{L}) = |e^{i2\varphi_L}|^2 P_\kappa(L) = P_\kappa(L)$$

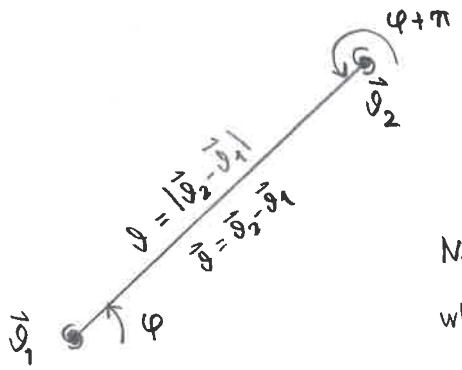
Here $L = \sqrt{L_1^2 + L_2^2} = |\vec{L}| = |L_1 + iL_2|$ is both the length of the vector $\vec{L}$ and the modulus (absolute value) of the complex number $L_1 + iL_2$.

- Thus we can relate the convergence power spectrum $P_\kappa(L)$ to the shear power spectrum $P_\gamma(L)$ – they are the same!

However, what we can get directly from observations, are the shear correlation functions rather than their power spectra

§5.4 Shear Correlation Functions

- We define the tangential and cross component of shear for a pair of points (galaxy images) using their separation line as the reference direction:



$$\boxed{\begin{aligned} \gamma_t &\equiv -\mathrm{Re}\left(\gamma e^{-i2\varphi}\right) \\ \gamma_\times &\equiv -\mathrm{Im}\left(\gamma e^{-i2\varphi}\right) \end{aligned}} \qquad (26)$$

Note that since $e^{-i2\varphi} = e^{-i2(\varphi+\pi)}$, it doesn't matter which way the separation direction is defined.

- We define the correlation functions

$$\langle \gamma_t(\vec{\vartheta}_1)\, \gamma_t(\vec{\vartheta}_2) \rangle = \langle \gamma_t \gamma_t \rangle (\vec{\vartheta}) = \langle \gamma_t \gamma_t \rangle (\vartheta)$$

$$\langle \gamma_\times(\vec{\vartheta}_1)\, \gamma_\times(\vec{\vartheta}_2) \rangle = \langle \gamma_\times \gamma_\times \rangle (\vec{\vartheta}) = \langle \gamma_\times \gamma_\times \rangle (\vartheta) \qquad (27)$$

$$\langle \gamma_t(\vec{\vartheta}_1)\, \gamma_\times(\vec{\vartheta}_2) \rangle \underset{\text{statistical homogeneity}}{=} \langle \gamma_t \gamma_\times \rangle (\vec{\vartheta}) \underset{\text{statistical isotropy}}{=} \langle \gamma_t \gamma_\times \rangle (\vartheta) \equiv \xi_\times(\vartheta)$$

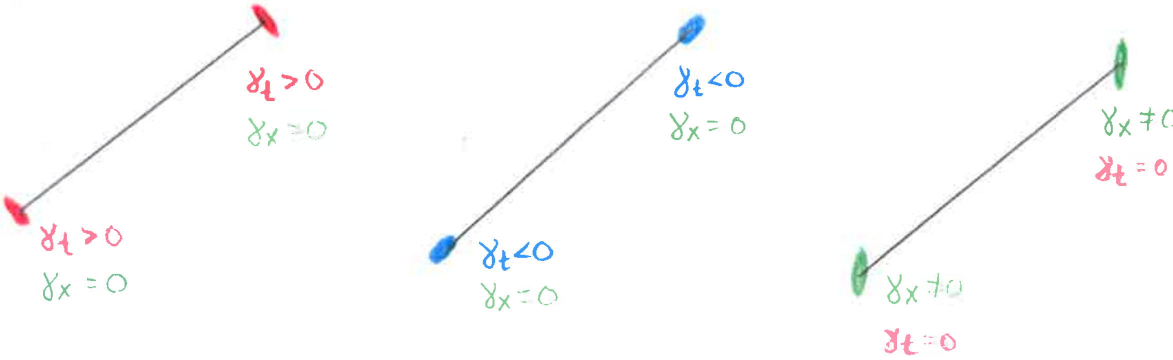
$$\xi_\pm(\vartheta) \equiv \langle \gamma_t \gamma_t \rangle (\vartheta) \pm \langle \gamma_\times \gamma_\times \rangle (\vartheta) \qquad (28)$$

- These are evaluated from the data simply by using galaxy image ellipticities as shear estimators: For each bin $\vartheta \pm \frac{1}{2}\Delta\vartheta$, find all N image pairs whose separation falls in this range, and calculate the average:

$$\hat{\xi}_\pm(\vartheta) = \frac{1}{N} \sum_{i<j} \left(\epsilon_{i,t}\, \epsilon_{j,t} \pm \epsilon_{i,\times}\, \epsilon_{j,\times} \right) \qquad (29)$$

Note that the tangential and cross components $\epsilon_t, \epsilon_\times$ for each galaxy image are defined differently for each pair that it is part of.

