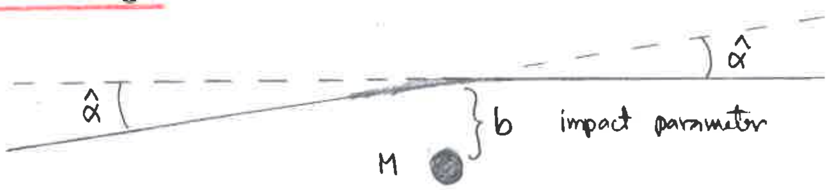
# 1. GRAVITATIONAL LENS THEORY [SKW Part I, Sec. 2]

## §1.1 Deflection Angle

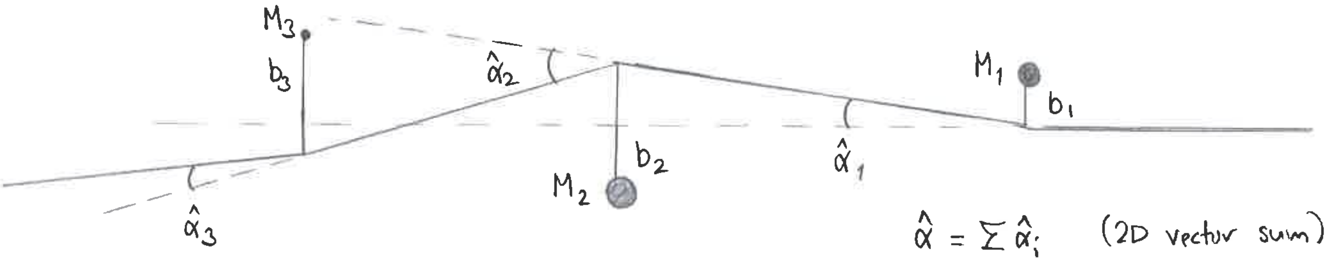



For $b \gg R_s = 2GM$ (Schwarzschild radius), GR: $\boxed{\hat{\alpha} = \frac{4GM}{b} \ll 1}$, (1)

twice the Newtonian result (for a mass moving at the speed of light $c = 1$).

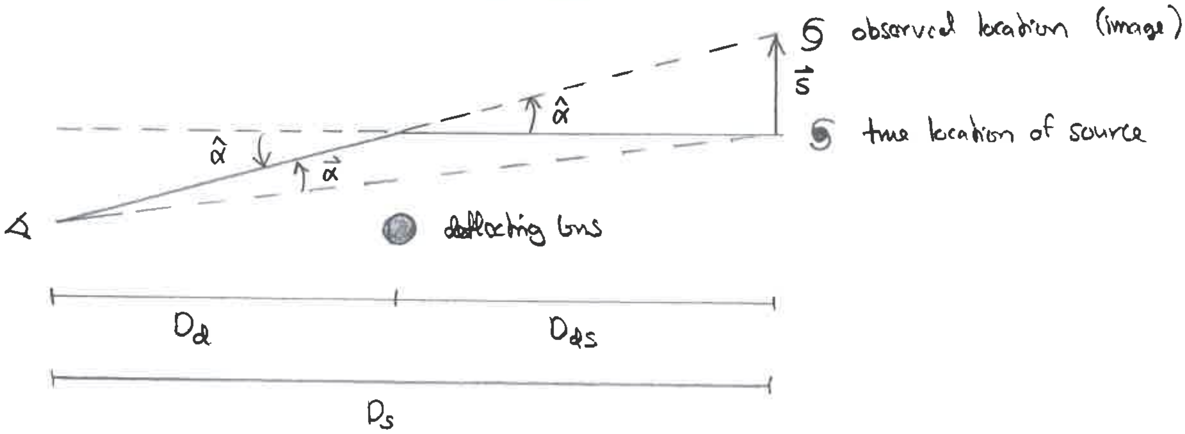
- For a weak gravitational field, GR may be linearized (1st order perturbation theory)

  $\Rightarrow$ deflection due to many mass points = sum of the individual deflections.




