Cosmological Perturbation Theory

Hannu Kurki-Suonio

21.1.2012

About these lecture notes:

I lectured a course on cosmological perturbation theory at the University of Helsinki in the spring of 2003, and then again in the fall of 2010. These are the lecture notes for the latter course.

There is an unfortunate variety in the notation employed by various authors. My notation is the result of first learning perturbation theory from Mukhanov, Feldman, and Brandenberger (Phys. Rep. **215**, 213 (1992)) and then from Liddle and Lyth (*Cosmological Inflation and Large-Scale Structure*, Cambridge University Press 2000, Chapters 14 and 15), and represents a compromise between their notations.

Contents

1 Perturbative General Relativity **1**

2 The Background Universe **2**

3 The Perturbed Universe **4**

4 Gauge Transformations **4**
4.1 Gauge Transformation of the Metric Perturbations 8

5 Separation into Scalar, Vector, and Tensor Perturbations **9**

6 Perturbations in Fourier Space **12**
6.1 Gauge Transformation in Fourier Space 14

7 Scalar Perturbations **15**
7.1 Bardeen Potentials 16

8 Conformal–Newtonian Gauge **17**
8.1 Perturbation in the Curvature Tensors 17

9 Perturbation in the Energy Tensor **19**
9.1 Separation into Scalar, Vector, and Tensor Parts 21
9.2 Gauge Transformation of the Energy Tensor Perturbations 21
9.2.1 General Rule 21
9.2.2 Scalar Perturbations 22
9.2.3 Conformal-Newtonian Gauge 22
9.3 Scalar Perturbations in the Conformal-Newtonian Gauge 22

10 Field Equations for Scalar Perturbations in the Newtonian Gauge **23**

11 Energy-Momentum Continuity Equations **24**

12 Perfect Fluid Scalar Perturbations in the Newtonian Gauge **25**
12.1 Field Equations 25
12.2 Adiabatic perturbations 26
12.3 Continuity Equations 26

13 Scalar Perturbations in the Matter-Dominated Universe **27**

14 Perfect Fluid Scalar Perturbations when $p = p(\rho)$ **29**

15 Other Gauges **30**
15.1 Slicing and Threading 31
15.2 Comoving Gauge 31
15.3 Mixing Gauges 33
15.4 Comoving Curvature Perturbation 34
15.5 Perfect Fluid Scalar Perturbations, Again 35
15.5.1 Adiabatic Perfect Fluid Perturbations at Superhorizon Scales 36
15.6 Spatially Flat Gauge 37

16 Synchronous Gauge **37**

17 Fluid Components **40**
17.1 Division into Components . . . 40
17.2 Gauge Transformations . . . 41
17.3 Equations . . . 42

18 Adiabatic and Isocurvature Perturbations in a Simplified Universe **44**
18.1 Background solution for radiation+matter . . . 44
18.2 Perturbations . . . 45
18.3 Initial Epoch . . . 47
18.3.1 Adiabatic modes . . . 48
18.3.2 Isocurvature Modes . . . 49
18.3.3 Other Perturbations . . . 49
18.4 Full evolution for large scales . . . 50

19 Effect of a Smooth Component **51**

20 The Real Universe **51**

21 Early Radiation-Dominated Era **53**
21.1 Neutrino Adiabaticity . . . 55
21.2 Matter . . . 56
21.2.1 The Completely Adiabatic Solution . . . 57
21.2.2 Baryon and CDM Entropy Perturbations . . . 57
21.3 Neutrino perturbations . . . 58

22 Superhorizon Evolution and Relation to Original Perturbations **59**

23 Gaussian Initial Conditions **59**

24 Large Scales **59**

25 Sachs–Wolfe Effect **59**

26 Matter Power Spectrum **61**

A General Perturbation **62**

1 Perturbative General Relativity

In the perturbation theory of general relativity one considers a spacetime, the *perturbed spacetime*, that is *close* to a simple, symmetric, spacetime, the *background spacetime*, that we already know. In the development of perturbation theory we keep referring to these *two different spacetimes* (see Fig. 1).

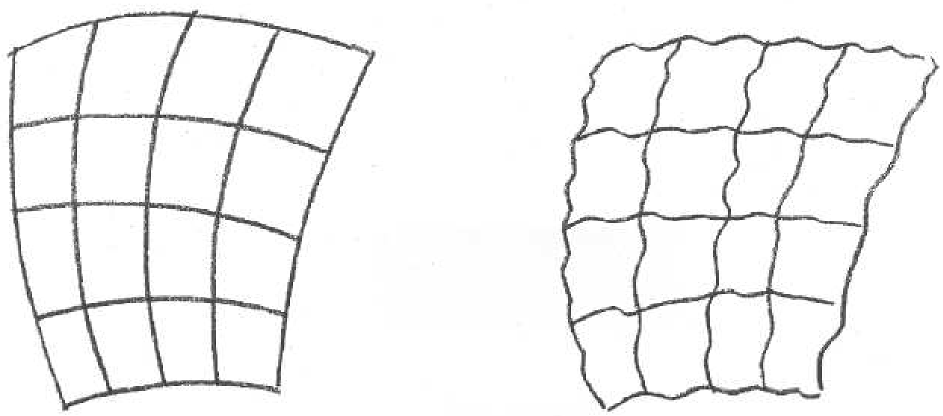

Figure 1: The background spacetime and the perturbed spacetime.

This means that there exists a coordinate system on the perturbed spacetime, where its metric can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \,, \tag{1.1}$$

where $\bar{g}_{\mu\nu}$ is the metric of the background spacetime (we shall refer to the *background* quantities with the overbar[1]), and $\delta g_{\mu\nu}$ is small. We also require that the first and second partial derivatives, $\delta g_{\mu\nu,\rho}$ and $\delta g_{\mu\nu,\rho\sigma}$ are small[2].

The curvature tensors and the energy tensor of the perturbed spacetime can then be written as

$$G^\mu_\nu = \bar{G}^\mu_\nu + \delta G^\mu_\nu \tag{1.2}$$
$$T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu \,, \tag{1.3}$$

where δG^μ_ν and δT^μ_ν are small. Subtracting the Einstein equations of the two spacetimes,

$$G^\mu_\nu = 8\pi G T^\mu_\nu \qquad \text{and} \qquad \bar{G}^\mu_\nu = 8\pi G \bar{T}^\mu_\nu \,, \tag{1.4}$$

from each other, we get the field equation for the perturbations

$$\delta G^\mu_\nu = 8\pi G \delta T^\mu_\nu \,. \tag{1.5}$$

The above discussion requires a pointwise correspondence between the two spacetimes, so that we can perform the comparisons and the subtractions. This correspondence is given by the coordinate system (x^0, x^1, x^2, x^3): the point $\bar{P}$ in the background spacetime and the point P in the perturbed spacetime which have the same coordinate values, correspond to each other. Now, given a coordinate system on the background spacetime, there exist many coordinate systems,

[1] Beware: the overbars will be dropped eventually.

[2] Actually it is not always necessary to require the second derivatives of the metric perturbation to be small; but then our development would require more care.

all close to each other, for the perturbed spacetime, for which (1.1) holds. The choice among these coordinate systems is called the *gauge choice*, to be discussed a little later.

In *first-order* (or *linear*) perturbation theory, we drop all terms from our equations which contain products of the small quantities $\delta g_{\mu\nu}$, $\delta g_{\mu\nu,\rho}$ and $\delta g_{\mu\nu,\rho\sigma}$. The field equation (1.5) becomes then a linear differential equation for $\delta g_{\mu\nu}$, making things much easier than in full GR. In *second-order* perturbation theory, one keeps also those terms with a product of two (but no more) small quantities. In these lectures we only discuss first-order perturbation theory.

The simplest case is the one where the background is the Minkowski space. Then $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, and $\bar{G}^\mu_\nu = \bar{T}^\mu_\nu = 0$.

In *cosmological perturbation theory* the background spacetime is the Friedmann–Robertson–Walker universe. Now the background spacetime is curved, and is not empty. While it is homogeneous and isotropic, it is *time-dependent*. In these lectures we shall only consider the case where the background is the *flat* FRW universe. This case is much simpler than the open and closed ones[3], since now the $t = \text{const}$ time slices ("space at time t") have Euclidean geometry. This will allow us to do 3-dimensional Fourier-transformations in space.

The separation into the background and perturbation is always done so that the spatial average of the perturbation is zero, i.e., the background value (at time t) is the spatial average of the full quantity over the time slice $t = \text{const}$.

2 The Background Universe

Our background spacetime is the flat Friedmann–Robertson–Walker universe FRW(0). The background metric in *comoving* coordinates (t, x, y, z) is

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)\,, \tag{2.1}$$

where $a(t)$ is the *scale factor*, to be solved from the (flat universe) Friedmann equations,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho} \tag{2.2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p}) \tag{2.3}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter,

$$\cdot \equiv \frac{d}{dt}\,,$$

and the $\bar{\rho}$ and $\bar{p}$ are the homogeneous background energy density and pressure. Another version of the second Friedmann equation is

$$\dot{H} \equiv \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G(\bar{\rho} + \bar{p})\,. \tag{2.4}$$

We shall find it more convenient to use as a time coordinate the *conformal time* η, defined by

$$d\eta = \frac{dt}{a(t)} \tag{2.5}$$

[3]For cosmological perturbation theory for open or closed FRW universe, see e.g. Mukhanov, Feldman, Brandenberger, Phys. Rep. **215**, 203 (1992).

so that the background metric is

$$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j\right] = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2)\,. \tag{2.6}$$

That is,

$$\bar{g}_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \quad \Rightarrow \quad \bar{g}^{\mu\nu} = a^{-2}(\eta)\eta^{\mu\nu}\,. \tag{2.7}$$

Using the conformal time, the Friedmann equations are (exercise)

$$\mathcal{H}^2 = \left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}a^2 \tag{2.8}$$

$$\mathcal{H}' = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p})a^2\,, \tag{2.9}$$

where

$$' \equiv \frac{d}{d\eta} = a\frac{d}{dt} = a(\;)^{\cdot} \tag{2.10}$$

and

$$\mathcal{H} \equiv \frac{a'}{a} = aH = \dot{a} \tag{2.11}$$

is the *conformal*, or *comoving*, Hubble parameter. Note that

$$\mathcal{H}' = \left(\frac{a'}{a}\right)' = \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = (a\dot{a})^{\cdot} - \dot{a}^2 = a\ddot{a} = a^2\frac{\ddot{a}}{a} = a^2(\dot{H} + H^2)\,. \tag{2.12}$$

The energy continuity equation

$$\dot{\bar{\rho}} = -3H(\bar{\rho} + \bar{p}) \tag{2.13}$$

becomes just

$$\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{p}) \equiv -3\mathcal{H}(1+w)\bar{\rho}\,. \tag{2.14}$$

For later convenience we define the equation-of-state parameter

$$w \equiv \frac{\bar{p}}{\bar{\rho}} \tag{2.15}$$

and the "speed of sound squared"[4]

$$c_s^2 \equiv \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} \equiv \frac{\bar{p}'}{\bar{\rho}'}\,. \tag{2.16}$$

These two quantities always refer to the background values.

From the Friedmann equations (2.8,2.9) and the continuity equation (2.14) one easily derives additional useful *background relations*, like

$$\mathcal{H}' = -\tfrac{1}{2}(1+3w)\mathcal{H}^2\,, \tag{2.17}$$

$$\frac{w'}{1+w} = 3\mathcal{H}(w - c_s^2)\,, \tag{2.18}$$

and

$$\bar{p}' = w\bar{\rho}' + w'\bar{\rho} = -3\mathcal{H}(1+w)c_s^2\bar{\rho}\,. \tag{2.19}$$

Eq. (2.17) shows that $w = -\frac{1}{3}$ corresponds to constant comoving Hubble length $\mathcal{H}^{-1} = \text{const.}$ For $w < -\frac{1}{3}$ the comoving Hubble length shrinks with time ("inflation"), whereas for $w > -\frac{1}{3}$ it grows with time ("normal" expansion). When $w = \text{const.}$, we have $c_s^2 = w$.

[4] This turns out to be the speed of sound if our ρ and p describe ordinary fluid. Even if they do not, we nevertheless define this quantity, although the name is then misleading.

3 The Perturbed Universe

We write the metric of the perturbed (around FRW(0)) universe as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu})\,, \tag{3.1}$$

where $h_{\mu\nu}$, as well as $h_{\mu\nu,\rho}$ and $h_{\mu\nu,\rho\sigma}$ are assumed small. Since we are doing first-order perturbation theory, we shall drop from all equations all terms with are of order $\mathcal{O}(h^2)$ or higher, and just write "=" to signify equality to first order in $h_{\mu\nu}$. Here the perturbation $h_{\mu\nu}$ is not a tensor in the perturbed universe, neither is $\eta_{\mu\nu}$, but we *define*

$$h^\mu_\nu \equiv \eta^{\mu\rho} h_{\rho\nu}\,, \qquad h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}\,. \tag{3.2}$$

One easily finds (exercise), that the inverse metric of the perturbed spacetime is

$$g^{\mu\nu} = a^{-2}(\eta^{\mu\nu} - h^{\mu\nu}) \tag{3.3}$$

(to first order).

We shall now give different names for the time and space components of the perturbed metric, defining

$$[h_{\mu\nu}] \equiv \begin{bmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix} \tag{3.4}$$

where

$$D = -\tfrac{1}{6} h^i_i \tag{3.5}$$

carries the trace of the spatial metric perturbation h_{ij}, and E_{ij} is traceless,

$$\delta^{ij} E_{ij} = 0\,. \tag{3.6}$$

Since indices on $h_{\mu\nu}$ are raised and lowered with $\eta_{\mu\nu}$, we immediately have

$$[h^{\mu\nu}] \equiv \begin{bmatrix} -2A & +B_i \\ +B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix} \tag{3.7}$$

On B_i and E_{ij} we do not raise/lower indices (or if we do, it is just the same thing, $B^i = \delta^{ij} B_j = B_i$).

The line element is thus

$$ds^2 = a^2(\eta)\left\{-(1+2A)d\eta^2 - 2B_i d\eta dx^i + \left[(1-2D)\delta_{ij} + 2E_{ij}\right] dx^i dx^j\right\}\,. \tag{3.8}$$

The function $A(\eta, x^i)$ is called the *lapse function*, and $B_i(\eta, x^i)$ the *shift vector*.

4 Gauge Transformations

The association between points in the background spacetime and the perturbed spacetime is via the coordinate system $\{x^\alpha\}$. As we noted earlier, for a given coordinate system in the background, there are many possible coordinate systems in the perturbed spacetime, all close to each other, that we could use. In GR perturbation theory, a *gauge transformation* means a coordinate transformation between such coordinate systems in the perturbed spacetime. (It may be helpful to temporarily forget at this point what you have learned about gauge transformations in other field theories, e.g. electrodynamics, so that you can learn the properties of this concept here with a fresh mind, without preconceptions.)

In this section, we denote the coordinates of the background by x^α, and two different coordinate systems in the perturbed spacetime (corresponding to two "gauges") by $\hat{x}^\alpha$ and $\tilde{x}^\alpha$.[5] The coordinates $\hat{x}^\alpha$ and $\tilde{x}^\alpha$ are related by a coordinate transformation

$$\tilde{x}^\alpha = \hat{x}^\alpha + \xi^\alpha \,, \tag{4.1}$$

where ξ^α and the derivatives $\xi^\alpha{}_{,\beta}$ are first-order small. The difference between $\frac{\partial \xi^\alpha}{\partial \hat{x}^\beta}$ and $\frac{\partial \xi^\alpha}{\partial \tilde{x}^\beta}$ is second-order small, and thus ignored, so we can write just $\xi^\alpha{}_{,\beta}$. In fact, we shall think of ξ^α as living on the background spacetime.

The situation is illustrated in Fig. 2. The coordinate system $\{\hat{x}^\alpha\}$ associates point $\bar{P}$ in the background with $\hat{P}$, whereas $\{\tilde{x}^\alpha\}$ associates the same background point $\bar{P}$ with another point $\tilde{P}$. The association is by

$$\tilde{x}^\alpha(\tilde{P}) = \hat{x}^\alpha(\hat{P}) = x^\alpha(\bar{P}) \,. \tag{4.2}$$

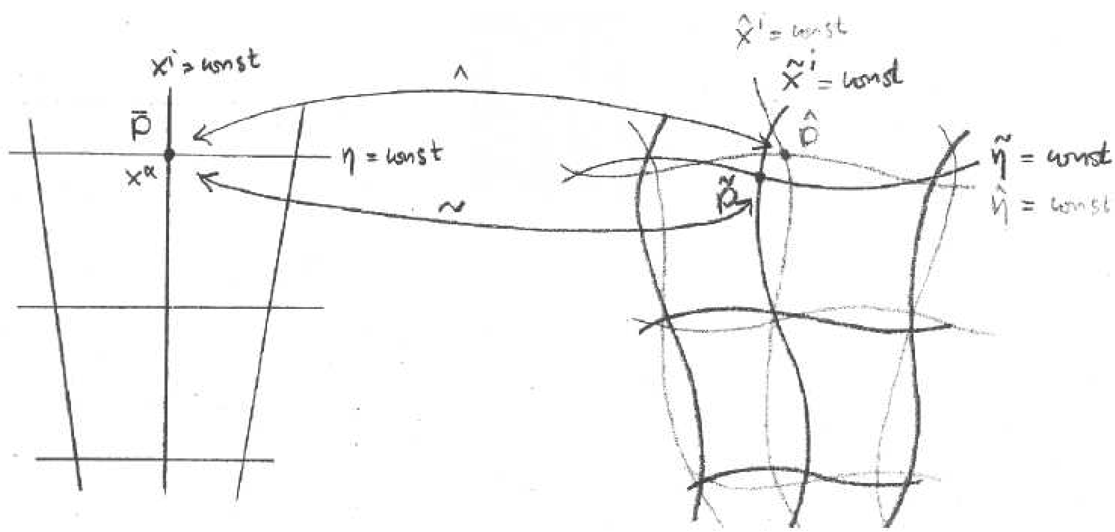

Figure 2: Gauge transformation. The background spacetime is on the left and the perturbed spacetime, with two different coordinate systems, is on the right. The "const" for the coordinate lines refer to the same constant for the two coordinate systems, i.e., $\tilde{\eta} = \hat{\eta} = \text{const} \equiv \eta(\bar{P})$ and $\tilde{x}^i = \hat{x}^i = \text{const} \equiv x^i(\bar{P})$.

The coordinate transformation relates the coordinates of the *same* point in the perturbed spacetime, i.e.,

$$\begin{aligned} \tilde{x}^\alpha(\tilde{P}) &= \hat{x}^\alpha(\tilde{P}) + \xi^\alpha \\ \tilde{x}^\alpha(\hat{P}) &= \hat{x}^\alpha(\hat{P}) + \xi^\alpha \,. \end{aligned} \tag{4.3}$$

Now the difference $\xi^\alpha(\tilde{P}) - \xi^\alpha(\hat{P})$ is second-order small. Thus we write just ξ^α and associate it with the background point:

$$\xi^\alpha = \xi^\alpha(\bar{P}) = \xi^\alpha(x^\beta) \,.$$

Using Eqs. (4.2) and (4.3), we get the relation between the coordinates of the two different points in a given coordinate system,

$$\begin{aligned} \hat{x}^\alpha(\tilde{P}) &= \hat{x}^\alpha(\hat{P}) - \xi^\alpha \\ \tilde{x}^\alpha(\tilde{P}) &= \tilde{x}^\alpha(\hat{P}) - \xi^\alpha \,. \end{aligned} \tag{4.4}$$

[5] Ordinarily (when not doing gauge transformations) we write just x^α for both the background and perturbed spacetime coordinates.

Let us now consider how various tensors transform in the gauge transformation. We have, of course, the usual GR transformation rules for scalars (s), vectors (w^α), and other tensors,

$$\begin{aligned} s &= s \\ w^{\tilde\alpha} &= X^{\tilde\alpha}_{\hat\beta} w^{\hat\beta} \\ A^{\tilde\alpha}{}_{\tilde\beta} &= X^{\tilde\alpha}_{\hat\gamma} X^{\hat\delta}_{\tilde\beta} A^{\hat\gamma}{}_{\hat\delta} \\ B_{\tilde\alpha\tilde\beta} &= X^{\hat\gamma}_{\tilde\alpha} X^{\hat\delta}_{\tilde\beta} B_{\hat\gamma\hat\beta} \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} X^{\tilde\alpha}_{\hat\beta} &\equiv \frac{\partial \tilde x^\alpha}{\partial \hat x^\beta} = \delta^\alpha_\beta + \xi^\alpha{}_{,\beta} \\ X^{\hat\delta}_{\tilde\beta} &\equiv \frac{\partial \hat x^\delta}{\partial \tilde x^\beta} = \delta^\delta_\beta - \xi^\delta{}_{,\beta}\,. \end{aligned} \tag{4.6}$$

These rules refer to the values of these quantities at a given point in the perturbed spacetime. However, this is not what we want now! Please, pay attention, since the following is central to understanding GR perturbation theory!

We shall be interested in perturbations of various quantities. In the background spacetime we may have various 4-scalar fields $\bar s$, 4-vector fields $\bar w^\alpha$ and tensor fields $\bar A^\alpha{}_\beta$, $\bar B_{\alpha\beta}$. In the perturbed spacetime we have corresponding perturbed quantities,

$$\begin{aligned} s &= \bar s + \delta s \\ w^\alpha &= \bar w^\alpha + \delta w^\alpha \\ A^\alpha{}_\beta &= \bar A^\alpha{}_\beta + \delta A^\alpha{}_\beta \\ B_{\alpha\beta} &= \bar B_{\alpha\beta} + \delta B_{\alpha\beta}\,. \end{aligned} \tag{4.7}$$

Consider first the 4-scalar s. The full quantity $s = \bar s + \delta s$ lives on the perturbed spacetime. However, we cannot assign a unique background quantity $\bar s$ to a point in the perturbed spacetime, because in different gauges this point is associated with different points in the background, with different values of $\bar s$. Therefore there is also no unique perturbation δs, but the perturbation is gauge-dependent. The perturbations in different gauges are defined as

$$\begin{aligned} \widehat{\delta s}(x^\alpha) &\equiv s(\hat P) - \bar s(\bar P) \\ \widetilde{\delta s}(x^\alpha) &\equiv s(\tilde P) - \bar s(\bar P)\,. \end{aligned} \tag{4.8}$$

The perturbation δs is obtained from a subtraction between two spacetimes, and we consider it as *living on the background spacetime*. It changes in the gauge transformation. Relate now $\widehat{\delta s}$ to $\widetilde{\delta s}$:

$$s(\tilde P) = s(\hat P) + \frac{\partial s}{\partial \hat x^\alpha}(\hat P)\left[\hat x^\alpha(\tilde P) - \hat x^\alpha(\hat P)\right] = s(\hat P) - \frac{\partial s}{\partial \hat x^\alpha}(\hat P)\xi^\alpha = s(\hat P) - \frac{\partial \bar s}{\partial x^\alpha}(\bar P)\xi^\alpha\,,$$

where we approximated $\frac{\partial s}{\partial \hat x^\alpha}(\hat P) \approx \frac{\partial \bar s}{\partial x^\alpha}(\bar P)$, since the difference[6] between them is a first order perturbation, and multiplication by ξ^α makes it second order.

Since our background is homogeneous, $\bar s = \bar s(\eta, x^i) = \bar s(\eta)$ only, and

$$\frac{\partial \bar s}{\partial x^\alpha}(\bar P)\xi^\alpha = \frac{\partial \bar s}{\partial \eta}(\bar P)\xi^0 = \bar s'\xi^0\,.$$

[6] This difference is the perturbation of the covariant vector $s_{,\alpha}$. We are assuming perturbations are first order small also in quantities derived by covariant derivation from the "primary" quantities.

Thus we get

$$s(\tilde{P}) = s(\hat{P}) - \bar{s}'\xi^0\,, \tag{4.9}$$

and our final result for the gauge transformation of δs is

$$\widetilde{\delta s}(x^\alpha) = s(\hat{P}) - \bar{s}'\xi^0 - \bar{s}(\bar{P}) = \widehat{\delta s}(x^\alpha) - \bar{s}'\xi^0\,. \tag{4.10}$$

In analogy with (4.8), the perturbations in vector and tensor fields in the two gauges are defined

$$\begin{aligned} \widehat{\delta w}^\alpha(x^\beta) &\equiv w^{\hat\alpha}(\hat{P}) - \bar{w}^\alpha(\bar{P}) \\ \widetilde{\delta w}^\alpha(x^\beta) &\equiv w^{\tilde\alpha}(\tilde{P}) - \bar{w}^\alpha(\bar{P})\,. \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \widehat{\delta A}^\alpha{}_\beta(x^\gamma) &\equiv A^{\hat\alpha}{}_{\hat\beta}(\hat{P}) - \bar{A}^\alpha{}_\beta(\bar{P}) \\ \widetilde{\delta A}^\alpha{}_\beta(x^\gamma) &\equiv A^{\tilde\alpha}{}_{\tilde\beta}(\tilde{P}) - \bar{A}^\alpha{}_\beta(\bar{P}) \\ \widehat{\delta B}_{\alpha\beta}(x^\gamma) &\equiv B_{\hat\alpha\hat\beta}(\hat{P}) - \bar{B}_{\alpha\beta}(\bar{P}) \\ \widetilde{\delta B}_{\alpha\beta}(x^\gamma) &\equiv B_{\tilde\alpha\tilde\beta}(\tilde{P}) - \bar{B}_{\alpha\beta}(\bar{P})\,. \end{aligned} \tag{4.12}$$

Consider the case of a type $(0,2)$ 4-tensor field. We have

$$B_{\hat\mu\hat\nu}(\tilde{P}) = B_{\hat\mu\hat\nu}(\hat{P}) + \frac{\partial B_{\hat\mu\hat\nu}}{\partial \hat{x}^\alpha}\left[\hat{x}^\alpha(\tilde{P}) - \hat{x}^\alpha(\hat{P})\right] = B_{\hat\mu\hat\nu}(\hat{P}) - \frac{\partial \bar{B}_{\mu\nu}}{\partial x^\alpha}(\bar{P})\xi^\alpha \tag{4.13}$$

and

$$\begin{aligned} B_{\tilde\mu\tilde\nu}(\tilde{P}) &= X^{\hat\rho}_{\tilde\mu} X^{\hat\sigma}_{\tilde\nu} B_{\hat\rho\hat\sigma}(\tilde{P}) = (\delta^\rho_\mu - \xi^\rho{}_{,\mu})(\delta^\sigma_\nu - \xi^\sigma{}_{,\nu})\left[B_{\hat\rho\hat\sigma}(\hat{P}) - \frac{\partial \bar{B}_{\rho\sigma}}{\partial x^\alpha}(\bar{P})\xi^\alpha\right] \\ &= B_{\hat\mu\hat\nu}(\hat{P}) - \xi^\rho{}_{,\mu} B_{\hat\rho\hat\nu}(\hat{P}) - \xi^\sigma{}_{,\nu} B_{\hat\mu\hat\sigma}(\hat{P}) - \frac{\partial \bar{B}_{\mu\nu}}{\partial x^\alpha}(\bar{P})\xi^\alpha \\ &= B_{\hat\mu\hat\nu}(\hat{P}) - \xi^\rho{}_{,\mu} \bar{B}_{\rho\nu}(\bar{P}) - \xi^\sigma{}_{,\nu} \bar{B}_{\mu\sigma}(\bar{P}) - \frac{\partial \bar{B}_{\mu\nu}}{\partial x^\alpha}(\bar{P})\xi^\alpha\,, \end{aligned} \tag{4.14}$$

where we can replace $B_{\hat\mu\hat\sigma}(\hat{P})$ with $\bar{B}_{\mu\sigma}(\bar{P})$ in the two middle terms, since it is multiplied by a first-order quantity $\xi^\sigma{}_{,\nu}$ and we can thus ignore the perturbation part, which becomes second order.