- Meaning of γ_t and $\gamma_\times$ in pictures.



The reason for the − sign in the definition of γ_t: This way it is typically expected to be positive: the more nearby of the pair of galaxies lenses the more distant one in the tangential direction. (Without the − sign we should call γ_t the radial component.)

- I didn't bother to check the sign of $\gamma_\times$ in the third picture; it will depend on how we define the coordinate axes: if we define a right-handed 3D coordinate system, it depends on whether we choose the third axis as pointing towards or away from the observer.

- Parity symmetry $\Rightarrow$ $\xi_\times(\vartheta)$ is expected to vanish, since in a parity transformation (mirror universe) $\gamma_t \to \gamma_t$ but $\gamma_\times \to -\gamma_\times$.

- We should now relate $\xi_\pm(\vartheta)$ to $P_\kappa(L)$.

I don't know if there's a clever way to do this directly with the complex shear; I did this by falling back to γ_1 and γ_2, defining first the correlation functions

$$\xi_1(\vec\vartheta) \equiv \langle \gamma_1(\vec\vartheta_1)\, \gamma_1(\vec\vartheta_2) \rangle$$

where $\vec\vartheta = \vec\vartheta_2 - \vec\vartheta_1$

$$\xi_2(\vec\vartheta) \equiv \langle \gamma_2(\vec\vartheta_1)\, \gamma_2(\vec\vartheta_2) \rangle$$

$$\xi_{12}(\vec\vartheta) \equiv \langle \gamma_1(\vec\vartheta_1)\, \gamma_2(\vec\vartheta_2) \rangle$$

These correlation functions are not expected to be isotropic, since the components depend on the orientation of the cd. system.

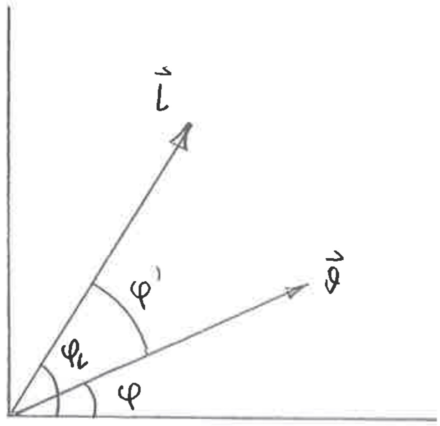
But statistical homogeneity $\Rightarrow$ they depend only on the separation $\vec\vartheta$.
(On next page I call $\vec\vartheta_1 = \vec\vartheta_0$ and $\vec\vartheta_2 = \vec\vartheta_0 + \vec\vartheta$).

The correlation functions $\xi_1(\vec{\vartheta})$ and $\xi_2(\vec{\vartheta})$ are Fourier transforms of their power spectra $P_{\gamma_1}(\vec{L})$ and $P_{\gamma_2}(\vec{L})$.

We have to deal with two directions of 2D vectors:

that of $\vec{L}$ (φ_L)

and that of $\vec{\vartheta}$ (φ)



$\varphi' \equiv \varphi_L - \varphi$ is the angle between $\vec{L}$ and $\vec{\vartheta}$

$$\Rightarrow \quad \vec{L}\cdot\vec{\vartheta} = L\vartheta\cos\varphi'$$

$$\xi_1(\vec{\vartheta}) \equiv \langle \gamma_1(\vec{\vartheta}_0)\,\gamma_1(\vec{\vartheta}_0+\vec{\vartheta})\rangle = \frac{1}{(2\pi)^2}\int d^2L\, P_{\gamma_1}(\vec{L})\, e^{i\vec{L}\cdot\vec{\vartheta}} = \frac{1}{(2\pi)^2}\int d^2L\, P_{\gamma_1}(\vec{L})\, e^{iL\vartheta\cos\varphi'}$$

$$= \frac{1}{(2\pi)^2}\int d^2L\, P_{\gamma}(L)\,\cos^2 2\varphi_L\, e^{iL\vartheta\cos\varphi'}$$