$\hat{\alpha} = \sum \hat{\alpha}_i$ (2D vector sum)

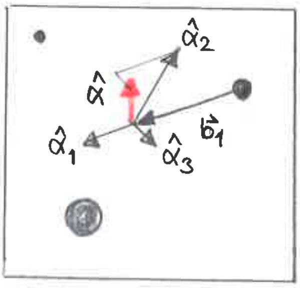
- The sign of $\hat{\alpha}$ and the scaled deflection angle $\vec{\alpha}$:




$$\vec{\alpha} \equiv \frac{\vec{s}}{D_s} = \frac{D_{ds}}{D_s}\frac{\vec{s}}{D_{ds}} = \frac{D_{ds}}{D_s}\hat{\alpha} \qquad (2)$$

The image is shifted away from the lens

$\vec{\alpha}$ and $\hat{\alpha}$ treated as 2D vectors; the direction is the direction the image has shifted, orthogonal to the line of sight.

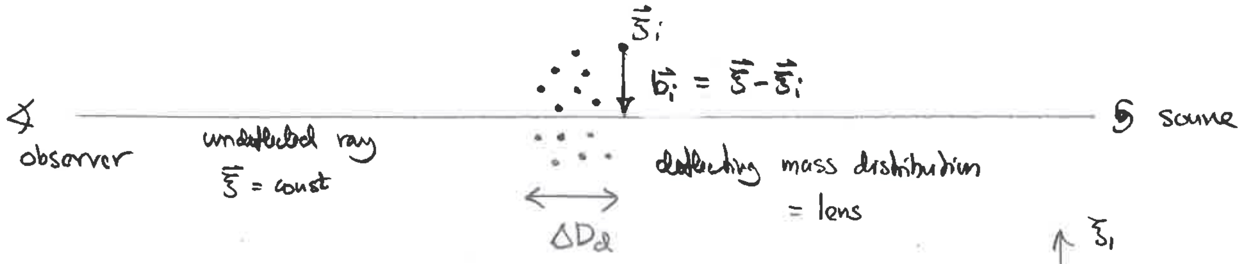


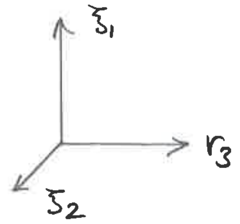

$$\hat{\alpha}_i = + \frac{4GM_i}{b_i^2}\vec{b}_i \qquad (3)$$

impact vector

Assume $\alpha \ll 1$ and the deflecting mass distribution located within small section $\Delta D_d \ll D_d, D_{ds}$ of separation between observer and source.

Born approximation: Find total deflection $\vec{\alpha}$ by calculating each component from the undeflected light ray.




impact vector $\vec{b}_i = \vec{\xi} - \vec{\xi}_i$

$$\hat{\alpha}(\vec{\xi}) = 4G \sum_i M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2}$$

Continuous mass distribution $\quad \sum M_i \rightarrow \int dm = \int \rho \, dV$

$$\hat{\alpha}(\vec{\xi}) = 4G \int d^2\xi' \underbrace{dr_3 \, \rho(\vec{\xi}', r_3)}_{\Sigma(\vec{\xi}')\ \text{surface mass density}} \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} = 4G \int d^2\xi' \, \Sigma(\vec{\xi}') \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} \qquad (4)$$

$\Delta D_d \ll D_d, D_{ds} \Rightarrow$ Idealize the lens as a flat plane, orthogonal to the line of sight, w surface mass density $\Sigma(\vec{\xi})$ : Lens plane