Subtracting the background value at $\bar{P}$ we get the gauge transformation rule for the tensor perturbation $\delta B_{\mu\nu}$,

$$\begin{aligned} \widetilde{\delta B}_{\mu\nu} &\equiv B_{\tilde\mu\tilde\nu}(\tilde{P}) - \bar{B}_{\mu\nu}(\bar{P}) \\ &= B_{\hat\mu\hat\nu}(\hat{P}) - \bar{B}_{\mu\nu}(\bar{P}) - \xi^\rho{}_{,\mu}\bar{B}_{\rho\nu}(\bar{P}) - \xi^\sigma{}_{,\nu}\bar{B}_{\mu\sigma}(\bar{P}) - \frac{\partial \bar{B}_{\mu\nu}}{\partial x^\alpha}(\bar{P})\xi^\alpha \\ &= \widehat{\delta B}_{\mu\nu} - \xi^\rho{}_{,\mu}\bar{B}_{\rho\nu} - \xi^\sigma{}_{,\nu}\bar{B}_{\mu\sigma} - \bar{B}_{\mu\nu,\alpha}\xi^\alpha\,. \end{aligned} \tag{4.15}$$

In a similar manner we obtain the gauge transformation rules for 4-vector perturbations,

$$\widetilde{\delta w}^\alpha = \widehat{\delta w}^\alpha + \xi^\alpha{}_{,\beta}\bar{w}^\beta - \bar{w}^\alpha{}_{,\beta}\xi^\beta \tag{4.16}$$

and perturbations of type $(1,1)$ 4-tensors (exercise),

$$\widetilde{\delta A}^\mu_\nu = \widehat{\delta A}^\mu_\nu + \xi^\mu{}_{,\rho}\bar{A}^\rho_\nu - \xi^\sigma{}_{,\nu}\bar{A}^\mu_\sigma - \bar{A}^\mu_{\nu,\alpha}\xi^\alpha\,. \tag{4.17}$$

Since the background is isotropic and homogeneous, and our background coordinate system fully respects these properties, the background 4-vectors and tensors must be of the form

$$\bar{w}^\alpha = (\bar{w}^0, \vec{0}) \qquad \bar{A}^\alpha_\beta = \begin{bmatrix} \bar{A}^0_0 & 0 \\ 0 & \frac{1}{3}\delta^i_j \bar{A}^k_k \end{bmatrix}, \tag{4.18}$$

and they depend only on the (conformal) time coordinate η. Using these properties we can write the gauge transformation rules for the individual components of 4-scalar, 4-vector and type (1,1) 4-tensor perturbations (we now drop the hats from the first gauge),

$$\begin{aligned}
\widetilde{\delta s} &= \delta s - \bar{s}'\xi^0 \\
\widetilde{\delta w}^0 &= \delta w^0 + \xi^0{}_{,0}\bar{w}^0 - \bar{w}^0{}_{,0}\xi^0 \\
\widetilde{\delta w}^i &= \delta w^i + \xi^i{}_{,0}\bar{w}^0 \\
\widetilde{\delta A}^0_0 &= \delta A^0_0 - \bar{A}^0_{0,0}\xi^0 \qquad (4.19) \\
\widetilde{\delta A}^0_i &= \delta A^0_i + \tfrac{1}{3}\xi^0{}_{,i}\bar{A}^k_k - \xi^0{}_{,i}\bar{A}^0_0 \\
\widetilde{\delta A}^i_0 &= \delta A^i_0 + \xi^i{}_{,0}\bar{A}^0_0 - \tfrac{1}{3}\xi^i{}_{,0}\bar{A}^k_k \\
\widetilde{\delta A}^i_j &= \delta A^i_j - \tfrac{1}{3}\delta^i_j\bar{A}^k_{k,0}\xi^0 \,. \qquad (4.20)
\end{aligned}$$

The following combinations (the trace and the traceless part of $\widetilde{\delta A}^i_j$) are also useful:

$$\begin{aligned}
\widetilde{\delta A}^k_k &= \delta A^k_k - \bar{A}^k_{k,0}\xi^0 \\
\widetilde{\delta A}^i_j - \tfrac{1}{3}\delta^i_j\widetilde{\delta A}^k_k &= \delta A^i_j - \tfrac{1}{3}\delta^i_j\delta A^k_k \,.
\end{aligned} \tag{4.21}$$

Thus the traceless part of δA^i_j is gauge-invariant!

4.1 Gauge Transformation of the Metric Perturbations

Applying the gauge transformation equation (4.15) to the metric perturbation, we have

$$\widetilde{\delta g}_{\mu\nu} = \delta g_{\mu\nu} - \xi^\rho{}_{,\mu}\bar{g}_{\rho\nu} - \xi^\sigma{}_{,\nu}\bar{g}_{\mu\sigma} - \bar{g}_{\mu\nu,0}\xi^0 \,, \tag{4.22}$$

where we have replaced the sum $\bar{g}_{\mu\nu,\alpha}\xi^\alpha$ with $\bar{g}_{\mu\nu,0}\xi^0$, since the background metric depends only on the time coordinate $x^0 = \eta$, and dropped the hats from the first gauge. Remembering $\bar{g}_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$ from Eq. (2.7), we have

$$\bar{g}_{\mu\nu,0} = 2a'a\eta_{\mu\nu} \tag{4.23}$$

and

$$\widetilde{\delta g}_{\mu\nu} = \delta g_{\mu\nu} + a^2\left[-\xi^\rho{}_{,\mu}\eta_{\rho\nu} - \xi^\sigma{}_{,\nu}\eta_{\mu\sigma} - 2\frac{a'}{a}\eta_{\mu\nu}\xi^0\right] . \tag{4.24}$$

From Eqs. (3.1) and (3.4) we have

$$[\delta g_{\mu\nu}] = a^2 \begin{bmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix} \tag{4.25}$$

Applying the gauge transformation law (4.24) now separately to the different metric perturbation components, we get first

$$\begin{aligned}\widetilde{\delta g}_{00} &\equiv -2a^2\tilde{A} \\ &= \delta g_{00} + a^2\left(-\xi^\rho{}_{,0}\eta_{\rho 0} - \xi^\sigma{}_{,0}\eta_{0\sigma} - 2\frac{a'}{a}\eta_{00}\xi^0\right) \\ &= -2a^2 A + a^2\left(+\xi^0{}_{,0} + \xi^0{}_{,0} + 2\frac{a'}{a}\xi^0\right), \end{aligned} \tag{4.26}$$

from which we obtain the gauge transformation law

$$\tilde{A} = A - \xi^0{}_{,0} - \frac{a'}{a}\xi^0 . \tag{4.27}$$

Similarly, from δg_{0i} we obtain

$$\tilde{B}_i = B_i + \xi^i{}_{,0} - \xi^0{}_{,i}, \tag{4.28}$$

and from δg_{ij},

$$-\tilde{D}\delta_{ij} + \tilde{E}_{ij} = -D\delta_{ij} + E_{ij} - \tfrac{1}{2}(\xi^i{}_{,j} + \xi^j{}_{,i}) - \frac{a'}{a}\xi^0\delta_{ij} . \tag{4.29}$$

The trace of $\frac{1}{2}(\xi^i{}_{,j} + \xi^j{}_{,i})$ is $\xi^k{}_{,k}$, so we can write

$$\tfrac{1}{2}(\xi^i{}_{,j} + \xi^j{}_{,i}) = \tfrac{1}{3}\delta_{ij}\xi^k{}_{,k} + \tfrac{1}{2}(\xi^i{}_{,j} + \xi^j{}_{,i}) - \tfrac{1}{3}\delta_{ij}\xi^k{}_{,k}, \tag{4.30}$$

where the last two terms are the traceless part, and we can separate Eq. (4.29) into

$$\begin{aligned}\tilde{D} &= D + \tfrac{1}{3}\xi^k{}_{,k} + \frac{a'}{a}\xi^0 \\ \tilde{E}_{ij} &= E_{ij} - \tfrac{1}{2}(\xi^i{}_{,j} + \xi^j{}_{,i}) + \tfrac{1}{3}\delta_{ij}\xi^k{}_{,k} . \end{aligned} \tag{4.31}$$

5 Separation into Scalar, Vector, and Tensor Perturbations

In GR perturbation theory there are two kinds of coordinate transformations of interest. One is the gauge transformation just discussed, where the coordinates of the background are kept fixed, but the coordinates in the perturbed spacetime are changed, changing the correspondence between the points in the background and the perturbed spacetime.

The other kind is one where we keep the gauge, i.e, the correspondence between the background and perturbed spacetime points, fixed, but do a coordinate transformation in the background spacetime. This then induces a corresponding coordinate transformation in the perturbed spacetime. Our background coordinate system was chosen to respect the symmetries of the background, and we do not want to lose this property. In cosmological perturbation theory we have chosen the background coordinates to respect its homogeneity property, which gives us a unique slicing of the spacetime into homogeneous $t =$ const. spacelike slices. Thus we do not want to change this slicing. This leaves us:

1. homogeneous transformations of the time coordinate, i.e., reparameterizations of time, of which we already had an example, when we switched from cosmic time t to conformal time η,

2. and transformations in the space coordinates[7]

$$x^{i'} = X^{i'}_{\ k} x^k \,, \tag{5.1}$$

where $X^{i'}_k$ is independent of time; which is the case we consider in this section.

We had chosen the coordinates for our background, FRW(0), so that the 3-metric was Euclidean,

$$g_{ij} = a^2 \delta_{ij} \,, \tag{5.2}$$

and we want to keep this property. This leaves us rotations. The full transformation matrices are then

$$X^{\mu'}_{\ \rho} = \begin{bmatrix} 1 & 0 \\ 0 & X^{i'}_{\ k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R^{i'}_{\ k} \end{bmatrix} \qquad \text{and} \qquad X^{\mu}_{\ \rho'} = \begin{bmatrix} 1 & 0 \\ 0 & R^{i}_{\ k'} \end{bmatrix} \,, \tag{5.3}$$

where $R^{i'}_{\ k}$ is a rotation matrix[8], with the property $R^T R = I$, or $R^{i'}_{\ k} R^{i'}_{\ l} = (R^T R)_{kl} = \delta_{kl}$. Thus $R^T = R^{-1}$ so that $R^{i'}_{\ k} = R^{k}_{\ i'}$.

This coordinate transformation in the background induces the corresponding transformation,

$$x^{\mu'} = X^{\mu'}_{\ \rho} x^\rho \,, \tag{5.4}$$

into the perturbed spacetime. Here the metric is

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1-2A & -B_i \\ -B_i & (1-2D)\delta_{ij} + 2E_{ij} \end{bmatrix} = a^2 \eta_{\mu\nu} + a^2 \begin{bmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix} . \tag{5.5}$$

Transforming the metric,

$$g_{\rho'\sigma'} = X^{\mu}_{\ \rho'} X^{\nu}_{\ \sigma'} g_{\mu\nu} \,, \tag{5.6}$$

we get for the different components

$$\begin{aligned} g_{0'0'} &= X^{\mu}_{\ 0'} X^{\nu}_{\ 0'} g_{\mu\nu} = X^{0}_{\ 0'} X^{0}_{\ 0'} g_{00} = g_{00} = a^2(-1-2A) \\ g_{0'l'} &= X^{\mu}_{\ 0'} X^{\nu}_{\ l'} g_{\mu\nu} = X^{0}_{\ 0'} X^{j}_{\ l'} g_{0j} = -a^2 R^{j}_{\ l'} B_j \\ g_{k'l'} &= X^{i}_{\ k'} X^{j}_{\ l'} g_{ij} = a^2 \left(-2D\delta_{ij} R^{i}_{\ k'} R^{j}_{\ l'} + 2E_{ij} R^{i}_{\ k'} R^{j}_{\ l'} \right) \\ &= a^2 \left(-2D\delta_{kl} + 2E_{ij} R^{i}_{\ k'} R^{j}_{\ l'} \right) \,, \end{aligned} \tag{5.7}$$

from which we identify the perturbations in the new coordinates,

$$\begin{aligned} A' &= A \\ D' &= D \\ B_{l'} &= R^{j}_{\ l'} B_j \\ E_{k'l'} &= R^{i}_{\ k'} R^{j}_{\ l'} E_{ij} \,. \end{aligned} \tag{5.8}$$

Thus A and D transform as scalars under rotations in the background spacetime coordinates, B_i transforms as a 3-vector, and E_{ij} as a 3-d tensor. While staying in a fixed gauge, we can thus think of them as scalar, vector, and tensor fields on the 3-d Euclidean background space. We

[7] In this section we use $'$ to denote the other coordinate system. Do not confuse with $' \equiv d/d\eta$ in the other sections.

[8] In this notation $R^{i'}_{\ j}$ and $R^{i}_{\ j'}$ are two different matrices, corresponding to opposite rotations; the position of the $'$ indicates which way we are rotating. We have put the first index upstairs to follow the Einstein summation convention—but we could have written $R_{i'j}$ and $R_{ij'}$ just as well.

are, however, not yet satisfied. We can extract two more scalar quantities and one more vector quantity from B_i and E_{ij}.

We know from Euclidean 3-d vector calculus, that a vector field can be divided into two parts, the first one with zero curl, the second one with zero divergence,

$$\vec{B} = \vec{B}^S + \vec{B}^V \,, \qquad \text{with} \qquad \nabla \times \vec{B}^S = 0 \qquad \text{and} \qquad \nabla \cdot \vec{B}^V = 0 \,, \tag{5.9}$$

and that the first one can be expressed as (minus) a gradient of some scalar field[9]

$$\vec{B}^S = -\nabla B \,. \tag{5.10}$$

In component notation,

$$B_i = -B_{,i} + B_i^V \,, \qquad \text{where} \qquad \delta^{ij} B^V_{i,j} = 0 \,. \tag{5.11}$$

In like manner, the symmetric traceless tensor field E_{ij} can be divided into three parts,

$$E_{ij} = E^S_{ij} + E^V_{ij} + E^T_{ij} \,, \tag{5.12}$$

where E^S_{ij} and E^V_{ij} can be expressed in terms of a scalar field E and a vector field E_i,

$$E^S_{ij} = \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right) E = E_{,ij} - \tfrac{1}{3}\delta_{ij}\delta^{kl}E_{,kl} \tag{5.13}$$

$$E^V_{ij} = -\tfrac{1}{2}(E_{i,j} + E_{j,i}) \,, \qquad \text{where} \qquad \delta^{ij}E_{i,j} = \nabla \cdot \vec{E} = 0 \tag{5.14}$$

$$\text{and} \qquad \delta^{ik}E^T_{ij,k} = 0, \qquad \delta^{ij}E^T_{ij} = 0 \,. \tag{5.15}$$

We see that E^S_{ij} is symmetric and traceless by construction. E^V_{ij} is symmetric by construction, and the condition on E_i makes it traceless. The tensor E^T_{ij} is assumed symmetric, and the two conditions on it make it *transverse* and traceless. The meaning of "transverse" and the nature of the above construction becomes clearer in the next section when we do this in Fourier space.

Under rotations in background space,

$$\begin{aligned} A' &= A \,, \qquad B' = B \,, \qquad D' = D \,, \qquad E' = E \,, \\ B^V_{l'} &= R^j{}_{l'} B^V_j \,, \qquad E_{l'} = R^j{}_{l'} E_j \,, \\ E^T_{k'l'} &= R^i{}_{k'} R^j{}_{l'} E^T_{ij} \,. \end{aligned} \tag{5.16}$$

The metric perturbation can thus be divided into

1. a *scalar* part, consisting of A, B, D, and E,

2. a *vector* part, consisting of B^V_i and E_i,

3. and a *tensor* part E^T_{ij}.

The names "scalar", "vector", and "tensor" refer to their *transformation properties under rotations in the background space.*[10]

The Einstein tensor perturbation δG^μ_ν and the energy tensor perturbation δT^μ_ν can likewise be divided into scalar+vector+tensor; the scalar part of δG^μ_ν coming only from the scalar part of $\delta g_{\mu\nu}$ and so on.

[9]This sign convention corresponds to thinking of the scalar function B as a "potential", where the vector field $\vec{B}^S$ "flows downhill". We use the same letter B here for both the original vector field B_i and this scalar potential B. There should be no confusion since the vector field always has an index (or $\vec{\ }$). Same goes for the E.

[10]Thus "scalar" does *not* mean, e.g., that the perturbation would be invariant under gauge transformations—scalar perturbations are *not gauge-invariant*, as we have already seen, e.g. in Eqs. (4.27) and (4.31).

The important thing about this division is that *the scalar, vector, and tensor parts do not couple to each other* (in first-order perturbation theory), but they evolve independently. This allows us to treat them separately: We can study, e.g., scalar perturbations as if the vector and tensor perturbations were absent. The total evolution of the full perturbation is just a linear superposition of the independent evolution of the scalar, vector, and tensor part of the perturbation.

We imposed one constraint on each of the 3-vectors B_i^V and E_i, and $3 + 1 = 4$ constraints on the symmetric 3-d tensor E_{ij}^T leaving each of them 2 independent components. Thus the 10 degrees of freedom corresponding to the 10 components of the metric perturbation $h_{\mu\nu}$ are divided into

$$\begin{aligned} 1+1+1+1 &= 4 \qquad \text{scalar} \\ 2+2 &= 4 \qquad \text{vector} \\ 2 &= 2 \qquad \text{tensor} \end{aligned} \tag{5.17}$$

degrees of freedom.

The scalar perturbations are the most important. They couple to density and pressure perturbations and exhibit gravitational instability: overdense regions grow more overdense. They are responsible for the formation of structure in the universe from small initial perturbations.

Vector perturbations couple to rotational velocity perturbations in the cosmic fluid. They tend to decay in an expanding universe, and are therefore probably not important in cosmology.

We have done all of the above in a fixed gauge. It turns out that gauge transformations affect scalar and vector perturbations, but tensor perturbations are gauge-invariant. Tensor perturbations are nothing but gravitational waves, this time[11] in an expanding universe. When they are extracted from the total perturbation by the above separation procedure, they are automatically in the "transverse traceless gauge". (This expression is clarified in the next section.) Tensor perturbations have cosmological importance since, if strong enough, they have an observable effect on the anisotropy of the cosmic microwave background.

6 Perturbations in Fourier Space

Because our background space is flat we can Fourier expand the perturbations. For an arbitrary perturbation $f = f(\eta, x^i) = f(\eta, \vec{x})$, we write

$$f(\eta, \vec{x}) = \sum_{\vec{k}} f_{\vec{k}}(\eta) e^{i\vec{k}\cdot\vec{x}} \,. \tag{6.1}$$

(Using a Fourier sum implies using a fiducial box with some volume V. At the end of the day we can let $V \to \infty$, and replace remaining Fourier sums with integrals.) In first-order perturbation theory each Fourier component evolves independently. We can thus just study the evolution of a single Fourier component, with some arbitrary wave vector $\vec{k}$, and we drop the subscript $_{\vec{k}}$ from the Fourier amplitudes.

Since $\vec{x} = (x^1, x^2, x^3)$ is a comoving coordinate, $\vec{k}$ is a *comoving wave vector*. The comoving (or coordinate) wave number $k \equiv |\vec{k}|$ and wavelength $\lambda = 2\pi/k$ are related to the physical wavelength and wave number of the Fourier mode by

$$k_{\text{phys}} = \frac{2\pi}{\lambda_{\text{phys}}} = \frac{2\pi}{a\lambda} = a^{-1}k \,. \tag{6.2}$$

[11] Contrasted to perturbation theory around Minkowski space, which is the way gravitational waves are usually introduced in GR.

Thus the wavelength $\lambda_{\rm phys}$ of the Fourier mode $\vec{k}$ grows in time as the universe expands.

In the separation into scalar+vector+tensor, we follow Liddle&Lyth and include an additional factor $k \equiv |\vec{k}|$ in the Fourier components of B and E_i, and a factor k^2 in E, so that we have, e.g.,

$$B(\eta, \vec{x}) = \sum_{\vec{k}} \frac{B_{\vec{k}}(\eta)}{k} e^{i\vec{k}\cdot\vec{x}} . \tag{6.3}$$

The purpose of this is to make them have the same dimension and magnitude as B_i^S, E_{ij}^S and E_{ij}^V. [12] That is,

$$B_i = B_i^S + B_i^V , \qquad \text{and} \qquad E_{ij} = E_{ij}^S + E_{ij}^V + E_{ij}^T , \tag{6.4}$$

where

$$\begin{aligned} B_i^S &= -B_{,i} \qquad \text{becomes} \qquad B_i^S = -i\frac{k_i}{k}B \\ E_{ij}^S &= \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right) E \qquad \text{becomes} \qquad E_{ij}^S = \left(-\frac{k_i k_j}{k^2} + \tfrac{1}{3}\delta_{ij}\right) E , \\ E_{ij}^V &= -\tfrac{1}{2}(E_{i,j} + E_{j,i}) \qquad \text{becomes} \qquad E_{ij}^V = -\frac{i}{2k}(k_i E_j + k_j E_i) , \end{aligned} \tag{6.5}$$

and the conditions

$$\delta^{ij} B_{i,j}^V = 0 , \qquad \delta^{ij} E_{i,j} = 0 , \qquad \text{and} \qquad \delta^{ik} E_{ij,k}^T = \delta^{ij} E_{ij}^T = 0 \tag{6.6}$$

become

$$\delta^{ij} k_j B_i^V = \vec{k}\cdot\vec{B}^V = 0 , \qquad \delta^{ij} k_j E_i = \vec{k}\cdot\vec{E} = 0 , \qquad \text{and} \qquad \delta^{ik} k_k E_{ij}^T = \delta^{ij} E_{ij}^T = 0 . \tag{6.7}$$

To make the separation into scalar+vector+tensor parts as clear as possible, rotate the background coordinates so that the z axis becomes parallel to $\vec{k}$,

$$\vec{k} = k\hat{z} = (0, 0, k) \tag{6.8}$$

($\hat{z}$ denoting the unit vector in z direction.) Then

$$B_i^S = (0, 0, -iB) \tag{6.9}$$

and

$$E_{ij}^S = \begin{bmatrix} 0 & & \\ & 0 & \\ & & -E \end{bmatrix} + \begin{bmatrix} \frac{1}{3}E & & \\ & \frac{1}{3}E & \\ & & \frac{1}{3}E \end{bmatrix} = \begin{bmatrix} \frac{1}{3}E & & \\ & \frac{1}{3}E & \\ & & -\frac{2}{3}E \end{bmatrix} \tag{6.10}$$

and we can write the scalar part of $\delta g_{\mu\nu}$ as

$$\delta g_{\mu\nu}^S = a^2 \begin{bmatrix} -2A & & & +iB \\ & 2(-D + \frac{1}{3}E) & & \\ & & 2(-D + \frac{1}{3}E) & \\ +iB & & & 2(-D - \frac{2}{3}E) \end{bmatrix} \tag{6.11}$$

For the vector part we have then

$$\vec{k}\cdot\vec{B}^V = 0 \quad \Rightarrow \quad \vec{B}^V = (B_1, B_2, 0) \tag{6.12}$$
$$\vec{k}\cdot\vec{E} = 0 \quad \Rightarrow \quad \vec{E} = (E_1, E_2, 0) \tag{6.13}$$

[12] Powers of k cancel in Eqs. (6.5). The metric perturbations, A, B, D, E, B_i^V, E_i, and E_{ij}^T will then all have the same dimension in Fourier space, which facilitates comparison of their magnitudes.

and

$$E^V_{ij} = \frac{-i}{2k}\left(k_i E_j + k_j E_i\right) = -\frac{i}{2}\begin{bmatrix} & & E_1 \\ & & E_2 \\ E_1 & E_2 & \end{bmatrix}, \tag{6.14}$$

so that the vector part of $\delta g_{\mu\nu}$ is

$$\delta g^V_{\mu\nu} = a^2 \begin{bmatrix} & -B_1 & -B_2 & \\ -B_1 & & & -iE_1 \\ -B_2 & & & -iE_2 \\ & -iE_1 & -iE_2 & \end{bmatrix}. \tag{6.15}$$

For the tensor part,

$$\delta^{ik} k_k E^T_{ij} \equiv \sum_i k_i E^T_{ij} = 0 \;\Rightarrow\; E^T_{3j} \equiv E^T_{j3} = 0 \tag{6.16}$$

so that

$$E^T_{ij} = \begin{bmatrix} E^T_{11} & E^T_{12} & \\ E^T_{12} & -E^T_{11} & \\ & & \end{bmatrix}. \tag{6.17}$$

The tensor part of $\delta g_{\mu\nu}$ becomes

$$\delta g^T_{\mu\nu} = a^2 \begin{bmatrix} & & & \\ & 2E^T_{11} & 2E^T_{12} & \\ & 2E^T_{12} & -2E^T_{11} & \\ & & & \end{bmatrix} = a^2 \begin{bmatrix} & & & \\ & h_+ & h_\times & \\ & h_\times & -h_+ & \\ & & & \end{bmatrix}, \tag{6.18}$$

where we have denoted the two gravitational wave polarization amplitudes by $E^T_{11} = \frac{1}{2}h_+$ and $E^T_{12} = \frac{1}{2}h_\times$.

We see how the scalar part of the perturbation is associated with the time direction, the wave direction $\vec{k}$ and the trace. The vector part is associated with the two remaining space directions, those *transverse* to the wave vector. Thus the vectors have only two independent components. The tensor part is also associated with these two transverse directions; being also symmetric and traceless, it thus has only two independent components.

Putting all together, the full metric perturbation (for a Fourier mode in the z direction) is

$$\delta g_{\mu\nu} = a^2 \begin{bmatrix} -2A & -B_1 & -B_2 & +iB \\ -B_1 & 2(-D + \frac{1}{3}E) + h_+ & h_\times & -iE_1 \\ -B_2 & h_\times & 2(-D + \frac{1}{3}E) - h_+ & -iE_2 \\ +iB & -iE_1 & -iE_2 & 2(-D - \frac{2}{3}E) \end{bmatrix}. \tag{6.19}$$

6.1 Gauge Transformation in Fourier Space

Consider then how the gauge transformation appears in Fourier space. We need now the Fourier transform of the gauge transformation vector

$$\xi^\mu(\eta, \vec{x}) = \sum_{\vec{k}} \xi^\mu_{\vec{k}}(\eta) e^{i\vec{k}\cdot\vec{x}}. \tag{6.20}$$

For a single Fourier mode, the gauge transformation Eqs. (4.27,4.28,4.31) become

$$\begin{aligned}\tilde{A} &= A-(\xi^0)'-\frac{a'}{a}\xi^0 &(6.21)\\ \tilde{B}_i &= B_i+(\xi^i)'-ik_i\xi^0 &(6.22)\\ \tilde{D} &= D+\tfrac{1}{3}ik_i\xi^i+\frac{a'}{a}\xi^0 &(6.23)\\ \tilde{E}_{ij} &= E_{ij}-\tfrac{1}{2}i\left(k_i\xi^j+k_j\xi^i\right)+\tfrac{1}{3}i\delta_{ij}k_k\xi^k\,. &(6.24)\end{aligned}$$

For illustration, consider again a mode in the z direction, $\vec{k}=(0,0,k)$. Now the new part added into the matrix of Eq. (6.19) is

$$a^2\begin{bmatrix} 2(\xi^0)'+2\frac{a'}{a}\xi^0 & -(\xi^1)' & -(\xi^2)' & -(\xi^3)'+ik\xi^0 \\ -(\xi^1)' & -2\frac{a'}{a}\xi^0 & & -ik\xi^1 \\ -(\xi^2)' & & -2\frac{a'}{a}\xi^0 & -ik\xi^2 \\ -(\xi^3)'+ik\xi^0 & -ik\xi^1 & -ik\xi^2 & -2\frac{a'}{a}\xi^0-2ik\xi^3 \end{bmatrix} \quad (6.25)$$

(note the cancelations on the diagonal from the D and E_{ij} parts). We see that no new tensor part is introduced. Thus *the tensor part of the metric perturbation is gauge-invariant*[13]. But we see that the components ξ^0 and ξ^3 are responsible for a new scalar part and the components ξ^1 and ξ^2 are responsible for a new vector part.

For Fourier modes in an arbitrary direction, the above means that the time component ξ^0 and the component of the space part $\vec{\xi}$ parallel to $\vec{k}$ are responsible for a change in the scalar perturbation and the *transverse* part of $\vec{\xi}$ is responsible for a change in the vector perturbation.

This freedom of doing gauge transformations can be used, e.g., to set 2 of the 4 scalar quantities of scalar perturbations and 2 of the 4 independent components of vector perturbations to zero. Thus only two of the degrees of freedom are real physical degrees of freedom in each case. Thus there are in total 6 physical degrees of freedom, 2 scalar, 2 vector, and 2 tensor. The other 4 (of the total 10) are just gauge degrees of freedom, representing perturbing just the coordinates, not the spacetime.

If the perturbation can be completely eliminated by a gauge transformation, we say the perturbation is "pure gauge", i.e., it is not a real perturbation of spacetime, just a perturbation in the coordinates.

7 Scalar Perturbations

From here on (except for the beginning of Sec. 9, where we discuss perturbations in the energy tensor) we shall consider scalar perturbations only. They are the ones responsible for the structure of the universe (i.e. the deviation from the homogeneous and isotropic FRW universe).

