12 Michael's Selection Theorem *Let X be a fully normal space, Y a Banach space and $\varphi : X \to \mathcal{P}(Y)$ an lsc carrier such that every $\varphi(x)$ is convex and closed. Then φ has a continuous selection.*

Proof. We denote by d the norm distance in Y. By induction on n, we define a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of continuous mappings $X \to Y$ such that, for all $n \in \mathbb{N}$ and $x \in X$, we have that $d(f_n(x), \varphi(x)) < 2^{-n-1}$ and, if $n > 1$, then $d(f_n(x), f_{n-1}(x)) \leq 2^{-n+1}$.

By Lemma 11, there exists a continuous mapping $f_1 : X \to Y$ such that we have $d(f(x), \varphi(x)) < \frac{1}{4}$ for every $x \in X$. Assume that $n > 1$ and that f_{n-1} has already been defined. To define f_n, we first note that the formula $\theta(x) = \varphi(x) \cap B_d(f_{n-1}(x), 2^{-n})$ defines a convex-valued carrier $\theta : X \to \mathcal{P}(Y)$. By Lemma 10 (and Example 9(a)), the carrier θ is lsc. By Lemma 11, there exists a continuous mapping $f_n : X \to Y$ such that we have $d(f_n(x), \theta(x)) < 2^{-n-1}$ for every $x \in X$. The definition of θ shows that f_n satisfies both required conditions. This completes the induction.

By completeness of Y, there exists a function $f : X \to Y$ such that $f_n(x) \to f(x)$ for every $x \in X$. The convergence is uniform, and hence the mapping f is continuous. For every $x \in X$, we have that $d(f(x), \varphi(x)) = 0$, and it follows, since $\varphi(x) \Subset Y$, that $f(x) \in \varphi(x)$. As a consequence, f is a continuous selection of φ.□

We shall now exhibit some consequence's of Michael's Theorem.

13 Corollary *Let X be a fully normal space, let $F \Subset X$ and let Y be a Banach space and f a continuous mapping $F \to Y$. Then f can be extended to a continuous mapping $X \to H$, where H is the closed convex hull of $f(F)$.*

Proof. We define $\varphi : X \to \mathcal{P}(Y)$ be setting $\varphi(x) = f\{x\}$ for $x \in F$ and $\varphi(x) = H$ for $x \in X \smallsetminus F$. For every $G \circledcirc Y$ with $G \cap H \neq \emptyset$, we have that $\{x \in X : \varphi(x) \cap G \neq \emptyset\} = (X \smallsetminus F) \cup f^{-1}(G)$ and this set is open in X since f is continuous. As a consequence, φ is lsc, and it follows by Theorem 12 that φ has a continuous selection g. Then g is a continuous mapping $X \to H$ and $f \subset g$. □

The above result is a significant extension (for fully normal spaces) of the Tietze-Urysohn Extension Theorem. We give two examples of the use of this result.

14 Corollary *Let X be a fully normal space. Then every continuous pseudometric of a closed subspace of X can be extended to a continuous pseudometric of X.*

Proof. Let $F \Subset X$, and let d be a continuous pseudometric of F. Let $x_0 \in F$. For every $x \in F$, define $f_x : F \to \mathbb{R}$ by the formula $f_x(y) = d(y, x) - d(y, x_0)$. As was noted after the proof of Theorem 2, the rule $h(x) = f_x$ defines an isometric mapping $h : (F, d) \to (\ell_F^\infty, \rho)$, where ρ denotes sup-norm distance in ℓ_F^∞. By Corollary 13, the mapping h has a continuous extension $\bar{h} : X \to \ell_F^\infty$. Now the formula $\bar{d}(x, y) = \rho(\bar{h}(x), \bar{h}(y))$ defines a continuous pseudometric $\bar{d}$ of X which extends the pseudometric d. □

A *retraction* is a continuous mapping $f : X \to X$ such that we have $f(f(x)) = f(x)$ for every $x \in X$. A subset A of X is a *retract* of X provided there exists a retraction $f : X \to X$ such that $f(X) = A$; in this situation, we have that $A = \{x \in X : f(x) = x\}$. It follows that every retract of a Hausdorff space is a closed subset.

Even in a Euclidean space, a closed subset may fail to be a retract. A well-known example of this is the closed subset $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ of $\mathbb{R}^2$. However, the following result obtains.

