

Due on Tuesday, January 15th, 2008, by 14:00 o'clock.

1. **Thomson scattering of circular polarization.** The differential cross section of Thomson scattering is

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi} |\hat{\epsilon}' \cdot \hat{\epsilon}^*|^2$$

where  $\hat{\epsilon}'$  and  $\hat{\epsilon}$  are the polarization vectors of the incoming and outgoing photons. In general they are complex unit vectors, and can be expressed, e.g., in the scattering plane basis  $\hat{\epsilon}_{\parallel}$ ,  $\hat{\epsilon}_{\perp}$  as  $a_1 e^{i\alpha_1} \hat{\epsilon}_{\parallel} + a_2 e^{i\alpha_2} \hat{\epsilon}_{\perp}$ . Define the Stokes parameters with respect to the scattering plane basis, and find out

- (a) how Stokes parameters  $I(\vec{q}')$ ,  $V(\vec{q}')$  scatter into  $I(\vec{q})$ ,  $V(\vec{q})$   
 (b) how Stokes parameters  $I(\vec{q}')$ ,  $Q(\vec{q}')$  scatter into  $I(\vec{q})$ ,  $V(\vec{q})$ .

2. **Scattering matrix** The scattering matrix for the Stokes parameter “vector”  $(I_{\parallel}, I_{\perp}, U)$  is

$$S(\beta) = \frac{3}{2} \begin{bmatrix} \cos^2 \beta & & \\ & 1 & \\ & & \cos \beta \end{bmatrix}$$

Convert it to the scattering matrix for  $\vec{T} \equiv (I, Q + iU, Q - iU)$  ( $Q$  and  $U$  defined with respect to the scattering plane).

3. **Scattering of anisotropic unpolarized radiation.** Consider Thomson scattering of unpolarized radiation. Express the gain terms  $dI(\vec{q})/dt$ ,  $dQ(\vec{q})/dt$ ,  $dU(\vec{q})/dt$  in terms of the multipoles of the directional dependence of the incoming radiation  $I(q, \hat{n}')$  in the coordinate system where  $\hat{z} \parallel \vec{q}$ . Show that the outgoing radiation is isotropic and unpolarized if the quadrupole of the incoming radiation vanishes.

4. **Symmetries of Wigner D-functions.** Derive the symmetry relation

$$D_{mm'}^{\ell}(\alpha, \beta, \gamma) = D_{-m, -m'}^{\ell*}(\alpha, -\beta, \gamma),$$

Eq. (11) of lecture notes Chapter C4, from the relations given before that in the lecture notes.

5. **Spin 2 spherical harmonics.** Derive the explicit forms of

$${}_2Y_2^m(\theta, \phi) \equiv \frac{N_2}{\sqrt{2}} (W_{2m} + iX_{2m}),$$

from the equations for tensor spherical harmonics in Chapter Y3 and the formulae for the ordinary  $Y_{\ell}^m$  given in Chapter F6. Verify that they agree with

$${}_sY_{\ell}^m(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} e^{im\phi} d_{m, -s}^{\ell}(\theta)$$

and the formulae for  $d_{mm'}^2(\theta)$  in Table 4.6 of Varshalovich et al. (reproduced on page C4.3 of lecture notes).

6. **Free streaming of polarization.** Find the polarization multipoles  $E_{\ell}^{m'}$  and  $B_{\ell}^{m'}$  of the free streaming equation

$$\frac{\partial \bar{T}(\eta, \hat{n})}{\partial \eta} = -ikn_3 \bar{T}(\eta, \hat{n}).$$

(Consider separately the case of  $\ell = 2$ ). Explain, using this result, why vector and tensor perturbations produce both E-mode and B-mode polarization, but scalar perturbations produce only E-mode polarization.