The metric is now

$$ds^2=a(\eta)^2\left\{-(1+2A)d\eta^2+2B_{,i}d\eta dx^i+\left[(1-2\psi)\delta_{ij}+2E_{,ij}\right]dx^idx^j\right\}\,, \quad (7.1)$$

where we have defined[14] the *curvature perturbation*

$$\psi\equiv D+\tfrac{1}{3}\nabla^2E\,. \quad (7.2)$$

[13] To make the meaning of this statement clear: Consider a perturbation that is initially purely tensor, and do an arbitrary gauge transformation. The parts that get added to the perturbation are of scalar and/or vector nature. Thus this perturbation is not gauge-invariant; but its *tensor part*—because of the way we have defined the tensor part of a perturbation—is.

[14] In my spring 2003 lecture notes (CMB Physics / Cosmological Perturbation Theory) I was using the symbol ψ for what I am now denoting D. The present notation is better in line with common usage.

In Fourier space this reads

$$\psi_{\vec{k}} = D_{\vec{k}} - \tfrac{1}{3}E_{\vec{k}}\,. \tag{7.3}$$

The components of $h_{\mu\nu}$ are

$$h_{\mu\nu} = \left[\begin{array}{ll} -2A & B_{,i} \\ B_{,i} & -2\psi\delta_{ij} + 2E_{,ij} \end{array}\right]. \tag{7.4}$$

Exercise: Curvature of the spatial hypersurface. The hypersurface η = const. is a 3-dimensional curved manifold. Calculate the connection coefficients ${}^{(3)}\Gamma^i_{jk}$ and the scalar curvature ${}^{(3)}R \equiv g^{ij\,(3)}R_{ij}$ of this 3-space for a scalar perturbation in terms of ψ and E.

Now if we start from a pure scalar perturbation and do an arbitrary gauge transformation, represented by the field $\xi^\mu = (\xi^0, \xi^i)$, we may introduce also a vector perturbation. This vector perturbation is however, pure gauge, and thus of no interest. Just like we did for the shift vector B_i earlier, we can divide ξ^i into a part with zero divergence (a transverse part) and a part with zero curl, expressible as a gradient of some function ξ,

$$\xi^i = \xi^i_{\rm tr} - \delta^{ij}\xi_{,j} = \vec{\xi}_{\rm tr} - \nabla\xi \qquad \text{where} \qquad \xi^i_{{\rm tr},i} = \nabla\cdot\vec{\xi}_{\rm tr} = 0\,. \tag{7.5}$$

The part $\xi^i_{\rm tr}$ is responsible for the spurious vector perturbation, whereas ξ^0 and $\xi_{,j}$ change the scalar perturbation. For our discussion of scalar perturbations we thus lose nothing, if we decide that we only consider gauge transformations, where the $\xi^i_{\rm tr}$ part is absent. These "scalar gauge transformations" are fully specified by two functions, ξ^0 and ξ,

$$\begin{aligned} \tilde{\eta} &= \eta + \xi^0(\eta, \vec{x}) \\ \tilde{x}^i &= x^i - \delta^{ij}\xi_{,j}(\eta, \vec{x}) \end{aligned} \tag{7.6}$$

and they preserve the scalar nature of the perturbation.

Applied to scalar perturbations and gauge transformations, our transformation equations (4.27,4.28,4.31) become

$$\begin{aligned} \tilde{A} &= A - \xi^{0\prime} - \frac{a'}{a}\xi^0 \\ \tilde{B} &= B + \xi' + \xi^0 \\ \tilde{D} &= D - \tfrac{1}{3}\nabla^2\xi + \frac{a'}{a}\xi^0 \\ \tilde{E} &= E + \xi\,, \end{aligned} \tag{7.7}$$

where we use the notation $' \equiv \partial/\partial\eta$ for quantities which depend on both η and $\vec{x}$. The quantity ψ defined in Eq. (7.2) is often used as the fourth scalar variable instead of D. For it, we get

$$\tilde{\psi} = \psi + \frac{a'}{a}\xi^0 = \psi + \mathcal{H}\xi^0\,. \tag{7.8}$$

7.1 Bardeen Potentials

We now define the following two quantities, called the *Bardeen potentials*,

$$\begin{aligned} \Phi &\equiv A + \mathcal{H}(B - E') + (B - E')' \\ \Psi &\equiv D + \tfrac{1}{3}\nabla^2 E - \mathcal{H}(B - E') \;=\; \psi - \mathcal{H}(B - E')\,. \end{aligned} \tag{7.9}$$

These quantities are *invariant* under gauge transformations (exercise).

8 Conformal–Newtonian Gauge

We can use the gauge freedom to set the scalar perturbations B and E equal to zero. From Eq. (7.7) we see that this is accomplished by choosing

$$\begin{aligned} \xi &= -E \\ \xi^0 &= -B + E' \,. \end{aligned} \tag{8.1}$$

Doing this gauge transformation we arrive at a commonly used gauge, which has many names: the *conformal-Newtonian* gauge (or sometimes, for short, just the *Newtonian* gauge), the *longitudinal* gauge, and the *zero-shear* gauge. We shall denote quantities in this gauge with the sub- or superscript N. Thus $B^N = E^N = 0$, whereas you immediately see that

$$\begin{aligned} A^N &= \Phi \\ D^N = \psi^N &= \Psi \,. \end{aligned} \tag{8.2}$$

Thus the Bardeen potentials are equal to the two nonzero metric perturbations in the conformal-Newtonian gauge.

From here on (until otherwise noted) we shall calculate in the conformal-Newtonian gauge. The metric is thus just

$$ds^2 = a(\eta)^2 \left[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\right] \,, \tag{8.3}$$

or

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1-2\Phi & \\ & (1-2\Psi)\delta_{ij} \end{bmatrix} \quad \text{and} \quad g^{\mu\nu} = a^{-2} \begin{bmatrix} -1+2\Phi & \\ & (1+2\Psi)\delta_{ij} \end{bmatrix} , \tag{8.4}$$

or

$$h_{\mu\nu} = \begin{bmatrix} -2\Phi & \\ & -2\Psi\delta_{ij} \end{bmatrix} \quad \text{and} \quad h^{\mu\nu} = \begin{bmatrix} -2\Phi & \\ & -2\Psi\delta_{ij} \end{bmatrix} . \tag{8.5}$$

8.1 Perturbation in the Curvature Tensors

From the conformal-Newtonian metric (8.3) we get the connection coefficients

$$\begin{array}{lll} \Gamma^0_{00} = \frac{a'}{a} + \Phi' & \Gamma^0_{0k} = \Phi_{,k} & \Gamma^0_{ij} = \frac{a'}{a}\delta_{ij} - \left[2\frac{a'}{a}(\Phi+\Psi) + \Psi'\right]\delta_{ij} \\ \Gamma^i_{00} = \Phi_{,i} & \Gamma^i_{0j} = \frac{a'}{a}\delta^i_j - \Psi'\delta^i_j & \Gamma^i_{kl} = -\left(\Psi_{,l}\delta^i_k + \Psi_{,k}\delta^i_l\right) + \Psi_{,i}\delta_{kl} \end{array} \tag{8.6}$$

and the sums

$$\begin{aligned} \Gamma^\alpha_{0\alpha} &= 4\frac{a'}{a} + \Phi' - 3\Psi' \\ \Gamma^\alpha_{i\alpha} &= \Phi_{,i} - 3\Psi_{,i} \end{aligned} \tag{8.7}$$

where we have dropped all terms higher than first order in the small quantities Φ and Ψ. Thus these expressions contain only 0^{th} and 1^{st} order terms, and separate into the background and perturbation, accordingly:

$$\Gamma^\alpha_{\beta\gamma} = \bar{\Gamma}^\alpha_{\beta\gamma} + \delta\Gamma^\alpha_{\beta\gamma} \,, \tag{8.8}$$

where

$$\begin{array}{lll} \bar{\Gamma}^0_{00} = \mathcal{H} & \bar{\Gamma}^0_{0k} = 0 & \bar{\Gamma}^0_{ij} = \mathcal{H}\delta_{ij} \\ \bar{\Gamma}^i_{00} = 0 & \bar{\Gamma}^i_{0j} = \mathcal{H}\delta^i_j & \bar{\Gamma}^i_{kl} = 0 \end{array} \tag{8.9}$$

and

$$\begin{aligned}
\delta\Gamma^0_{00} &= \Phi' & \delta\Gamma^0_{0k} &= \Phi_{,k} & \delta\Gamma^0_{ij} &= -\left[2\mathcal{H}(\Phi+\Psi)+\Psi'\right]\delta_{ij} \\
\delta\Gamma^i_{00} &= \Phi_{,i} & \delta\Gamma^i_{0j} &= -\Psi'\delta^i_j & \delta\Gamma^i_{kl} &= -\left(\Psi_{,l}\delta^i_k+\Psi_{,k}\delta^i_l\right)+\Psi_{,i}\delta_{kl}\,.
\end{aligned} \tag{8.10}$$

The Ricci tensor is

$$\begin{aligned}
R_{\mu\nu} &= \Gamma^\alpha_{\nu\mu,\alpha}-\Gamma^\alpha_{\alpha\mu,\nu}+\Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\nu\mu}-\Gamma^\alpha_{\nu\beta}\Gamma^\beta_{\alpha\mu} \\
&= \bar{R}_{\mu\nu}+\delta\Gamma^\alpha_{\nu\mu,\alpha}-\delta\Gamma^\alpha_{\alpha\mu,\nu}+\bar{\Gamma}^\alpha_{\alpha\beta}\delta\Gamma^\beta_{\nu\mu}+\bar{\Gamma}^\beta_{\nu\mu}\delta\Gamma^\alpha_{\alpha\beta}-\bar{\Gamma}^\alpha_{\nu\beta}\delta\Gamma^\beta_{\alpha\mu}-\bar{\Gamma}^\beta_{\alpha\mu}\delta\Gamma^\alpha_{\nu\beta}\,.
\end{aligned} \tag{8.11}$$

Calculation gives

$$\begin{aligned}
R_{00} &= -3\mathcal{H}'+3\Psi''+\nabla^2\Phi+3\mathcal{H}(\Phi'+\Psi') \\
R_{0i} &= 2(\Psi'+\mathcal{H}\Phi)_{,i} \\
R_{ij} &= (\mathcal{H}'+2\mathcal{H}^2)\delta_{ij} \\
&\quad +\left[-\Psi''+\nabla^2\Psi-\mathcal{H}(\Phi'+5\Psi')-(2\mathcal{H}'+4\mathcal{H}^2)(\Phi+\Psi)\right]\delta_{ij} \\
&\quad +(\Psi-\Phi)_{,ij}
\end{aligned} \tag{8.12}$$

Next we raise an index to get R^μ_ν. Note that we can not just raise the index of the background and perturbation parts separately, since

$$R^\mu_\nu = g^{\mu\alpha}R_{\alpha\nu} = (\bar{g}^{\mu\alpha}+\delta g^{\mu\alpha})(\bar{R}_{\alpha\nu}+\delta R_{\alpha\nu}) = \bar{R}^\mu_\nu+\delta g^{\mu\alpha}\bar{R}_{\alpha\nu}+\bar{g}^{\mu\alpha}\delta R_{\alpha\nu}\,. \tag{8.13}$$

We get

$$\begin{aligned}
R^0_0 &= 3a^{-2}\mathcal{H}'+a^{-2}\left[-3\Psi''-\nabla^2\Phi-3\mathcal{H}(\Phi'+\Psi')-6\mathcal{H}'\Phi\right] \\
R^0_i &= -2a^{-2}\left(\Psi'+\mathcal{H}\Phi\right)_{,i} \\
R^i_0 &= -R^0_i \;=\; 2a^{-2}\left(\Psi'+\mathcal{H}\Phi\right)_{,i} \\
R^i_j &= a^{-2}(\mathcal{H}'+2\mathcal{H}^2)\delta^i_j \\
&\quad +a^{-2}\left[-\Psi''+\nabla^2\Psi-\mathcal{H}(\Phi'+5\Psi')-(2\mathcal{H}'+4\mathcal{H}^2)\Phi\right]\delta_{ij} \\
&\quad +a^{-2}(\Psi-\Phi)_{,ij}\,.
\end{aligned} \tag{8.14}$$

and summing for the curvature scalar

$$\begin{aligned}
R &= R^0_0+R^i_i \\
&= 6a^{-2}(\mathcal{H}'+\mathcal{H}^2) \\
&\quad +a^{-2}\left[-6\Psi''+2\nabla^2(2\Psi-\Phi)-6\mathcal{H}(\Phi'+3\Psi')-12(\mathcal{H}'+\mathcal{H}^2)\Phi\right]\,.
\end{aligned} \tag{8.15}$$

And, finally, the Einstein tensor

$$\begin{aligned}
G^0_0 &= R^0_0-\tfrac{1}{2}R \\
&= -3a^{-2}\mathcal{H}^2+a^{-2}\left[-2\nabla^2\Psi+6\mathcal{H}\Psi'+6\mathcal{H}^2\Phi\right] \\
G^0_i &= R^0_i \\
G^i_0 &= R^i_0 \;=\; -R^0_i \;=\; -G^0_i \\
G^i_j &= R^i_j-\tfrac{1}{2}\delta^i_jR \\
&= a^{-2}(-2\mathcal{H}'-\mathcal{H}^2)\delta^i_j \\
&\quad +a^{-2}\left[2\Psi''+\nabla^2(\Phi-\Psi)+\mathcal{H}(2\Phi'+4\Psi')+(4\mathcal{H}'+2\mathcal{H}^2)\Phi\right]\delta^i_j \\
&\quad +a^{-2}(\Psi-\Phi)_{,ij}\,.
\end{aligned} \tag{8.16}$$

Note the background (written first) and perturbation parts in all these quantities. Since the background $\bar{R}^\mu_\nu$ and $\bar{G}^\mu_\nu$ are diagonal, the off-diagonals contain just the perturbation, and we have

$$R^0_i = G^0_i = \delta R^0_i = \delta G^0_i \,. \tag{8.17}$$

9 Perturbation in the Energy Tensor

Consider then the energy tensor[15].

The background energy tensor is necessarily of the perfect fluid form[16]

$$\begin{aligned} \bar{T}^{\mu\nu} &= (\bar\rho + \bar p)\bar u^\mu \bar u^\nu + \bar p \bar g^{\mu\nu} \\ \bar{T}^\mu_\nu &= (\bar\rho + \bar p)\bar u^\mu \bar u_\nu + \bar p \delta^\mu_\nu \,. \end{aligned} \tag{9.1}$$

Because of homogeneity, $\bar\rho = \bar\rho(\eta)$ and $\bar p = \bar p(\eta)$. Because of isotropy, the fluid is at rest, $\bar u^i = 0 \;\Rightarrow\; \bar u^\mu = (\bar u^0, 0, 0, 0)$ in the background universe. Since

$$\bar u_\mu \bar u^\mu = \bar g_{\mu\nu}\bar u^\mu \bar u^\nu = a^2 \eta_{\mu\nu}\bar u^\mu \bar u^\nu = -a^2(\bar u^0)^2 = -1 \,, \tag{9.2}$$

we have

$$\bar u^\mu = \frac{1}{a}(1, \vec 0) \qquad \text{and} \qquad \bar u_\mu = a(-1, \vec 0) \,. \tag{9.3}$$

The energy tensor of the perturbed universe is

$$T^\mu_\nu = \bar T^\mu_\nu + \delta T^\mu_\nu \,. \tag{9.4}$$

Just like the metric perturbation, the energy tensor perturbation has 10 degrees of freedom, of which 6 are physical and 4 are gauge. It can likewise be divided into scalar+vector+tensor, with 4+4+2 degrees of freedom, of which 2+2+2 are physical. The perturbation can also be divided into perfect fluid + non-perfect, with 5+5 degrees of freedom.

The perfect fluid degrees of freedom in δT^μ_ν are those which keep T^μ_ν in the perfect fluid form

$$T^\mu_\nu = (\rho + p)u^\mu u_\nu + p\delta^\mu_\nu \,. \tag{9.5}$$

Thus they can be taken as the density perturbation, pressure perturbation, and velocity perturbation

$$\rho = \bar\rho + \delta\rho \,, \qquad p = \bar p + \delta p \,, \qquad \text{and} \qquad u^i = \bar u^i + \delta u^i = \delta u^i \equiv \frac{1}{a} v_i \,. \tag{9.6}$$

The δu^0 is not an independent degree of freedom, because of the constraint $u_\mu u^\mu = -1$. We shall call

$$v_i \equiv a u^i \tag{9.7}$$

the velocity perturbation. It is equal to the coordinate velocity, since (to first order)

$$\frac{dx^i}{d\eta} = \frac{u^i}{u^0} = \frac{u^i}{\bar u^0} = a u^i = v_i \,. \tag{9.8}$$

[15] This section could actually have been earlier. We do not specify a gauge here, and the restriction to scalar perturbations is done only in the end.

[16] The "imperfections" can only show up in the energy tensor if there is inhomogeneity or anisotropy. Whether an observer would "feel" the $\bar p$ as pressure is another matter, which depends on the interactions of the fluid particles. But gravity only cares about the energy tensor.

It is also equal to the fluid velocity observed by a comoving (i.e., one whose x^i = const.) observer, since the ratio of change in comoving coordinate dx^i to change in conformal time $d\eta$ equals the ratio of the corresponding physical distance adx^i to the change in cosmic time $dt = ad\eta$.

We also define the *relative energy density perturbation*

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}}, \tag{9.9}$$

which is a dimensionless quantity in coordinate space (but not in Fourier space).

To express u^μ and u_ν in terms of v_i, write them as

$$\begin{aligned} u^\mu = \bar{u}^\mu + \delta u^\mu &\equiv \left(a^{-1} + \delta u^0, a^{-1}v_1, a^{-1}v_2, a^{-1}v_3\right) \\ u_\nu = \bar{u}_\nu + \delta u_\nu &\equiv \left(-a + \delta u_0, \delta u_1, \delta u_2, \delta u_3\right). \end{aligned} \tag{9.10}$$

These are related by $u_\nu = g_{\mu\nu}u^\nu$ and $u_\mu u^\mu = -1$. Using

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1-2A & -B_i \\ -B_i & (1-2D)\delta_{ij} + 2E_{ij} \end{bmatrix}, \tag{9.11}$$

we get

$$\begin{aligned} u_0 = g_{0\mu}u^\mu &= a^2(-1-2A)(a^{-1} + \delta u^0) - \delta^{ij}a^2 B_i a^{-1} v_j \\ &= -a - a^2\delta u^0 - 2aA \end{aligned} \tag{9.12}$$

(where we dropped higher than 1[st] order quantities, like $B_i v_j$), from which follows

$$\delta u_0 = -a^2\delta u^0 - 2aA. \tag{9.13}$$

Likewise

$$\delta u_i = u_i = g_{i\mu}u^\mu = -aB_i + av_i. \tag{9.14}$$

We solve the remaining unknown, δu^0 from

$$u_\mu u^\mu = \ldots = -1 - 2a\delta u^0 - 2A = -1 \quad \Rightarrow \quad \delta u^0 = -\frac{1}{a}A \tag{9.15}$$

Thus we have for the 4-velocity

$$u^\mu = \frac{1}{a}(1-A, v_i) \qquad \text{and} \qquad u_\mu = a(-1-A, v_i - B_i). \tag{9.16}$$

Inserting this into Eq. (9.5) we get

$$\begin{aligned} T^\mu_\nu &= \bar{T}^\mu_\nu + \delta T^\mu_\nu \\ &= \begin{bmatrix} -\bar{\rho} & 0 \\ 0 & \bar{p}\delta^i_j \end{bmatrix} + \begin{bmatrix} -\delta\rho & (\bar{\rho}+\bar{p})(v_i - B_i) \\ -(\bar{\rho}+\bar{p})v_i & \delta p \delta^i_j \end{bmatrix}. \end{aligned} \tag{9.17}$$

There are 5 remaining degrees of freedom in the space part, δT^i_j, corresponding to perturbations away from the perfect fluid from. We write them as

$$\delta T^i_j = \delta p \delta^i_j + \Sigma_{ij} \equiv \bar{p}\left(\frac{\delta p}{\bar{p}} + \Pi_{ij}\right). \tag{9.18}$$

Here Σ_{ij} and $\Pi_{ij} \equiv \Sigma_{ij}/\bar{p}$ are symmetric and traceless, which makes the separation into

$$\delta p \equiv \delta T^k_k \tag{9.19}$$

and

$$\Sigma_{ij} \equiv \delta T^i_j - \tfrac{1}{3}\delta^i_j \delta T^k_k \tag{9.20}$$

unique (the trace and the traceless part of δT^i_j). Σ_{ij} is called anisotropic stress (or anisotropic pressure). For a perfect fluid $\Sigma_{ij} = 0$.

9.1 Separation into Scalar, Vector, and Tensor Parts

The energy tensor perturbation δT^μ_ν is built out of the scalar perturbations $\delta\rho$, δp, the 3-vector $\vec{v} = v_i$ and the traceless 3-tensor Π_{ij}. Just like for the metric perturbations, we can extract a scalar perturbation out of $\vec{v}$:

$$v_i = v_i^S + v_i^V\,, \qquad \text{where} \quad v_i^S = -v_{,i} \quad \text{and} \quad \nabla\cdot\vec{v}^V = 0\,. \tag{9.21}$$

and a scalar + a vector perturbation out of Π_{ij}:

$$\Pi_{ij} = \Pi_{ij}^S + \Pi_{ij}^V + \Pi_{ij}^T\,, \tag{9.22}$$

where

$$\begin{aligned} \Pi_{ij}^S &= \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\Pi \\ \Pi_{ij}^V &= -\tfrac{1}{2}(\Pi_{i,j} + \Pi_{j,i}) \qquad \text{and} \end{aligned} \tag{9.23}$$

$$\delta^{ik}\Pi_{ij,k}^T = 0\,. \tag{9.24}$$

We see that perfect fluid perturbations ($\Pi_{ij} = 0$) do not have a tensor perturbation component.

For Fourier components of v_i and Π_{ij} we use the same (Liddle&Lyth) convention as for B_i and E_{ij} (see Sec. (6)).

In the early universe, we have anisotropic pressure from the cosmic neutrino background during and after neutrino decoupling, and from the cosmic microwave background during and after photon decoupling. Perturbations in the metric will make the momentum distribution of noninteracting particles anisotropic (this is anisotropic pressure). If there are sufficient interactions among the particles, these will isotropize the momentum distribution. Decoupling means that the interactions become too weak for this. If we aim for high precision in our calculations, we need to take this anisotropic pressure into account. For a more approximate treatment, the perfect fluid approximation can be made, which simplifies the calculations significantly.

9.2 Gauge Transformation of the Energy Tensor Perturbations

9.2.1 General Rule

Using the gauge transformation rules from Sect. 4 we have

$$\begin{aligned} \widetilde{\delta T}{}^0_0 &= -\widetilde{\delta\rho} \;=\; \delta T^0_0 - \bar{T}^0_{0,0}\xi^0 \;=\; -\delta\rho + \bar{\rho}'\xi^0 \\ \widetilde{\delta T}{}^i_0 &= -(\bar{\rho}+\bar{p})\tilde{v}_i \;=\; \delta T^i_0 + \xi^i{}_{,0}(\bar{T}^0_0 - \tfrac{1}{3}\bar{T}^k_k) \\ &= -(\bar{\rho}+\bar{p})v_i - \xi^i{}_{,0}(\bar{\rho}+\bar{p}) \\ \tfrac{1}{3}\widetilde{\delta T}{}^k_k &= \widetilde{\delta p} \;=\; \tfrac{1}{3}(\delta T^k_k - \bar{T}^k_{k,0}\xi^0) \;=\; \delta p - \bar{p}'\xi^0 \\ \widetilde{\delta T}{}^i_j - \tfrac{1}{3}\delta^i_j\widetilde{\delta T}{}^k_k &= \bar{p}\tilde{\Pi}_{ij} \;=\; \delta T^i_j - \tfrac{1}{3}\delta^i_j\delta T^k_k \;=\; \bar{p}\Pi_{ij} \end{aligned} \tag{9.25}$$

and we get the gauge transformation laws for the different parts of the energy tensor perturbation:

$$\begin{aligned} \widetilde{\delta\rho} &= \delta\rho - \bar{\rho}'\xi^0 && (9.26) \\ \widetilde{\delta p} &= \delta p - \bar{p}'\xi^0 && (9.27) \\ \tilde{v}_i &= v_i + \xi^i_{,0} && (9.28) \\ \tilde{\Pi}_{ij} &= \Pi_{ij} && (9.29) \\ \tilde{\delta} &= \delta - \frac{\bar{\rho}'}{\bar{\rho}}\xi^0 = \delta + 3\mathcal{H}(1+w)\xi^0 \,. && (9.30) \end{aligned}$$

Thus the anisotropic stress is gauge-invariant (being the traceless part of δT^i_j). Note that the $\delta\rho$ and δp equations are those of a perturbation of a 4-scalar, as they should be, as ρ and p are, indeed, 4-scalars.

9.2.2 Scalar Perturbations

For scalar perturbations, $v_i = -v_{,i}$ and $\xi^i = -\xi_{,i}$, so that we have

$$\begin{aligned} \tilde{v} &= v + \xi' \\ \tilde{\Pi} &= \Pi \,. \end{aligned} \qquad (9.31)$$

These hold both in coordinate space and Fourier space (we use the same Fourier convention for ξ as for v and B).

9.2.3 Conformal-Newtonian Gauge

We get to the conformal-Newtonian gauge by $\xi^0 = -B + E'$ and $\xi = -E$. Thus

$$\begin{aligned} \delta\rho^N &= \delta\rho + \bar{\rho}'(B - E') = \delta\rho - 3\mathcal{H}(1+w)\bar{\rho}(B - E') \\ \delta p^N &= \delta p + \bar{p}'(B - E') = \delta p - 3\mathcal{H}(1+w)c_s^2\bar{\rho}(B - E') \\ v^N &= v - E' \\ \Pi &= \Pi \,. \end{aligned} \qquad (9.32)$$

9.3 Scalar Perturbations in the Conformal-Newtonian Gauge

From here on we shall (unless otherwise noted)

1. consider scalar perturbations only, so that $v_i = -v_{,i}$ and $B_i = -B_{,i}$
2. use the conformal-Newtonian gauge, so that $B = 0$.

Thus the energy tensor perturbation has the form

$$\delta T^\mu_\nu = \begin{bmatrix} -\delta\rho^N & -(\bar{\rho}+\bar{p})v^N_{,i} \\ (\bar{\rho}+\bar{p})v^N_{,i} & \delta p^N \delta^i_j + \bar{p}(\Pi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\Pi) \end{bmatrix} \,. \qquad (9.33)$$

10 Field Equations for Scalar Perturbations in the Newtonian Gauge

We can now write the Einstein equations

$$\delta G^\mu_\nu = 8\pi G \delta T^\mu_\nu \tag{10.1}$$

for scalar perturbations in the conformal-Newtonian gauge. We have the left-hand side δG^μ_ν from Sect. 8.1 and the right-hand side δT^μ_ν from Sect. 9.3:

$$\begin{aligned}
\delta G^0_0 &= a^{-2}\left[-2\nabla^2\Psi + 6\mathcal{H}(\Psi' + \mathcal{H}\Phi)\right] = -8\pi G\delta\rho^N \\
\delta G^0_i &= -2a^{-2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = -8\pi G(\bar\rho + \bar p)v^N_{,i} \\
\delta G^i_0 &= 2a^{-2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = 8\pi G(\bar\rho + \bar p)v^N_{,i} \\
\delta G^i_j &= a^{-2}\left[2\Psi'' + \nabla^2(\Phi - \Psi) + \mathcal{H}(2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2)\Phi\right]\delta^i_j \\
&\quad + a^{-2}(\Psi - \Phi)_{,ij} = 8\pi G\left[\delta p^N\delta^i_j + \bar p(\Pi_{,ij} - \tfrac{1}{3}\delta_{ij}\nabla^2\Pi)\right] .
\end{aligned} \tag{10.2}$$

Separating the δG^i_j equation into its trace and traceless part (the trace of δ^i_j is 3, and the trace of $(\Psi - \Phi)_{,ij}$ is $\nabla^2(\Psi - \Phi)$) the full set of Einstein equations is

$$3\mathcal{H}(\Psi' + \mathcal{H}\Phi) - \nabla^2\Psi = -4\pi Ga^2\delta\rho^N \tag{10.3}$$

$$(\Psi' + \mathcal{H}\Phi)_{,i} = 4\pi Ga^2(\bar\rho + \bar p)v^N_{,i} \tag{10.4}$$

$$\Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \tfrac{1}{3}\nabla^2(\Phi - \Psi) = 4\pi Ga^2\delta p^N \tag{10.5}$$

$$(\partial_i\partial_j - \tfrac{1}{3}\delta^i_j\nabla^2)(\Psi - \Phi) = 8\pi Ga^2\bar p(\partial_i\partial_j - \tfrac{1}{3}\delta^i_j\nabla^2)\Pi . \tag{10.6}$$

The off-diagonal part of the last equation gives

$$(\Psi - \Phi)_{,ij} = 8\pi Ga^2\bar p\Pi_{,ij} \qquad \text{for} \qquad i \neq j . \tag{10.7}$$

In Fourier space this reads

$$-k_ik_j(\Psi_{\vec k} - \Phi_{\vec k}) = -\frac{k_ik_j}{k^2}8\pi Ga^2\bar p\Pi_{\vec k} \qquad \text{for} \qquad i \neq j . \tag{10.8}$$

(with the Liddle&Lyth Fourier convention for Π). Since we can always rotate the background coordinate system so that more than one of the components of $\vec k$ are non-zero, this means that

$$k^2(\Psi_{\vec k} - \Phi_{\vec k}) = 8\pi Ga^2\bar p\Pi_{\vec k} \qquad \text{for} \qquad \vec k \neq \vec 0 . \tag{10.9}$$

The 0^{th} Fourier component represents a constant offset. But the split into a background and a perturbation is always chosen so that the spatial average of the perturbation vanishes (and the background value thus represents the spatial average of the full perturbed quantity).