15 Corollary *In a Banach space, every closed and convex subset is a retract.*

Proof. Let H be a closed convex subset of a Banach space Y. Denote by f the identity mapping on H. By Theorem 1.7, the metric space Y is fully normal. It follows, by Corollary 13, that the mapping f has a continuous extension $g : Y \to H$. The mapping g is a retraction and $g(Y) = H$. □

Next we use the selection theorem to show that every closed linear subspace of a Banach space has a "topological complement".

16 Corollary *(Bartle-Graves) Let F be a closed linear subspace of a Banach space Y, and let ϕ be the quotient mapping $Y \to Y/F$. Then there exists $Z \subset Y$ such that the restriction of ϕ to Z is a homeomorphism $Z \to Y/F$.*

Proof. The mapping ϕ is continuous, and the Open Mapping Theorem shows that ϕ is an open map. It follows that the carrier $\varphi : Y/F \to \mathcal{P}(Y)$, defined by $\varphi(\bar{y}) = \phi^{-1}\{\bar{y}\}$, is lsc (see Exercise 5/5). Moreover, for every $\bar{y} = \phi(y) \in Y/F$, we have that $\varphi(\bar{y}) = y + F$, and hence the set $\varphi(\bar{y})$ is closed and convex. By Michael's Theorem, the carrier φ has a continuous selection $f : Y/F \to Y$. Let $Z = f(Y/F)$, and denote the restriction of ϕ to Z by ψ. Then ψ is a homeomorphism $Z \to Y/F$, because we have that $f(\psi(y)) = y$ for each $y \in Z$ and $\psi(f(\bar{y})) = \bar{y}$ for every $\bar{y} \in Y/F$. □

Sometimes the above result is stated by saying that there exists a continuous "lifting" for the quotient mapping.

For a Banach space of type $C(K)$ (equipped with the supremum-norm), we have the following special case of the Bartle-Graves Theorem.

17 Corollary *Let K be a compact Hausdorff space, and let $F \Subset K$. Then, to each $f \in C(F)$, we can assign an extension $\bar{f} \in C(K)$ in such a way that the mapping $f \mapsto \bar{f}$ is an embedding of $C(F)$ into $C(K)$.*

Proof. The formula $R(f) = f|F$ defines a bounded linear mapping $R : C(K) \to C(F)$. Since $F \Subset K$, the mapping R is onto. Denote by H the kernel $\{f \in C(K) : f|F \equiv 0\}$ of R. Then the formula $\psi(f) = T^{-1}(f)$ defines an isomorphism $C(F) \to C(K)/H$.

Let ϕ be the quotient mapping $C(K) \to C(K)/H$. By Corollary 16, there exists $Z \subset C(K)$ such that the mapping $\theta = \phi|Z$ is a homeomorphism $Z \to C(K)/H$. Now the mapping $\theta \circ \psi$ is an embedding $C(F) \to C(K)$ and for every $f \in C(F)$, the function $\bar{f} = \theta(\psi(f))$ belongs to the set $R^{-1}\{f\}$ and hence $\bar{f}$ extends f. □

We say that the mapping $f \mapsto \bar{f}$ above is a *continuous extender* for F in K.

We shall next show that quotient mappings do *not* always have continuous *linear* liftings and that closed subspaces of compact Hausdorff spaces do *not* always admit *linear* continuous extenders.

A subset A of a normed space is *bounded* if the set $\{\|a\| : a \in A\}$ is bounded in $\mathbb{R}$.

18 Lemma *Let E be a Banach space and let A be a subset of E such that the set* $\{a_1 + \cdots + a_n : n \in \mathbb{N}$ and $a_1, ..., a_n \in A$ are distinct$\}$ *is bounded. Then, for every $\phi \in E^*$, the set $\{a \in A : \varphi(a) \neq 0\}$ is countable.*