Likewise, the extent of the source $\ll D_d, D_{ds} \Rightarrow$ Source plane

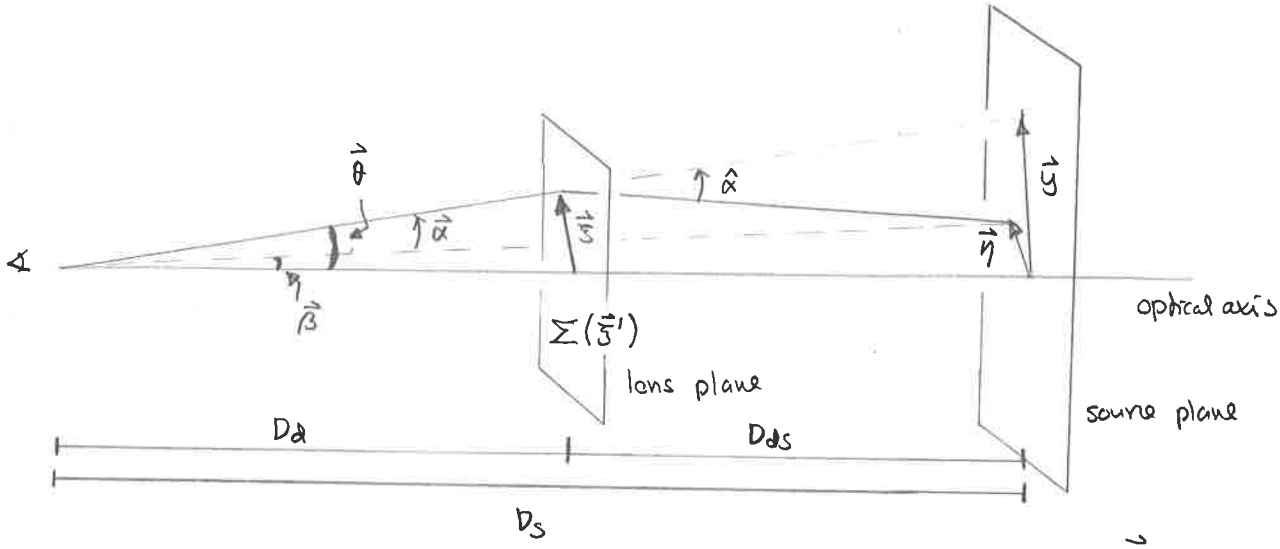
## §1.2 Lens Equation

- Consider different locations $\vec{\xi}$ on the lens plane

All angles small $\Rightarrow$ $\sin\vartheta = \tan\vartheta = \vartheta$

all rays can be considered parallel, and orthogonal to the lens plane (i.e. they have fixed $\vec{\xi}$), while they are traversing the lens, for the purpose of calculating the deflection angle

$\Rightarrow$ angles as 2D vectors $= \dfrac{\text{vector on the plane}}{\text{distance of the plane}}$




$\vec{\eta}$ = position of source in source plane, corresponding to angle $\vec{\beta} \equiv \dfrac{\vec{\eta}}{D_s}$

$\vec{\xi}$ = position of its image on lens plane, angle $\vec{\vartheta} \equiv \dfrac{\vec{\xi}}{D_d} = \dfrac{\vec{y}}{D_s}$

$$\therefore \quad \vec{y} = \vec{\eta} + D_{ds}\hat{\alpha} \quad \Rightarrow \quad \vec{\eta} = \vec{y} - D_{ds}\hat{\alpha} = \frac{D_s}{D_d}\vec{\xi} - D_s\vec{\alpha} \quad \Big| \cdot \frac{1}{D_s}$$

$$\Rightarrow \quad \boxed{\vec{\beta} = \vec{\vartheta} - \vec{\alpha}(\vec{\vartheta})} \quad (4)$$

Lens equation, relates true ($\vec{\beta}$) and observed ($\vec{\vartheta}$) position of source on the sky.