Thus we have (going back to $\vec x$-space)

$$\Psi - \Phi = 8\pi Ga^2\bar p\Pi . \tag{10.10}$$

Likewise, since the spatial average of a perturbation is always zero, the equality of gradients of two perturbations means the equality of those perturbations themselves. Thus Eq. (10.4) says that

$$\Psi' + \mathcal{H}\Phi = 4\pi Ga^2(\bar\rho + \bar p)v^N = \tfrac{3}{2}\mathcal{H}^2(1 + w)v^N . \tag{10.11}$$

Inserting this into Eq. (10.3) gives

$$\nabla^2\Psi = 4\pi G a^2\bar\rho\left[\delta^N + 3\mathcal{H}(1+w)v^N\right] . \tag{10.12}$$

The final form of the Einstein equations can be divided into two *constraint equations*

$$\begin{aligned}\nabla^2\Psi &= 4\pi G a^2\bar\rho\left[\delta^N + 3\mathcal{H}(1+w)v^N\right] & (10.13)\\ \Psi - \Phi &= 8\pi G a^2\bar p\Pi & (10.14)\end{aligned}$$

that apply to any given time slice, and to two *evolution equations*

$$\begin{aligned}\Psi' + \mathcal{H}\Phi &= \tfrac{3}{2}\mathcal{H}^2(1+w)v^N & (10.15)\\ \Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \tfrac{1}{3}\nabla^2(\Phi - \Psi) &= 4\pi G a^2\delta p^N . & (10.16)\end{aligned}$$

that determine how the metric perturbation evolves in time.

In Fourier space the Einstein equations can be written as

$$\begin{aligned}\mathcal{H}^{-1}\Psi' + \Phi + \tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Psi &= -\tfrac{1}{2}\delta^N\\ \mathcal{H}^{-1}\Psi' + \Phi &= \tfrac{3}{2}(1+w)\frac{\mathcal{H}}{k}v^N\\ \mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}\left(\Phi' + 2\Psi'\right) + \left(1 + \frac{2\mathcal{H}'}{\mathcal{H}^2}\right)\Phi - \tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\Phi - \Psi) &= \tfrac{3}{2}\frac{\delta p^N}{\bar\rho}\\ \left(\frac{k}{\mathcal{H}}\right)^2(\Psi - \Phi) &= 3w\Pi , & (10.17)\end{aligned}$$

where the powers of $\mathcal{H}$ are arranged so that the distance scales k^{-1} and time scales $d\eta$ are always related to the conformal Hubble scale $\mathcal{H}^{-1}$, and we used the background relation

$$4\pi G a^2\bar\rho = \tfrac{3}{2}\mathcal{H}^2 , \tag{10.18}$$

which follows directly from the Friedmann equation (2.2).

11 Energy-Momentum Continuity Equations

We know that from the Einstein equation, $G^\mu_\nu = 8\pi G T^\mu_\nu$, the energy-momentum continuity equations,

$$T^\mu_{\nu;\mu} = 0 , \tag{11.1}$$

follow. Just like for the background universe, we may use the energy-momentum continuity equations instead of some of the Einstein equations.

Calculating

$$T^\mu_{\nu;\mu} = T^\mu_{\nu,\mu} + \Gamma^\mu_{\alpha\mu}T^\alpha_\nu - \Gamma^\alpha_{\nu\mu}T^\mu_\alpha = 0 \tag{11.2}$$

to first order in perturbations, one obtains the 0^{th}order (background) equation

$$\bar\rho' = -3\mathcal{H}(\bar\rho + \bar p) \tag{11.3}$$

and the 1^{st}order (perturbation) equations, which for scalar perturbations in the conformal-Newtonian gauge are (exercise)

$$\begin{aligned}(\delta\rho^N)' &= -3\mathcal{H}(\delta\rho^N + \delta p^N) + (\bar\rho + \bar p)(\nabla^2 v^N + 3\Psi') & (11.4)\\ (\bar\rho + \bar p)(v^N)' &= -(\bar\rho + \bar p)'v^N - 4\mathcal{H}(\bar\rho + \bar p)v^N + \delta p^N + \tfrac{2}{3}\bar p\nabla^2\Pi + (\bar\rho + \bar p)\Phi & (11.5)\end{aligned}$$

Note that v^N is the velocity potential, $\vec{v}^N = -\nabla v^N$. It is easy to interpret the various terms in these equations.

In the energy perturbation equation (11.4), we have first the effect of the background expansion, then the effect of velocity divergence (local fluid expansion) and then the effect of the expansion/contraction in the metric perturbation.

In the momentum perturbation equation (11.5), the lhs and the first term on the right represent the change in inertia×velocity. The second on the right is the effect of background expansion. The third and last terms represent forces due to gradients in pressure and gravitational potential.

With manipulations involving background relations, these can be worked (exercise) into the form

$$
\begin{aligned}
(\delta^N)' &= (1+w)\left(\nabla^2 v^N + 3\Psi'\right) + 3\mathcal{H}\left(w\delta^N - \frac{\delta p^N}{\bar\rho}\right) && (11.6)\\
(v^N)' &= -\mathcal{H}(1-3w)v^N - \frac{w'}{1+w}v^N + \frac{\delta p^N}{\bar\rho+\bar p} + \frac{2}{3}\frac{w}{1+w}\nabla^2\Pi + \Phi\,. && (11.7)
\end{aligned}
$$

12 Perfect Fluid Scalar Perturbations in the Newtonian Gauge

12.1 Field Equations

For a perfect fluid, things simplify a lot, since now $\Pi = 0$ and thus *for a perfect fluid*

$$\Psi = \Phi\,, \tag{12.1}$$

and we have *only one degree of freedom in the scalar metric perturbation.* We can now replace Ψ with Φ in the field equations. The original set becomes

$$
\begin{aligned}
\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) &= 4\pi G a^2 \delta\rho^N && (12.2)\\
\left(\Phi' + \mathcal{H}\Phi\right)_{,i} &= 4\pi G a^2 (\bar\rho + \bar p) v^N_{,i} && (12.3)\\
\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi &= 4\pi G a^2 \delta p^N\,, && (12.4)
\end{aligned}
$$

and the reworked set becomes

$$
\begin{aligned}
\nabla^2\Phi &= 4\pi G a^2 \bar\rho \left[\delta^N + 3\mathcal{H}(1+w)v^N\right] \\
&= \tfrac{3}{2}\mathcal{H}^2\left[\delta^N + 3\mathcal{H}(1+w)v^N\right] && (12.5)\\
\Phi' + \mathcal{H}\Phi &= 4\pi G a^2 (\bar\rho + \bar p) v^N \;=\; \tfrac{3}{2}\mathcal{H}^2 (1+w) v^N && (12.6)\\
\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi &= 4\pi G a^2 \delta p^N \;=\; \tfrac{3}{2}\mathcal{H}^2 \delta p^N/\bar\rho\,, && (12.7)
\end{aligned}
$$

where we have used Eq. (2.8).

If we change the time variable from conformal time η to cosmic time t, they read

$$
\begin{aligned}
\nabla^2\Phi &= 4\pi G a^2 \bar\rho \left[\delta^N + 3aH(1+w)v^N\right] && (12.8)\\
\dot\Phi + H\Phi &= 4\pi G a (\bar\rho + \bar p) v^N && (12.9)\\
\ddot\Phi + 4H\dot\Phi + \left(2\dot H + 3H^2\right)\Phi &= 4\pi G \delta p^N\,. && (12.10)
\end{aligned}
$$

We define the *total entropy perturbation* as

$$\mathcal{S} \equiv \mathcal{H}\left(\frac{\delta p}{\bar p'} - \frac{\delta\rho}{\bar\rho'}\right) \;\equiv\; H\left(\frac{\delta p}{\dot{\bar p}} - \frac{\delta\rho}{\dot{\bar\rho}}\right)\,. \tag{12.11}$$

From the gauge transformation equations (9.26,9.27) we see that it is *gauge invariant*.

Using the background relations $\bar{\rho}' = -3\mathcal{H}(1+w)\bar{\rho}$ and $\bar{p}' = c_s^2\bar{\rho}'$ we can also write

$$\mathcal{S} = \frac{1}{3(1+w)}\left(\frac{\delta\rho}{\bar{\rho}} - \frac{1}{c_s^2}\frac{\delta p}{\bar{\rho}}\right), \tag{12.12}$$

from which we get

$$\delta p = c_s^2\left[\delta\rho - 3(\bar{\rho}+\bar{p})\mathcal{S}\right], \tag{12.13}$$

which holds in any gauge.

Using the entropy perturbation, we can now write in the rhs of Eq. (12.7)

$$\delta p^N/\bar{\rho} \;=\; c_s^2\left[\delta^N - 3(1+w)\mathcal{S}\right], \tag{12.14}$$

where, from (12.5) and (12.6),

$$\delta^N \;=\; -3\mathcal{H}(1+w)v^N + \frac{2}{3\mathcal{H}^2}\nabla^2\Phi \;=\; -\frac{2}{\mathcal{H}}(\Phi' + \mathcal{H}\Phi) + \frac{2}{3\mathcal{H}^2}\nabla^2\Phi\,. \tag{12.15}$$

With some use of background relations, the evolution equation (12.7) becomes

$$\mathcal{H}^{-2}\Phi'' + 3(1+c_s^2)\mathcal{H}^{-1}\Phi' + 3(c_s^2 - w)\Phi \;=\; c_s^2\mathcal{H}^{-2}\nabla^2\Phi - \tfrac{9}{2}c_s^2(1+w)\mathcal{S}\,. \tag{12.16}$$

12.2 Adiabatic perturbations

Perturbations where the total entropy perturbation vanishes,[17]

$$\mathcal{S} = 0 \quad\Leftrightarrow\quad \delta p = c_s^2\delta\rho \tag{12.17}$$

are called *adiabatic perturbations*.[18] For adiabatic perturbations, Eq. (12.16) becomes (going to Fourier space)

$$\mathcal{H}^{-2}\Phi''_{\vec{k}} + 3(1+c_s^2)\mathcal{H}^{-1}\Phi'_{\vec{k}} + 3(c_s^2 - w)\Phi_{\vec{k}} = -\left(\frac{c_s k}{\mathcal{H}}\right)^2\Phi_{\vec{k}}\,, \tag{12.18}$$

from which $\Phi_{\vec{k}}(\eta)$ can be solved, given the initial conditions. From Eq. (12.6) we then get $v^N_{\vec{k}}(\eta)$, and after that, from Eq. (12.5), $\delta^N_{\vec{k}}(\eta)$.

12.3 Continuity Equations

For a perfect fluid, Eqs. (11.6) and (11.7) become

$$(\delta^N)' \;=\; (1+w)\left(\nabla^2 v^N + 3\Phi'\right) + 3\mathcal{H}\left(w\delta^N - \frac{\delta p^N}{\bar{\rho}}\right) \tag{12.19}$$

$$(v^N)' \;=\; -\mathcal{H}(1-3w)v^N - \frac{w'}{1+w}v^N + \frac{\delta p^N}{\bar{\rho}+\bar{p}} + \Phi\,. \tag{12.20}$$

[17] For pressureless matter, where $\delta p = \bar{p} = w = c_s^2 = 0$, Eq. (12.11) is not defined, but $\delta p = c_s^2\delta\rho$ holds always ($0 = 0$). We use then this latter definition for adiabaticity; and the perturbations are necessarily adiabatic.

[18] Warning: In cosmology, it is customary to speak about adiabatic perturbations, when one means perturbations which were *initially* adiabatic. Such perturbations do not usually stay adiabatic as the universe evolves.

13 Scalar Perturbations in the Matter-Dominated Universe

Let us now consider density perturbations in the simplest case, the *matter-dominated universe*. By "matter" we mean here non-relativistic matter, whose pressure is so small compared to energy density, that we can ignore it here.[19][20] In general relativity this is often called "dust".

According to our present understanding, the universe was radiation-dominated for the first few ten thousand years, after which it became matter-dominated. For our present discussion, we take "matter-dominated" to mean that matter dominates energy density to the extent that we can ignore the other components. This approximation becomes valid after the first few million years.

Until late 1990's it was believed that this matter-dominated state persists until (and beyond) the present time. But new observational data points towards another component in the energy density of the universe, with a large negative pressure, resembling vacuum energy, or a cosmological constant. This component is called "dark energy". The dark energy seems to have become dominant a few billion (10^9) years ago. Thus the validity of the matter-dominated approximation is not as extensive as was thought before; but anyway there was a significant period in the history of the universe, when it holds good.

We now make the matter-dominated approximation, i.e., we ignore pressure,

$$\bar{p} = w = 0 \qquad \text{and} \qquad \delta p = \Pi = 0\,. \tag{13.1}$$

This is our first example of solving a perturbation theory problem. The order of work is always:

1. Solve the background problem.
2. Using the background quantities as known functions of time, solve the perturbation problem.

In the present case, the background equations are

$$\mathcal{H}^2 = \left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}a^2 \tag{13.2}$$

$$\mathcal{H}' = -\frac{4\pi G}{3}\bar{\rho}a^2\,, \tag{13.3}$$

from which we have

$$2\mathcal{H}' + \mathcal{H}^2 = 0\,. \tag{13.4}$$

The background solution is the familiar $k = 0$ matter-dominated Friedmann model, $a \propto t^{2/3}$. But let us review the solution in terms of conformal time. Since $\bar{\rho} \propto a^{-3(1+w)} \propto a^{-3}$, the solution of Eq. (13.2) gives

$$a(\eta) \propto \eta^2\,. \tag{13.5}$$

Since $dt = a d\eta$, or $dt/d\eta = a$, we get $t(\eta) \propto \eta^3 \propto a^{3/2}$ or $a \propto t^{2/3}$.

From $a \propto \eta^2$ we get

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{2}{\eta} \qquad \text{and} \qquad \mathcal{H}' = -\frac{2}{\eta^2} \tag{13.6}$$

[19] Likewise, we can ignore its anisotropic stress. Thus nonrelativistic matter is a perfect fluid for our purposes.

[20] Note that there are situations where we can make the matter-dominated approximation at the background level, but for the perturbations pressure gradients are still important at small distance scales (large k). Here we, however, we make the approximation that also pressure perturbations can be ignored.

Thus, from Eq. (13.2),

$$4\pi G a^2 \bar\rho \;=\; \frac{3}{2}\mathcal{H}^2 \;=\; \frac{6}{\eta^2}\,. \tag{13.7}$$

The perturbation equations (12.5), (12.6), and (12.7) with $\bar p = w = \delta p = 0$, are

$$\begin{aligned} \nabla^2\Phi &= 4\pi G a^2\bar\rho\left[\delta^N + 3\mathcal{H}v^N\right] && (13.8)\\ \Phi' + \mathcal{H}\Phi &= 4\pi G a^2\bar\rho v^N && (13.9)\\ \Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi &= 0\,. && (13.10)\end{aligned}$$

Using Eq. (13.4), Eq. (13.10) becomes

$$\Phi'' + 3\mathcal{H}\Phi' = \Phi'' + \frac{6}{\eta}\Phi' \;= 0\;\;, \tag{13.11}$$

whose solution is

$$\Phi(\eta,\vec x) \;=\; C_1(\vec x) + C_2(\vec x)\eta^{-5}\,. \tag{13.12}$$

The second term, $\propto \eta^{-5}$ is the *decaying part.* We get $C_1(\vec x)$ and $C_2(\vec x)$ from the initial values $\Phi_{\rm in}(\vec x)$, $\Phi'_{\rm in}(\vec x)$ at some initial time $\eta = \eta_{\rm in}$,

$$\begin{aligned} \Phi_{\rm in}(\vec x) &= C_1(\vec x) + C_2(\vec x)\eta_{\rm in}^{-5} && (13.13)\\ \Phi'_{\rm in}(\vec x) &= -5C_2(\vec x)\eta_{\rm in}^{-6} && (13.14)\end{aligned}$$

as

$$\begin{aligned} C_1(\vec x) &= \Phi_{\rm in}(\vec x) + \tfrac{1}{5}\eta_{\rm in}\Phi'_{\rm in}(\vec x) && (13.15)\\ C_2(\vec x) &= -\tfrac{1}{5}\eta_{\rm in}^6\Phi'_{\rm in}(\vec x) && (13.16)\end{aligned}$$

Unless we have very special initial conditions, conspiring to make $C_1(\vec x)$ vanishingly small, the decaying part soon becomes $\ll C_1(\vec x)$ and can be ignored. Thus we have the important result

$$\Phi(\eta,\vec x) = \Phi(\vec x)\,, \tag{13.17}$$

i.e., the Bardeen potential Φ is *constant in time* for perturbations in the flat ($k = 0$) matter-dominated universe.

Ignoring the decaying part, we have $\Phi' = 0$ and we get for the velocity perturbation from Eq. (13.9)

$$v^N \;=\; \frac{\mathcal{H}\Phi}{4\pi G a^2\bar\rho} \;=\; \frac{2\Phi}{3\mathcal{H}} \;=\; \tfrac{1}{3}\Phi\eta \;\propto\; \eta \;\propto\; a^{1/2} \;\propto\; t^{1/3}\,. \tag{13.18}$$

and Eq. (13.8) becomes

$$\nabla^2\Phi \;=\; 4\pi G a^2\bar\rho\left[\delta^N + 2\Phi\right] \;=\; \frac{3}{2}\mathcal{H}^2\left[\delta^N + 2\Phi\right] \tag{13.19}$$

or

$$\delta^N \;=\; -2\Phi + \frac{2}{3\mathcal{H}^2}\nabla^2\Phi\,. \tag{13.20}$$

In Fourier space this reads

$$\delta^N_{\vec k}(\eta) \;=\; -\left[2 + \frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\right]\Phi_{\vec k}\,. \tag{13.21}$$

Thus we see that for *superhorizon* scales, $k \ll \mathcal{H}$, or $k_{\rm phys} \ll H$, the density perturbation stays constant,

$$\delta^N_{\vec k} \;=\; -2\Phi_{\vec k} \;=\; \text{const.} \tag{13.22}$$

whereas for *subhorizon* scales, $k \gg \mathcal{H}$, or $k_{\rm phys} \gg H$, they grow proportional to the scale factor,

$$\delta_{\vec{k}} \;=\; -\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2 \Phi_{\vec{k}} \;\propto\; \eta^2 \;\propto\; a \;\propto\; t^{2/3}\,. \tag{13.23}$$

Since the comoving Hubble scale $\mathcal{H}^{-1}$ grows with time, various scales k are superhorizon to begin with, but later become subhorizon as $\mathcal{H}^{-1}$ grows past k^{-1}. (In physical terms, in the expanding universe $\lambda_{\rm phys}/2\pi = k_{\rm phys}^{-1} \propto a$ grows slower than the Hubble scale $H^{-1} \propto a^{3/2}$.) We say that the scale in question "enters the horizon". (The word "horizon" in this context refers just to the Hubble scale $1/\mathcal{H}$, and not to other definitions of "horizon".) We see that *density perturbations begin to grow when they enter the horizon*, and after that they grow proportional to the scale factor. Thus the present magnitude of the density perturbation at comoving scale k should be a_0/a_k times its primordial value

$$\delta_{\vec{k}}(t_0) \sim \frac{a_0}{a_k}\delta^N_{\vec{k},\rm pr}\,, \tag{13.24}$$

where a_0 is the present value of the scale factor, and a_k is its value at the time the scale k "entered horizon". The "primordial" density perturbation $\delta^N_{\vec{k},\rm pr}$ refers to the constant value it had at superhorizon scales, after the decaying part of Φ had died out. Of course Eq. (13.24) is valid only for those (large) scales where the perturbation is still small today. Once the perturbation becomes large, $\delta \sim 1$, perturbation theory is no more valid. We say the scale in question "goes nonlinear"[21].

If we take into account the presence of "dark energy", the above result is modified to

$$\delta_{\vec{k}}(t_0) \sim \frac{a_{DE}}{a_k}\delta^N_{\vec{k},\rm pr}\,, \tag{13.25}$$

where $a_{DE} \gtrsim a_0/2$ is the value of a when dark energy became dominant, since that stops the growth of density perturbations.

One has to remember that these results refer to the density and velocity perturbations in the conformal-Newtonian gauge only. In some other gauge these perturbations, and their growth laws would be different. However, for subhorizon scales general relativistic effects become unimportant and a Newtonian description becomes valid. In this limit, the issue of gauge choice becomes irrelevant as all "sensible gauges" approach each other, and the conformal-Newtonian density and velocity perturbations become those of a Newtonian description. The Bardeen potential can then be understood as a Newtonian gravitational potential due to density perturbations. (Eq. (13.8) acquires the form of the Newtonian law of gravity as the second term on the right becomes small compared to the first term, δ^N. The factor a^2 appears on the right since ∇^2 on the left refers to comoving coordinates.)

14 Perfect Fluid Scalar Perturbations when $p = p(\rho)$

We shall next discuss a somewhat more complicated case, where pressure can not be ignored, but the equation of state has the form

$$p = p(\rho)\,, \tag{14.1}$$

[21] Since in our universe this happens only at subhorizon scales, the nonlinear growth of perturbations can be treated with Newtonian physics. First the growth of the density perturbation (for overdensities) becomes much faster than in linear perturbation theory, but then the system "virializes", settling into a relatively stable structure, a galaxy or a galaxy cluster, where further collapse is prevented by the conservation of angular momentum, as the different parts of the system begin to orbit the center of mass of the system. For underdensities, we have of course $\rho \geq 0 \Rightarrow \delta \geq -1$ always, so the underdensity cannot "grow" beyond that.

i.e., there are no other thermodynamic variables than ρ that the pressure would depend on. In this case the perturbations are guaranteed to be adiabatic, since now

$$\frac{\delta p}{\delta \rho} = \frac{\bar{p}'}{\bar{\rho}'} = \frac{dp}{d\rho} = c_s^2 \,. \tag{14.2}$$

(Also, we keep assuming the fluid is perfect.)

This discussion will also apply to adiabatic perturbations of general perfect fluids.[22] That is, when the fluid in principle may have more state variables, but these other degrees of freedom are not "used". Let us show that the adiabaticity of perturbations,

$$\delta p = c_s^2 \delta\rho \equiv \frac{\bar{p}'}{\bar{\rho}'}\delta\rho \,, \tag{14.3}$$

implies that a unique relation (14.1) holds everywhere and -when:

Note first, that in the background solution, where $\bar{p} = \bar{p}(t)$ and $\bar{\rho} = \bar{\rho}(t)$, we can (assuming $\bar{\rho}$ decreases monotonously with time) invert for $t(\bar{\rho})$, and thus $\bar{p} = \bar{p}(t(\bar{\rho}))$, defining a function $\bar{p}(\bar{\rho})$, whose derivative is c_s^2. Now Eq. (14.3) guarantees that also $p = \bar{p} + \delta p$ and $\rho = \bar{\rho} + \delta\rho$ satisfy this same relation.

We note a property, which illuminates the nature of adiabatic perturbations: A small region of the perturbed universe is just like the background universe at a slightly earlier or later time. We can thus think of adiabatic perturbations as a perturbation in the "timing" of the different parts of the universe. (In adiabatic oscillations, this "corresponding background solution time" may oscillate back and forth.)

(Perhaps I will finish this section some day...)

15 Other Gauges

Different gauges are good for different purposes, and therefore it is useful to be able to work in different gauges, and to switch from one gauge to another in the course of a calculation.[23] When one uses more than one gauge it is important to be clear about to which gauge each quantity refers. One useful gauge is the *comoving gauge*. Particularly useful quantities which refer to the comoving gauge, are the comoving density perturbation δ^C and the *comoving curvature perturbation*

$$\mathcal{R} \equiv -\psi^C \,. \tag{15.1}$$

This is often just called the *curvature perturbation*, however that term is also used to refer to ψ in any other gauge, so beware! (There are different sign conventions for $\mathcal{R}$ and ψ. In my sign conventions, positive $\mathcal{R}$, but negative ψ, correspond to positive curvature of the 3-dimensional $\eta = \text{const.}$ slice.

Gauges are usually specified by giving gauge conditions. These may involve the metric perturbation variables A, D, B, E (e.g., the Newtonian gauge condition $E = B = 0$), the energy-momentum perturbation variables $\delta\rho$, δp, v, or both kinds.

[22] Thus we could have called this section "Adiabatic perfect fluid scalar perturbations". The reason we did not, is that mentioned in the earlier footnote. In this section we require the perturbations stay adiabatic the whole time. Perturbations of general fluids, which are initially (when they are at superhorizon scales) adiabatic, acquire entropy perturbations when they approach and enter the horizon.

[23] One may even combine quantities of different gauges in the same equation to simplify the equations. This may sound dangerous but is not, when one knows what one is doing.

15.1 Slicing and Threading

The gauge corresponds to the coordinate system $\{x^\mu\} = \{\eta, x^i\}$ in the perturbed spacetime. The conformal time η gives the *slicing* of the perturbed spacetime into $\eta =$ const. time slices (3-dim spacelike hypersurfaces). The spatial coordinates x^i give the *threading* of the perturbed spacetime into $x^i =$ const. threads (1-dim timelike curves). See Fig. 3. Slicing and threading are orthogonal to each other if and only if the shift vector vanishes, $B^i = 0$.

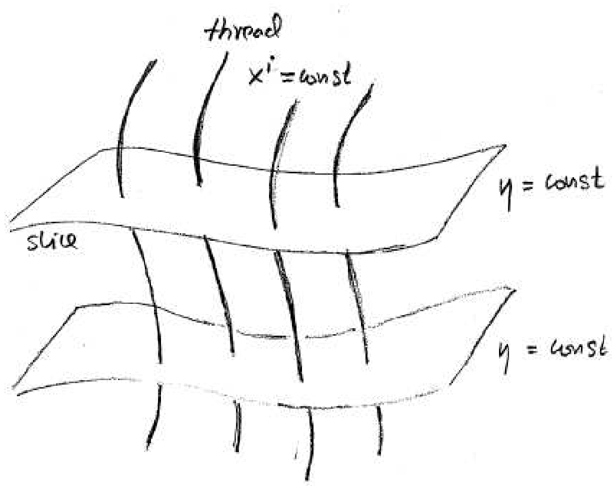

Figure 3: Slicing and threading the perturbed spacetime.

In the gauge transformation,

$$\tilde{x}^\alpha = x^\alpha + \xi^\alpha\,, \tag{15.2}$$

or

$$\begin{aligned} \tilde{\eta} &= \eta + \xi^0 \\ x^i &= x^i + \xi^i\,, \end{aligned} \tag{15.3}$$

ξ^0 changes the slicing, and ξ^i changes the threading. From the 4-scalar transformation law, $\delta\tilde{s} = \delta s - \bar{s}'\xi^0$, we see that perturbations in 4-scalars, e.g., $\delta\rho$ and δp depend only on the slicing.