Using $\int d^2L = \int_0^\infty L\,dL \int_0^{2\pi} d\varphi_L = \int_0^\infty L\,dL\int_0^{2\pi} d\varphi'$ and the integral representation of

Bessel functions
$$J_n(x) = \frac{(-i)^n}{\pi}\int_0^\pi e^{ix\cos\varphi}\cos n\varphi\, d\varphi = \frac{(-i)^n}{2\pi}\int_0^{2\pi} e^{ix\cos\varphi}\cos n\varphi\, d\varphi \qquad (30)$$

one finds (exercise)

$$\boxed{\begin{aligned}\xi_1(\vec{\vartheta}) &= \frac{1}{4\pi}\int_0^\infty L\,dL\, P_\gamma(L)\left[J_0(L\vartheta) + (\cos^2 2\varphi - \sin^2 2\varphi)\, J_4(L\vartheta)\right]\\ \xi_2(\vec{\vartheta}) &= \frac{1}{4\pi}\int_0^\infty L\,dL\, P_\gamma(L)\left[J_0(L\vartheta) + (\sin^2 2\varphi - \cos^2 2\varphi)\, J_4(L\vartheta)\right]\end{aligned}} \qquad (31)$$

Also

$$\xi_{12}(\vec{\vartheta}) \equiv \langle \gamma_1(\vec{\vartheta}_0)\,\gamma_2(\vec{\vartheta}_0+\vec{\vartheta})\rangle = \frac{1}{(2\pi)^4}\int d^2L\, d^2L'\, e^{-i\vec{L}\cdot\vec{\vartheta}_0}\, e^{+i\vec{L}'\cdot(\vec{\vartheta}_0+\vec{\vartheta})}\,\langle \gamma_1(\vec{L})^*\gamma_2(\vec{L}')\rangle$$

$$= \dots = \frac{1}{4\pi}\int_0^\infty L\,dL\, P_\gamma(L)\cdot 2\sin 2\varphi\cos 2\varphi\cdot J_4(L\vartheta) \qquad (32)$$

statistical homogeneity

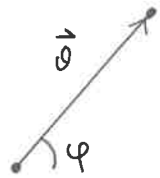
and $\xi_{21}(\vec{\vartheta}) \equiv \langle \gamma_2(\vec{\vartheta}_0)\,\gamma_1(\vec{\vartheta}_0+\vec{\vartheta})\rangle = \langle \gamma_1(\vec{\vartheta}_0+\vec{\vartheta})\,\gamma_2(\vec{\vartheta}_0)\rangle = \langle \gamma_1(\vec{\vartheta}_0)\,\gamma_2(\vec{\vartheta}_0-\vec{\vartheta})\rangle$

$= \xi_{12}(-\vec{\vartheta}) = \xi_{12}(\vec{\vartheta})$ since $\sin 2\varphi\cos 2\varphi$ is invariant in $\varphi \to \varphi+\pi$ $(\vec{\vartheta} \to -\vec{\vartheta})$

- Return now to γ_t and $\gamma_\times$:

$$\gamma e^{-i2\varphi} = (\gamma_1 + i\gamma_2)(\cos 2\varphi - i\sin 2\varphi) = \gamma_1 \cos 2\varphi + \gamma_2 \sin 2\varphi + i\left[-\gamma_1 \sin 2\varphi + \gamma_2 \cos 2\varphi\right]$$

$$\Rightarrow \boxed{\begin{aligned} \gamma_t &\equiv -\mathrm{Re}\left(\gamma e^{-i2\varphi}\right) = -\gamma_1 \cos 2\varphi - \gamma_2 \sin 2\varphi \\ \gamma_\times &\equiv -\mathrm{Im}\left(\gamma e^{-i2\varphi}\right) = \gamma_1 \sin 2\varphi - \gamma_2 \cos 2\varphi \end{aligned}} \qquad (33)$$



$$\Rightarrow \langle \gamma_t \gamma_t \rangle(\vec{\vartheta}) = \ldots = \frac{1}{4\pi} \int_0^\infty l\,dl\, P_\kappa(l) \cdot \left[J_0(l\vartheta) + J_4(l\vartheta)\right]$$

$$\langle \gamma_\times \gamma_\times \rangle(\vec{\vartheta}) = \ldots = \frac{1}{4\pi} \int_0^\infty l\,dl\, P_\kappa(l) \cdot \left[J_0(l\vartheta) - J_4(l\vartheta)\right]$$