$$\vec{\alpha}(\vec{\vartheta}) \overset{(2,4)}{=} \frac{D_{ds}}{D_s} 4G \int d^2\xi' \, \Sigma(\vec{\xi}') \frac{\vec{\xi}-\vec{\xi}'}{|\vec{\xi}-\vec{\xi}'|^2} = \frac{D_d D_{ds}}{D_s} 4G \int d^2\vartheta' \, \Sigma(D_d\vec{\vartheta}') \frac{\vec{\vartheta}-\vec{\vartheta}'}{|\vec{\vartheta}-\vec{\vartheta}'|^2}$$

$$= \frac{1}{\pi} \int d^2\vartheta' \, \kappa(\vec{\vartheta}') \frac{\vec{\vartheta}-\vec{\vartheta}'}{|\vec{\vartheta}-\vec{\vartheta}'|^2} \quad \text{where} \quad \kappa(\vec{\vartheta}) \equiv 4\pi G \frac{D_d D_{ds}}{D_s} \Sigma(D_d\vec{\vartheta}) \equiv \frac{\Sigma(D_d\vec{\vartheta})}{\Sigma_{cr}} \quad (5)$$

dimensionless surface mass density = convergence

$$\Sigma_{cr} \equiv \frac{1}{4\pi G} \frac{D_s}{D_d D_{ds}}$$

critical surface mass density, dividing line between weak and strong lenses (which can produce multiple images)

- Units: In relativistic units ($c=1$), $[G] = \frac{\text{distance}}{\text{mass}}$ ; $[\Sigma] = \frac{\text{mass}}{\text{area}}$ $\therefore$ $\varkappa$ is dimensionless

- $$\boxed{\vec{\alpha} = \frac{1}{\pi}\int d^2\vartheta' \, \varkappa(\vec{\vartheta}') \frac{\vec{\vartheta}-\vec{\vartheta}'}{|\vec{\vartheta}-\vec{\vartheta}'|^2}} \quad (9) \qquad \boxed{\vec{\beta} = \vec{\vartheta} - \vec{\alpha}(\vec{\vartheta})} \quad \text{Lens Equation} \quad (8)$$

mapping from the lens plane to the source plane, $\vec{\vartheta} \to \vec{\beta}$

observed location $\to$ true location

- Rewrite $\vec{\alpha}$ in terms of a potential, $\vec{\alpha} = \nabla\psi$ :

$$\psi(\vec{\vartheta}) \equiv \frac{1}{\pi}\int d^2\vartheta' \, \varkappa(\vec{\vartheta}') \ln|\vec{\vartheta}-\vec{\vartheta}'| \quad \Rightarrow \quad \nabla\psi = \frac{1}{\pi}\int d^2\vartheta' \, \varkappa(\vec{\vartheta}') \underbrace{\nabla \ln|\vec{\vartheta}-\vec{\vartheta}'|}_{\text{what is this?}} = \vec{\alpha}$$

deflection potential

For a 2D vector $\vec{r} = (x,y)$

$$\nabla \ln|\vec{r}| = \nabla \ln\sqrt{x^2+y^2} = \left(\partial_x \ln\sqrt{x^2+y^2}, \partial_y \ln\sqrt{x^2+y^2}\right) = \frac{(x,y)}{r^2} = \frac{\vec{r}}{r^2}$$

$$\partial_x \ln\sqrt{x^2+y^2} = \frac{\partial_x\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \frac{2x}{2(x^2+y^2)} = \frac{x}{r^2}$$

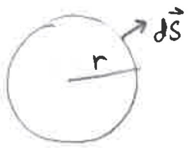
$$\nabla\cdot\vec{\alpha} = \nabla^2\psi = \frac{1}{\pi}\int d^2\vartheta' \, \varkappa(\vec{\vartheta}') \nabla^2 \ln|\vec{\vartheta}-\vec{\vartheta}'| = 2\int d^2\vartheta' \, \varkappa(\vec{\vartheta}') \delta_D(\vec{\vartheta}-\vec{\vartheta}') = 2\varkappa(\vec{\vartheta}) \quad (13)$$