15.2 Comoving Gauge

We say that the *slicing is comoving*, if the time slices are orthogonal to the fluid 4-velocity. For scalar perturbations such a slicing always exists. This condition turns out to be equivalent to the condition that the fluid velocity perturbation v^i equals the shift vector B^i. For scalar perturbations,

$$\text{Comoving slicing} \quad \Leftrightarrow \quad v = B\,. \tag{15.4}$$

From the gauge transformation equations (7.7) and (9.31),

$$\begin{aligned} \tilde{v} &= v + \xi' \\ \tilde{B} &= B + \xi' + \xi^0 \end{aligned} \tag{15.5}$$

we see that we get to comoving slicing by $\xi^0 = v - B$.

We say that the threading is comoving, if the threads are world lines of comoving (with the fluid) observers, i.e., the velocity perturbation vanishes, $v^i = 0$. For scalar perturbations,

$$\text{Comoving threading} \quad \Leftrightarrow \quad v = 0\,. \tag{15.6}$$

We get to comoving threading by the gauge transformation $\xi' = -v$. Note that comoving threads are usually not geodesics, since pressure gradients cause the fluid flow to deviate from free fall.

The *comoving gauge* is defined by requiring both comoving slicing and comoving threading. Thus

$$\text{Comoving gauge} \quad \Leftrightarrow \quad v = B = 0\,. \tag{15.7}$$

(We assume that we are working with scalar perturbations.) The threading is now orthogonal to the slicing. We denote the comoving gauge by the sub- or superscript C. Thus the statement $v^C = B^C = 0$ is generally true, whereas the statement $v = B = 0$ holds only in the comoving gauge.

We get to the comoving gauge from an arbitrary gauge by the gauge transformation

$$\begin{aligned} \xi' &= -v \\ \xi^0 &= v - B\,. \end{aligned} \tag{15.8}$$

This does not fully specify the coordinate system in the perturbed spacetime, since only ξ' is specified, not ξ. Thus we remain free to do time-independent transformations

$$\tilde{x^i} = x^i - \xi(\vec{x})_{,i}\,, \tag{15.9}$$

while staying in the comoving gauge. This, however, does not change the way the spacetime is sliced and threaded by the coordinate system, it just relabels the threads with different coordinate values x^i.

Applying Eq. (15.8) to the general scalar gauge transformation Eqs. (7.7,15.47,9.26,9.27,9.31), we get the rules to relate the comoving gauge perturbations to perturbations in an arbitrary gauge. For the metric:

$$\begin{aligned} A^C &= A - (v-B)' - \mathcal{H}(v-B) \\ B^C &= B - v + (v-B) = 0 \\ D^C &= -\tfrac{1}{3}\nabla^2\xi + \mathcal{H}(v-B) \\ E^C &= E + \xi \\ \psi^C &\equiv -\mathcal{R} \;=\; \psi + \mathcal{H}(v-B)\,. \end{aligned} \tag{15.10}$$

For the energy tensor:

$$\begin{aligned} \delta\rho^C &= \delta\rho - \bar\rho'(v-B) = \delta\rho + 3\mathcal{H}(1+w)\bar\rho(v-B) \\ \delta p^C &= \delta p - \bar p'(v-B) = \delta p + 3\mathcal{H}(1+w)c_s^2\bar\rho(v-B) \\ \delta^C &= \delta + 3\mathcal{H}(1+w)(v-B) \\ v^C &= v - v = 0 \\ \Pi^C &= \Pi\,. \end{aligned} \tag{15.11}$$

Because of the remaining gauge freedom (relabeling the threading) left by the comoving gauge condition (only ξ' is fixed, not ξ), D^C and E^C are not fully fixed. However, ψ^C is, and likewise

$$E^{C\prime} = E' - v\,. \tag{15.12}$$

In particular, we get the transformation rule from the Newtonian gauge (where $A = \Phi$, $B = 0$, $\psi = D = \Psi$, $E = 0$) to the comoving gauge. For the metric:

$$\begin{aligned} A^C &= \Phi - v^{N\prime} - \mathcal{H}v^N \\ \mathcal{R} &= -\Psi - \mathcal{H}v^N \\ E^{C\prime} &= -v^N \,. \end{aligned} \tag{15.13}$$

For the energy tensor:

$$\begin{aligned} \delta\rho^C &= \delta\rho^N + 3\mathcal{H}(1+w)\bar{\rho}v^N \\ \delta p^C &= \delta p^N + 3\mathcal{H}(1+w)c_s^2\bar{\rho}v^N \\ \delta^C &= \delta^N + 3\mathcal{H}(1+w)v^N \,. \end{aligned} \tag{15.14}$$

Thus, for example, we see that in the matter-dominated universe discussed in Sect. 13, Eqs. (13.18,13.20) lead to

$$\delta^C \;=\; \delta^N + 3\mathcal{H}v^N \;=\; -2\Phi + \frac{2}{3\mathcal{H}^2}\nabla^2\Phi + 2\Phi \;=\; \frac{2}{3\mathcal{H}^2}\nabla^2\Phi \;\propto\; \mathcal{H}^{-2} \;\propto\; a \tag{15.15}$$

both inside and outside (and through) the horizon.

Note that in Fourier space we have included an extra factor of k in v, so that, e.g., Eq. (15.14) reads in Fourier space as

$$\delta^C \;=\; \delta^N + 3\mathcal{H}(1+w)k^{-1}v^N \,. \tag{15.16}$$

15.3 Mixing Gauges

We see that Eq. (15.14c) is the rhs of Eq. (10.13), which we can thus write in the shorter form:

$$\nabla^2\Psi = 4\pi G a^2\bar{\rho}\delta^C \,. \tag{15.17}$$

This is an equation which has a Newtonian gauge metric perturbation on the lhs, but a comoving gauge density perturbation on the rhs. Is it dangerous to mix gauges like this? No, if we know what we are doing, i.e., in which gauge each quantity is, and the equations were derived correctly for this combination of quantities.

We could also say, that instead of working in any particular gauge, we work with *gauge-invariant quantities*, that the quantity that we denote by δ^C is the gauge-invariant quantity defined by

$$\delta^C = \delta + 3\mathcal{H}(1+w)(v - B)$$

which just happens to coincide with the density perturbation in the comoving gauge. Just like Ψ happens equal the metric perturbation ψ in the Newtonian gauge.

Switching to the comoving gauge for δ and δp, but keeping the velocity perturbation in the Newtonian gauge, the Einstein and continuity equations can be rewritten as (**Exercise**):

$$\nabla^2\Psi \;=\; \tfrac{3}{2}\mathcal{H}^2\delta_C \tag{15.18}$$

$$\Psi - \Phi \;=\; 3\mathcal{H}^2 w\Pi \tag{15.19}$$

$$\Psi' + \mathcal{H}\Phi \;=\; \tfrac{3}{2}\mathcal{H}^2(1+w)v_N \tag{15.20}$$

$$\Psi'' + (2 + 3c_s^2)\mathcal{H}\Psi' + \mathcal{H}\Phi' + 3(c_s^2 - w)\mathcal{H}^2\Phi + \tfrac{1}{3}\nabla^2(\Phi - \Psi) \;=\; \tfrac{3}{2}\mathcal{H}^2\frac{\delta p_C}{\bar{\rho}} \tag{15.21}$$

$$\delta_C' - 3\mathcal{H}w\delta_C \;=\; (1+w)\nabla^2 v_N + 2\mathcal{H}w\nabla^2\Pi \tag{15.22}$$

$$v_N' + \mathcal{H}v_N \;=\; \frac{\delta p_C}{\bar{\rho} + \bar{p}} + \tfrac{2}{3}\frac{w}{1+w}\nabla^2\Pi + \Phi \,. \tag{15.23}$$

15.4 Comoving Curvature Perturbation

The comoving curvature perturbation

$$\mathcal{R} \equiv -\psi^C = -\psi - \mathcal{H}(v - B) \tag{15.24}$$

turns out to be a useful quantity for discussing superhorizon perturbations.

From the second Einstein equation in the Newtonian gauge, Eq. (10.15), we have that

$$v^N = \frac{2}{3\mathcal{H}^2(1+w)}(\Psi' + \mathcal{H}\Phi), \tag{15.25}$$

so that the relation of the comoving curvature perturbation and the Bardeen potentials is

$$\mathcal{R} = -\Psi - \frac{2}{3(1+w)}\left(\mathcal{H}^{-1}\Psi' + \Phi\right) = -\Psi - \frac{2}{3(1+w)}\Phi - \frac{2}{3(1+w)}\mathcal{H}^{-1}\Psi'. \tag{15.26}$$

Derivating Eq. (15.26),

$$\begin{aligned}\mathcal{R}' &= -\Psi' + \frac{2w'}{3(1+w)^2}\left(\mathcal{H}^{-1}\Psi' + \Phi\right) - \frac{2}{3(1+w)}\left(-\frac{\mathcal{H}'}{\mathcal{H}^2}\Psi' + \mathcal{H}^{-1}\Psi'' + \Phi'\right) \\ &= -\frac{2\mathcal{H}^{-1}}{3(1+w)}\Psi'' - \frac{4+6c_s^2}{3(1+w)}\Psi' - \frac{2}{3(1+w)}\Phi' + 2\mathcal{H}\frac{w-c_s^2}{1+w}\Phi,\end{aligned}$$

(where we used some background relations), and using the Einstein equations (10.17) and (10.17), we get an evolution equation for $\mathcal{R}$,

$$\begin{aligned}-\tfrac{3}{2}(1+w)\mathcal{H}^{-1}\mathcal{R}' &= \mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}\left(\Phi' + 2\Psi'\right) + 3c_s^2\left(\mathcal{H}^{-1}\Psi' + \Phi\right) - 3w\Phi \\ &= \mathcal{H}^{-2}\Psi'' + (2+3c_s^2)\mathcal{H}^{-1}\Psi' + \mathcal{H}^{-1}\Phi' + 3(c_s^2 - w)\Phi \\ &= 3c_s^2\left(\mathcal{H}^{-1}\Psi' + \Phi\right) + \tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\Phi - \Psi) + \tfrac{3}{2}\frac{\delta p^N}{\bar\rho} \\ &= -c_s^2\left(\frac{k}{\mathcal{H}}\right)^2\Psi + \tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\Phi - \Psi) + \tfrac{3}{2}\left(\frac{\delta p^N}{\bar\rho} - c_s^2\delta^N\right).\end{aligned} \tag{15.27}$$

From Sec. 12 we have

$$\mathcal{S} = \frac{1}{3(1+w)}\left(\delta - \frac{1}{c_s^2}\frac{\delta p}{\bar\rho}\right) \Rightarrow \delta p = c_s^2[\delta\rho - 3(\bar\rho + \bar p)\mathcal{S}] \tag{15.28}$$

(valid in any gauge), so that Eq. (15.27) becomes the important result

$$-\tfrac{3}{2}(1+w)\mathcal{H}^{-1}\mathcal{R}' = \left(\frac{k}{\mathcal{H}}\right)^2\left[c_s^2\Psi + \tfrac{1}{3}(\Psi - \Phi)\right] + \tfrac{9}{2}c_s^2(1+w)\mathcal{S}. \tag{15.29}$$

As an evolution equation, this does not appear very useful, since it mixes metric perturbations from two different gauges. However, the importance of this equation comes from the two observations we can now make immediately: 1) For adiabatic perturbations, $\mathcal{S} = 0$, the second term on the rhs vanishes. 2) For superhorizon perturbations, i.e., for Fourier modes whose wavelength is much larger than the Hubble distance, $k \ll \mathcal{H}$, we can ignore the first term on the rhs. Thus:

For adiabatic perturbations, the comoving curvature perturbation stays constant outside the horizon.

Adiabatic (scalar) perturbations have only one degree of freedom, and thus their full evolution is captured in the evolution of $\mathcal{R}$.

A general perturbation at a given time can be decomposed into an adiabatic component, which has $\mathcal{S}=0$, and an isocurvature component, which has $\mathcal{R}=0$. Because of the linearity of first order perturbation theory, these components evolve independently, and the evolution of the general perturbation is just the superposition of the evolution of these two components, and thus they can be studied separately. Beware, however, that the "adiabatic" component does not necessarily remain adiabatic in its evolution, and the "isocurvature" component does not necessarily maintain zero comoving curvature. We later show that adiabatic perturbations stay adiabatic while they are well outside the horizon.

Adiabatic perturbations are important, since the simplest theory for the origin of structure of the universe, single-field inflation, produces adiabatic perturbations. Using the constancy of $\mathcal{R}$, it is easy to follow the evolution of these perturbations while they are well outside the horizon.

Using Eq. (15.21) we can write the second line of Eq. (15.27) as

$$
\begin{aligned}
-\tfrac{3}{2}(1+w)\mathcal{H}\mathcal{R}' &= \Psi'' + (2+3c_s^2)\mathcal{H}\Psi' + \mathcal{H}\Phi' + 3(c_s^2-w)\mathcal{H}^2\Phi \\
&= \tfrac{3}{2}\mathcal{H}^2\frac{\delta p_C}{\bar\rho} - \tfrac{1}{3}\nabla^2(\Phi-\Psi) \qquad (15.30)
\end{aligned}
$$

or (using Eq. 15.19)

$$
\mathcal{H}^{-1}\mathcal{R}' = -\frac{\delta p^C}{\bar\rho+\bar p} - \frac{2}{3}\frac{w}{1+w}\nabla^2\Pi = -c_s^2\left(\frac{\delta^C}{1+w} - 3\mathcal{S}\right) - \frac{2}{3}\frac{w}{1+w}\nabla^2\Pi \qquad (15.31)
$$

which is completely in the comoving gauge.

15.5 Perfect Fluid Scalar Perturbations, Again

With $\Pi = 0 \;\Rightarrow\; \Psi = \Phi$, Eqs. (15.18,15.19,15.20,15.21,15.22,15.23) become

$$
\begin{aligned}
\nabla^2\Phi &= \tfrac{3}{2}\mathcal{H}^2\delta_C \qquad (15.32)\\
\Phi' + \mathcal{H}\Phi &= \tfrac{3}{2}\mathcal{H}^2(1+w)v_N \qquad (15.33)
\end{aligned}
$$

$$
\Phi'' + 3(1+c_s^2)\mathcal{H}\Phi' + 3(c_s^2-w)\mathcal{H}^2\Phi = \tfrac{3}{2}\mathcal{H}^2\frac{\delta p_C}{\bar\rho} \qquad (15.34)
$$

$$
\begin{aligned}
\delta_C' - 3\mathcal{H}w\delta_C &= (1+w)\nabla^2 v_N \qquad (15.35)\\
v_N' + \mathcal{H}v_N &= \frac{\delta p_C}{\bar\rho+\bar p} + \Phi\,. \qquad (15.36)
\end{aligned}
$$

and Eqs. (15.26,15.31) become

$$
\begin{aligned}
\mathcal{R} &= -\Phi - \frac{2}{3(1+w)\mathcal{H}}\left(\Phi' + \mathcal{H}\Phi\right) \qquad (15.37)\\
\mathcal{H}^{-1}\mathcal{R}' &= -\frac{\delta p^C}{\bar\rho+\bar p} = -c_s^2\left(\frac{\delta^C}{1+w} - 3\mathcal{S}\right) \qquad (15.38)
\end{aligned}
$$

and Eq. (15.34) as

$$
\mathcal{H}^{-2}\Phi'' + 3(1+c_s^2)\mathcal{H}^{-1}\Phi' + 3(c_s^2-w)\Phi = \tfrac{3}{2}c_s^2[\delta^C - 3(1+w)\mathcal{S}]\,. \qquad (15.39)
$$

which is Eq. (12.16), with just $\nabla^2\Phi$ replaced by δ^C using Eq. (15.32). (I seem to be repeating here some material from Sec. 12.) Our aim is to get a differential equation with preferably just one perturbation quantity to solve from, so this might seem a step backwards, replaceing one of the Φ with δ^C, but we can now go ahead and replace also all the other Φ with δ^C:

Taking the Laplacian of this (**Exercise**), we get the *Bardeen equation*

$$\mathcal{H}^{-2}\delta_C{}'' + \left(1 - 6w + 3c_s^2\right)\mathcal{H}^{-1}\delta_C{}' - \tfrac{3}{2}\left(1 + 8w - 6c_s^2 - 3w^2\right)\delta_C \;=\; c_s^2\mathcal{H}^{-2}\nabla^2\left[\delta_C - 3(1+w)\mathcal{S}\right] , \tag{15.40}$$

a differential equation from which we can solve the evolution of the comoving density perturbation for superhorizon scales (when one can ignore the rhs) and for adiabatic perturbations at all scales (when $\mathcal{S} = 0$).

For the general case we need also an equation for $\mathcal{S}$. To be able to do this, we need to take a closer look at the fluid, see Sec. 17 and beyond.

15.5.1 Adiabatic Perfect Fluid Perturbations at Superhorizon Scales

Rewrite Eq. (15.37) as

$$\tfrac{2}{3}\mathcal{H}^{-1}\Phi' + \frac{5+3w}{3}\Phi = -(1+w)\mathcal{R} . \tag{15.41}$$

If we have a period in the history of the universe, where we can approximate $w =$ const., then, for adiabatic perturbations at superhorizon scales, Eq. (15.41) is a differential equation for Φ with $w = \mathcal{R} =$ const. for that period. This equation has a special solution

$$\Phi = -\frac{3+3w}{5+3w}\mathcal{R} .$$

The corresponding homogeneous equation is

$$\begin{aligned}
\mathcal{H}^{-1}\Phi' + \frac{5+3w}{2}\Phi &= 0 \\
\Rightarrow a\frac{d\Phi}{da} &= -\frac{5+3w}{2}\Phi \\
\Rightarrow \Phi &= Ca^{-\frac{5+3w}{2}}
\end{aligned}$$

so that the general solution to Eq. (15.41) is

$$\Phi = -\frac{3+3w}{5+3w}\mathcal{R} + Ca^{-\frac{5+3w}{2}} . \tag{15.42}$$

If $w \approx$ const. for a long enough time, the second part becomes negligible, and we have

$$\Phi = \Psi = -\frac{3+3w}{5+3w}\mathcal{R} = \text{const.} \tag{15.43}$$

In particular, we have the relations

$$\Phi_{\boldsymbol{k}} = -\frac{2}{3}\mathcal{R}_{\boldsymbol{k}} \qquad (\text{adiab., rad.dom, } w = \tfrac{1}{3},\ k \ll \mathcal{H}) \tag{15.44}$$

$$\Phi_{\boldsymbol{k}} = -\frac{3}{5}\mathcal{R}_{\boldsymbol{k}} \qquad (\text{adiab., mat.dom, } w = 0,\ k \ll \mathcal{H}) . \tag{15.45}$$

While the universe goes from radiation domination to matter domination, w is not a constant, so Eq. (15.43) does not apply, but we know from Eq. (15.45) that, $\Phi_{\boldsymbol{k}}$ changes from $-\frac{2}{3}\mathcal{R}_{\boldsymbol{k}}$ to $-\frac{3}{5}\mathcal{R}_{\boldsymbol{k}}$, i.e, changes by a factor 9/10 (assuming $k \ll \mathcal{H}$ the whole time, so that $\mathcal{R}_{\boldsymbol{k}}$ stays constant.

15.6 Spatially Flat Gauge

The spatially flat gauge, denoted by the sub/superscript Q, is defined by the condition that the curvature perturbation ψ vanishes, i.e.,

$$\psi^Q = 0. \tag{15.46}$$

Since the metric perturbation ψ transforms as

$$\tilde{\psi} = \psi + \mathcal{H}\xi^0. \tag{15.47}$$

we get to the spatially flat gauge by the gauge transformation

$$\xi^0 = -\mathcal{H}^{-1}\psi. \tag{15.48}$$

From the gauge transformation rule of the relative density perturbation (9.30),

$$\tilde{\delta} = \delta + 3\mathcal{H}(1+w)\xi^0, \tag{15.49}$$

we get that

$$\delta^Q = \delta - 3(1+w)\psi. \tag{15.50}$$

In particular,

$$\delta^Q = \delta^C + 3(1+w)\mathcal{R}, \tag{15.51}$$

so that the comoving curvature perturbation $\mathcal{R}$ is proportional to the difference between the relative density perturbations in the spatially flat and comoving gauges,

$$\mathcal{R} = \frac{1}{3(1+w)}\left(\delta^Q - \delta^C\right) = \frac{1}{3(1+w)}\left[\delta^Q + \frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Psi\right], \tag{15.52}$$

where we used Eq. (15.18) for the latter equality. Thus for superhorizon scales we have the correspondence

$$\mathcal{R} \approx \frac{1}{3(1+w)}\delta^Q \tag{15.53}$$

between the comoving gauge curvature perturbation and the flat gauge density perturbation.

We shall later use the spatially flat gauge to solve perturbation equations for scalar fields.

16 Synchronous Gauge

The synchronous gauge is defined by the requirement $A = B_i = 0$. Synchronous gauge can be used both for scalar and vector perturbations. In this section we consider only scalar perturbations, where it means that

$$A^Z = B^Z = 0. \tag{16.1}$$

(We use Z instead of S to denote synchronous gauge, so as not to confuse it with the scalar part of a perturbation.)

One gets to synchronous gauge from an arbitrary gauge with a gauge transformation ξ^μ that satisfies

$$\xi^{0\prime} + \mathcal{H}\xi^0 = A \tag{16.2}$$
$$\xi' = -\xi^0 - B. \tag{16.3}$$

This is a differential equation from which to solve ξ^0. Thus it is not easy to switch from another gauge to synchronous gauge. We see that, like for comoving gauge, only the time derivative of ξ is determined.

In the synchronous gauge the threads are orthogonal to the slices and the metric perturbation is only in the space part of the metric,

$$\begin{aligned} h_{ij} &= -2D^Z\delta_{ij} + 2\left(E^Z_{,ij} - \tfrac{1}{3}\nabla^2 E^Z\delta_{ij}\right) \\ &= -2\psi^Z\delta_{ij} + 2E^Z_{,ij}\,. \end{aligned} \tag{16.4}$$

From Eq. (A.2), setting $A = B = 0$, the Christoffel symbols are

$$\begin{aligned} \Gamma^0_{00} &= \mathcal{H} \\ \Gamma^0_{0i} &= \Gamma^i_{00} = 0 \\ \Gamma^0_{ij} &= \mathcal{H}\left[(1-2\psi)\delta_{ij} + 2E_{,ij}\right] - \delta_{ij}\psi' + E'_{,ij} \\ \Gamma^i_{0j} &= \mathcal{H}\delta_{ij} - \delta_{ij}\psi' + E'_{,ij} \\ \Gamma^i_{jk} &= -\delta_{ij}\psi_{,k} - \delta_{ik}\psi_{,j} + \delta_{jk}\psi_{,i} + E_{,ijk}\,. \end{aligned} \tag{16.5}$$

From $\Gamma^0_{0i} = \Gamma^i_{00} = 0$ follows that *the space coordinate lines are geodesics*. Namely, the geodesic equations

$$\ddot{x}^0 + \Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \ddot{x}^0 + \mathcal{H}\dot{x}^0\dot{x}^0 + \Gamma^0_{ij}\dot{x}^i\dot{x}^j = 0 \tag{16.6}$$

$$\ddot{x}^i + \Gamma^i_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \ddot{x}^i + 2\Gamma^i_{0j}\dot{x}^0\dot{x}^j + \Gamma^i_{jk}\dot{x}^j\dot{x}k = 0 \tag{16.7}$$

are satisfied by $x^i = \text{const.} \Rightarrow \dot{x}^i = 0$ and $\dot{x}^0 \equiv d\eta/d\tau \equiv dt/ad\tau = a \Rightarrow dt = d\tau$. Thus one can construct the synchronous gauge coordinate system by choosing an initial spacelike hypersurface, distributing observers carrying space coordinate values on that hypersurface, with their initial 4-velocities orthogonal to the hypersurface (i.e., they are at rest with respect to that surface), synchronizing the clocks of the observers, and letting the observers then fall freely. The world lines of these observers are then the space coordinate lines and their clocks show the time coordinate (t, not η).

We use Ma & Bertschinger[2] (hereafter MB) as our main reference for synchronous gauge and adopt from their notation

$$\begin{aligned} h &\equiv -6D^Z \equiv h^{iZ}_i \\ \eta &\equiv \psi^Z = D^Z + \tfrac{1}{3}\nabla^2 E^Z \\ \mu &\equiv 2E^Z\,, \end{aligned} \tag{16.8}$$

so that

$$\begin{aligned} h_{ij} &= \tfrac{1}{3}h\delta_{ij} + (\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2)\mu \\ &= -2\eta\delta_{ij} + \mu_{,ij}\,. \end{aligned} \tag{16.9}$$

Unfortunately, the MB notation fro the synchronous curvature perturbation is the same as our notation for conformal time (MB use τ for the latter). Let's hope this does not lead to confusion. MB discuss the synchronous metric perturbation in terms of the variables h and η, instead of the pair h,μ or η,μ. In coordinate space this appears impractical, since to solve μ from h and η requires integration ($\eta = \frac{1}{6}(-h + \nabla^2\mu)$). However, MB work entirely in Fourier space, where $\eta = -\frac{1}{6}(h+\mu)$ or $\mu = -h - 6\eta$. Thus, in Fourier space, the metric is

$$\begin{aligned} h_{ij} &= -2D^Z\delta_{ij} - 2\hat{k}^i\hat{k}^jE^Z + \tfrac{2}{3}E^Z\delta_{ij} \\ &= \tfrac{1}{3}h\delta_{ij} - \hat{k}^i\hat{k}^j\mu + \tfrac{1}{3}\mu\delta_{ij} \\ &= \hat{k}^i\hat{k}^jh + (\hat{k}^i\hat{k}^j - \tfrac{1}{3}\delta_{ij})6\eta\,, \end{aligned} \tag{16.10}$$

which is MB Eq. (4).

From Eq. (8.1), we get from the synchronous gauge to the Newtonian gauge by

$$\begin{aligned} \xi_{Z\to N} &= -E^Z = -\tfrac{1}{2}\mu \\ \xi^0_{Z\to N} &= E^{Z\prime} = -\xi'_{Z\to N}\,. \end{aligned} \tag{16.11}$$

From Eq. (7.9), the Bardeen potentials are

$$\begin{aligned} \Phi &= -\mathcal{H}E^{Z\prime} - E^{Z\prime\prime} = -\tfrac{1}{2}\mathcal{H}\mu' - \tfrac{1}{2}\mu'' \\ \Psi &= \psi^Z + \mathcal{H}E^{Z\prime} = \eta + \tfrac{1}{2}\mathcal{H}\mu'\,. \end{aligned} \tag{16.12}$$

One has to be careful when Fourier transforming equations which employ different Fourier conventions for different terms. Thus, in Fourier space, Eq. (16.11) reads

$$\xi_{Z\to N} = -\frac{1}{2k}\mu = \frac{1}{2k}(h+6\eta) \tag{16.13}$$

$$\xi^0_{Z\to N} = -\frac{1}{k}\xi'_{Z\to N} \tag{16.14}$$

and Eq. (16.12) reads

$$\begin{aligned} \Phi &= \frac{1}{2k^2}\left(-\mathcal{H}\mu' - \mu''\right) = \frac{1}{2k^2}\left[h'' + 6\eta'' + \mathcal{H}(h'+6\eta')\right] \\ \Psi &= \eta - \frac{1}{2k^2}\mathcal{H}(h'+6\eta')\,, \end{aligned} \tag{16.15}$$

which is MB Eq. (18).

The opposite transformation, from Newtonian to synchronous gauge is, of course, just

$$\xi^0_{N\to Z} = -\xi^0_{Z\to N} \qquad \xi_{N\to Z} = -\xi_{Z\to N}\,, \tag{16.16}$$

so we can easily express it in synchronous gauge quantities, which is actually what we typically want.