Proof. Assume on the contrary that there exists a functional $\varphi \in E^*$ such that the set $B = \{a \in A : \varphi(a) \neq 0\}$ is uncountable. Then there exists $r > 0$ such that the set $C = \{a \in A : |\varphi(a)| \geq r\}$ is infinite. At least one of the sets $C_+ = \{a \in A : \varphi(a) \geq r\}$ or $C_- = \{a \in A : \varphi(a) \leq -r\}$ is infinite. By replacing φ with $-\varphi$ if necessary, we can assume that the set C_+ is infinite. Now, for all distinct elements $a_1, ..., a_n$ of C_+ we have that $\varphi(a_1 + \cdots + a_n) = \varphi(a_1) + \cdots + \varphi(a_n) \geq nr$. This leads to a contradiction, since the functional φ and the set $\{a_1 + \cdots + a_n : n \in \mathbb{N}$ and $a_1, ..., a_n \in C_+$ are distinct$\}$ are bounded. □

19 Proposition *Let E be a Banach space which has an uncountable subset A such that the set* $\{a_1 + \cdots + a_n : n \in \mathbb{N}$ and $a_1, ..., a_n \in A$ are distinct$\}$ *is bounded. Then there is no bounded linear* 1-1 *mapping $E \to \ell^\infty$.*

Proof. Let $T : E \to \ell^\infty$ be a bounded linear mapping. We show that T is not 1-1. For every $n \in \mathbb{N}$, denote by e_n the bounded linear functional $\langle x_i \rangle \mapsto x_n$ of ℓ^∞, and denote by φ_n the mapping $e_n \circ T$; note that $\varphi_n \in C(F)^*$. It follows from Lemma 18 that, for every

$n \in \mathbb{N}$, the set $A_n = \{a \in A : \varphi_n(a) \neq 0\}$ is countable. Since A is uncountable, there exists $a \in A$ such that $a \neq \mathbf{0}$ and $a \notin \bigcup_{n\in\mathbb{N}} A_n$. For every $n \in \mathbb{N}$, since $a \notin A_n$, we have that $T(a)_n = e_n(T(a)) = \varphi_n(a) = 0$. As a consequence, we have that $T(a) = \mathbf{0}$. It follows, since $a \neq \mathbf{0}$, that T is not 1-1. $\square$

20 Corollary *(Phillips) There exists no bounded linear* 1-1 *mapping* $\ell^\infty/c_0 \to \ell^\infty$.

Proof. We can represent the Banach space ℓ^∞ as $C(\beta\mathbb{N})$ (see Example II.4.19). In this representation, the subspace c_0 of ℓ^∞ corresponds to the subspace $F = \{f \in C(\beta\mathbb{N}) : f|N^* \equiv 0\}$ of $C(\beta\mathbb{N})$ (recall that $\mathbb{N}^*$ is the remainder $\beta\mathbb{N} \setminus \mathbb{N}$). Since $\mathbb{N}^* \Subset \beta\mathbb{N}$, we see as in the proof of Corollary 17, that the quotient space $C(\beta\mathbb{N})/F$ is isomorphic with $C(\mathbb{N}^*)$.

By the foregoing, it suffices to show that there exists no bounded linear 1-1 mapping $C(\mathbb{N}^*) \to \ell^\infty$. By Problem 2 of Exercise set 2, there exists an uncountable family $\mathcal{H}$ of subsets of $\mathbb{N}$ such that we have $|H \cap J| < \omega$ for any two distinct $H, J \in \mathcal{H}$. For every $H \in \mathcal{H}$, let $H^* = \check{H} \setminus \mathbb{N} = \{\mathcal{U} \in \mathbb{N}^* : H \in \mathcal{U}\}$. Then the family $\mathcal{H}^* = \{H^* : H \in \mathcal{H}\}$ is uncountable, disjoint and consists of clopen subsets of $\mathbb{N}^*$. The uncountable subset $A = \{\chi_G : G \in \mathcal{H}^*\}$ of $C(\mathbb{N}^*)$ satisfies the condition mentioned in Proposition 19, and the proposition shows that there exists no bounded linear 1-1 mapping $C(\mathbb{N}^*) \to \ell^\infty$. $\square$

The above proof shows that we also have an example of the non-existence of a continuous linear extender.

21 Corollary *There exists no continuous linear extender for the closed subset* $\mathbb{N}^*$ *of* $\beta\mathbb{N}$.

We mention another way to interpret Phillips' Theorem. Let H be a closed subspace of a Banach space Y. Then the existence of a continuous and linear lifting for the quotient mapping $Y \to Y/H$ is equivalent with the condition that H is *complemented* in Y, in other words, with the existence of a closed linear subspace J of Y such that $J \cap H = \{\mathbf{0}\}$ and $J+H = Y$. Corollary 20 shows that c_0 is not complemented in ℓ^∞. This is in contrast with Sobczyk's Theorem which says that, in a separable Banach space, any subspace isomorphic with c_0 is complemented.