$$\therefore \boxed{\begin{aligned} \xi_+(\vartheta) &\equiv \langle \gamma_t \gamma_t \rangle + \langle \gamma_\times \gamma_\times \rangle = \frac{1}{2\pi} \int_0^\infty l\,dl\, J_0(l\vartheta)\, P_\kappa(l) \\ \xi_-(\vartheta) &\equiv \langle \gamma_t \gamma_t \rangle - \langle \gamma_\times \gamma_\times \rangle = \frac{1}{2\pi} \int_0^\infty l\,dl\, J_4(l\vartheta)\, P_\kappa(l) \end{aligned}} \qquad (34)$$

- Also $\xi_\times(\vec{\vartheta}) \equiv \langle \gamma_t \gamma_\times \rangle(\vec{\vartheta}) = \ldots = 0$

- We can invert these equations using the Bessel function closure (orthogonality) equation

$$\int_0^\infty J_n(\alpha x) J_n(\alpha' x)\, x\,dx = \frac{1}{\alpha} \delta_D(\alpha - \alpha') \qquad (35)$$

$$2\pi \int_0^\infty d\vartheta\, \vartheta\, \xi_+(\vartheta) J_0(l\vartheta) = \int_0^\infty d\vartheta \int_0^\infty dl'\, l' \vartheta\, J_0(l\vartheta) J_0(l'\vartheta) P_\kappa(l')$$

$$= \int_0^\infty dl'\, l' P_\kappa(l') \underbrace{\int_0^\infty J_0(l\vartheta) J_0(l'\vartheta)\, \vartheta\, d\vartheta}_{\frac{1}{l}\delta_D(l - l')} = P_\kappa(l) \qquad (36a)$$

likewise

$$2\pi \int_0^\infty d\vartheta\, \vartheta\, \xi_-(\vartheta) J_4(l\vartheta) = \ldots = P_\kappa(l) \qquad (36b)$$

Thus the two shear correlation functions $\xi_+(\vartheta)$ and $\xi_-(\vartheta)$ are not independent (as the two components of shear are both derived from the same scalar quantity, the deflection potential ψ). We can express them in terms of each other by inserting (36b) into (34a) and (36a) into (34b). Schneider (p.364) gives the results

$$\xi_+(\vartheta) = \frac{1}{2\pi}\int_0^\infty L\,dL\, J_0(L\vartheta) P_\kappa(L) \overset{(36b)}{=} \int_0^\infty L\,dL \int_0^\infty \vartheta' d\vartheta'\, J_0(L\vartheta) J_4(L\vartheta')\, \xi_-(\vartheta') \tag{37a}$$

$$= \int_0^\infty \vartheta' d\vartheta'\, \xi_-(\vartheta') \int_0^\infty L\,dL\, J_0(L\vartheta) J_4(L\vartheta') = \dots = \xi_-(\vartheta) + \int_\vartheta^\infty \frac{d\vartheta'}{\vartheta'}\, \xi_-(\vartheta')\left(4 - 12\frac{\vartheta^2}{\vartheta'^2}\right)$$

$$\xi_-(\vartheta) = \frac{1}{2\pi}\int_0^\infty L\,dL\, J_4(L\vartheta) P_\kappa(L) = \dots = \xi_+(\vartheta) + \int_0^\vartheta \frac{d\vartheta'\,\vartheta'}{\vartheta^2}\, \xi_+(\vartheta')\left(4 - 12\frac{\vartheta'^2}{\vartheta^2}\right) \tag{37b}$$

I haven't so far managed to do these. Tools for doing the integral over the Bessel function product include:

- Bessel function recursion formulae, e.g., $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$
- The closure relation (35), which can be used to give the first term in (37a,b)
- Gradshteyn & Ryzhik integral 6.512.3:

$$\int_0^\infty J_n(\alpha x) J_{n-1}(\beta x) = \begin{cases} 0 & \text{if } \alpha < \beta \\ \frac{1}{2\beta} & \text{if } \alpha = \beta \\ \frac{\beta^{n-1}}{\alpha^n} & \text{if } \alpha > \beta \end{cases}$$

which can be used to cut the integral $\int_0^\infty d\vartheta'$ to $\int_\vartheta^\infty d\vartheta'$ or $\int_0^\vartheta d\vartheta'$

(The exact forms of (37a,b) are maybe not that important; just the principle is.)