We get the Einstein tensor perturbations from Eq. (A.7), setting $A = B = 0$,

$$\begin{aligned} \delta G^0_0 &= a^{-2}\left[2k^2\psi + 6\mathcal{H}D'\right] \\ \delta G^0_i &= a^{-2}\left[-2ik_i\psi'\right] \\ \delta G^i_0 &= a^{-2}\left[2ik_i\psi'\right] \\ \delta G^i_j &= a^{-2}\left[2D'' + k^2 D + 4\mathcal{H}D' - \tfrac{1}{3}k^2 E + \tfrac{1}{3}E'' + \tfrac{2}{3}\mathcal{H}E'\right]\delta^i_j \\ &\quad -k_i k_j a^{-2}\left[D - \tfrac{1}{3}E + \frac{1}{k^2}(E'' + 2\mathcal{H}E')\right] \\ \delta G^i_i &= a^{-2}\left[6D'' + 2k^2\psi + 12\mathcal{H}D'\right]\,. \end{aligned} \tag{16.17}$$

In the MB η, h notation,

$$\begin{aligned} \delta G^0_0 &= a^{-2}\left[2k^2\eta - \mathcal{H}h'\right] \\ \delta G^0_i &= a^{-2}\left[-2ik_i\eta'\right] \\ \delta G^i_0 &= a^{-2}\left[2ik_i\eta'\right] \\ \delta G^i_j &= a^{-2}\left[-\tfrac{1}{2}h'' - \eta'' - \mathcal{H}(h'+2\eta') + k^2\eta\right]\delta^i_j \\ &\quad -k_i k_j a^{-2}\left[\eta - \frac{1}{k^2}(\tfrac{1}{2}h'' + 3\eta'' + \mathcal{H}h' + 6\mathcal{H}\eta')\right] \\ \delta G^i_i &= a^{-2}\left[-h'' + 2k^2\eta - 2\mathcal{H}h'\right]\,. \end{aligned} \tag{16.18}$$

The Einstein equations are thus

$$\begin{aligned} k^2\eta - \tfrac{1}{2}\mathcal{H}h' &= -4\pi Ga^2\delta\rho^Z = -\tfrac{3}{2}\mathcal{H}^2\delta^Z \\ k^2\eta' &= 4\pi Ga^2(\bar\rho+\bar p)kv^Z = \tfrac{3}{2}\mathcal{H}^2(1+w)kv^Z \\ h'' + 2\mathcal{H}h' - 2k^2\eta &= -24\pi Ga^2\delta p^Z = -9\mathcal{H}^2\frac{\delta p^Z}{\bar\rho} \\ h'' + 6\eta'' + 2\mathcal{H}h' + 12\mathcal{H}\eta' - 2k^2\eta &= -16\pi Ga^2\bar p\Pi = -6\mathcal{H}^2 w\Pi\,, \end{aligned} \tag{16.19}$$

which is MB Eq. (21). Note that MB uses the notation

$$\theta \equiv \nabla\cdot\vec v = -\nabla^2 v = kv \qquad \text{and} \qquad (\bar\rho+\bar p)\sigma \equiv \tfrac{2}{3}\bar p\Pi\,. \tag{16.20}$$

We get the continuity equations from Eq. (A.15), setting $A = B = 0$ and $D = -\frac{1}{6}h$,

$$\begin{aligned} \delta\rho^{Z\prime} &= -3\mathcal{H}(\delta\rho^Z + \delta p^Z) - (\bar\rho+\bar p)(\tfrac{1}{2}h' + kv^Z) \\ (\bar\rho+\bar p)v^{Z\prime} &= -(\bar\rho+\bar p)'v^Z - 4\mathcal{H}(\bar\rho+\bar p)v^Z + k\delta p^Z - \tfrac{2}{3}k\bar p\Pi \\ \delta^{Z\prime} &= -(1+w)(kv^Z + \tfrac{1}{2}h') + 3\mathcal{H}\left(w\delta^Z - \frac{\delta p^Z}{\bar\rho}\right) \\ v^{Z\prime} &= -\mathcal{H}(1-3w)v^Z - \frac{w'}{1+w}v^Z + \frac{k\delta p^Z}{\bar\rho+\bar p} - \tfrac{2}{3}\frac{w}{1+w}k\Pi\,. \end{aligned} \tag{16.21}$$

The two last equations are MB Eq. (30).

Exercise: Derive the synchronous gauge Einstein equations and continuity equations from the corresponding Newtonian gauge equations by gauge transformation .

17 Fluid Components

17.1 Division into Components

In practice, the cosmological fluid consists of many components (e.g., particle species: photons, baryons, CDM, neutrinos, ...). It is useful to divide the energy tensor into such components:

$$T^\mu_\nu = \sum_i T^\mu_{\nu(i)}\,, \tag{17.1}$$

which have their corresponding background and perturbation parts

$$\bar T^\mu_\nu = \sum_i \bar T^\mu_{\nu(i)} \qquad \text{and} \qquad \delta T^\mu_\nu = \sum_i \delta T^\mu_{\nu(i)}\,. \tag{17.2}$$

Here i labels the different components (so I'll use l and m for space indices).

Often the component pressure is a unique function of the component energy density, $p_i = p_i(\rho_i)$, or can be so approximated (common cases are $p_i = 0$ and $p_i = \rho_i/3$), although this is not true for the total pressure and energy density.

The total fluid and component quantities for the background are related as

$$\begin{aligned} \bar\rho &= \sum \bar\rho_i \\ \bar p &= \sum \bar p_i = \sum w_i\bar\rho_i \\ w &\equiv \frac{\bar p}{\bar\rho} = \sum \frac{\bar\rho_i}{\bar\rho}w_i \\ c_s^2 &\equiv \frac{\bar p'}{\bar\rho'} = \frac{\sum \bar p_i'}{\bar\rho'} = \sum \frac{\bar\rho_i'}{\bar\rho'}c_i^2\,, \end{aligned} \tag{17.3}$$

where

$$w_i \equiv \frac{\bar{p}_i}{\bar{\rho}_i} \qquad \text{and} \qquad c_i^2 \equiv \frac{\bar{p}_i'}{\bar{\rho}_i'}\,. \tag{17.4}$$

The total fluid and component quantities for the perturbations are related as

$$\begin{aligned} \delta\rho &= \sum \delta\rho_i \\ \delta p &= \sum \delta p_i \\ \delta &\equiv \frac{\delta\rho}{\bar{\rho}} = \sum \frac{\bar{\rho}_i}{\bar{\rho}}\delta_i\,, \end{aligned} \tag{17.5}$$

where $\delta_i \equiv \delta\rho_i/\bar{\rho}_i$. From

$$(\bar{\rho}+\bar{p})v_l = \sum(\bar{\rho}_i+\bar{p}_i)v_{l(i)} \tag{17.6}$$

we get that

$$v_l = \sum \frac{\bar{\rho}_i+\bar{p}_i}{\bar{\rho}+\bar{p}}v_{l(i)} \tag{17.7}$$

and from

$$\Sigma_{lm} = \sum \Sigma_{lm(i)} = \sum \bar{p}_i\Pi_{lm(i)}$$

we get that

$$\Pi_{lm} \equiv \frac{\Sigma_{lm}}{\bar{p}} = \sum \frac{\bar{p}_i}{\bar{p}}\Pi_{lm(i)} = \sum \frac{w_i\bar{\rho}_i}{w\bar{\rho}}\Pi_{lm(i)}\,. \tag{17.8}$$

17.2 Gauge Transformations

Perhaps I'll write more here someday ...

Note that gauge conditions that refer to fluid perturbations, refer usually to the total fluid. Thus, e.g., in the comoving gauge the total velocity perturbation v vanishes, but the component velocities v_i do not vanish (unless they happen to be all equal). Thus, the velocity v that appears in the gauge transformation equations to comoving gauge refers to the total density perturbation. For example the component gauge tranformation equations that correspond to Eq. (15.14) read

$$\begin{aligned} \delta\rho_i^C &= \delta\rho_i^N - \bar{\rho}_i'v^N \\ \delta p_i^C &= \delta p_i^N - \bar{p}_i'v^N \\ \delta_i^C &= \delta_i^N - \frac{\bar{\rho}_i'}{\bar{\rho}}v^N\,. \end{aligned} \tag{17.9}$$

Since the gauge transformations are the same for all components, we find some gauge invariances. The *relative velocity perturbation* between two components i and j,

$$v_i - v_j \qquad \text{is gauge invariant.} \tag{17.10}$$

Like the total anisotropic stress, also the component anisotropic stresses

$$\Pi_i \qquad \text{are gauge invariant.} \tag{17.11}$$

We can also define a kind of *entropy perturbation*

$$S_{ij} \equiv -3\mathcal{H}\left(\frac{\delta\rho_i}{\bar{\rho}_i'} - \frac{\delta\rho_j}{\bar{\rho}_j'}\right) \tag{17.12}$$

between two fluid components i and j which turns out to be gauge invariant due to the way the density perturbations $\delta\rho_i$ transform. The above is a special case of a *generalized entropy perturbation*

$$S_{xy} \equiv \mathcal{H}\left(\frac{\delta x}{\bar{x}'} - \frac{\delta y}{\bar{y}'}\right) \tag{17.13}$$

between any two 4-scalar quantities x and y, which is gauge invariant due to the way 4-scalar perturbations transform. Beware of the many different quantities called "entropy perturbation"! Some of them can be interpreted as perturbations in some entropy/particle ratio. What is common to all of them, is that they all vanish in the case of adiabatic perturbations.

17.3 Equations

The Einstein equations (both background and perturbation) involve the total fluid quantities. The metric perturbations are not divided into components due to different fluid components! There is a single gravity, due to the total fluid, which each fluid component obeys.

If there is no energy transfer between the fluid components in the background universe, the background energy continuity equation is satisfied separately by such independent components,

$$\bar{\rho}_i' = -3\mathcal{H}(\bar{\rho}_i + \bar{p}_i)\,, \tag{17.14}$$

and in that case we can write Eq. (17.3) as

$$c_s^2 = \sum \frac{\bar{\rho}_i + \bar{p}_i}{\bar{\rho} + \bar{p}} c_i^2\,, \tag{17.15}$$

and Eq. (17.12) as

$$S_{ij} = \frac{\delta\rho_i}{(1+w_i)\bar{\rho}_i} - \frac{\delta\rho_j}{(1+w_j)\bar{\rho}_j} = \frac{\delta_i}{1+w_i} - \frac{\delta_j}{1+w_j} \tag{17.16}$$

Likewise, the gauge transformation equations (17.9) become

$$\begin{aligned} \delta\rho_i^C &= \delta\rho_i^N + 3\mathcal{H}(1+w_i)\bar{\rho}_i v^N \\ \delta p_i^C &= \delta p_i^N + 3\mathcal{H}(1+w_i)c_i^2\bar{\rho}_i v^N \\ \delta_i^C &= \delta_i^N + 3\mathcal{H}(1+w_i)v^N\,. \end{aligned} \tag{17.17}$$

But if there is energy transfer between two fluid components, then their component energy continuity equations acquire an interaction term.

Even if there is no energy transfer between fluid components at the background level, the perturbations often introduce energy and momentum transfer between the components. It may also be the case that the energy transfer can be neglected in practice, but the momentum transfer remains important.

In the case of noninteracting fluid components (no energy or momentum transfer), we have the perturbation energy and momentum continuity equations separately for each such fluid component,

$$T^{\mu}_{\nu(i);\mu} = 0\,. \tag{17.18}$$

For the case of scalar perturbations in the conformal-Newtonian gauge, they read

$$(\delta_i^N)' = (1+w_i)\left(\nabla^2 v_i^N + 3\Psi'\right) + 3\mathcal{H}\left(w_i\delta_i^N - \frac{\delta p_i^N}{\bar{\rho}_i}\right) \tag{17.19}$$

$$(v_i^N)' = -\mathcal{H}(1-3w_i)v_i^N - \frac{w_i'}{1+w_i}v_i^N + \frac{\delta p_i^N}{\bar{\rho}_i + \bar{p}_i} + \frac{2}{3}\frac{w_i}{1+w_i}\nabla^2\Pi_i + \Phi\,. \tag{17.20}$$

In Fourier space they become

$$(\delta_i^N)' = (1+w_i)\left(-kv_i^N + 3\Psi'\right) + 3\mathcal{H}\left(w_i\delta_i^N - \frac{\delta p_i^N}{\bar\rho_i}\right) \tag{17.21}$$

$$(v_i^N)' = -\mathcal{H}(1-3w_i)v_i^N - \frac{w_i'}{1+w_i}v_i^N + \frac{k\delta p_i^N}{\bar\rho_i+\bar p_i} - \frac{2}{3}\frac{w_i}{1+w_i}k\Pi_i + \Phi\,. \tag{17.22}$$

If there are interactions between the fluid components, there will be interaction terms ("collision terms") in these equations.

Note that one cannot write the mixed gauge equations for fluid components by just replacing the fluid quantities in equations (15.22,15.23) with component quantities, since these equations were derived by making a gauge transformation between the comoving and Newtonian gauges, which involved the total fluid velocity.

For the case where energy transfer between components can be neglected both at the background and perturbation level, we can use Eqs. (17.14) and (17.19) to find the time derivative of the entropy perturbation S_{ij},

$$\begin{aligned} S_{ij}' &= \nabla^2\left(v_i^N - v_j^N\right) + 3\mathcal{H}\left(\frac{c_i^2\delta\rho_i^N - \delta p_i^N}{\bar\rho_i+\bar p_i} - \frac{c_j^2\delta\rho_j^N - \delta p_j^N}{\bar\rho_j+\bar p_j}\right) \\ &= \nabla^2(v_i - v_j) - 9\mathcal{H}\left(c_i^2\mathcal{S}_i - c_j^2\mathcal{S}_j\right)\,, \end{aligned} \tag{17.23}$$

where

$$\mathcal{S}_i \equiv \mathcal{H}\left(\frac{\delta p_i}{\bar p_i'} - \frac{\delta\rho_i}{\bar\rho_i'}\right) \tag{17.24}$$

is the (gauge invariant) internal entropy perturbation of component i. If we have in addition, that for both components the equation of state has the form $p_i = f_i(\rho_i)$, then the internal entropy perturbations vanish[24], and we have simply

$$S_{ij}' = \nabla^2(v_i - v_j)\,. \tag{17.25}$$

In Fourier space Eq. (17.25) reads

$$S_{ij}' = -k(v_i - v_j) \qquad \text{or} \qquad \mathcal{H}^{-1}S_{ij}' = -\frac{k}{\mathcal{H}}(v_i - v_j)\,, \tag{17.26}$$

showing that entropy perturbations remain constant at superhorizon scales $k \ll \mathcal{H}$. In other words, *adiabatic perturbations ($S_{ij} = 0$) stay adiabatic while outside the horizon.*

We can also obtain an equation for S_{ij}'' from Eq. (17.20), if also momentum transfer between components can be neglected.

In the synchronous gauge, Eqs. (17.21,17.22) become

$$(\delta_i^Z)' = -(1+w_i)\left(kv_i^Z + \tfrac{1}{2}h'\right) + 3\mathcal{H}\left(w_i\delta_i^Z - \frac{\delta p_i^Z}{\bar\rho_i}\right) \tag{17.27}$$

$$(v_i^Z)' = -\mathcal{H}(1-3w_i)v_i^Z - \frac{w_i'}{1+w_i}v_i^Z + \frac{k\delta p_i^Z}{\bar\rho_i+\bar p_i} - \frac{2}{3}\frac{w_i}{1+w_i}k\Pi_i\,. \tag{17.28}$$

Since E does not appear in the general gauge scalar continuity equations (A.15), the only difference in them when going from Newtonian to synchronous gauge (as both have $B = 0$) is that $\Psi = D^N$ is replaced by $-\frac{1}{6}h = D^Z$ and $\Phi = A^N$ is dropped as $A^Z = 0$.

[24]For baryons this requires that we ignore baryon pressure, since $p_b = p_b(n_b, T) = n_b T$, and $\rho_b = \rho_b(n_b, T) = n_b(m_b + \frac{3}{2}T)$.

18 Adiabatic and Isocurvature Perturbations in a Simplified Universe

Consider the case where the energy tensor consists of two perfect fluid components, matter with $p = 0$ and radiation with $p = \frac{1}{3}\rho$, that do not interact with each other, i.e., there is no energy or momentum transfer between them. (Compared to the real universe, this is a simplification since, while cold dark matter does not much interact with the other fluid components, the baryonic matter does interact with photons. Also, the radiation components of the real universe, neutrinos and photons, behave like a perfect fluid only until their decoupling.)

18.1 Background solution for radiation+matter

We have now two fluid components,

$$\rho = \rho_r + \rho_m \,,$$

where

$$\rho_r \propto a^{-4} \qquad \text{and} \qquad \rho_m \propto a^{-3} \,,$$

and

$$p_m = 0 \qquad \text{and} \qquad p_r = \tfrac{1}{3}\rho_r \,.$$

The equation of state and sound speed parameters are

$$w_m = c_m^2 = 0 \qquad \text{and} \qquad w_r = c_r^2 = \tfrac{1}{3} \,. \tag{18.1}$$

We define

$$y \equiv \frac{a}{a_{\text{eq}}} = \frac{\rho_m}{\rho_r} \tag{18.2}$$

so that

$$\frac{\rho_r}{\rho} = \frac{1}{1+y} \qquad \frac{\rho_m}{\rho} = \frac{y}{1+y} \qquad \frac{\rho_r + p_r}{\rho + p} = \frac{4}{4+3y} \qquad \frac{\rho_m + p_m}{\rho + p} = \frac{3y}{4+3y} \tag{18.3}$$

and

$$w = \frac{1}{3(1+y)} \qquad 1 + w = \frac{4+3y}{3(1+y)} \qquad c_s^2 = \frac{4}{3(4+3y)} \qquad 1 - 3c_s^2 = \frac{3y}{4+3y} \,. \tag{18.4}$$

The Friedmann equation is

$$\begin{aligned} \mathcal{H}^2 \equiv \frac{1}{a^2}\left(\frac{da}{d\eta}\right)^2 &= \frac{8\pi G}{3}\rho a^2 = \frac{8\pi G}{3}(1+y)\rho_{r0} a_0^4 a^{-2} \\ \Rightarrow \quad \frac{dy}{\sqrt{1+y}} &= 2C d\eta \,, \end{aligned}$$

where

$$C \equiv \sqrt{\frac{2\pi G}{3}\rho_{r0}\frac{a_0^2}{a_{\text{eq}}}} \,.$$

The solution is

$$y = C^2\eta^2 + 2C\eta \,. \tag{18.5}$$

At the time of matter-radiation equality,

$$y = y_{\text{eq}} = C^2\eta_{\text{eq}}^2 + 2C\eta_{\text{eq}} = 1 \quad \Rightarrow \quad C\eta_{\text{eq}} = \sqrt{2} - 1 \,, \tag{18.6}$$

so we can write the solution as

$$y = \left(\frac{\eta}{\eta_3}\right)^2 + 2\left(\frac{\eta}{\eta_3}\right), \tag{18.7}$$

where

$$\eta_3 \equiv \frac{\eta_{\rm eq}}{\sqrt{2}-1} = \left(\sqrt{2}+1\right)\eta_{\rm eq} \tag{18.8}$$

is the time when $y = 3$ ($\rho_m = 3\rho_r$).

The Hubble parameter is

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{y'}{y} = \frac{\eta+\eta_3}{\eta_3\eta + \frac{1}{2}\eta^2}. \tag{18.9}$$

At matter-radiation equality it has the value

$$\mathcal{H}_{\rm eq} = \frac{2\sqrt{2}}{\sqrt{2}+1}\frac{1}{\eta_{\rm eq}} = \frac{4-2\sqrt{2}}{\eta_{\rm eq}} = \frac{2\sqrt{2}}{\eta_3}. \tag{18.10}$$

At early times, $\eta \ll \eta_3$, the universe is radiation dominated, so that

$$y \approx \frac{2\eta}{\eta_3} \ll 1 \quad \Rightarrow \quad a \propto \eta \quad \Rightarrow \quad \mathcal{H} = \frac{1}{\eta} \propto a^{-1}. \tag{18.11}$$

At late times, $\eta \gg \eta_3$, the universe is matter dominated, so that

$$y \approx \left(\frac{\eta}{\eta_3}\right)^2 \gg 1 \quad \Rightarrow \quad a \propto \eta^2 \quad \Rightarrow \quad \mathcal{H} = \frac{2}{\eta} \propto a^{-1/2}. \tag{18.12}$$

When solving for perturbations it turns out to be more convenient to use y (or $\log y$) as time coordinate instead of η. Inverting Eq. (18.7), we have that

$$\eta = \left(\sqrt{1+y}-1\right)\eta_3 = \frac{\sqrt{1+y}-1}{\sqrt{2}-1}\eta_{\rm eq}, \tag{18.13}$$

and

$$\mathcal{H} = \frac{\sqrt{1+y}}{y}\frac{2}{\eta_3} = \frac{\sqrt{1+y}}{y}\frac{\mathcal{H}_{\rm eq}}{\sqrt{2}}. \tag{18.14}$$

18.2 Perturbations

In terms of the component perturbations the total perturbations are now

$$\delta = \frac{1}{1+y}\delta_r + \frac{y}{1+y}\delta_m \tag{18.15}$$

$$v = \frac{4}{4+3y}v_r + \frac{3y}{4+3y}v_m \tag{18.16}$$

and the relative entropy perturbation is

$$S \equiv S_{mr} = \delta_m - \tfrac{3}{4}\delta_r. \tag{18.17}$$

From the pair (18.15,18.17) we can solve δ_m and δ_r in terms of δ and S:

$$\delta_m = \frac{3+3y}{4+3y}\delta + \frac{4}{4+3y}S \tag{18.18}$$

$$\delta_r = \frac{4+4y}{4+3y}\delta - \frac{4y}{4+3y}S. \tag{18.19}$$

Likewise we can express v_m and v_r in terms of the total and relative velocity perturbations, v and $v_m - v_r$:

$$v_m = v + \frac{4}{4+3y}(v_m - v_r) \tag{18.20}$$

$$v_r = v - \frac{3y}{4+3y}(v_m - v_r). \tag{18.21}$$

We can now also relate the total entropy perturbation $\mathcal{S}$ to S:

$$\mathcal{S} = \frac{1}{3(1+w)}\left(\frac{\delta\rho}{\bar\rho} - \frac{1}{c_s^2}\frac{\delta p}{\bar\rho}\right) = \ldots = \frac{y}{4+3y}S = \tfrac{1}{3}(1-3c_s^2)S. \tag{18.22}$$

The Bardeen equation (15.40) becomes now

$$\begin{aligned}\mathcal{H}^{-2}\delta_C{}'' + \left(1-6w+3c_s^2\right)\mathcal{H}^{-1}\delta_C{}' - \tfrac{3}{2}\left(1+8w-6c_s^2-3w^2\right)\delta_C \\ = -c_s^2\left(\frac{k}{\mathcal{H}}\right)^2\left[\delta_C - (1+w)(1-3c_s^2)S\right] \\ = -c_s^2\left(\frac{k}{\mathcal{H}}\right)^2\left(\delta_C - \frac{y}{1+y}S\right).\end{aligned} \tag{18.23}$$

We get the entropy evolution equation by derivation Eq. (17.25),

$$S' = -k(v_m - v_r) \quad\Rightarrow\quad S'' = -k(v_m' - v_r'). \tag{18.24}$$

Using the v_i^N evolution equations (17.22), this becomes (**Exercise**)

$$S'' = \mathcal{H}k(v_m - v_r) + \mathcal{H}kv_r + \tfrac{1}{4}k^2\delta_r^N. \tag{18.25}$$

Here the δ_r^N is converted to the comoving gauge using the *total* fluid velocity (see Eq. 17.17),

$$\delta_r^N = \delta_r^C - 3\mathcal{H}(1+w_r)k^{-1}v^N \tag{18.26}$$

so that Eq. (18.24) becomes

$$S'' = \mathcal{H}k\left(\frac{4}{4+3y}\right)(v_m - v_r) + \tfrac{1}{4}k^2\delta_r^C. \tag{18.27}$$

Replacing $v_m - v_r$ by $-k^{-1}S'$ we get (**Exercise**) the Kodama-Sasaki equation

$$\mathcal{H}^{-2}S'' + \frac{4}{4+3y}\mathcal{H}^{-1}S' = \left(\frac{k}{\mathcal{H}}\right)^2\left(\frac{1+y}{4+3y}\delta^C - \frac{y}{4+3y}S\right). \tag{18.28}$$

or

$$\mathcal{H}^{-2}S'' + 3c_s^2\mathcal{H}^{-1}S' = \frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2\left(\frac{1}{1+w}\delta - (1-3c_s^2)S\right),$$

where $\delta \equiv \delta^C$.

The two equations (18.23) and (18.28) form a pair of ordinary differential equations, from which we can solve the evolution of the perturbations $\delta_{\vec k}^C(\eta)$ and $S_{\vec k}(\eta)$. Since the coefficient functions of these equations can more easily be expressed in terms of the scale factor y, it may be more convenient to use y as the time coordinate instead of η. The time derivatives are converted with (**Exercise**)

$$\mathcal{H}^{-1}f' = y\frac{df}{dy} \qquad\text{and}\qquad \mathcal{H}^{-2}f'' = y^2\frac{d^2f}{dy^2} + \tfrac{1}{2}(1-3w)y\frac{df}{dy} \tag{18.29}$$

and the equations become

$$y^2\frac{d^2\delta}{dy^2} + \tfrac{3}{2}(1-5w+2c_s^2)y\frac{d\delta}{dy} - \tfrac{3}{2}(1+8w-6c_s^2-3w^2)\delta = -\left(\frac{k}{\mathcal{H}}\right)^2 c_s^2\left(\delta - \frac{y}{1+y}S\right) \quad (18.30)$$

$$y^2\frac{d^2S}{dy^2} + \tfrac{1}{2}(1-3w+6c_s^2)y\frac{dS}{dy} = \left(\frac{k}{\mathcal{H}}\right)^2\left(\frac{1+y}{4+3y}\delta^C - \frac{y}{4+3y}S\right).$$

For solving the other perturbation quantities, we collect here the relevant equations:

$$\Phi = -\frac{3}{2}\left(\frac{\mathcal{H}}{k}\right)^2\delta \quad (18.31)$$

$$v^N = \frac{2}{3(1+w)}\left(\frac{k}{\mathcal{H}}\right)(\mathcal{H}^{-1}\Phi' + \Phi)$$

$$\delta^N = \delta - 3\left(\frac{\mathcal{H}}{k}\right)(1+w)v^N$$

$$\mathcal{R} = -\Phi - \frac{2}{3(1+w)\mathcal{H}}(\Phi' + \mathcal{H}\Phi)$$

When judging which quantities are negligible at superhorizon scales ($k \ll \mathcal{H}$), one has to exrcise some care, and not just look blindly at powers of $k/\mathcal{H}$ in equations which contain different perturbation quantities. From Eq. (18.31a) one sees that at superhorizon scales, a small comoving δ can still be important and cause a large gravitational potential perturbation Φ. Eq. (15.38),

$$\mathcal{H}^{-1}\mathcal{R}' = -c_s^2\left(\frac{\delta}{1+w} - 3\mathcal{S}\right),$$

may seem to contradict the statement that for adiabatic perturbations $\mathcal{R}$ stays constant at superhorizon scales, but the explanation is, that $\mathcal{R}$ is then of the same order of magnitude as Φ, whereas δ is suppressed by two powers of $k/\mathcal{H}$ compared to Φ. Using Eqs. (18.31a,18.22) we rewrite (15.38) as

$$\mathcal{H}^{-1}\mathcal{R}' = c_s^2\left[\frac{1}{1+w}\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Phi + (1-3c_s^2)S\right]. \quad (18.32)$$

18.3 Initial Epoch

For $y \ll 1$, the equations (18.23,18.28,18.30) can be approximated by

$$\mathcal{H}^{-2}\delta'' - 2\delta = -\tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\delta - yS) \quad (18.33)$$

$$\mathcal{H}^{-2}S'' + \mathcal{H}^{-1}S' = \tfrac{1}{4}\left(\frac{k}{\mathcal{H}}\right)^2(\delta - yS) \quad (18.34)$$

or

$$y^2\frac{d^2\delta}{dy^2} - 2\delta = -\tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\delta - yS) = -\frac{2}{3}\left(\frac{k}{\mathcal{H}_{\mathrm{eq}}}\right)^2(y^2\delta - y^3S) \quad (18.35)$$

$$y^2\frac{d^2S}{dy^2} + y\frac{dS}{dy} = \tfrac{1}{4}\left(\frac{k}{\mathcal{H}}\right)^2(\delta - yS) = \frac{1}{2}\left(\frac{k}{\mathcal{H}_{\mathrm{eq}}}\right)^2(y^2\delta - y^3S). \quad (18.36)$$

At early times, all cosmologically intersecting scales are outside the horizon, and we start by making the approximation, that the rhs of these equations can be ignored. Then the evolution of δ and S decouple and the general solutions are

$$\delta_{\vec{k}} = A_{\vec{k}} y^2 + B_{\vec{k}} y^{-1} \tag{18.37}$$

$$S_{\vec{k}} = C_{\vec{k}} + D_{\vec{k}} \ln y\,. \tag{18.38}$$

Thus, for each Fourier component $\vec{k}$, there are four independent modes. We identify the adiabatic growing ($A_{\vec{k}}$) and decaying ($B_{\vec{k}}$) modes, and the isocurvature[25] "growing" ($C_{\vec{k}}$) and decaying ($D_{\vec{k}}$) modes. The D-mode is indeed decaying, since $0 < y \ll 1$.