The proof of Corollary 20 shows that there is no continuous linear 1-1 mapping $C(\mathbb{N}^*) \to \mathbb{C}(\beta\mathbb{N})$. This is a particular case of a more general result.

Recall that a topological space satisfies the *countable chain condition (ccc)* if every disjoint family of open subsets of X is countable.

22 Proposition *Let K and L be compact Hausdorff space. Assume that K is separable and there exists a* 1-1 *bounded linear mapping $C(L) \to C(K)$. Then L is ccc.*

Proof. Exercise. $\square$

IV Paracompact spaces

In this chapter, we consider the covering property of paracompactness. We start by establishing A.H. Stone's Coincidence Theorem according to which paracompactness and full normality are mutually equivalent properties in a Hausdorff space. This theorem shows that many of the results in the preceding chapter can be stated as results on paracompact spaces. We provide many characterizations for paracompactness, and we apply our results to study metrizability and the behaviour of normality in products.

1. Definition and basic properties.

1 Definition A topological space is *paracompact* if every open cover of the space has a locally finite open refinement.

Every compact space is paracompact. In studying compactness, it is often useful to restrict the study to Hausdorff spaces, because every compact Hausdorff space is normal. We shall show that the corresponding result holds for paracompact spaces. First we establish a property of locally finite families.

A family $\mathcal{L}$ of subsets of a space X is *closure-preserving* provided that we have $\overline{\bigcup_{N\in\mathcal{N}} N} = \bigcup_{N\in\mathcal{N}} \overline{N}$ for every $\mathcal{N} \subset \mathcal{L}$. Note that a closed family is closure-preserving iff the union of any subfamily is closed.

2 Lemma *Every locally finite family is closure-preserving.*

Proof. Exercise. □

3 Proposition *A paracompact Hausdorff space is normal.*

Proof. Let X be a paracompact Hausdorff space. To show that X is normal, we first show that X is regular. Let $F \subset X$ and $x \in X \setminus F$. Since X is Hausdorff there exists, for every $z \in F$, an open nbhd U_z of z in X such that $x \notin \overline{U_z}$. The family $\mathcal{U} = \{U_z : z \in F\} \cup \{X \setminus F\}$ is an open cover X. Let $\mathcal{V}$ be a locally finite open refinement of $\mathcal{U}$. For every $V \in (\mathcal{V})_F$, there exists $z \in F$ such that $V \subset U_z$, and hence we have that $\overline{V} \subset \overline{U}_z \subset X \setminus \{x\}$. By Lemma 2, we have that $\overline{\mathrm{St}(F,\mathcal{V})} = \overline{\bigcup(\mathcal{V})_F} = \bigcup\{\overline{V} : V \in (\mathcal{V})_F\} \subset X \setminus \{x\}$. By the foregoing, the sets $\mathrm{St}(F,\mathcal{V})$ and $X \setminus \overline{\mathrm{St}(F,\mathcal{V})}$ are disjoint open nbhds of F and x, respectively. We have shown that X is regular.

To show that X is normal, let $F \Subset X$ and $H \Subset X$ be disjoint. By the first part of this proof, for every $x \in F$, there exists an open nbhd G_x of x in X such that $\overline{G_x} \cap H = \emptyset$.

Let $\mathcal{V}$ be a locally finite open refinement of the open cover $\mathcal{G} = \{G_x : x \in F\} \cup \{X \setminus F\}$ of X. A similar argument as above shows that now the sets $\mathrm{St}(F, \mathcal{V})$ and $X \setminus \overline{\mathrm{St}(F, \mathcal{V})}$ are disjoint open nbhds of F and H, respectively. We have shown that X is normal. □

4 A.H. Stone's Coincidence Theorem *A Hausdorff space is paracompact iff the space is fully normal.*

Proof. *Necessity.* Let X be a paracompact Hausdorff space. By Proposition 3, X is normal. It follows by Proposition III.1.13 that every open cover of X is d-uniform for some continuous pseudometric d of X, and it follows further, by Theorem III.1.8, that X is fully normal.
Sufficiency. Let X be a fully normal space. By Theorem III.3.4 and Proposition III.3.7, every open cover of X has a subordinated locally finitely supported partition of unity. As a consequence, X is paracompact. □

We can now restate Theorem III.1.7 in the following form.