(2D Poisson equation)

$$\nabla^2 \ln\vec{r} = \nabla\cdot\left(\frac{\vec{r}}{r^2}\right) = 2\pi\delta_D^2(\vec{r}), \quad \text{since}$$

$$\nabla\cdot\left(\frac{\vec{r}}{r^2}\right) = \partial_x\left(\frac{x}{x^2+y^2}\right) + \partial_y\left(\frac{y}{x^2+y^2}\right) = \frac{2}{x^2+y^2} - \frac{x\cdot 2x + y\cdot 2y}{(x^2+y^2)^2} = 0 \quad \text{for } \vec{r}\neq 0$$

$$\text{But } \int_A \nabla\cdot\left(\frac{\vec{r}}{r^2}\right) = \int_S \frac{\vec{r}}{r^2}\cdot d\vec{S} = \int_S \frac{1}{r}\cdot r\,d\varphi = 2\pi \quad \text{for a disk enclosing } \vec{r}=0$$




Note the similarity w 3D Newtonian gravity: $\varkappa$ is like mass density $\rho$, deflection potential $\psi$ is like gravitational potential $\phi$, and $\vec{\alpha}$ like acceleration (gravitational field) $\vec{g}$

## Comparison to 3D Newtonian gravity

$$\phi(\vec r) = -G\int d^3 r' \, \rho(\vec r') \frac{1}{|\vec r - \vec r'|} \qquad \psi(\vec\theta) = \frac{1}{\pi}\int d^2\theta' \, \kappa(\vec\theta') \ln|\vec\theta - \vec\theta'|$$

$$\vec a(\vec r) = -\nabla\phi = -G\int d^3 r' \, \rho(\vec r') \frac{\vec r - \vec r'}{|\vec r - \vec r'|^3} \qquad \vec\alpha(\vec\theta) = \nabla\psi = \frac{1}{\pi}\int d^2\theta' \, \kappa(\vec\theta') \frac{\vec\theta - \vec\theta'}{|\vec\theta - \vec\theta'|^2}$$

$$\nabla^2\phi = 4\pi G\rho = -\nabla\cdot\vec a \qquad \nabla^2\psi = 2\kappa = \nabla\cdot\vec\alpha$$

- The differences come from 2D vs 3D. We get to 2D by integrating over the 3rd direction.

$$\nabla\left(-\frac{1}{r}\right) = +\frac{\hat r}{r^2} = +\frac{\vec r}{|\vec r|^3} \qquad \nabla \ln r = \frac{\hat r}{r} = \frac{\vec r}{|\vec r|^2}$$

$$\nabla^2\left(-\frac{1}{r}\right) = \nabla\cdot\left(\frac{\vec r}{r^3}\right) = +4\pi\delta_D^3(\vec r) \qquad \nabla^2 \ln r = \nabla\cdot\left(\frac{\vec r}{r^2}\right) = 2\pi\delta_D^2(\vec r)$$

- Signs: the deflection angle $\vec\alpha$ corresponds to the acceleration $\vec a$, as it is caused by gravitational acceleration of light. But we defined it w the opposite sign: to correspond to the shift in the image position. Thus it is $-\vec\alpha$ that corresponds to $\vec a$

| | |
|---|---|
| $\frac{-1}{\lvert\vec r - \vec r'\rvert}$ | $+\ln\lvert\vec\theta - \vec\theta'\rvert$ |
| $G\rho$ | $\frac{\kappa}{\pi} = 4G\frac{D_d D_{ds}}{D_s}\Sigma$ |
| $\phi$ | $\psi$ |
| $\vec a$ | $-\vec\alpha$ |

factor 2 from GR vs NG, leaves $\frac{2 D_d D_{ds}}{D_s}$ from integration acceleration → angle

$$\nabla^2\phi = 4\pi\cdot G\rho \qquad \nabla^2\psi = 2\pi\cdot\frac{\kappa}{\pi}$$

difference $4\pi$ vs $2\pi$ is 3D vs 2D