The decaying modes diverge as $y \to 0$, but we can suppose that our description of the universe is not valid all the way to $y = 0$, but at very early times there is some process that is responsible for fixing the initial values of $\delta_{\vec{k}}$, $\delta'_{\vec{k}}$, $S_{\vec{k}}$, and $S'_{\vec{k}}$ at some early time $y_{\rm init} > 0$, and fixing thus $A_{\vec{k}}$, $B_{\vec{k}}$, $C_{\vec{k}}$, and $D_{\vec{k}}$.

Let us now consider the effect of the rhs of the eqs. These couple the δ and S. Thus it is not consistent to assume that the other is exactly zero.

18.3.1 Adiabatic modes

Consider first the adiabatic modes ($C_{\vec{k}} = D_{\vec{k}} = 0$). The coupling forces the existence of a small nonzero S, which at first is $\ll \delta$. We can thus keep ignoring S on the rhs, and for the δ equation we can keep ignoring the whole rhs. But for the S equation we now have very small S on the lhs, and therefore we can not ignore the large δ on the rhs, even though it is suppressed by $(k/\mathcal{H})^2$, or y^2. Thus for adiabatic modes the pair of equations can be approximated as

$$\mathcal{H}^{-2}\delta'' - 2\delta = 0 \tag{18.39}$$

$$\mathcal{H}^{-2}S'' + \mathcal{H}^{-1}S' = \tfrac{1}{4}\left(\frac{k}{\mathcal{H}}\right)^2 \delta \tag{18.40}$$

or

$$y^2\frac{d^2\delta}{dy^2} - 2\delta = 0 \tag{18.41}$$

$$y^2\frac{d^2S}{dy^2} + y\frac{dS}{dy} = \frac{1}{2}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)^2 y^2\delta\,. \tag{18.42}$$

The solution for δ remains Eq. (18.37), but we also get that

$$S_{\vec{k}} = \frac{1}{32}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)^2 A_{\vec{k}} y^4 + \frac{1}{2}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)^2 B_{\vec{k}} y\,. \tag{18.43}$$

Thus for the growing adiabatic mode

$$S_{\vec{k}} = \frac{1}{64}\left(\frac{k}{\mathcal{H}}\right)^2 \delta_{\vec{k}} \tag{18.44}$$

and for the decaying adiabatic mode

$$S_{\vec{k}} = \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)^2 \delta_{\vec{k}}\,. \tag{18.45}$$

So you see that, although these are called adiabatic modes, the perturbations are not exactly adiabatic! The name just means that $S \to 0$ as $y \to 0$. The entropy perturbations remain small compared to the density perturbation while the Fourier mode is outside the horizon, but can become large near and after horizon entry.

[25] The term "isocurvature" will be explained later.

18.3.2 Isocurvature Modes

Consider then the isocurvature modes ($A_{\vec{k}} = B_{\vec{k}} = 0$). Now the coupling causes a small $\delta \ll S$. We can ignore the rhs of the S equation, but the large S cannot be ignored on the rhs of the δ equation. Thus for isocurvature modes the pair of equations can be approximated by

$$\mathcal{H}^{-2}\delta'' - 2\delta = +\tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2 yS \qquad (18.46)$$

$$\mathcal{H}^{-2}S'' + \mathcal{H}^{-1}S' = 0 \qquad (18.47)$$

or

$$y^2\frac{d^2\delta}{dy^2} - 2\delta = +\frac{2}{3}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)^2 y^3 S \qquad (18.48)$$

$$y^2\frac{d^2S}{dy^2} + y\frac{dS}{dy} = 0\,. \qquad (18.49)$$

The solution for S remains Eq. (18.38), but for the density perturbation we get

$$\delta_{\vec{k}} = \frac{1}{6}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)^2 C_{\vec{k}}y^3 + \frac{1}{6}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)^2 D_{\vec{k}}\left(y^3\ln y - \frac{5}{4}y^3\right)\,. \qquad (18.50)$$

(For the decaying isocurvature mode the differential equation for δ is a little more difficult, but you can check (**Exercise**) that the above is the correct solution.) For the growing isocurvature mode we have thus that

$$\delta_{\vec{k}} = \frac{1}{12}\left(\frac{k}{\mathcal{H}}\right)^2 yS_{\vec{k}}\,. \qquad (18.51)$$

So although the relative entropy perturbation stays constant at early times, the density perturbation is growing in this mode. For the decaying isocurvature mode

$$\delta_{\vec{k}} = \frac{1}{12}\left(\frac{k}{\mathcal{H}}\right)^2\left(y - \frac{5}{4}\frac{y}{\ln y}\right)S_{\vec{k}} \approx \frac{1}{12}\left(\frac{k}{\mathcal{H}}\right)^2 yS_{\vec{k}}\,. \qquad (18.52)$$

18.3.3 Other Perturbations

For the initial eboch, $w = \frac{1}{3}$, and Eqs. (18.31) become

$$\begin{aligned}\Phi &= -\frac{3}{2}\left(\frac{\mathcal{H}}{k}\right)^2\delta \\ v^N &= \frac{1}{2}\left(\frac{k}{\mathcal{H}}\right)\left(\mathcal{H}^{-1}\Phi' + \Phi\right) \\ \delta^N &= \delta - 4\left(\frac{\mathcal{H}}{k}\right)v^N \\ \mathcal{R} &= -\tfrac{3}{2}\Phi - \tfrac{1}{2}\mathcal{H}^{-1}\Phi'\end{aligned} \qquad (18.53)$$

For the growing adiabatic mode, we have

$$\Phi = -\tfrac{3}{2}\left(\frac{\mathcal{H}_{\rm eq}}{k}\right)^2 A = \text{const.} \qquad (18.54)$$

$$\mathcal{R} = -\tfrac{3}{2}\Phi = \text{const.} \qquad (18.55)$$

$$v^N = -\frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)\mathcal{R} = -\frac{\sqrt{2}}{3}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)y\mathcal{R} \qquad (18.56)$$

$$\delta^N = -2\Phi = \tfrac{4}{3}\mathcal{R}\,. \qquad (18.57)$$

For the growing isocurvature mode, where $S=$ const. we have

$$\Phi = -\tfrac{1}{8}yS = -\frac{\mathcal{H}_{\rm eq}}{8\sqrt{2}}\eta S \tag{18.58}$$

$$\mathcal{R} = \frac{1}{4\sqrt{2}}\mathcal{H}_{\rm eq}S\eta = \tfrac{1}{4}Sy = -2\Phi \tag{18.59}$$

$$v^N = -\frac{1}{4\sqrt{2}}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)Sy^2 \tag{18.60}$$

$$\delta^N = \frac{1}{6}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)^2 Sy^3 + \tfrac{1}{2}Sy \approx \tfrac{1}{2}Sy = 2\mathcal{R}\,. \tag{18.61}$$

Thus the growing adiabatic mode is characterized by a constant $\mathcal{R}$ and the growing isocurvature mode by a constant S. If both modes are present, and these constants are of a similar magnitude, then the growing S associated with the adiabatic mode is negligible as long as $k \ll \mathcal{H}$, and the growing $\mathcal{R}$ associated with the isocurvature mode is negligible as long as $y \ll 1$.

Thus, after the decaying modes have died out, the general (adiabatic+isocurvature) mode is characterized by these two constants (for each Fourier mode), which we denote by $\mathcal{R}_{\vec{k}}(\text{rad})$ and $S_{\vec{k}}(\text{rad})$. Including the two small growing contributions we have, during the initial epoch,

$$\mathcal{R}_{\vec{k}} = \frac{9}{8}\left(\frac{\mathcal{H}_{\rm eq}}{k}\right)^2 A_{\vec{k}} + \tfrac{1}{4}yC_{\vec{k}} = \mathcal{R}_{\vec{k}}(\text{rad}) + \tfrac{1}{4}yS_{\vec{k}}(\text{rad}) \tag{18.62}$$

$$\begin{aligned} S_{\vec{k}} &= \frac{1}{32}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)^2 A_{\vec{k}}y^4 + C_{\vec{k}} = S_{\vec{k}}(\text{rad}) + \frac{1}{36}\left(\frac{k}{\mathcal{H}_{\rm eq}}\right)^4 \mathcal{R}_{\vec{k}}(\text{rad})y^4 \\ &= S_{\vec{k}}(\text{rad}) + \frac{1}{9}\left(\frac{k}{\mathcal{H}}\right)^4 \mathcal{R}_{\vec{k}}(\text{rad})\,. \end{aligned} \tag{18.63}$$

18.4 Full evolution for large scales

The full evolution in the general case is not amenable to analytic solution, and has to be solved numerically. This is not too difficult since we just have a pair of ordinary differential equations to solve for each k. The results from Sect. 18.3 can be used to set initial values at some small y.

For large scales ($k \ll k_{\rm eq} \equiv \mathcal{H}_{\rm eq}$) we can, however, solve the evolution analytically. We now drop the decaying modes and consider what happens to the growing modes after the radiation-dominated epoch.

Our basic equations are (18.32) and (18.28):

$$\mathcal{H}^{-1}\mathcal{R}'_{\vec{k}} = c_s^2\left[\frac{1}{1+w}\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Phi_{\vec{k}} + (1-3c_s^2)S_{\vec{k}}\right] \tag{18.64}$$

$$\mathcal{H}^{-2}S''_{\vec{k}} + 3c_s^2\mathcal{H}^{-1}S'_{\vec{k}} = \frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2\left[\frac{1}{1+w}\delta_{\vec{k}} - (1-3c_s^2)S_{\vec{k}}\right]\,. \tag{18.65}$$

We see that for superhorizon scales, $S_{\vec{k}} =$ const. remains a solution even when the universe is no longer radiation dominated. Since the other solution has decayed away, we conclude that

$$S_{\vec{k}} = \text{const.} = S_{\vec{k}}(\text{rad}) \qquad \text{for} \qquad k \ll \mathcal{H}\,. \tag{18.66}$$

We can also see that, for the *adiabatic mode*, $\mathcal{R}_{\vec{k}}$ stays constant for superhorizon scales, since on the rhs, $S_{\vec{k}}$ remains negligible for $k \ll \mathcal{H}$. For scales that enter the horizon during the matter-dominated epoch, the $\mathcal{R}_{\vec{k}}$ of adiabatic perturbations stays constant even through and after horizon entry, since c_s^2 becomes negligibly small, before $k/\mathcal{H}$ and $S_{\vec{k}}$ become large. We can thus conclude that

$$\mathcal{R}_{\vec{k}} \;=\; \text{const.} \;=\; \mathcal{R}_{\vec{k}}(\text{rad}) \qquad \text{for adiabatic modes with} \qquad k \ll k_{\text{eq}}\,. \tag{18.67}$$

If the isocurvature mode is present, however, we cannot assume that $S_{\vec{k}}$ is negligible for superhorizon scales; it's just constant, and we have, for $k \ll \mathcal{H}$,

$$\mathcal{H}^{-1}\mathcal{R}'_{\vec{k}} \;=\; c_s^2(1-3c_s^2)S_{\vec{k}}\,, \tag{18.68}$$

or

$$\frac{d\mathcal{R}_{\vec{k}}}{dy} \;=\; \frac{4}{(4+3y)^2}S_{\vec{k}}\,. \tag{18.69}$$

Integrating this, we get

$$\mathcal{R}_{\vec{k}} \;=\; \mathcal{R}_{\vec{k}}(\text{rad}) + \frac{y}{4+3y}S_{\vec{k}}(\text{rad}) \qquad \text{for} \qquad k \ll \mathcal{H}\,. \tag{18.70}$$

For $k \ll k_{\text{eq}}$, this has reached the final value $\mathcal{R}_{\vec{k}}(\text{rad}) + \frac{1}{3}S_{\vec{k}}(\text{rad})$ when the universe has become matter dominated, before horizon entry. After that, $\mathcal{R}_{\vec{k}}$ stays constant even through and after horizon entry by the same argument as for the adiabatic mode. Thus we conclude that

$$\mathcal{R}_{\vec{k}} \;=\; \text{const.} \;=\; \mathcal{R}_{\vec{k}}(\text{rad}) + \tfrac{1}{3}S_{\vec{k}}(\text{rad}) \qquad \text{for} \quad k \ll k_{\text{eq}} \quad \text{and} \quad \eta \gg \eta_{\text{eq}}\,. \tag{18.71}$$

For smaller scales, the exact solution has to be obtained numerically. It is customary to represent the solution in terms of transfer functions $T_{RR}(k)$ and $T_{RS}(k)$, which we can define by

$$\mathcal{R}_{\vec{k}} \;=\; \text{const.} \;\equiv\; T_{RR}(k)\mathcal{R}_{\vec{k}}(\text{rad}) + \tfrac{1}{3}T_{RS}(k)S_{\vec{k}}(\text{rad}) \qquad \text{for} \quad \eta \gg \eta_{\text{eq}}\,, \tag{18.72}$$

so that $T_{RR}(k) = T_{RS}(k) = 1$ for $k \ll k_{\text{eq}}$.

Finally, we can ask what happens to $S_{\vec{k}}$ after horizon entry (**Exercise**). ...

19 Effect of a Smooth Component

Should add here a section about the effect of a fluid component, whose perturbations we ignore, so that it affects only the background solution. The main application is dark energy. In case dark energy is just a cosmological constant, it has no perturbations. If the dark energy is close to a cosmological constant ($w \sim -1$), its perturbations should be small.

20 The Real Universe

According to present understanding, the universe contains 5 major "fluid" components: "baryons" (including electrons), cold dark matter, photons, neutrinos, and the mysterious dark energy,

$$\rho \;=\; \rho_b + \rho_c + \rho_\gamma + \rho_\nu + \rho_{DE}\,. \tag{20.1}$$

We shall here assume there are no perturbations in the dark energy.

We make the approximation that the pressure of baryons and cold dark matter can be ignored.[26] Thus $p_b = p_c = 0$ (both for background and perturbations). For photons, $p_\gamma = \rho_\gamma/3$. We assume massless neutrinos, so the same relation holds for them. Thus we have

$$\begin{aligned} w_b &= w_c = c_b^2 = c_c^2 = 0 \\ w_\gamma &= w_\nu = c_\gamma^2 = c_\nu^2 = \tfrac{1}{3}\,. \end{aligned} \tag{20.2}$$

[26] For small distance scales the baryon pressure is important after photon decoupling. If we were interested in small-scale structure formation, we should include it. Now our main interest is the CMB, where we are not interested in as small distance scales.

Moreover,

$$\begin{aligned} \delta p_b &= \delta p_c = 0 \\ \delta p_\gamma &= \tfrac{1}{3}\delta\rho_\gamma \\ \delta p_\nu &= \tfrac{1}{3}\delta\rho_\nu \,. \end{aligned} \tag{20.3}$$

Thus we have the happy situation, that for each component we have a unique relation $p_i = p_i(\rho_i)$, which moreover is very simple, either $p_i = 0$ or $p_i = \rho_i/3$. (Also, the simplest kind of dark energy has $p_{DE} = -\rho_{DE}$.

The components are, however, not all independent. Cold dark matter does not interact with the other components. We can ignore the interactions of neutrinos, since we are now only interested in times much after neutrino decoupling. But the baryons and photons interact via Thomson scattering.

Our continuity equations for perturbations are thus (for scalar perturbations in the conformal-Newtonian gauge)

$$\begin{aligned} \delta_c' &= \nabla^2 v_c + 3\Psi' \qquad (20.4) \\ v_c' &= -\mathcal{H}v_c + \Phi \\ \delta_b' &= \nabla^2 v_b + 3\Psi' + (\text{collision term}) \\ v_b' &= -\mathcal{H}v_b + \Phi + \text{collision term} \\ \delta_\gamma' &= \tfrac{4}{3}\nabla^2 v_\gamma + 4\Psi' + (\text{collision term}) \\ v_\gamma' &= \tfrac{1}{4}\delta_\gamma + \tfrac{1}{6}\nabla^2\Pi_\gamma + \Phi + \text{collision term} \\ \delta_\nu' &= \tfrac{4}{3}\nabla^2 v_\nu + 4\Psi' \\ v_\nu' &= \tfrac{1}{4}\delta_\nu + \tfrac{1}{6}\nabla^2\Pi_\nu + \Phi \,. \end{aligned}$$

We have put the collision terms for δ_b' and δ_γ' in parenthesis, since it will turn out that they can be neglected, and only momentum transfer between photons and baryons is important.

In Fourier space these equations read

$$\begin{aligned} \delta_c' &= -kv_c + 3\Psi' \qquad (20.5) \\ v_c' &= -\mathcal{H}v_c + k\Phi \\ \delta_b' &= -kv_b + 3\Psi' + (\text{collision term}) \\ v_b' &= -\mathcal{H}v_b + k\Phi + \text{collision term} \\ \delta_\gamma' &= -\tfrac{4}{3}kv_\gamma + 4\Psi' + (\text{collision term}) \\ v_\gamma' &= \tfrac{1}{4}k\delta_\gamma - \tfrac{1}{6}k\Pi_\gamma + k\Phi + \text{collision term} \\ \delta_\nu' &= -\tfrac{4}{3}kv_\nu + 4\Psi' \\ v_\nu' &= \tfrac{1}{4}k\delta_\nu - \tfrac{1}{6}k\Pi_\nu + k\Phi \,. \end{aligned}$$

(Remember our Fourier convention for v and Π.)

These equations are supplemented by 2 Einstein equations (there are 4 Einstein equations for perturbations, but since we are also using continuity equations, only two of them remain independent). Thus we have 10 perturbation equations, but there are 12 perturbation quantities to solve. (If we think of the Einstein equations as the equations for Φ and Ψ, the "extra" quantities lacking an equation of their own are the anisotropic stresses Π_γ and Π_ν. In the perfect fluid approximation these vanish, and the number of quantities equals the number of equations.) Also, we do not yet have the collision terms.

Thus more work is needed. This will lead us to the Boltzmann equations which employ a more detailed description of the fluid components, in terms of *distribution functions*.

In synchronous gauge one can simplify the cold dark matter equations, since cold dark matter falls freely (in our approximation). Thus we can use cold dark matter particles as the freely falling observers that define the synchronous space coordinate, so that

$$v_c^Z = 0\,. \tag{20.6}$$

In synchronous gauge Eqs. (20.5) become thus

$$\begin{aligned}
\delta_c' &= -\tfrac{1}{2}h' \\
v_c &= 0 \\
\delta_b' &= -kv_b - \tfrac{1}{2}h' + (\text{collision term}) \\
v_b' &= -\mathcal{H}v_b + \text{collision term} \\
\delta_\gamma' &= -\tfrac{4}{3}kv_\gamma - \tfrac{2}{3}h' + (\text{collision term}) \\
v_\gamma' &= \tfrac{1}{4}k\delta_\gamma - \tfrac{1}{6}k\Pi_\gamma + \text{collision term} \\
\delta_\nu' &= -\tfrac{4}{3}kv_\nu - \tfrac{2}{3}h' \\
v_\nu' &= \tfrac{1}{4}k\delta_\nu - \tfrac{1}{6}k\Pi_\nu\,.
\end{aligned} \tag{20.7}$$

21 Early Radiation-Dominated Era

The initial conditions for the evolution of the large scale structure and the cosmic microwave background can be specified during the radiation-dominated epoch, sufficiently early that all scales k of interest are outside the horizon. We do not want to deal with the electron-positron annihilation or big-bang nucleosynthesis (BBN), so we limit this time period to start after BBN. We assume it is sufficiently far after whatever event created the perturbations in the first place, so that we can assume that all decaying modes have already died out.

We are not just specifying "initial values" at some particular instant of time. Rather, we solve the perturbation equations for this particular epoch, and find that the solutions are characterized by quantities that remain constant for the whole epoch. Other perturbations quantities are related to these constant quantities by some powers of $k\eta$. These perturbations during this epoch we call the "primordial perturbations".

There are different modes of primordial perturbations. In *adiabatic perturbations* all fluid perturbations are determined by the metric perturbations. However, the metric perturbations depend only on the total fluid perturbations δ, δ_p, v, and Π. Thus there are additional degrees of freedom in the component fluids: *entropy perturbations*. For adiabatic perturbations, all component velocity perturbations are equal, $v_i = v$, and the density perturbations are related

$$\frac{\delta_i}{1+w_i} = \frac{\delta}{1+w}\,. \tag{21.1}$$

The (relative) entropy perturbations are defined

$$S_{ij} \equiv -3\mathcal{H}\left(\frac{\delta\rho_i}{\bar\rho_i'} - \frac{\delta\rho_j}{\bar\rho_j'}\right) = \frac{\delta_i}{1+w_i} - \frac{\delta_j}{1+w_j} \tag{21.2}$$

(we assume we can ignore energy transfer between fluid components).

For N fluid components, there are $N-1$ independent entropy perturbations. Often photons are taken as the reference fluid component for entropy perturbations, so that the independent entropy perturbations are taken to be

$$S_i \equiv \frac{\delta_i}{1+w_i} - \frac{3}{4}\delta_\gamma\,, \qquad i \neq \gamma\,. \tag{21.3}$$

For this section, we shall work mainly in the Newtonian gauge. Unless otherwise specified, $\delta \equiv \delta^N$ and $v = v^N$(!!!). The relevant equations are the Einstein equations (10.17):

$$\mathcal{H}^{-1}\Psi' + \Phi + \tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2 \Psi = -\tfrac{1}{2}\delta \tag{21.4}$$

$$\mathcal{H}^{-1}\Psi' + \Phi = \tfrac{3}{2}(1+w)\frac{\mathcal{H}}{k}v \tag{21.5}$$

$$\mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}\left(\Phi' + 2\Psi'\right) + \left(1 + \frac{2\mathcal{H}'}{\mathcal{H}^2}\right)\Phi - \tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2 (\Phi - \Psi) = \tfrac{3}{2}\frac{\delta p}{\bar\rho} \tag{21.6}$$

$$\left(\frac{k}{\mathcal{H}}\right)^2 (\Psi - \Phi) = 3w\Pi\,, \tag{21.7}$$

the fluid equations (20.5), and we also want to refer to the comoving curvature perturbation

$$\begin{aligned} \mathcal{R} &= -\Psi - \frac{2}{3(1+w)}\Phi - \frac{2}{3(1+w)}\mathcal{H}^{-1}\Psi' \\ \Rightarrow \quad \frac{2}{3}\mathcal{H}^{-1}\Psi' + \frac{5+3w}{3}\Psi &= -(1+w)\mathcal{R} + \frac{2}{3}(\Psi - \Phi) \end{aligned} \tag{21.8}$$

(from Eq. 15.26).

The early radiation-dominated era has 4 properties which simplify the solution of the perturbation equations:

1. All scales of interest are outside the horizon, $k \ll \mathcal{H}$. This allows us to drop some (but not necessarily all) of the gradient terms (those with k) from the perturbation equations.

2. Radiation domination, $\rho_\gamma\,,\ \rho_\nu \gg \rho_b\,,\ \rho_c\,,\ \rho_{\rm DE}$

$$\Rightarrow w = c_s^2 = \tfrac{1}{3} \quad \Rightarrow \quad \text{the background solution is} \qquad \mathcal{H} = \frac{1}{\eta}\,, \tag{21.9}$$

   and we can ignore the baryon, CDM, and DE contributions to the total fluid perturbation. We can also ignore the collision term in the photon velocity equation (but not in the baryon velocity equation) since the momentum the baryonic fluid can transfer to the photon fluid is negligible compared to the inertia density of the photon fluid.

3. This era is before recombination and photon decoupling, so baryons and photons are tightly coupled

$$\Rightarrow \quad v_b = v_\gamma \tag{21.10}$$

   and the continuous interaction with baryons (really the electrons) keeps the photon distribution isotropic

$$\Rightarrow \quad \Pi_\gamma = 0\,. \tag{21.11}$$

4. $m_\nu \ll T \quad \Rightarrow$ We can approximate neutrinos to be massless. This helps in solving the evolution of the neutrino momentum distribution. We do not discuss this here; this belongs to the course of CMB Physics, and we need to take one result from there.

Thus we have

$$\begin{aligned}\delta &= (1-f_\nu)\delta_\gamma + f_\nu\delta_\nu \\ \Pi &= f_\nu\Pi_\nu\,.\end{aligned} \tag{21.12}$$

where

$$f_\nu \equiv \frac{\rho_\nu}{\rho_\gamma+\rho_\nu} = \text{const.} \sim 0.405\,. \tag{21.13}$$

The relevant equations thus become

$$\begin{aligned}\mathcal{H}^{-1}\delta'_\gamma &= -\frac{4}{3}\left(\frac{k}{\mathcal{H}}\right)v_\gamma + 4\mathcal{H}^{-1}\Psi' \\ \mathcal{H}^{-1}v'_\gamma &= \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)\delta_\gamma + \left(\frac{k}{\mathcal{H}}\right)\Phi \\ \mathcal{H}^{-1}\delta'_\nu &= -\frac{4}{3}\left(\frac{k}{\mathcal{H}}\right)v_\nu + 4\mathcal{H}^{-1}\Psi' \\ \mathcal{H}^{-1}v'_\nu &= \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)\delta_\nu - \frac{1}{6}\left(\frac{k}{\mathcal{H}}\right)\Pi_\nu + \left(\frac{k}{\mathcal{H}}\right)\Phi\,.\end{aligned}$$

and

$$\mathcal{H}^{-1}\Psi' + \Phi = -\tfrac{1}{2}\delta \tag{21.14}$$

$$\mathcal{H}^{-1}\Psi' + \Phi = 2\frac{\mathcal{H}}{k}v \tag{21.15}$$

$$\mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}\left(\Phi' + 2\Psi'\right) - \Phi = \tfrac{1}{2}\delta \tag{21.16}$$

$$\left(\frac{k}{\mathcal{H}}\right)^2(\Psi-\Phi) = f_\nu\Pi_\nu \tag{21.17}$$

$$\frac{2}{3}\mathcal{H}^{-1}\Psi' + 2\Psi = -\frac{4}{3}\mathcal{R} + \frac{2}{3}(\Psi-\Phi)\,, \tag{21.18}$$

21.1 Neutrino Adiabaticity

We consider now the simpler case, when there are no neutrino entropy perturbations,

$$S_\nu = 0 \quad\Rightarrow\quad \delta_\nu = \delta_\gamma = \delta \qquad \text{and} \qquad v_\nu = v_\gamma = v\,. \tag{21.19}$$

We allow for the presence of baryon and CDM entropy perturbations. However, during the radiation-dominated epoch their effect on metric perturbations is negligible. Thus the evolution of metric perturbations are as if the perturbations were adiabatic.

In the simpler case discussed in Sec. (18), where we had $\Phi = \Psi$, we found that the growing adiabatic mode had $\Phi = \Psi = \text{const.}$ Guided by that, we now try the ansatz

$$\Phi = \text{const.} \quad \text{and} \quad \Psi = \text{const.} \tag{21.20}$$

and check that it is a solution. The Einstein and $\mathcal{R}$ equations are satisfied with

$$\delta_\nu = \delta_\gamma = \delta = -2\Phi = \text{const.} \tag{21.21}$$

$$v_\nu = v_\gamma = v = \frac{1}{2}\left(\frac{k}{\mathcal{H}}\right)\Phi = -\tfrac{1}{4}\left(\frac{k}{\mathcal{H}}\right)\delta = \tfrac{1}{2}k\eta\Phi \tag{21.22}$$

$$\Pi_\nu = \frac{1}{f_\nu}\left(\frac{k}{\mathcal{H}}\right)^2(\Psi-\Phi) \tag{21.23}$$

$$\mathcal{R} = -(\Psi + \tfrac{1}{2}\Phi) = \text{const.} \tag{21.24}$$

In the δ'_γ and δ'_ν equations we can now ignore the $(k/\mathcal{H})$ terms, and we see that they are also satisfied. Using $\mathcal{H} = 1/\eta$ the velocity equations become

$$\begin{aligned}\eta v'_\gamma &= \frac{1}{4}k\eta\delta_\gamma + k\eta\Phi \\ \eta v'_\nu &= \frac{1}{4}k\eta\delta_\nu - \frac{1}{6}k\eta\Pi_\nu + k\eta\Phi\,.\end{aligned}$$

Here the Π_ν term can be ignored since Π_ν is suppressed by $(k/\mathcal{H})^2$ compared to Ψ and Φ, and we then see that these equations are also satisfied.