5 Corollary *Every pseudometrizable space is paracompact.*

Another important class of paracompact spaces is indicated in the following result.

6 Theorem *Every regular Lindelöf space is paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of a regular Lindelöf space X. By regularity, X has an open cover $\mathcal{V}$ such that the family $\{\overline{V} : V \in \mathcal{V}\}$ refines $\mathcal{U}$. Since X is Lindelöf, $\mathcal{V}$ has a countable subcover $\{V_1, V_2, ...\}$. For every $n \in \mathbb{N}$, let $U_n \in \mathcal{U}$ be such that $\overline{V_n} \subset U_n$. For every $n \in \mathbb{N}$, set $W_n = U_n \setminus \bigcup_{i<n} \overline{V_n}$. It is easy to see that the family $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ is an open refinement of $\mathcal{U}$. Moreover, $\mathcal{W}$ is locally finite, since the open sets V_K, $k \in \mathbb{N}$ cover X, and we have that $W_n \cap V_k = \emptyset$ whenever $n > k$. □

An uncountable discrete space shows that a paracompact space may fail to be Lindelöf. Nevertheless, as we will show in the following, there are several partial converses to the above result.

7 Lemma *(a) In a compact space, every locally finite family of subsets is finite.*
(b) In a Lindelöf space, every locally finite family of subsets is countable.

Proof. Exercise. □

Since a dense subset of a space intersects every non-empty open subset, it follows from (b) above that if X has a dense Lindelöf-subspace, then every locally finite family of open subsets of X is countable. As a consequence, we have the following result.

8 Proposition *If a paracompact space has a dense Lindelöf-subspace, then the space is Lindelöf.*

9 Corollary *Every separable paracompact space is Lindelöf.*

The result of the corollary also follows the observation that, in a separable space, every point-finite family of open sets is countable.

A space X is ω_1-*compact* provided that every closed and discrete subset of the space is countable. Since closed subspaces of Lindelöf spaces are Lindelöf, every Lindelöf space is ω_1-compact. We shall next show that the converse result holds in the class of paracompact T_1-spaces: every ω_1-compact paracompact T_1-space is Lindelöf. To prove this, we need some auxiliary results.

We say that a family $\mathcal{L}$ of sets is *irreducible* provided that we have $\bigcup\mathcal{N} \subsetneqq \bigcup\mathcal{L}$ for every $\mathcal{N} \subsetneqq \mathcal{L}$.

10 Lemma *Every point-finite cover contains an irreducible subcover.*

Proof. Let $\mathcal{L}$ be a point-finite cover of a set E. Let $\mathcal{H} = \{\mathcal{K} \subset \mathcal{L} : \mathcal{K} \text{ covers } E\}$. By the Hausdorff Maximality Principle, there exists a maximal chain $\mathcal{J}$ in the partially ordered set $(\mathcal{H}, \subset)$. We show that the family $\mathcal{M} = \bigcap\{\mathcal{K} : \mathcal{K} \in \mathcal{J}\}$ is an irreducible cover of E.

To show that $\mathcal{M}$ covers E, let $x \in E$. Then $\{(\mathcal{K})_x : \mathcal{K} \in \mathcal{J}\}$ is a $\subset$-chain of non-empty finite families, and hence the family $\mathcal{A} = \bigcap\{(\mathcal{K})_x : \mathcal{K} \in \mathcal{J}\}$ is non-empty. Let $L \in \mathcal{A}$. Then we have that $x \in L \in \bigcap\{\mathcal{K} : \mathcal{K} \in \mathcal{J}\} = \mathcal{M}$. We have shown that $\mathcal{M}$ covers E.

