5. COSMIC SHEAR [SKW Part III, Sec. 6]

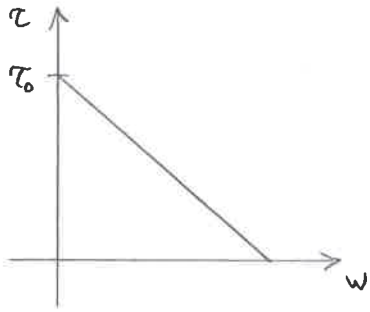
§5.1 Light Propagation in an Inhomogeneous Universe

- The metric of an FRW universe with scalar perturbations, approximating the Bardeen potentials to be equal, $\phi = \psi$ (good at least in the ΛCDM model during matter domination), can be written

$$ds^2 = a(\tau)^2 \left\{ -(1+2\phi)d\tau^2 + (1-2\phi)\left[dw^2 + f_K(w)^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]\right\} \qquad (1)$$

where the Bardeen potential ϕ is a function of space and time.

Here τ denotes conformal time and w is the radial comoving distance (in the unperturbed FRW universe).



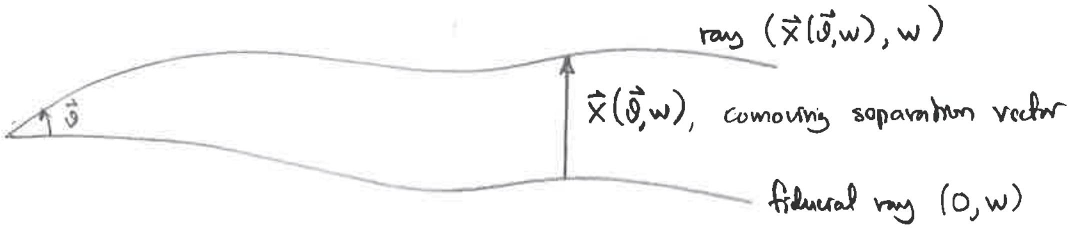
Our observations lie on our past light cone $w(\tau) = \tau_0 - \tau$.

Thus we need ϕ only on this light cone: $\tau(w) = \tau_0 - w$

$$\phi(\vec{x}, w) = \phi(\tau_0 - w, \vec{x}, w)$$

↑ 2D coordinate vector in the direction transverse to the line of sight.

- Consider two light rays arriving at the observer:
  - the fiducial light ray corresponding to $\vec{\vartheta} = 0$
  - another ray observed at an angle $\vec{\vartheta}$ from the fiducial ray



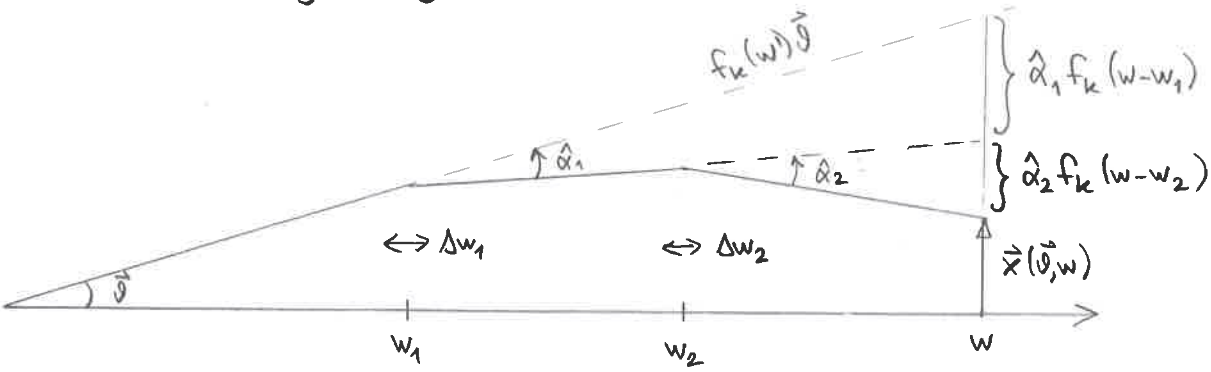
- Without lensing ($\phi = 0$), the two light rays would be separated by $\vec{x}(w) = f_K(w)\vec{\vartheta}$

From homework 10.2, the deflection angle accumulates as $d\hat{\alpha} = 2\nabla_\perp \phi \, dr_3$

In the homework both $\nabla_\perp$ and dr_3 were in local coordinates; we can replace them with comoving coordinates by scaling both with $1+z$; these cancel

$\Rightarrow$ $d\hat{\alpha} = 2\nabla_\perp \phi \, dw$ (2) where $\nabla_\perp$ is now wrt to the comoving transverse coordinates $\vec{x}$: $\nabla_\perp \phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\right)$

Suppose we have have a finite number (in the figure below, two, at w_1 and w_2) of thin (thickness Δw) along the way:



$$\therefore \quad \vec{x}(\vec{\vartheta},w) = f_K(w)\vec{\vartheta} - \sum_i \hat{\alpha}_i f_K(w-w_i)$$

$$= f_K(w)\vec{\vartheta} - 2\sum_i f_K(w-w_i) \int_{\Delta w_i} \nabla_\perp \phi(\vec{x}(\vec{\vartheta},w'),w')\, dw'$$

$$\rightarrow f_K(w)\vec{\vartheta} - 2\int_0^w dw' f_K(w-w') \nabla_\perp \phi(\vec{x}(\vec{\vartheta},w'),w')$$

when we consider the entire distance from 0 to w as consisting of adjacent thin lenses, i.e. the light ray is being lensed all the time by the local $\nabla_\perp \phi$.