We are still missing a piece of information that would tell us what $\Phi - \Psi$, or, in other words, what Π_ν is. The neutrino anisotropy Π_ν depends on the neutrino momentum distribution. Before neutrino decoupling, interactions kept $\Pi_\nu = 0$. After neutrinos decoupled, neutrinos have been "freely streaming", i.e., moving without collisions through the perturbed universe. In CMB Physics we derive a so-called Boltzmann hierarchy of equations that relates the evolution of the different moments of the neutrino momentum distribution to each other. Using these one can show that, starting from $\Pi_\nu = 0$, one obtains for superhorizon scales a "decreasing hierarchy", where the effect of the higher moments on lower moments can be ignored. The evolution of the "second moment" Π_ν depends then only on the "first moment" v_ν, and the relevant equation is

$$\mathcal{H}^{-1}\Pi'_\nu = \frac{8}{5}\left(\frac{k}{\mathcal{H}}\right)v_\nu\,. \tag{21.25}$$

This finally allows us to solve (**Exercise**):

$$\begin{aligned}\Psi &= (1 + \tfrac{2}{5}f_\nu)\Phi \approx 1.162\Phi & (21.26)\\ \Pi_\nu &= \frac{2}{5}(k\eta)^2\Phi & (21.27)\\ \mathcal{R} &= -\frac{3}{2}\left(1 + \frac{4}{15}f_\nu\right)\Phi = \text{const.} & (21.28)\\ \Phi &= -\frac{2}{3}\frac{1}{1+\frac{4}{15}f_\nu}\mathcal{R} \approx -0.6017\mathcal{R} & (21.29)\\ \Psi &= -\frac{2}{3}\frac{1+\frac{2}{5}}{1+\frac{4}{15}f_\nu}\mathcal{R} \approx -0.6992\mathcal{R}\,. & (21.30)\end{aligned}$$

(We see that the perfect fluid approximation, which gave $\Psi = \Phi = -\frac{2}{3}\mathcal{R}$, led to a 10% error in Φ and to a 5% error in Ψ.)

21.2 Matter

During the early radiation-dominated era, the metric and the radiation perturbations do not care about matter perturbations, but matter perturbations will become important later, and therefore we are interested in their "initial" behavior in the radiation-dominated era. The continuity equations for baryons and CDM are

$$\begin{aligned}
\mathcal{H}^{-1}\delta_c' + \left(\frac{k}{\mathcal{H}}\right) v_c - 3\mathcal{H}^{-1}\Psi' &= 0 \\
\mathcal{H}^{-1}\delta_b' + \left(\frac{k}{\mathcal{H}}\right) v_b - 3\mathcal{H}^{-1}\Psi' &= 0 \\
\mathcal{H}^{-1}v_c' + v_c - \left(\frac{k}{\mathcal{H}}\right)\Phi &= 0 \\
\mathcal{H}^{-1}v_b' + v_b - \left(\frac{k}{\mathcal{H}}\right)\Phi &= a n_e \sigma_T \frac{4\rho_\gamma}{3\rho_b}(v_\gamma - v_b)\,,
\end{aligned} \tag{21.31}$$

where the collision term in the last equation is derived in CMB Physics, σ_T is the Thomson cross section for photon-electron scattering, and n_e is the free electron number density. Well before photon decoupling, $a n_e \sigma_T (4\rho_\gamma)/(3\rho_b)$ is very large, and the collision term forces $v_b = v_\gamma$ (baryons are tightly coupled to photons).

For the above photon+neutrino adiabatic growing mode solution, these become

$$\begin{aligned}
\eta\delta_c' + k\eta v_c &= 0 \\
\eta\delta_b' + k\eta v_b &= 0 \\
\eta v_c' + v_c - k\eta\Phi &= 0 \\
\eta v_b' + v_b - k\eta\Phi &= a n_e \sigma_T \frac{4\rho_\gamma}{3\rho_b}(v - v_b)\,.
\end{aligned} \tag{21.32}$$

21.2.1 The Completely Adiabatic Solution

One solution for Eq. (21.32) is the completely adiabatic solution:

$$\begin{aligned}
\delta_c \;=\; \delta_b \;&=\; \tfrac{3}{4}\delta \;=\; -\tfrac{3}{2}\Phi \;=\; \text{const.} \\
v_c \;=\; v_b \;&=\; v \;=\; \tfrac{1}{2}k\eta\Phi\,.
\end{aligned} \tag{21.33}$$

To check this, substitute Eq. (21.33) into Eq. (21.32). This gives

$$0 + \tfrac{1}{2}(k\eta)^2\Phi \;=\; 0$$

for the δ equations, and

$$\tfrac{1}{2}k\eta\Phi + \tfrac{1}{2}k\eta\Phi - k\eta\Phi = 0\,,$$

so the equations are indeed satisfied. The δ equations are satisfied to accuracy $(k\eta)^2 \ll 1$, i.e., in a Hubble time, δ_i will deviate from its initial value $-\frac{3}{2}\Phi$ by about $-\frac{1}{2}(k\eta)^2\Phi$, a negligible change.

21.2.2 Baryon and CDM Entropy Perturbations

There are three independent entropy perturbations: the neutrino, baryon, and CDM entropy perturbations,

$$S_\nu \equiv \tfrac{3}{4}(\delta_\nu - \delta_\gamma) \qquad S_b \equiv \delta_b - \tfrac{3}{4}\delta_\gamma \qquad S_c \equiv \delta_c - \tfrac{3}{4}\delta_\gamma\,. \tag{21.34}$$

Their evolution equations are

$$S_\nu' = -k(v_\nu - v_\gamma) \qquad S_b' = -k(v_b - v_\gamma) \qquad S_c' = -k(v_c - v_\gamma)\,. \tag{21.35}$$

The relative entropy perturbation stays constant unless there is a corresponding relative velocity perturbation.

Entropy perturbations also tend to stay constant at superhorizon scales[27] , as

$$\mathcal{H}^{-1}S_i' = -\left(\frac{k}{\mathcal{H}}\right)(v_i - v_\gamma)\,. \tag{21.36}$$

Assume now the neutrino-adiabatic growing mode solution of Sec. 21.1. This assumes $S_\nu = 0$, but we may still have baryon and CDM entropy perturbations.

The baryon and neutrino density perturbations are

$$\delta_b = \tfrac{3}{4}\delta + S_b \qquad \text{and} \qquad \delta_c = \tfrac{3}{4}\delta + S_c\,. \tag{21.37}$$

Since baryons are tightly coupled to photons, $v_b = v_\gamma = v$, we have

$$S_b' = 0 \quad \Rightarrow \quad S_b = \text{const.} \quad \Rightarrow \quad \delta_b = \text{const.} \tag{21.38}$$

For CDM we do not have this constraint. Write

$$v_{\rm rel} \equiv v_c - v \quad \Rightarrow \quad v_c = \tfrac{1}{2}k\eta\Phi + v_{\rm rel}\,. \tag{21.39}$$

Eq. (21.32c) becomes

$$\begin{aligned} \eta\tfrac{1}{2}k\Phi + \eta v_{\rm rel}' + \tfrac{1}{2}k\eta\Phi + v_{\rm rel} - k\eta\Phi &= 0 \\ \Rightarrow \quad \eta v_{\rm rel}' &= -v_{\rm rel} \quad \Rightarrow \quad v_{\rm rel} \propto \eta^{-1}\,. \end{aligned}$$

Thus we have

$$v_c = \tfrac{1}{2}(k\eta)\Phi + C\eta^{-1}\,. \tag{21.40}$$

from which we identify a growing mode and a decaying mode. As time goes on, the decaying mode decays away, and $v_c \to v$. Ignoring the decaying mode, we have

$$v_c = v \quad \Rightarrow \quad S_c = \text{const.} \quad \Rightarrow \quad \delta_c = \text{const.} \tag{21.41}$$

Thus (assuming neutrino adiabaticity), the "initial conditions" at the early radiation-dominated epoch can be specified by giving three constants for each Fourier mode $\vec{k}$: $\Phi_k(\text{rad})$, $S_{c\vec{k}}(\text{rad})$, and $S_{b\vec{k}}(\text{rad})$. The general perturbation is a superposition of three modes, where two of these constants are zero:

$$\begin{aligned} (\Phi, S_c, S_b) &= (\Phi, 0, 0) && \text{adiabatic mode (AD)} \\ (\Phi, S_c, S_b) &= (0, S_c, 0) && \text{CDM isocurvature mode (CDI)} \\ (\Phi, S_c, S_b) &= (0, 0, S_b) && \text{baryon isocurvature mode (BI)}\,. \end{aligned} \tag{21.42}$$

In the following we shall use $\mathcal{R}$ instead of Φ (see Eq. 21.28) as the first initial value constant (since it is better in staying constant also later).

21.3 Neutrino perturbations

Perhaps I will do this someday ...

[27] This property is not as general as the constancy of $\mathcal{R}$ at superhorizon scales: The result (21.36) relies on two assumptions: 1) no interaction terms in the component energy continuity equations, and 2) the component fluids have a unique relation $p_i = p_i(\rho_i)$. Note also that this does not hold for the "total entropy perturbation" $\mathcal{S}$.

22 Superhorizon Evolution and Relation to Original Perturbations

See Section I3.2 of CMB Physics 2004.

23 Gaussian Initial Conditions

See Section I4 of CMB Physics 2004.

24 Large Scales

See Section I5 of CMB Physics 2004.

25 Sachs–Wolfe Effect

Consider photon travel in the perturbed universe. The geodesic equation is

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du} = 0\,, \tag{25.1}$$

where u is an affine parameter of the geodesic. For photons, we choose u so that the photon 4-momentum is

$$p^\mu = \frac{dx^\mu}{du}\,, \tag{25.2}$$

which allows us to write the geodesic equation as

$$\frac{dp^\mu}{du} + \Gamma^\mu_{\alpha\beta}p^\alpha p^\beta = 0\,. \tag{25.3}$$

Dividing by $p^0 = d\eta/du$, this becomes

$$\frac{dp^\mu}{d\eta} + \Gamma^\mu_{\alpha\beta}\frac{p^\alpha p^\beta}{p^0} = 0\,. \tag{25.4}$$

In the following, we need only the time component of this equation,

$$\frac{dp^0}{d\eta} + \Gamma^0_{00}p^0 + 2\Gamma^0_{0k}p^k + \Gamma^0_{ij}\frac{p^i p^j}{p^0} = 0\,. \tag{25.5}$$

Assuming scalar perturbations and using the Newtonian gauge (the $\Gamma^\mu_{\alpha\beta}$ from Eq. (8.6)), this becomes

$$\frac{dp^0}{d\eta} + \left(\mathcal{H} + \Phi'\right)p^0 + 2\Phi_{,k}p^k + \left[\mathcal{H} - 2\mathcal{H}(\Phi+\Psi) - \Psi'\right]\frac{\delta_{ij}p^i p^j}{p^0} = 0\,. \tag{25.6}$$

These 4-momentum components p^μ are in the coordinate frame. What the observer interprets as the photon energy and momentum are the components $p^{\hat\mu}$ in his local orthonormal frame. Since the metric is diagonal, the conversion is easy, $p^{\hat\mu} = \sqrt{|g_{\mu\mu}|}p^\mu$ (for a comoving observer):

$$\begin{aligned} E \equiv p^{\hat 0} &= a\sqrt{1+2\Phi}\,p^0 = a(1+\Phi)p^0 \\ p^{\hat i} &= a\sqrt{1-2\Psi}\,p^i = a(1-\Psi)p^i\,. \end{aligned} \tag{25.7}$$

Since photons are massless, $E^2 = \delta_{ij}p^{\hat i}p^{\hat j}$.

In the background universe, the photon energy redshifts as $\bar{E} \propto a^{-1} \Leftrightarrow a\bar{E} = \text{const}$. In the presence of perturbations, $q \equiv aE \neq \text{const}$. Thus we define q and $\vec{q}$,

$$\begin{aligned} q &\equiv aE = a^2(1+\Phi)p^0 \quad \Rightarrow \quad p^0 = a^{-2}(1-\Phi)q \\ q^i &\equiv ap^{\hat{i}} = a^2(1-\Psi)p^i \quad \Rightarrow \quad p^i = a^{-2}(1+\Psi)q^i \end{aligned} \tag{25.8}$$

where $q^2 = \delta_{ij}q^iq^j$, as suitable quantities to track the perturbation in the redshift.

Rewriting Eq. (25.6) in terms of q and $\vec{q}$ (and dropping 2nd order terms) gives (exercise)

$$(1-\Phi)\frac{dq}{d\eta} = q\frac{d\Phi}{d\eta} - q\Phi' - 2q^k\Phi_{,k} + q\Psi' \,. \tag{25.9}$$

Here the rhs is 1st order small, therefore $dq/d\eta$ is also 1st order small, and we can drop the factor $(1-\Phi)$. Dividing by q we get

$$\frac{1}{q}\frac{dq}{d\eta} = \frac{d\Phi}{d\eta} - \Phi' + \Psi' - 2\frac{\vec{q}\cdot\nabla\Phi}{q} \,. \tag{25.10}$$

Here the total derivative along the photon geodesic is

$$\frac{d}{d\eta} = \frac{\partial}{\partial\eta} + \frac{dx^k}{d\eta}\frac{\partial}{\partial x^k} \tag{25.11}$$

and

$$\frac{\vec{q}}{q} = \frac{(1-\Psi)p^k}{(1+\Phi)p^0} \approx \frac{p^k}{p^0} = \frac{dx^k}{d\eta} \qquad \text{to 0th order} \tag{25.12}$$

so that

$$-2\frac{\vec{q}\cdot\nabla\Phi}{q} = -2\frac{dx^k}{d\eta}\frac{\partial\Phi}{\partial x^k} = -2\left(\frac{d\Phi}{d\eta} - \frac{\partial\Phi}{\partial\eta}\right) , \tag{25.13}$$

so that Eq. (25.10) becomes

$$\frac{1}{q}\frac{dq}{d\eta} = -\frac{d\Phi}{d\eta} + \Phi' + \Psi' \tag{25.14}$$

The relative perturbation in the photon energy, $\delta E/\bar{E}$, that the photon accumulates when traveling from $\boldsymbol{x}_*$ to $\boldsymbol{x}_{\text{obs}}$ in the perturbed universe is thus

$$\begin{aligned} \frac{\delta E}{\bar{E}} &= \frac{\delta q}{\bar{q}} = \int\frac{dq}{q} = -\int d\Phi + \int\left(\Phi' + \Psi'\right)d\eta \\ &= \Phi(\boldsymbol{x}_*) - \Phi(\boldsymbol{x}_{\text{obs}}) + \int_{\eta_*}^{\eta_{\text{obs}}}\left(\frac{\partial\Phi}{\partial\eta} + \frac{\partial\Psi}{\partial\eta}\right)d\eta \,, \end{aligned} \tag{25.15}$$

where the integrals are along the photon path.

For a thermal distribution of photons, a uniform relative photon energy perturbation corresponds to a temperature perturbation of the same amount:

$$\left(\frac{\delta T}{T}\right)_{\text{jour}} = \frac{\delta E}{\bar{E}} \,. \tag{25.16}$$

Here "jour" refers to the temperature perturbation the photon distribution accumulates on the journey between $\boldsymbol{x}_*$ and $\boldsymbol{x}_{\text{obs}}$.

The other contributions to the observed CMB temperature anisotropy are due to the local photon energy density perturbation and photon velocity perturbation at the origin of the photon, on the last scattering surface:

$$\left(\frac{\delta T}{T}\right)_{\text{intr}} = \tfrac{1}{4}\delta_\gamma(\boldsymbol{x}_*) - \vec{v}(\boldsymbol{x}_*)\cdot\hat{n} \,, \tag{25.17}$$

where $\hat{n}$ is the direction the observer is looking at.

For a given observer, the $\Phi(\boldsymbol{x}_{\text{obs}})$ part is common to photons from all directions, and the observer interprets it as part of the mean (background) photon temperature. Thus the observed CMB temperature anisotropy is

$$\frac{\delta T}{T} = \tfrac{1}{4}\delta_\gamma^N(\boldsymbol{x}_*) - \vec{v}^N(\boldsymbol{x}_*)\cdot\hat{n} + \Phi(\boldsymbol{x}_*) + \int_{\eta_*}^{\eta_{\text{obs}}}\left(\frac{\partial\Phi}{\partial\eta} + \frac{\partial\Psi}{\partial\eta}\right)d\eta\,. \tag{25.18}$$

(There is also a contribution from the motion of the observer that causes a dipole pattern in the observed anisotropy. To get rid of that, the dipole of the observed anisotropy is subtracted away from the observations before any cosmological analysis.)

For large scales, much larger than the horizon size at photon decoupling, the Doppler effect $-\vec{v}^N(\boldsymbol{x}_*)\cdot\hat{n}$ is small compared to the other terms. The contribution

$$\left(\frac{\delta T}{T}\right)_{\text{SW}} = \tfrac{1}{4}\delta_\gamma^N(\boldsymbol{x}_*) + \Phi(\boldsymbol{x}_*) \tag{25.19}$$

is called the *ordinary Sachs–Wolfe effect*, and the contribution

$$\left(\frac{\delta T}{T}\right)_{\text{ISW}} = \int_{\eta_*}^{\eta_{\text{obs}}}\left(\frac{\partial\Phi}{\partial\eta} + \frac{\partial\Psi}{\partial\eta}\right)d\eta \tag{25.20}$$

is called the *integrated Sachs–Wolfe effect.*

For more, see Section I5.3 of CMB Physics 2004.

26 Matter Power Spectrum

See Chapter M of CMB Physics 2004, which is based on Chapter 7 of Dodelson[3]. Note that the multipole moments of the photon brightness function Θ_ℓ are related to the Θ_ℓ^m of CMB Physics 2007 and to perturbations of the photon energy tensor by

$$\Theta_0 \equiv \Theta_0^0 \equiv \tfrac{1}{4}\delta_\gamma \qquad \Theta_1 \equiv \tfrac{1}{3}\Theta_1^0 \equiv \tfrac{1}{3}v_\gamma\,. \tag{26.1}$$

A General Perturbation

From Eq. (3.8) we have that the general perturbed metric (around the flat Friedmann model) is

$$ds^2 = a^2(\eta)\left\{-(1+2A)d\eta^2 - 2B_i d\eta dx^i + \left[(1-2D)\delta_{ij} + 2E_{ij}\right]dx^i dx^j\right\} . \tag{A.1}$$

The Christoffel symbols are

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} + A' \\
\Gamma^0_{0i} &= -\mathcal{H}B_i + A_{,i} \\
\Gamma^0_{ij} &= \mathcal{H}\left[(1-2A-2D)\delta_{ij} + 2E_{ij}\right] + \tfrac{1}{2}(B_{i,j} + B_{j,i}) - \delta_{ij}D' + E'_{ij} \\
\Gamma^i_{00} &= -\mathcal{H}B_i - B'_i + A_{,i} \\
\Gamma^i_{0j} &= \mathcal{H}\delta_{ij} + \tfrac{1}{2}(B_{j,i} - B_{i,j}) - D'\delta_{ij} + E'_{ij} \\
\Gamma^i_{jk} &= \mathcal{H}\delta_{jk}B_i - \delta^i_j D_{,k} - \delta^i_k D_{,j} + \delta_{jk}D_{,i} + E_{ij,k} + E_{ik,j} - E_{jk,i} \, .
\end{aligned} \tag{A.2}$$

and we have the Christoffel sums

$$\begin{aligned}
\Gamma^\mu_{0\mu} &= 4\mathcal{H} + A' - 3D' \\
\Gamma^\mu_{i\mu} &= A_{,i} - 3D_{,i} \, .
\end{aligned} \tag{A.3}$$

Note that the Christoffel sums contain only scalar perturbations. Thus for vector and tensor perturbations, these sums contain only the background value $\Gamma^\mu_{0\mu} = 4\mathcal{H}$.

The Einstein tensor is

$$\begin{aligned}
G^0_0 &= -3a^{-2}\mathcal{H}^2 + a^{-2}\left[-2\nabla^2 D + 6\mathcal{H}D' + 6\mathcal{H}^2 A - 2\mathcal{H}B_{i,i} - E_{ik,ik}\right] \\
G^0_i &= a^{-2}\left[-2D'_{,i} - 2\mathcal{H}A_{,i} - \tfrac{1}{2}(B_{i,kk} - B_{k,ik}) - E'_{ik,k}\right] \\
G^i_0 &= a^{-2}\left[2D'_{,i} + 2\mathcal{H}A_{,i} + \tfrac{1}{2}(B_{i,kk} - B_{k,ik}) + 2\mathcal{H}'B_i - 2\mathcal{H}^2 B_i + E'_{ik,k}\right] \\
G^i_j &= a^{-2}\left(-2\mathcal{H}' - \mathcal{H}^2\right)\delta_{ij} \\
&\quad + a^{-2}\left[2D'' - \nabla^2(D-A) + \mathcal{H}(2A' + 4D') + (4\mathcal{H}' + 2\mathcal{H}^2)A - B'_{k,k} - 2\mathcal{H}B_{k,k} - E_{kl,kl}\right]\delta^i_j \\
&\quad + a^{-2}\left[(D-A)_{,ij} + \tfrac{1}{2}(B'_{i,j} + B'_{j,i}) + \mathcal{H}(B_{i,j} + B_{j,i}) + E''_{ij} - \nabla^2 E_{ij} + E_{ik,jk} + E_{jk,ik} + 2\mathcal{H}E'_{ij}\right]
\end{aligned} \tag{A.4}$$

For scalar perturbations, the perturbation in the Einstein tensor becomes

$$\begin{aligned}
\delta G^0_0 &= a^{-2}\left[-2\nabla^2\psi + 6\mathcal{H}D' + 6\mathcal{H}^2 A + 2\mathcal{H}\nabla^2 B\right] \\
\delta G^0_i &= a^{-2}\left[-2\psi'_{,i} - 2\mathcal{H}A_{,i}\right] \\
\delta G^i_0 &= a^{-2}\left[2\psi'_{,i} + 2\mathcal{H}A_{,i} - 2\mathcal{H}'B_{,i} + 2\mathcal{H}^2 B_{,i}\right] \\
\delta G^i_j &= a^{-2}\left[2D'' - \nabla^2(D-A) + \mathcal{H}(2A' + 4D') + (4\mathcal{H}' + 2\mathcal{H}^2)A + \nabla^2 B' + 2\mathcal{H}\nabla^2 B\right]\delta^i_j \\
&\quad + a^{-2}\left[-\tfrac{1}{3}\nabla^2(\nabla^2 E) - \tfrac{1}{3}\nabla^2 E'' - \tfrac{2}{3}\mathcal{H}\nabla^2 E'\right]\delta^i_j \\
&\quad + a^{-2}\left(D - A - B' - 2\mathcal{H}B + E'' + \tfrac{1}{3}\nabla^2 E + 2\mathcal{H}E'\right)_{,ij} .
\end{aligned} \tag{A.5}$$

The trace of the space part is

$$\delta G^i_i = a^{-2}\left[6D'' - 2\nabla^2\psi + 2\nabla^2 A + 3\mathcal{H}(2A' + 4D') + (12\mathcal{H}' + 6\mathcal{H}^2)A + 2\nabla^2 B' + 4\mathcal{H}\nabla^2 B\right] . \tag{A.6}$$

In Fourier space these read as

$$\begin{aligned}
\delta G^0_0 &= a^{-2}\left[2k^2\psi + 6\mathcal{H}D' + 6\mathcal{H}^2 A - 2\mathcal{H}kB\right] \\
\delta G^0_i &= a^{-2}\left[-2ik_i\psi' - 2i\mathcal{H}k_i A\right] \\
\delta G^i_0 &= a^{-2}\left[2ik_i\psi' + 2i\mathcal{H}k_i A - 2i\mathcal{H}'k_i B + 2i\mathcal{H}^2 k_i B\right] \\
\delta G^i_j &= a^{-2}\left[2D'' + k^2(D-A) + \mathcal{H}(2A' + 4D') + (4\mathcal{H}' + 2\mathcal{H}^2)A - kB' - 2\mathcal{H}kB\right]\delta^i_j \\
&\quad + a^{-2}\left[-\tfrac{1}{3}k^2 E + \tfrac{1}{3}E'' + \tfrac{2}{3}\mathcal{H}E'\right]\delta^i_j \\
&\quad - k_i k_j a^{-2}\left[D - A - \frac{1}{k}B' - 2\frac{\mathcal{H}}{k}B - \tfrac{1}{3}E + \frac{1}{k^2}(E'' + 2\mathcal{H}E')\right] \\
\delta G^i_i &= a^{-2}\left[6D'' + 2k^2\psi - 2k^2 A + \mathcal{H}(6A' + 12D') + (12\mathcal{H}' + 6\mathcal{H}^2)A - 2kB' - 4\mathcal{H}kB\right] .
\end{aligned} \tag{A.7}$$

The general perturbed energy tensor is

$$\begin{aligned} T^0_0 &= -\bar\rho - \delta\rho \\ T^0_i &= (\bar\rho + \bar p)(v_i - B_i) \\ T^i_0 &= -(\bar\rho + \bar p)v_i \\ T^i_j &= \bar p\delta^i_j + \delta p\delta^i_j + \bar p\Pi_{ij} \end{aligned} \tag{A.8}$$

The energy continuity equations

$$T^\mu_{\nu;\mu} \equiv T^\mu_{\nu,\mu} + \Gamma^\mu_{\alpha\nu}T^\alpha_\nu - \Gamma^\alpha_{\nu\mu}T^\mu_\alpha = 0 \tag{A.9}$$

become the background equation

$$\bar\rho' + 3\mathcal{H}(\bar\rho + \bar p) = 0 \tag{A.10}$$

and the fluid perturbation equations

$$\begin{aligned} \delta\rho' &= -3\mathcal{H}(\delta\rho + \delta p) + (\bar\rho + \bar p)(3D' - \nabla\cdot\vec v) \\ (\bar\rho + \bar p)(v_i - B_i)' &= -(\bar\rho + \bar p)'(v_i - B_i) - 4\mathcal{H}(\bar\rho + \bar p)(v_i - B_i) \\ &\quad -\delta p_{,i} - \bar p\Pi_{ij,j} - (\bar\rho + \bar p)A_{,i} \end{aligned} \tag{A.11}$$

For scalar perturbations, the fluid perturbation equations become

$$\begin{aligned} \delta\rho' &= -3\mathcal{H}(\delta\rho + \delta p) + (\bar\rho + \bar p)(3D' + \nabla^2 v) \\ (\bar\rho + \bar p)(v - B)' &= -(\bar\rho + \bar p)'(v - B) - 4\mathcal{H}(\bar\rho + \bar p)(v - B) \\ &\quad +\delta p + \tfrac{2}{3}\bar p\nabla^2\Pi + (\bar\rho + \bar p)A\,. \end{aligned} \tag{A.12}$$

Using $\delta \equiv \delta\rho/\bar\rho$ and background relations, these can be written

$$\begin{aligned} \delta' &= (1+w)\left(\nabla^2 v + 3D'\right) + 3\mathcal{H}\left(w\rho - \frac{\delta p}{\bar\rho}\right) \\ (v - B)' &= -\mathcal{H}(1 - 3c_s^2)(v - B) + \frac{\delta p}{\bar\rho + \bar p} + \tfrac{2}{3}\frac{w}{1+w}\nabla^2\Pi + A\,. \end{aligned} \tag{A.13}$$

The factor $\mathcal{H}(1 - 3c_s^2)$ can also be written as $(1 - 3w)\mathcal{H} + w'/(1+w)$.

In Fourier space these are

$$\begin{aligned} \delta\rho' &= -3\mathcal{H}\,(\delta\rho + \delta p) + (\bar\rho + \bar p)(3D' - kv) \\ (\bar\rho + \bar p)(v - B)' &= -(\bar\rho + \bar p)'(v - B) - 4\mathcal{H}(\bar\rho + \bar p)(v - B) \\ &\quad +k\delta p - \tfrac{2}{3}k\bar p\Pi + k(\bar\rho + \bar p)A \\ \delta' &= (1+w)(-kv + 3D') + 3\mathcal{H}\left(w\delta - \frac{\delta p}{\bar\rho}\right) \\ (v - B)' &= -\mathcal{H}(1 - 3w)(v - B) - \frac{w'}{1+w}(v - B) + \frac{k\delta p}{\bar\rho + \bar p} - \tfrac{2}{3}\frac{w}{1+w}k\Pi + kA\,. \end{aligned} \tag{A.14}$$

References

[1] A.R. Liddle and D.H. Lyth: Cosmological Inflation and Large-Scale Structure (Cambridge University Press 2000).

[2] C.-P. Ma and E. Bertschinger, ApJ 455, 7 (1995).

[3] S. Dodelson: Modern Cosmology (Academic Press 2003).