Since $\mathcal{M}$ covers E, we have that $\mathcal{M} \in \mathcal{H}$, and hence $\mathcal{M}$ is a $\subset$-minimal element of $\mathcal{H}$. It follows that $\mathcal{M}$ is irreducible: for every $L \in \mathcal{M}$, minimality of $\mathcal{M}$ shows that $\mathcal{M} \setminus \{L\} \notin \mathcal{H}$; hence the family $\mathcal{M} \setminus \{L\}$ does not cover E. □

11 Lemma *Let $\mathcal{U}$ be an open cover of a T_1-space X, let $\mathcal{V} \subset \mathcal{U}$, and for every $V \in \mathcal{V}$, let $x_V \in V \setminus \bigcup(\mathcal{U} \setminus \{V\})$. Then the subset $\{x_V : V \in \mathcal{V}\}$ of X is closed and discrete.*

Proof. Exercise. □

12 Proposition *Every ω_1-compact paracompact T_1-space is Lindelöf.*

Proof. It suffices to show that, in an ω_1-compact T_1-space X, every locally finite open cover $\mathcal{U}$ has a countable subcover. This follows easily from the previous results. By Lemma 10, $\mathcal{U}$ has an irreducible subcover $\mathcal{V}$. For every $V \in \mathcal{V}$, there exists a point $x_V \in V \setminus \bigcup(\mathcal{V} \setminus \{V\})$. By Lemma 11, the subset $F = \{x_V : V \in \mathcal{V}\}$ of X is closed and discrete. As a consequence, the set F is countable. It follows, since we clearly have that $x_V \neq x_W$ for any two distinct members V and W of $\mathcal{V}$, that the family $\mathcal{V}$ is countable. □

Recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. Note that a countably compact Lindelöf space is compact.

13 Corollary *Every countably compact paracompact T_1-space is compact.*

Proof. This follows from Proposition 12 and the result that a T_1-space X is countably compact iff every closed discrete subspace of X is finite (we leave the verification of this result as an exercise). □

We shall exhibit one more sufficient condition for a paracompact space to be Lindelöf.

14 Lemma *Let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ has a disjoint open partial refinement $\mathcal{G}$ such that the set $\bigcup\mathcal{G}$ is dense in X.*

Proof. Denote by $\mathcal{H}$ the collection of all disjoint open partial refinements of $\mathcal{U}$. For any chain $\mathcal{J}$ in $(\mathcal{H}, \subset)$, we have that $\bigcup\mathcal{J} \in \mathcal{H}$ and hence $\bigcup\mathcal{J}$ is an upper bound of $\mathcal{J}$ in $(\mathcal{H}, \subset)$. It follows by Zorn's Lemma that $\mathcal{H}$ contains an $\subset$-maximal member $\mathcal{G}$. To see that $\bigcup\mathcal{G}$ is dense, assume that there exists $x \in X \setminus \overline{\bigcup\mathcal{G}}$. Let $U \in \mathcal{U}$ be such that $x \in U$, and set $V = U \setminus \overline{\bigcup\mathcal{G}}$. Now $\mathcal{G} \cup \{V\}$ is a disjoint open partial refinement of $\mathcal{U}$, but this contradicts the maximality of $\mathcal{G}$. As a consequence, $\bigcup\mathcal{G}$ is dense in X. □

15 Lemma *Every locally finite family of open subsets of a ccc-space is countable.*

Proof. Let X be ccc, and let $\mathcal{U}$ be a locally finite family of open subsets of X. Every point $x \in X$ has an open nbhd V_x such that the family $(\mathcal{U})_{V_x}$ is finite. By Lemma 13, there exists a disjoint open partial refinement $\mathcal{G}$ of the open cover $\{V_x : x \in X\}$ of X such that the set $\bigcup\mathcal{G}$ is dense in X. Since X is ccc, the family $\mathcal{G}$ is countable. Since $\bigcup\mathcal{G}$ is dense in X, every member of the open family $\mathcal{U}$ meets some set of $\mathcal{G}$. As a consequence, we have that $\bigcup\{(\mathcal{U})_G : G \in \mathcal{G}\} = \mathcal{U}$. Moreover, for every $G \in \mathcal{G}$, there exists $x \in X$ such that $G \subset V_x$, and it follows that the family $(\mathcal{U})_G$ is contained in the finite family $(\mathcal{U})_{V_x}$. It follows, since $\bigcup\{(\mathcal{U})_G : G \in \mathcal{G}\} = \mathcal{U}$ and $\mathcal{G}$ is countable, that $\mathcal{U}$ is countable. □

Lemma 15 has the following consequence, which strengthens the result of Corollary 9.

16 Proposition *Every paracompact ccc-space is Lindelöf.*