- The preceding ignored the lensing of the fiducial ($\vec{\vartheta}=0$) ray. For $\vec{x}(\vec{\vartheta},w)$ to represent the (comoving) separation between these two rays, subtract the effect on the fiducial ray

$$\Rightarrow \boxed{\vec{x}(\vec{\vartheta},w) = f_K(w)\vec{\vartheta} - 2\int_0^w dw' f_K(w-w')\left[\nabla_\perp\phi(\vec{x}(\vec{\vartheta},w'),w') - \nabla_\perp\phi(0,w')\right]} \qquad (3)$$

$\therefore$ If there is a source at comoving distance w that we see at an angle $\vec{\vartheta}$, then its angular location on the source plane is

$$\vec{\beta} = \frac{\vec{x}(\vec{\vartheta},w)}{f_K(w)}$$

$\Rightarrow$ The Jacobian for the total lensing effect from observer to comoving distance w is

$$A(\vec{\vartheta},w) \equiv \frac{\partial\vec{\beta}}{\partial\vec{\vartheta}} = \frac{1}{f_K(w)}\frac{\partial\vec{x}}{\partial\vec{\vartheta}} = \delta_{ij} - \frac{2}{f_K(w)}\int_0^w dw' f_K(w-w')\underbrace{\frac{\partial}{\partial\vec{\vartheta}}\nabla_\perp\phi}_{\frac{\partial}{\partial\vartheta_j}\frac{\partial\phi}{\partial x_i}}$$

Express $\frac{\partial}{\partial\vartheta_j}\frac{\partial\phi}{\partial x_i}$ in terms of $\phi_{,ij} \equiv \frac{\partial^2\phi}{\partial x_i\partial x_j}$

dx_i and $d\vartheta_j$ are related at w' by $\quad A_{ij}(\vec{\vartheta},w') \equiv \frac{1}{f_K(w')}\frac{\partial x_i}{\partial\vartheta_j} \qquad (4)$

$$\Rightarrow \frac{\partial}{\partial\vartheta_j}\frac{\partial\phi}{\partial x_i} = \frac{\partial x_k}{\partial\vartheta_j}\frac{\partial^2\phi}{\partial x_k\partial x_i} = f_K(w')A_{kj}(\vec{\vartheta},w')\cdot\frac{\partial^2\phi}{\partial x_k\partial x_i} \qquad (5)$$

$$\boxed{A_{ij}(\vec{\vartheta},w) \equiv \frac{\partial\beta_i}{\partial\vartheta_j} = \delta_{ij} - 2\int_0^w dw' \frac{f_K(w-w')f_K(w')}{f_K(w)}\phi_{,ik}(\vec{x}(\vec{\vartheta},w'),w')A_{kj}(\vec{\vartheta},w')} \qquad (6)$$

This equation can be derived directly from the geodesic deviation equation of General Relativity, and is exact "in the limit of validity of the weak-field metric (1)" [Schneider p. 358].

Because of the recursion, $A_{ij}(\vec{\vartheta},w)$ depending on $A_{ij}(\vec{\vartheta},w')$ of all $w'<w$, it is not easy to use as is, so we need to make an approximation.

- Now we make the "Born approximation", where we calculate the integral (6) instead along the unlensed ray, replacing inside the integral

$$\vec{x}(\vec{\vartheta}, w') \simeq f_k(w')\vec{\vartheta} \quad \Rightarrow \quad \underline{\frac{\partial}{\partial x_j} \simeq \frac{1}{f_k(w')}\frac{\partial}{\partial \vartheta_j}} \quad \Rightarrow \quad A_{ij}(\vec{\vartheta}, w') \simeq \delta_{ij} \qquad (7).$$

$$\Rightarrow \quad \boxed{A_{ij}(\vec{\vartheta}, w) \simeq \delta_{ij} - 2\int_0^w dw' \, \frac{f_k(w-w') f_k(w')}{f_k(w)} \, \phi_{,ij}\left(f_k(w')\vec{\vartheta}, w'\right)} \qquad (8)$$

(Born apx)

Corrections to the Born apx are of order ϕ^2. Since $\phi \ll 1$, we can use (8).

- We can now define the deflection potential for cosmological shear

$$\boxed{\psi(\vec{\vartheta}, w) \equiv 2\int_0^w dw' \, \frac{f_k(w-w')}{f_k(w) f_k(w')} \, \phi\left(f_k(w')\vec{\vartheta}, w'\right)} \qquad (9)$$

so that $\quad A_{ij} = \delta_{ij} - \psi_{,ij}$

Note Eq. (7b) for moving between

$$\psi_{,ij} = \frac{\partial^2 \psi}{\partial \vartheta_i \partial \vartheta_j} \quad \text{and} \quad \phi_{,ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

and defining $\quad A \equiv \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix}$

we have

$$\kappa = \tfrac{1}{2}\nabla^2\psi = \tfrac{1}{2}(\psi_{,11} + \psi_{,22})$$

$$\gamma_1 = \tfrac{1}{2}(\psi_{,11} - \psi_{,22})$$

$$\gamma_2 = \psi_{,12}$$

$$\gamma \equiv \gamma_1 + i\gamma_2 = \tfrac{1}{2}(\psi_{,11} - \psi_{,22}) + i\psi_{,12}$$

as before.